ANNULI IN GENERALIZED HEEGAARD SPLITTINGS AND DEGENERATION OF TUNNEL NUMBER

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ABSTRACT. We analyze how a family of essential annuli in a compact 3-manifold will induce, from a strongly irreducible generalized Heegaard splitting of the ambient manifold, generalized Heegaard splittings of the complementary components. There are specific applications to the subadditivity of tunnel number of knots, improving somewhat bounds of Kowng [Kw]. For example, in the absence of 2-bridge summands, the tunnel number of the sum of n knots is no less than $\frac{2}{5}$ the sum of the tunnel numbers.

1. INTRODUCTION

The *tunnel number* $t(K)$ of a knot (or link) K in S^3 is the minimal number of arcs that, when attached to the knot, gives a graph whose complement is the interior of a handlebody. It was once naively hoped that this knot invariant might be additive under connected sum of knots, but Morimoto ([M2]) has found counterexamples in which tunnel number *degenerates* under connected sum. That is, there are knots K^1 and K^2 for which $t(K^1 \# K^2) < t(K^1) + t(K^2)$. The degree of degeneration possible was soon shown by Kobayashi ([Ko]) to be arbitrarily high. That is, given $d \geq 0$ there exist knots K^1 and K^2 for which $t(K^1) + t(K^2) - t(K^1 \# K^2) > d$.

Although Kobayashi's examples show that there is no uniform bound on the degeneration number, it is still natural to ask if there is a uniform bound less than 1 on the *degeneration ratio*

$$d(K^1, K^2) = \frac{t(K^1) + t(K^2) - t(K^1 \# K^2)}{t(K^1) + t(K^2)}.$$

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More generally, for knot summands $K^1, \ldots, K^n$ we can define the degeneration ratio

$$d(K^1, \ldots, K^n) = \frac{t(K^1) + \ldots + t(K^n) - t(K^1 \# \ldots \# K^n)}{t(K^1) + \ldots + t(K^n)}.$$

Kobayashi's methods show that the degeneration ratio for each of his examples is at least $\frac{1}{9}$ and Morimoto's specific example has degeneration ratio $\frac{1}{3}$. If we also consider links, Morimoto has shown [M1] that the connected sum of a knot K and a 2-component link L has tunnel number one exactly if K is a 2-bridge knot and L is the Hopf link. Such examples have degeneration ratio $\frac{1}{2}$.

In the other direction, Kowng showed in [Kw] the more general claim that if an irreducible orientable compact 3-manifold M contains a collection of tori $\mathcal{T}$, then M has a generalized Heegaard splitting of genus at most $3(genus(M) + |\mathcal{T}|) - 2$ for which each component of $\mathcal{T}$ is contained in a thin level (see Definition 2.9). When applied to $M = S^3 - \eta(K^1 \# \ldots \# K^n)$, this has the corollary that

$$\frac{1}{3}(t(K^1) + \ldots + t(K^n)) \leq t(K^1 \# \ldots \# K^n) + (n-1).$$

So Kowng showed that, in general,

$$d(K^1, \ldots, K^n) \leq \frac{2}{3} + \frac{n-1}{t(K^1) + \ldots + t(K^n)}.$$

Here we improve this aspect of Kowng's results, using a somewhat different approach growing out of the investigations in [SS1], [SS2]. The Main Theorem shows how to cut a strongly irreducible generalized Heegaard splitting of M along a family of essential annuli and create generalized Heegaard splittings for the resulting manifolds. Analysis of the result shows a bit more than the following: for K^j prime, $d(K^1, \ldots, K^n) \leq \frac{2}{3}$; if none of the K^j is 2-bridge, then $d(K^1, \ldots, K^n) \leq \frac{3}{5}$.

2-bridge knots seem to play a special role in the theory of tunnel numbers. Morimoto's original example of degeneration used a 2-bridge summand, and Kobayashi's (non-prime) examples each have for one of their summands a sum of 2-bridge knots. So it is intriguing that here too the bounds on degeneration are better if there are no 2-bridge knots among the summands.

Although we focus on cutting a manifold apart by annuli, the same sort of analysis extends to families of tori, in a way we only briefly describe.

After some preliminaries and a simplified but indicative special case, the outline of the proof is as follows:

Step 1 (Section 4): We examine annuli in, and on the boundary of, compression bodies. The goal is to develop criteria that allow us to cut open compression bodies along annuli and glue compression bodies together along annuli so that at the end, the result is still a collection of compression bodies.

Step 2 (Section 5): We prove that a strongly irreducible generalized Heegaard splitting can be cut along annuli and augmented in such a way as to yield generalized Heegaard splittings for the complementary components. In this construction a bound on the degeneration of the "index" is given by the number of "dipping annuli".

Step 3 (Section 6): We show that the generalized Heegaard splittings of the complementary components obtained in this way can be destabilized under certain conditions.

Step 4 (Sections 7 and 8): We find bounds on the number of dipping annuli. Section 7 is quite general, whereas Section 8 achieves better bounds in the context of a knot complement, when we can assume any surface with meridinal boundary has an even number of boundary components.

Step 5 (Section 9): We apply the derived inequalities to the study of tunnel numbers.

The paper concludes with an appendix by Andrew Casson. He constructs an example which demonstrates that the bound on the number of dipping annuli given in Section 7 is, in some sense, best possible.

We would like to thank Kanji Morimoto for his careful reading and useful comments.

2. Preliminaries

For standard definitions concerning knots, see [BZ] or [R] and for those concerning 3-manifolds, see [H] or [J]. All manifolds will be orientable.

Definition 2.1. *For N a properly embedded submanifold of M, we denote an open regular neighborhood of N in M by $\eta(N)$.*

Definition 2.2. *Let K be a knot in S^3. Denote the complement of K, $S^3 - \eta(K)$, by $C(K)$.*

Remark 2.3. *Let $K = K^1 \# K^2$ be the sum of two knots. Then the decomposing sphere gives rise to a decomposing annulus A properly embedded in $C(K)$ such that $C(K) = C(K^1) \cup_A C(K^2)$. If $K = K^1 \# \dots \# K^n$, then we may assume that the decomposing spheres are nested, so that $C(K) = C(K^1) \cup_{A_1} \cdots \cup_{A_{n-1}} C(K^n)$.*

Definition 2.4. *A tunnel system for a knot K is a collection of disjoint arcs $T = t_1 \cup \cdots \cup t_n$ properly embedded in $C(K)$ such that $C(K) - \eta(T)$ is a* *handlebody. The tunnel number of K, denoted by* $t(K)$, *is the least number of arcs required in a tunnel system for K.*

Definition 2.5. *A compression body is a 3-manifold W obtained from a connected closed orientable surface S by attaching 2-handles to* $S \times \{0\} \subset S \times I$ *and capping off any resulting 2-sphere boundary components. We denote* $S \times \{1\}$ *by* $\partial_+ W$ *and* $\partial W - \partial_+ W$ *by* $\partial_- W$. *Dually, a compression body is a connected orientable 3-manifold obtained from a (not necessarily connected) closed orientable surface* $\partial_- W \times I$ *by attaching 1-handles. Define the index of W by* $J(W) = \chi(\partial_- W) - \chi(\partial_+ W) \geq 0$.

Definition 2.6. *A set of defining disks for a compression body W is a set of disks* $\{D_1, \ldots, D_n\}$ *properly embedded in W with* $\partial D_i \subset \partial_+ W$ *for* $i = 1, \ldots, n$ *such that the result of cutting W along* $D_1 \cup \cdots \cup D_n$ *is homeomorphic to* $\partial_- W \times I$.

Definition 2.7. *A Heegaard splitting of a 3-manifold M is a decomposition* $M = V \cup_S W$ *in which V, W are compression bodies such that* $V \cap W = \partial_+ V = \partial_+ W = S$ *and* $M = V \cup W$. *We call S the splitting surface or Heegaard surface.*

Definition 2.8. *A Heegaard splitting is reducible (resp. weakly reducible) if there are essential disks* D_1 *and* D_2, *such that* $\partial D_1 = \partial D_2$ *(resp.* $\partial D_1 \cap \partial D_2 = \emptyset$*). A Heegaard splitting which is not (weakly) reducible is (strongly) irreducible.*

Definition 2.9. *A generalized Heegaard splitting of a compact orientable 3-manifold M is a structure* $M = (V_1 \cup_{S_1} W_1) \cup_{F_1} (V_2 \cup_{S_2} W_2) \cup_{F_2} \cdots \cup_{F_{m-1}} (V_m \cup_{S_m} W_m)$. *Each of the* V_i *and* W_i *is a union of compression bodies,* $\partial_+ V_i = S_i = \partial_+ W_i$, *(i.e.,* $V_i \cup_{S_i} W_i$ *is a union of Heegaard splittings of a submanifold of M) and* $\partial_- W_i = F_i = \partial_- V_{i+1}$. *We say that a generalized Heegaard splitting is strongly irreducible if each Heegaard splitting* $V_i \cup_{S_i} W_i$ *is strongly irreducible and each* F_i *is incompressible in M. We will denote*

$\cup_i F_i$ by $\mathcal{F}$ and $\cup_i S_i$ by $\mathcal{S}$. The surfaces in $\mathcal{F}$ are called the thin levels and the surfaces in $\mathcal{S}$ the thick levels.

Let $M = V \cup_S W$ be an irreducible Heegaard splitting. We may think of M as being obtained from $\partial_- V \times I$ by attaching all 1-handles in V followed by all 2-handles in W, followed, perhaps, by 3-handles. An untelescoping of $M = V \cup_S W$ is a rearrangement of the order in which the 1-handles of V and the 2-handles of W are attached. This rearrangement yields a generalized Heegaard splitting. For convenience, we will occasionally denote $\partial_- V = \partial_- V_1$ by F_0.

The Main Theorem in [ST1] together with the calculation [SS2, Lemma 2] implies the following:

Theorem 2.10. *Suppose M is an irreducible compact 3-manifold and $\partial_0 M$ is a collection of its boundary components. Then M has a Heegaard splitting $M = V \cup_S W$ with $\partial_0 M = \partial_- V$ if and only if M has a strongly irreducible generalized Heegaard splitting $(V_1 \cup_{S_1} W_1) \cup_{F_1} (V_2 \cup_{S_2} W_2) \cup_{F_2} \cdots \cup_{F_{m-1}} (V_m \cup_{S_m} W_m)$ such that $\partial_0 M = F_0 = \partial_- V_1$ and $\sum_{i=1}^{m} J(V_i) = J(V) = \chi(F_0) + 2g - 2$.*

One implication comes from untelescoping a Heegaard splitting of genus g, the other from thinking of a generalized Heegaard splitting as an untelescoping of some Heegaard splitting. The latter process is sometimes called the *amalgamation* of the generalized splitting into a standard splitting.

A strongly irreducible Heegaard splitting can be isotoped so that its splitting surface, S intersects an incompressible surface, P, only in curves essential in both S and P. This is a deep fact and is proven, for instance, in [Sc, Lemma 6]. This fact, together with the fact that incompressible surfaces can be isotoped to meet only in essential curves, establishes the following:

Lemma 2.11. *Let P be a properly embedded incompressible surface in an irreducible 3-manifold M and let $M = (V_1 \cup_{S_1} W_1) \cup_{F_1} \cdots \cup_{F_{m-1}} (V_m \cup_{S_m} W_m)$ be a strongly irreducible generalized Heegaard splitting of M. Then $\mathcal{F} \cup \mathcal{S}$ can be isotoped to intersect P only in curves that are essential in both P and $\mathcal{F} \cup \mathcal{S}$.*

3. An indicative first example

We will first point out a special simplified case in which an upper bound to the degeneration ratio is fairly easily found, and the ideas that are used are indicative of ideas that will be important in the general case.

The special case is this:

Theorem 3.1. *Let K^1 and K^2 be prime knots and assume that $M = C(K^1 \# K^2)$ possesses a minimal genus Heegaard splitting that is strongly irreducible. Then $t(K^1 \# K^2) \geq \frac{1}{2}(t(K^1) + t(K^2))$. Hence $d(K^1, K^2) \leq \frac{1}{2}$.*

Proof: Let $M = V \cup_S W$ be the strongly irreducible Heegaard splitting, with $\partial M = \partial_- V$ and W a handlebody. The hypothesis implies that $2t(K^1 \# K^2) = -\chi(S)$. Let $(A, \partial A) \subset (M, \partial M)$ denote the annulus in $M = C(K^1 \# K^2)$ that, when completed by a pair of meridian disks, constitutes a decomposing sphere for $K^1 \# K^2 \subset S^3$. Following Lemma 2.11 we can isotope S and A until their intersection consists of circles essential in A and in S. Isotope them further to reduce, as much as possible, the number of such components of $A \cap S$.

Let $A_V \subset A \cap V$ and $A_W = A \cap W$ be the essential non-spanning annuli of $A - S$ in V and W respectively. A sequence of ∂-compressions of $A_W \subset W$ could turn each annulus into a disk; similarly a sequence of ∂-compressions of all but the spanning annulus of $A_V \subset V$ could turn each annulus into a disk. Suppose that these sets of ∂-compressions could be done simultaneously. Do so on all but one annulus in A_W. The result would be that $A \cap S$ would become a pair of circles that are parallel and essential in A and bound a boundary compressible annulus in W. This annulus would divide W into two handlebodies, whose genus would add to $genus(W)+1$. The spanning annuli $A \cap V$ would divide V into two components, each of the form $annulus \times I \cup 1 - handles$. If, in each component of $M - A$, we thicken $A \cap W$ and regard this collar as in V, the result would be induced Heegaard splittings of $C(K^1)$ and $C(K^2)$. It would follow that $t(K_1) + t(K_2) \leq t(K^1 \# K^2)$ so there would be no degeneration.

In general, the boundary compressions of A_V and A_W cannot be done simultaneously, since the arcs to which they ∂-compress in S may intersect. This problem can be avoided if S were first stabilized by attaching a tube parallel to a spanning arc of each annulus in A_V. This would allow ∂-compressions of A_V onto arcs that run along the tubes, without affecting the ∂-compressions of A_W to the original S. So we see immediately that $|A_V|$, the number of tubes used, is an upper bound on degeneration. The first goal is then to get a bound on $|A_V|$.

Since any surface in S^3 is separating, any component of $S - A$ has an even number of boundary components, since it can be completed to become a closed surface in S^3 by attaching meridian disks of either K^1 or K^2. Assume, for initial simplification, that no component of $S - A$ is an annulus.

Let $S' = M - \eta(A)$ and notice then that each component S_0 of S' has at most $2 - \chi(S_0) \leq -2\chi(S_0)$ boundary components, since each component has non-trivial even Euler characteristic. It follows that $|\partial S'| \leq -2\chi(S') = -2\chi(S)$, so the number of essential curves $|A \cap S| \leq -\chi(S)$. Since two of these curves come from spanning annuli in V, this implies that the number of non-spanning annuli $|A_V| \leq -\frac{\chi(S)}{2} - 1 = t(K^1\#K^2) - 1$. Combining this argument with the previous one, we get

$$t(K_1) + t(K_2) \leq t(K^1\#K^2) + |A_V| \leq 2t(K^1\#K^2) - 1$$

or

$$t(K^1\#K^2) > \frac{1}{2}(t(K_1) + t(K_2)),$$

an inequality better than required.

Now examine the annulus components in $S - A$. Note first of all that any such annulus component B is necessarily ∂-parallel in the component $C(K^1)$ (say) of $M - A$ in which it lies, since K^1 is prime. Since the number of (essential) circles $|S \cap A|$ has been minimized, B is not parallel to a subannulus of A. So B must be parallel to the annulus component of $\partial M - \partial A$ that lies in $C(K^1)$, an annulus on ∂M which we denote by B_∂.

It follows that there cannot be two annulus components of $S - A$ that are adjacent in S, since one would be parallel to each of the two annulus components of $\partial M - \partial A$ and so the union of the two annuli in S would provide a way to "spin" a collar of ∂A in a way that would reduce the number of components of $S \cap A$. Since no two annuli in $S - A$ are adjacent in S, the total number of these annuli can be no larger than the number of circles $|A \cap S|$ that we calculated above in the absence of annuli. That is, there are no more than $-\chi(S)$ annuli among the components of $S - A$. Hence, even allowing annuli components in $S - A$, $|A \cap S| \leq -2\chi(S)$ so $|A_V| \leq \chi(S) - 1 = 2t(K^1\#K^2) - 1$ and we get

$$t(K_1) + t(K_2) \leq t(K^1\#K^2) + |A_V| \leq 3t(K^1\#K^2) - 1$$

or

$$t(K^1\#K^2) > \frac{1}{3}(t(K_1) + t(K_2)).$$

But we can do better. The region lying between the annuli B and B_∂ in $C(K^1)$ is homeomorphic to *annulus* $\times I$, i. e. a solid torus, and it is known how a strongly irreducible splitting surface like S can intersect a solid torus (cf [Sh]). The upshot is that all components of $S - A$ lying between B and B_∂ in $C(K^1)$ are annuli parallel to B, except possibly one component (an *exceptional* component) which consists of a pair of annuli parallel to B but then tubed together by a vertical tube. (See Figure 1.) An exceptional component can be ignored since, for example, the vertical tube could be slid across A into $C(K^2)$. Notice that, by choosing B outermost (i. e. furthest from $B_\partial \subset \partial M$), all annuli components of $S \cap C(K^1)$ lie between B and B_∂. By the analogous argument in $C(K^2)$ no component of $S \cap C(K^2)$ can be an annulus, since an innermost one would be adjacent in S to an innermost one

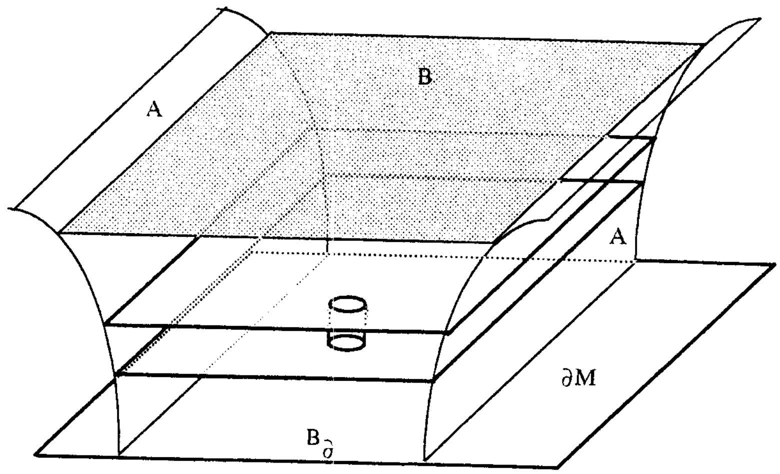

FIGURE 1. An exceptional component between B and B_∂

of $S \cap C(K^1)$, forming a ∂-parallel torus component of S, a contradiction. (This argument is complicated only a little by the possible presence of an exceptional component.)

So we may suppose that all w annulus components of $S - A$ lie in $C(K^1)$ and lie between and are parallel to the annuli B and B_∂. Then at least $\frac{w-2}{2}$ pairs of these annuli each cut off a parallel pair of annuli in A_V. For each of these pairs, only one stabilization is required in the process described above, since the annuli are parallel. (Imagine placing the added tube so that it lies in the collar between the annuli in V). So if $S - A$ contains w annuli there are two new effects: First, the count of $|\partial S'|$ is changed to $|\partial S'| \leq 2w - 2\chi(S') = 2w - 2\chi(S)$, so the number of essential curves $|A \cap S| \leq w - \chi(S)$. This implies that

$$|A_V| \leq \frac{w - \chi(S)}{2} - 1.$$

On the other hand, the second effect is that the number of stabilizations required is at most

$$|A_V| - \frac{w-2}{2} \leq \frac{-\chi(S)}{2} = t(K^1 \# K^2).$$

This implies that

$$t(K_1) + t(K_2) \leq t(K^1 \# K^2) + t(K^1 \# K^2) \leq 2t(K^1 \# K^2)$$

or

$$t(K^1 \# K^2) \geq \frac{1}{2}(t(K_1) + t(K_2)),$$

as required. □

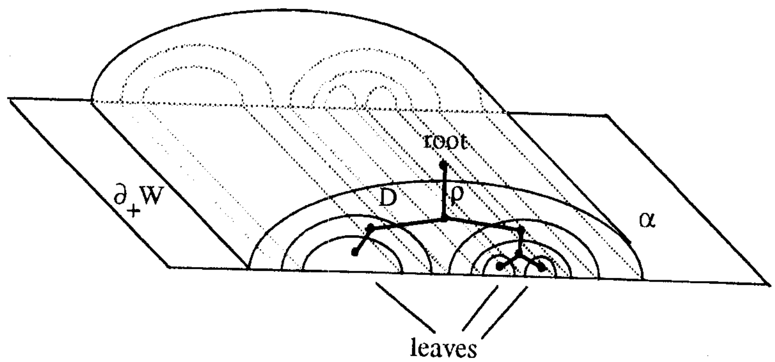

FIGURE 2. A tree of complexity $\{4, 4, 3\}$.

4. ANNULI IN AND ON COMPRESSION BODIES

A properly imbedded annulus $A \subset M$ is *essential* if it is incompressible and not ∂-parallel. In this section we study finite sets of disjoint essential annuli in a compression body W.

Lemma 4.1. *If A is an essential annulus in a compression body W, then either one component of ∂A lies on each of $\partial_+ W$ and $\partial_- W$ (i. e. A is* spanning*) or $\partial A \subset \partial_+ W$. In the latter case, A is boundary compressible.*

Proof: See [BnO, Lemma 9]. □

Suppose $\mathcal{A}$ is a properly imbedded collection of essential annuli in a compression body W so, in particular, the boundary of any non-spanning annulus A lies in $\partial_+ W$. Suppose D is a ∂-compressing disk for the non-spanning annulus A, i. e. the interior of D is disjoint from $\mathcal{A}$, and ∂D is the union of a spanning arc α of A and an arc in $\partial_+ W$. The arc components of $D \cap \mathcal{A}$ divide D into subdisks. This naturally gives rise to a tree in D, in which a vertex is chosen inside each subdisk and two vertices are connected if they abut the same annulus. (We can ignore closed components of intersection. Indeed, since $\mathcal{A}$ is incompressible, closed components of $D \cap \mathcal{A}$ can be removed by an isotopy with support disjoint from the arcs of intersection.) It will be useful to extend this tree by attaching, to the vertex corresponding to the subdisk that abuts α, an edge ρ that crosses α. The other end of ρ is called the root of the resulting tree τ; the other valence one edges are called the leaves of τ. Each leaf corresponds to a disk cut off by an outermost arc of $\mathcal{A} \cap D$ in D. (See Figure 2.)

Recall that there is a natural order on the set of finite sets of integers (see [Ga, Definition 4.3]). One set of integers is compared to another by arranging each in descending (or at least never-ascending) order and then comparing them lexicographically. This ordering has the property that if a subset of a set of integers is replaced by a subset of lower order then the resulting set is of lower order.

Definition 4.2. *The complexity of the tree τ is the set of distances (measured in edges traversed) from the root of τ to the leaves in τ. The complexity of a ∂-compressing disk for $A \in \mathcal{A}$ is the complexity of the associated tree. A minimal ∂-compressing disk for A is a ∂-compressing disk of minimal complexity (using the above order on sets of integers).*

If an annulus A has minimal disks abutting it on both sides, we say that A is ambivalent.

Lemma 4.3. *If D_A is a minimal ∂-compressing disk for $A \in \mathcal{A} \subset W$ then no component of $D_A \cap \mathcal{A}$ is an inessential arc in $\mathcal{A}$. In particular, D_A is disjoint from all spanning components of $\mathcal{A}$. Furthermore, if A' is another annulus in $\mathcal{A}$ and A' intersects D_A, then A' is not ambivalent and any arc component of $A' \cap D_A$ cuts off from D_A a subdisk $D'_A \subset D_A$ that is also a minimal disk.*

Proof: Immediate, since otherwise replacing D'_A by a minimal disk (on the other side of A' if possible) would decrease the complexity of D_A. □

Definition 4.4. *The side opposite to the side on which a minimal disk abuts a non-spanning annulus $A \in \mathcal{A}$ is called the root side of A. If A is ambivalent, arbitrarily choose a side to call the root side.*

Definition 4.5. *Let $\mathcal{A} \subset \partial_+ W$ be a collection of essential non-spanning annuli in the boundary of a compression body W. Then $\mathcal{A}$ is independently longitudinal if for each $A \in \mathcal{A}$ there is a meridian disk D_A for W such that $\partial D_A \cap \mathcal{A}$ consists precisely of a single spanning arc of A.*

An easy outermost arc argument shows that we can take the collection of meridians that arise in Definition 4.5 to be disjoint.

Lemma 4.6. *Suppose V and W are compression bodies, and a collection of essential non-spanning annuli $\mathcal{A} \subset \partial_+ W$ is independently longitudinal.*

Then if V is attached to W by identifying $\mathcal{A}$ to any collection of annuli in $\partial_+ V$, the result is a compression body.

Proof: $\mathcal{A}$ becomes a collection of boundary compressible annuli in $V \cup_{\mathcal{A}} W$. In this collection each annulus has a boundary compressing disk of complexity $\{1\}$ (i. e. a disk with interior disjoint from $\mathcal{A}$). Performing these boundary compressions turn $\mathcal{A}$ into a collection of disks. So an alternate and equivalent construction for $V \cup_{\mathcal{A}} W$ would be to first compress W along the meridians for $\mathcal{A} \subset \partial_+ W$ and then attach the resulting compression body W' to $\partial_+ V$ along the disks in $\partial_+ W'$ that are the remnants of $\mathcal{A}$. This alternate construction clearly gives a compression body. □

Lemma 4.7. *Let P be a (not necessarily connected) incompressible surface in a compression body W such that $\partial P \subset \partial_+ W$. Then the result of cutting W along P is a collection of compression bodies.*

Proof: See for instance [Sc, Lemma 2]. □

Lemma 4.8. *Suppose $\mathcal{A} \subset W$ is a set of essential annuli in the compression body W. Let $\mathcal{A}' \subset \mathcal{A}$ be the subcollection of non-spanning annuli. Cut out from W a collection of tunnels, one parallel to a spanning arc of each annulus A in $\mathcal{A}'$ and lying on the root side of A. Call the resulting compression body W'. Then the closure W_0' of any component of $W' - \mathcal{A}'$ is a compression body on which $\mathcal{A}' \cap \partial W_0$ is a collection of independently longitudinal annuli.*

Proof: Let A be any annulus in $\mathcal{A}' \cap \partial W_0'$ and let W_0 denote the component of $W - \mathcal{A}$ from which tunnels were removed to get W_0'. If, in W, W_0 lay on the root side of A then in constructing W_0' a tunnel has been removed that runs parallel to a spanning arc of A. The disk that defines the parallelism is a meridian for W_0' that intersects A precisely in a spanning arc, as required.

If, on the other hand, W_0 is not on the root side of A then consider a minimal ∂-compressing disk D_A for A in W. The component D_0 of $D_A - \mathcal{A}$ that abuts A lies in W_0 and abuts any other annulus in $\mathcal{A}$, if at all, only on its root side (see Lemma 4.3). In particular, for each such annulus, the tunnels that are removed to create W_0' can be positioned so that the component D_0' of $D_0 \cap W_0'$ that abuts A runs over the tunnels instead of across the other annuli. See Figure 3. D_0' is then a meridian for W_0' that intersects A only in a single spanning arc of A, as required. □

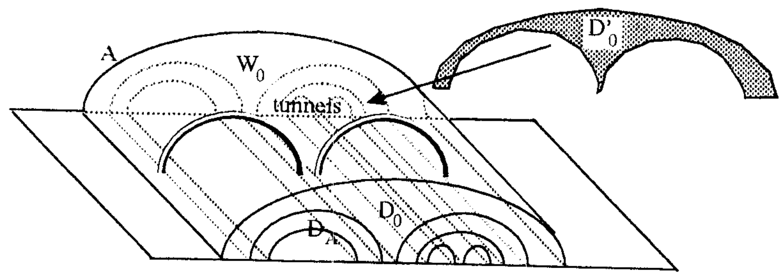

FIGURE 3. Placing D_0' as a meridian of W_0'.

5. Breaking up generalized Heegaard splittings by transverse annuli

A tunnel system for a knot corresponds to a Heegaard splitting of the knot complement. A Heegaard splitting, in turn, can be untelescoped to produce a strongly irreducible generalized Heegaard splitting, and vice versa. When we consider degeneration of tunnel number, we shall be concerned with constructing generalized Heegaard splittings for each $C(K^j), j = 1, ..., n$ from a strongly irreducible generalized Heegaard splitting for $C(K^1\#...\#K^n)$.

The more general context for this section is this: Let M be a compact orientable 3-manifold and $\mathcal{A}$ a properly imbedded collection of essential annuli in M. Suppose

$$(V_1 \cup_{S_1} W_1) \cup_{F_1} \cdots \cup_{F_{m-1}} (V_m \cup_{S_m} W_m)$$

is a strongly irreducible generalized Heegaard splitting of M, chosen so that $F_0 = \partial_- V_1$ contains $\partial\mathcal{A}$ and isotoped so that $(\mathcal{F} \cup \mathcal{S}) \cap \mathcal{A}$ consists only of curves essential in both $\mathcal{F} \cup \mathcal{S}$ and $\mathcal{A}$, and such that this number is minimal.

Definition 5.1. *An annulus component (or its closure) A of $\mathcal{A} - (\mathcal{F} \cup \mathcal{S})$ is called a* dipping *annulus if, for some $1 \leq i \leq m$, $A \subset V_i$ and $\partial A \subset S_i = \partial_+ V_i$.*

Theorem 5.2. *Let $M^1, ..., M^n$ be the components into which M is divided by the family of annuli $\mathcal{A}$. Then for each $1 \leq j \leq n$ there is a generalized Heegaard splitting $(V_1^j \cup_{S_1^j} W_1^j) \cup_{F_1^j} \cdots \cup_{F_{m-1}^j} (V_m^j \cup_{S_m^j} W_m^j)$ of M^j such that*

$$\sum_{i=1}^{m} J(V_i) \geq \sum_{j=1}^{n} \sum_{i=1}^{m} J(V_i^j) - 2k,$$

where k is the number of dipping annuli among the components of $\mathcal{A} - (\mathcal{F} \cup \mathcal{S})$.

Proof: The central problem in discerning Heegaard splittings in the M^j is that cutting V_i or W_i along $\mathcal{A}$ does not necessarily create compression bodies. For example, cutting $\mathcal{F} \cup \mathcal{S}$ along $\mathcal{A}$ does not even produce closed surfaces. We endeavor to remedy this fact by longitudinally attaching a solid torus (which we could view as a collar of $\mathcal{A}^j = \mathcal{A} \cap M^j$) to each of the annuli in ∂M^j and imbedding in each torus certain annuli with longitudinal boundary. These annuli, when attached to surfaces $S_i \cap M^j$ (suitably stabilized) and $F_i \cap M^j$ will be shown to yield generalized Heegaard splittings of the M^j.

The first step will be to describe how the new annuli are to be imbedded in the solid torus collar T^A of each $A \in \mathcal{A}^j \subset \partial M^j$. In M itself, it's natural to define the "distance" between two of the surfaces in $(\mathcal{F} \cup \mathcal{S}) \subset M$ as the smallest number of compression bodies one needs to pass through to get from a point in one surface to a point in the other. So, for example, the distance from $\partial_- V_1 = F_0$ to F_i is $2i$ and from F_0 to S_i the distance is $2i-1$.

$(\mathcal{F} \cup \mathcal{S}) \cap A$ is a collection of parallel essential curves in the annulus A. Let α be a spanning arc of A that meets each component of $(\mathcal{F} \cup \mathcal{S}) \cap A$ exactly once. Parameterize α by $0 \leq t \leq 1$. That is, choose a homeomorphism $h_1 : \alpha \to [0, 1]$. Let $h_2 : \alpha \to [0, 2m]$ be a continuous extension of the function that assigns to each point in $\alpha \cap (\mathcal{F} \cup \mathcal{S})$ its distance from F_0. We may as well take h_2 to be as simple as possible. For example, on a segment of α that runs between F_i and S_{i+1}, say, define h_2 to monotonically run from $2i$ to $2i+1$. On a segment of α that lies in W_i and has both ends on S_i, define h_2 so that it has a single maximum.

Let $h = (h_1, h_2) : \alpha \to [0, 1] \times [0, 2m]$ be the corresponding imbedding of α in the first quadrant of $\mathbb{R}^2$, with the endpoints of α on the x-axis. Informally, h identifies α with the graph of its distance from F_0. Let D be the disk that lies below $h(\alpha)$ in $[0, 1] \times [0, 2m] \subset \mathbb{R}^2$.

Each line $y = 2i$ (respectively $y = 2i - 1$) intersects $h(\alpha) \subset \mathbb{R}^2$ at points at which F_i (respectively S_i) intersects α. In view of this correspondence, let $\mathcal{L}_{F_i}$ denote the intersection of the line $y = 2i$ with D, $\mathcal{L}_{S_i}$ denote the intersection of the line $y = 2i - 1$ with D, $\mathcal{L}_F = \cup_i \mathcal{L}_{F_i}$ and $\mathcal{L}_S = \cup_i \mathcal{L}_{S_i}$. See Figure 4. Consider a component R of $D - (\mathcal{L}_F \cup \mathcal{L}_S)$. R is a polygon with an even number of sides. The sides lie alternately in $h(\alpha)$ and $\mathcal{L}_F \cup \mathcal{L}_S$.

If R is a bigon, one of its sides is a subarc of $h(\alpha)$ and one of its sides is a component of $\mathcal{L}_S$, since an incompressible annulus with both ends on $\partial_- H$ in a compression body H is ∂-parallel. Moreover h_2 has a maximum on the corresponding subarc of α, since D lies below the graph. That is, the annulus on which the corresponding arc of α lies is in some W_i.

If R is a quadrilateral, two opposite sides are subarcs of $h(\alpha)$, one side is a component of $\mathcal{L}_F$ and one a component of $\mathcal{L}_S$. If R has n sides with $n > 4$, then $\frac{n}{2}$ sides will be subarcs of $h(\alpha)$, one side will be a component of $\mathcal{L}_F$ and all other sides will be components of $\mathcal{L}_S$. Thus there will be $\frac{n}{2} - 1$ sides that are subarcs of $h(\alpha)$ and that connect two sides that are components of $\mathcal{L}_S$; these correspond to spanning arcs of dipping annuli in some V_i. Let Γ be the collection of subarcs of α that, in the boundary of some R, connect two

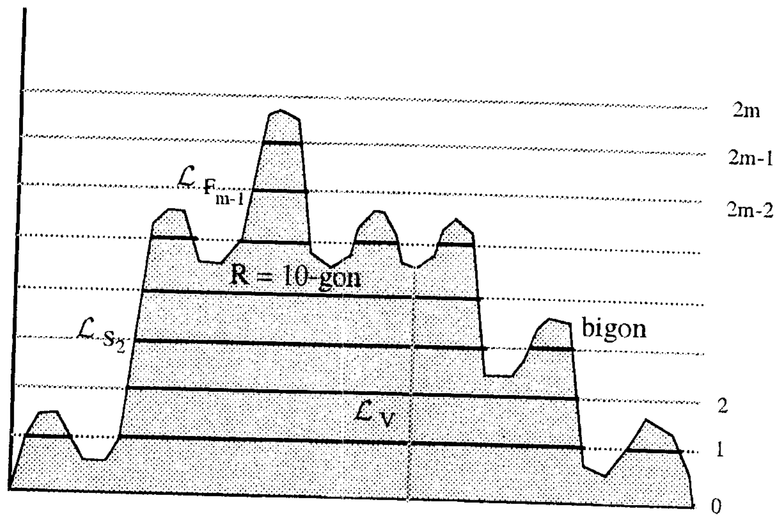

FIGURE 4. The lines $\mathcal{L}_F, \mathcal{L}_S, \mathcal{L}_V \subset V$.

sides that are components of $\mathcal{L}_S$. Then there is a correspondence between the components of Γ and the collection of dipping annuli that lie in A.

Now set $T^A = D \times S^1$. T^A will be attached to each of the two copies of A in $\cup_j(\partial M^j)$ by the obvious identification $h^{-1} \times \mathbf{1}_{S^1}$ of $h(\alpha) \times S^1$ with $A \cong (\alpha \times S^1)$. The surfaces $(\mathcal{F} \cup \mathcal{S}) \cap M^j$ can then be extended into T^A by just attaching the collection of annuli $(\mathcal{L}_F \cup \mathcal{L}_S) \times S^1$. This operation will not necessarily create compression bodies, but it will do so if we first modify $\mathcal{S}$ as described below.

Stabilize $\mathcal{S}$ by attaching tubes, one running parallel to a spanning arc on each dipping annulus and lying on the root side of the annulus in the compression body V_i in which it lies. Denote the resulting generalized Heegaard splitting of M, now of genus k higher than originally, by

$$(V_1' \cup_{S_1'} W_1') \cup_{F_1} \cdots \cup_{F_{m-1}} (V_m' \cup_{S_m'} W_m').$$

Denote $\cup_i S_i'$ by $\mathcal{S}'$.

It follows from Lemma 4.8 that each component of the complement in V_i' of the dipping annuli is a compression body on whose boundary the collection of incident dipping annuli are independently longitudinal.

Claim: Attaching each $T^A, A \subset \partial M^j$ to M^j and capping off the surfaces $(\mathcal{F} \cup \mathcal{S}') \cap M^j$ by the annuli $(\mathcal{L}_F \cup \mathcal{L}_S) \times S^1 \subset T^A$ yields a generalized Heegaard splitting of M^j.

For any i, cutting along non-spanning components of $A \cap V_i'$ (or $A \cap W_i'$) yields compression bodies by Lemma 4.7. Cutting along spanning annuli yields $(Q \times I) \cup (1 - handles)$ for some compact orientable surface Q. When

T^A is attached to M^j, then for R a region as above, manifolds of the form $R \times S^1$ are attached to $(Q \times I) \cup (1 - handles)$.

Case 1: R is a bigon.

Note that a bigon corresponds precisely to a non-spanning annulus component of $\mathcal{A} \cap W_i$. Then $R \times S^1$ is a solid torus that is attached to $(Q \times I) \cup (1 - handles)$ along a longitudinal annulus of $R \times S^1$. This does not change the homeomorphism type of $(Q \times I) \cup (1 - handles)$.

Case 2: R is a quadrilateral.

Attaching $R \times S^1$ to $(Q \times I) \cup (1 - handles)$ yields $(Q' \times I) \cup (1 - handles)$, where Q' is the compact surface obtained by connecting two boundary components of Q by an annulus.

Case 3: R is an n-gon with $n > 4$.

Here attaching $R \times S^1$ to $(Q \times I) \cup (1 - handles)$ has the same effect as attaching a $(quadrilateral) \times S^1$, but in addition, attachments are also made along dipping annuli in the corresponding V_i. S' has been constructed so that each dipping annulus is independently longitudinal in the component of $V_i' - \mathcal{A}$ on which it lies, so the result is still of the form $(Q' \times I) \cup (1 - handles)$, essentially by Lemma 4.6.

Since all components of ∂Q are eventually connected by annuli in this process, the result is a union V_i^j (or W_i^j) of compression bodies. □

Remark 5.3. *Theorem 5.2 will eventually be applied to the family of annuli* $\mathcal{A} = \{A^1, ..., A^{n-1}\}$ *that decompose* $C(K^1 \# \dots \# K^n)$ *into* $C(K^1), \dots, C(K^n)$.

Certain annuli in the tori T^A, now described, will be useful in the next section.

Definition 5.4. *Let* $D \subset \mathbb{R}^2$ *be the disk described in the proof of Theorem 5.2. Let* $\mathcal{L}_v$ *be the intersection of a vertical line* $x = x_0$ *with* D. *Then the annulus* $\mathcal{L}_v \times S^1$ *is called a plumbline annulus in* $T^A = D \times S^1$. *See Figure 4.*

Remark 5.5. *Let* $M_+^j \cong M^j$ *denote the manifold obtained from* M^j *by attaching the solid tori* $\{T^A | A \subset \partial M^j\}$ *to* M^j. *Note that each plumbline annulus in* $T^A \subset M_+^j$ *will intersect each splitting surface* S_i^j *in at most one component.*

If we replace the collection of annuli in Theorem 5.2 by a collection of essential (= incompressible and not ∂-parallel) tori, then an analogous proof

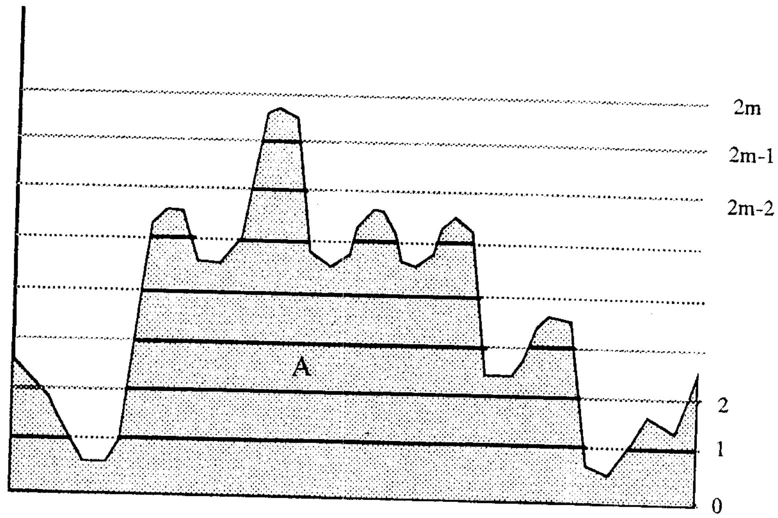

FIGURE 5. Identify left and right sides so the x-axis becomes S^1.

still applies. The spanning arc α must be replaced by an appropriate essential curve on T. More specifically, note that the components of $(\mathcal{F} \cup \mathcal{S}) \cap T$ are all parallel. The curve that replaces α must be an essential curve on T that intersects each component of $(\mathcal{F} \cup \mathcal{S}) \cap T$ exactly once. The graph of this curve in $S^1 \times \mathbb{R}^+$, constructed in analogy to $h(\alpha)$ in the proof of Theorem 5.2, will cut out an annulus. This annulus replaces D in the construction. See Figure 5. This yields the following result.

Theorem 5.6. *Let M be an orientable 3-manifold containing a family of essential tori $\mathcal{T}$. Suppose $(V_1 \cup_{S_1} W_1) \cup_{F_1} \cdots \cup_{F_{m-1}} (V_m \cup_{S_m} W_m)$ is a strongly irreducible generalized Heegaard splitting of M isotoped so that $(\mathcal{F} \cup \mathcal{S}) \cap \mathcal{T}$ consists only of curves essential in both $\mathcal{F} \cup \mathcal{S}$ and $\mathcal{T}$, and such that this number is minimal. Let $M^1, \ldots, M^n$ be the components into which M is divided by $\mathcal{T}$.*

Then for each $1 \leq j \leq n$ there is a generalized Heegaard splitting $(V_1^j \cup_{S_1^j} W_1^j) \cup_{F_1^j} \cdots \cup_{F_{m-1}^j} (V_m^j \cup_{S_m^j} W_m^j)$ of M^j such that

$$\sum_{i=1}^{m} J(V_i) \geq \sum_{j=1}^{n} \sum_{i=1}^{m} J(V_i^j) - 2k,$$

where k is the number of dipping annuli in $\mathcal{T} \cup (\cup_i V_i)$.

6. Destabilizations

The generalized Heegaard splitting of each M^j constructed in Theorem 5.2 may not be strongly irreducible and, even if it is, it might still be simplified. Recall

Definition 6.1. *A Heegaard splitting $M = V \cup_S W$ is* stabilized *if there are properly imbedded disks $D_1 \subset V$ and $D_2 \subset W$ such that $|\partial D_1 \cap \partial D_2| = 1$.*

Remark 6.2. *In this case, cutting V along D_1 (or D_2) yields another Heegaard splitting of lower genus.*

A strongly irreducible splitting may amalgamate to a stabilized or reducible splitting, so (to account for this) the definition of a stabilized generalized Heegaard splitting is necessarily a bit more complicated.

Definition 6.3. *A generalized Heegaard splitting $M = (V_1 \cup_{S_1} W_1) \cup_{F_1} \cdots \cup_{F_{m-1}} (V_m \cup_{S_m} W_m)$ is* stabilized *if there are disks D_1 and D_2 such that D_1 is properly embedded in $(V_1 \cup_{S_1} W_1) \cup_{F_1} \cdots \cup_{F_{i-1}} V_i$, for some i, and D_2 is properly embedded in $W_i \cup_{F_i} \cdots \cup_{F_{m-1}} (V_m \cup_{S_m} W_m)$. Furthermore, $|\partial D_1 \cap \partial D_2| = 1$ and each component of $D_j - (\mathcal{F} \cup \mathcal{S})$ is either a disk or an annulus spanning the compression body in which it lies.*

If a generalized Heegaard splitting is stabilized (i. e. satisfies Definition 6.3) the associated amalgamated Heegaard splitting is stabilized. It is easy to see that we may assume the annuli components of $D_j - (\mathcal{F} \cup \mathcal{S})$ are essential. It furthermore follows from [CG] that if the generalized splitting is strongly irreducible, then we may assume each annulus component of $V_i \cap D_k$ or $W_i \cap D_k$ is a spanning annulus in a component of V_i or W_i that is a trivial compression body.

Note that if a generalized splitting is stabilized we can create a generalized Heegaard splitting of lower genus (i. e. for which $\sum_i J(V_i)$ is reduced) by amalgamating, reducing the genus as in Remark 6.2, and then untelescoping again.

Here is an ad hoc criterion, useful in the present context, for showing that a given generalized splitting is stabilized.

Definition 6.4. *A surface $G \subset (D^2 \times S^1)$ is a* tubed product *if it can be described as follows: Start with a properly imbedded 1-manifold $\mathcal{L} \subset D^2$, and a collection τ of t disjoint arcs in the interior of D such that $\tau \cap \mathcal{L} = \partial\tau$.*

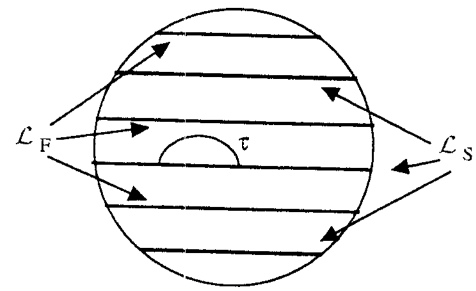

FIGURE 6

Then $\mathcal{L} \times S^1 \subset D^2 \times S^1$ is a union of tori and annuli. Create G by attaching unnested tubes to $\mathcal{L} \times S^1$ along the arcs $\tau \times \{point\} \subset D^2 \times \{point\}$.

Viewed dually, $\mathcal{L} \times S^1$ is obtained from G by simultaneously compressing t disks. For G_0 a component of G, each component of $\mathcal{L} \times S^1$ that results from compressing G_0 is said to come from G_0.

Lemma 6.5. *Suppose that a generalized Heegaard splitting $(V_1 \cup_{S_1} W_1) \cup_{F_1} \cdots \cup_{F_{m-1}} (V_m \cup_{S_m} W_m)$ of a compact 3-manifold M intersects an essential solid torus $D^2 \times S^1 \subset interior(M)$ so that the surface $G = (\mathcal{F} \cup \mathcal{S}) \cap (D^2 \times S^1)$ can be described as a tubed product with t tubes as in Definition 6.4. Suppose further that, for each $1 \leq i \leq m$, at most one arc component of $\mathcal{L} \times S^1$ comes from $S_i \cap (D^2 \times S^1)$. Then the splitting is stabilized at least t times.*

Proof: The proof is by induction on $|\mathcal{L}|$. If $\mathcal{L} = \emptyset$ then there is nothing on which to attach tubes, so $t = 0$ and there is nothing to prove. We may as well also assume that no component of τ is parallel to a subarc of $\mathcal{L}$ (that is, no disk component of $D - (\mathcal{L} \cup \tau)$ has boundary the union of a component of τ and a subarc of $\mathcal{L}$). For the corresponding tube is clearly a stabilization, so it can be removed without affecting the truth of the lemma.

Suppose $\mathcal{L}$ consists entirely of arcs. Since each F_i is incompressible in both W_i and V_{i+1}, tubes can only have been attached to components of $\mathcal{L} \times S^1$ coming from the thick surfaces $\mathcal{S}$. By assumption, there is at most one component of $\mathcal{L} \times S^1$ coming from any given S_i and distinct S_i's are separated by components of $\mathcal{F}$. It follows that at least one component of τ is parallel to a subsegment of $\mathcal{L}$, a contradiction. See Figure 6.

We proceed next to the case where there are closed components of $\mathcal{L}$. Each closed component $\lambda \subset \mathcal{L} \subset D^2$ abuts two components of $D^2 - \mathcal{L}$; call them the *inside* and *outside* regions neighboring λ, depending on whether or not λ separates the region from ∂D^2. If the outside neighboring region of each

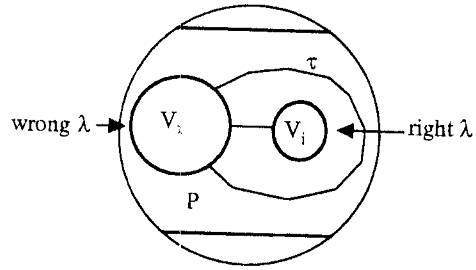

FIGURE 7

closed component is an annulus, and the annulus spans some V_i or W_i then it follows, much as when there are no closed components, that each tube attached is a stabilization. So we can focus on a closed component $\lambda \subset \mathcal{L}$ which is innermost among those whose outside neighboring component is not a spanning annulus. It follows that the components of $D^2 - \mathcal{L}$ lying inside λ consist exactly of spanning annuli in the compression bodies, together with a single compressing disk for some S_i. Then all tubes coming from arcs in τ lying inside λ in D^2 are stabilizations. This means that all components of $\mathcal{F} \cup \mathcal{S}$ lying inside the solid torus U bounded by $\lambda \times S^1$ can be removed, and both t and the number of stablizations will be reduced by the number of tubes lying therein. Indeed, $\lambda \times S^1$ itself can be removed, if it comes from $\mathcal{F}$.

So, by induction, we are left with the case in which λ comes from some S_i, cuts off from D^2 a compressing disk in V_i or W_i, say, V_i, and the outside neighbor of λ is not a spanning annulus of W_i. By judicious choice of curves satisfying these properties we may further assume that no arc in τ has both ends on λ. (For such an arc is either parallel to a subarc of λ, and so violates our assumption above, or cuts off from the outside neighboring region another family of circles among which a substitute λ can be found. See Figure 7.) If no tube is attached to λ in the outside neighboring region, the component P of W_i containing that region will be a product, and so $\partial P \cup \partial U$ can be removed from $\mathcal{F} \cup \mathcal{S}$ and still leave a generalized Heegaard splitting.

So we may as well assume that a tube is attached along an arc α in the outside neighboring region of λ, and one end of α lies on λ and the other end on another component λ' of $\mathcal{L}$. Necessarily λ' comes from S_i, but possibly λ' is an arc component of $\mathcal{L}$. Whether λ' is an arc or is closed, the complement of the tube in the annulus $\alpha \times S^1$ is a disk which, together with a meridian disk μ of U form a stabilizing pair for the splitting. Compressing S_i along μ then leaves a Heegaard splitting, still intersecting $\mathcal{F} \cup \mathcal{S}$ in a tubed product, but with both t and the number of stabilizations reduced by one. Since also $|\mathcal{L}|$ is reduced by one, the result follows by induction. □

We now turn to the setting of Theorem 5.2 in which M has a strongly irreducible generalized Heegaard splitting and will henceforth assume that the family of annuli $\mathcal{A}$, separating M into components $M^1, ..., M^n$, is *complete.* That is, if $(A', \partial A') \subset (M, \partial M)$ is a properly imbedded incompressible annulus disjoint from $\mathcal{A}$, then A' is ∂-parallel in the component M^j in which it lies. For example, if $M = C(K^1 \# ... \# K^n)$, then the knot summation defines a collection $\mathcal{A}$ of $n-1$ essential annuli in M and this is a complete collection if and only if each knot is prime. Indeed, an annulus A' in a knot complement $C(K^j)$, with $\partial A'$ a pair of meridian disks, can be extended to become a decomposing sphere in S^3 by capping off $\partial A'$ with meridian disks. A resulting summand is trivial if and only if A' is ∂-parallel in $C(K^j)$.

Note that the condition that $\mathcal{A}$ is complete is weaker than the assumption that each M^j is acylindrical, since it says nothing about incompressible annuli in M^j whose boundaries cross the curves $\partial \mathcal{A} \cap \partial M^j$.

Definition 6.6. *Suppose that the surface $\mathcal{F} \cup \mathcal{S}$ has been isotoped to intersect $\mathcal{A}$ only in curves essential in both $\mathcal{F} \cup \mathcal{S}$ and $\mathcal{A}$ and so that the number of components of $(\mathcal{F} \cup \mathcal{S}) \cap \mathcal{A}$ is minimal subject to this condition. Then an annular component ω of $\mathcal{S} - \mathcal{A}$ is called a* <u>*wide*</u> *annulus if it is adjacent to (that is, shares a boundary component with) a dipping annulus in $\mathcal{A} - (\mathcal{F} \cup \mathcal{S})$.*

A component U of $W_i - \mathcal{A}$ (resp $V_i - \mathcal{A}$) is <u>*exceptional*</u> *if, for an annulus B, there is an embedding $(B, \partial B) \times I \subset W_i$ and an open disk D in B so that U is the image of $(B - D) \times I$, with $U \cap \mathcal{A} = \partial(B) \times I$.*

Exceptional components were first described near the end of the proof of Theorem 3.1. See Figure 1. Note that $U \cap S_i$ is a 4-punctured sphere and the closure of $D \times \{point\}$ is a compressing disk for $U \cap S_i$ in V_i (resp W_i).

Definition 6.7. *For U an exceptional component of $W_i - \mathcal{A}$ (resp $V_i - \mathcal{A}$), the 4-punctured sphere $U_S = \partial U \cap S_i$ will be called an* <u>*exceptional*</u> *component of $S_i - \mathcal{A}$ and the annuli obtained by compressing U_s into V_i (resp W_i) will be called* <u>*virtual annuli*</u> *of S_i.*

Lemma 6.8. *Let M be a compact 3-manifold with strongly irreducible generalized Heegaard splitting $M = (V_1 \cup_{S_1} W_1) \cup_{F_1} (V_2 \cup_{S_2} W_2) \cup_{F_2} \cdots \cup_{F_{m-1}} (V_m \cup_{S_m} W_m)$. Let $M^1, ..., M^n$ be the components into which M is divided by the complete collection of annuli $\mathcal{A}$. Suppose there are w wide annuli among the components of $\mathcal{S} - \mathcal{A}$ and e exceptional components. Suppose* *further that each Heegaard splitting* $(V_1^j \cup_{S_1^j} W_1^j) \cup_{F_1^j} \cdots \cup_{F_{m-1}^j} (V_m^j \cup_{S_m^j} W_m^j)$ *of* M^j *in the construction in Theorem 5.2 can be destabilized* d_j *times. Then* $\Sigma_j d_j \geq \frac{w}{2} + e$.

Proof: Let $B \subset M^j$ be an annulus disjoint from $\mathcal{F} \cup \mathcal{S}$ whose ends are essential (hence core curves) in components A_0 and A_1 of $\mathcal{A}$. Here we will allow $A_0 = A_1$ but not if B is parallel to a subannulus of $A_0 = A_1$. Unless $\mathcal{S} \cap M^j$ contains no annuli or exceptional components, such an annulus can be found, e. g. parallel to an annulus component of $\mathcal{S} \cap M^j$ (or a virtual annulus). Since $\mathcal{A}$ is complete, B is boundary parallel in M^j and the annulus $B_\partial^+ \subset \partial M^j$ to which it is parallel contains as collars of its ends subannuli $A_0' \subset A_0$ and $A_1' \subset A_1$ with $A_0' \neq A_1'$. Denote the annulus $B_\partial^+ \cap \partial M = B_\partial^+ - (A_0' \cup A_1')$ by $B_\partial \subset \partial M$. It is known (see for example [Sh]) how a strongly irreducible Heegaard splitting can intersect a solid torus under the conditions here, so we know that each S_i intersects the solid torus T lying between B and B_∂^+ in M^j in a collection of ∂-parallel annuli, plus possibly a component in which two such annuli are tubed together by a ∂-parallel tube, i. e. an exceptional component.

Since the number of curves in $\mathcal{A} \cap (\mathcal{F} \cup \mathcal{S})$ has been minimized, none of the annuli of $(\mathcal{F} \cup \mathcal{S}) \cap T$ (and none of the virtual annuli) has both its ends on the same component of $\partial T \cap \mathcal{A}$. Thus each annulus component of $(\mathcal{F} \cup \mathcal{S}) \cap T$ (or virtual annulus) is an annulus much like B. So with no loss of generality, we may as well assume that B is outermost, i. e. no other annulus component of $(\mathcal{F} \cup \mathcal{S}) \cap M^j$ (or virtual annulus) cuts off a solid torus containing B. Also, T can be parameterized as $disk \times S^1$ so that the annuli of $(\mathcal{F} \cup \mathcal{S}) \cap T$ plus the virtual annuli are just the product of a collection of proper arcs in $disk$ with S^1.

Now attach to M^j the solid tori described in Theorem 5.2 to get M_+^j. Assume, for initial simplicity, that neither end of B abuts a dipping annulus on its root side. Extend B by attaching to the ends of B at A_0 and A_1 the plumbline annuli in the tori T^{A_0} and T^{A_1} described in Definition 5.4. Call the resulting annulus B_+ and denote by T_+ the solid torus that B_+ cuts off from M_+^j. By construction, the Heegaard splitting $(V_1^j \cup_{S_1^j} W_1^j) \cup_{F_1^j} \cdots \cup_{F_{m-1}^j} (V_m^j \cup_{S_m^j} W_m^j)$ intersects T_+ in a tubed product, where the tubes are attached in the manner described in Theorem 5.2 or, in the case of exceptional components, by the compressions that create the virtual annuli. By Lemma 6.5 it suffices to show that the number of tubes not coming from exceptional components in this tubed product is at least half as large as the number of wide annuli among the components of $\mathcal{S} \cap T$. The argument is little changed by assuming, as we henceforth do, that there are no exceptional components.

Under the homeomorphism $T \cong disk \times S^1$, any component T_0 of $T - (\mathcal{F} \cup \mathcal{S})$ is the product of a $2p$-gon with S^1. The sides of the $2p$-gon become annuli

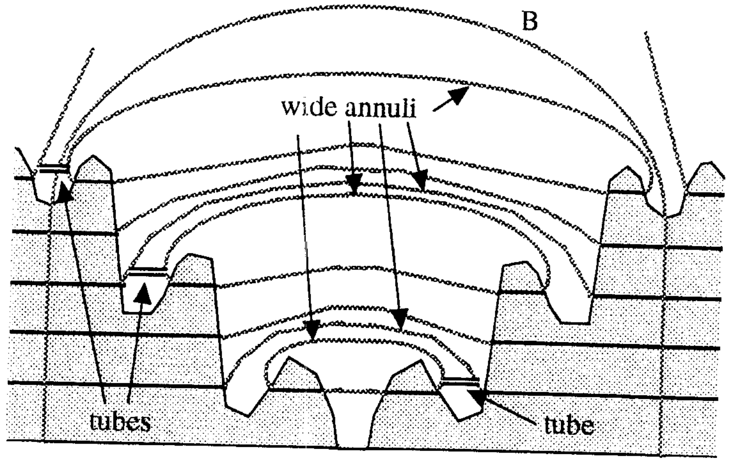

FIGURE 8. Wide annuli and stabilizing tubes

components lying in alternately $(\mathcal{F} \cup \mathcal{S} \cup \partial M) - \mathcal{A}$ and in $\mathcal{A} - (\mathcal{F} \cup \mathcal{S})$. At most one of the sides in each $2p$-gon lies in $\mathcal{F} \cup \partial M$. If T_0 lies in some V_i then either $p-2$ or p of its sides are dipping annuli, depending on whether or not T_0 abuts $\mathcal{F} \cup \partial M$. When T_0 abuts $\mathcal{F} \cup \partial M$ then the number of wide annuli is correspondingly 0 if $p = 2$ and $p-1$ if $p > 2$. When T_0 does not abut $\mathcal{F} \cup \partial M$ it is simply p.

It similarly follows from the construction in Theorem 5.2 that the number of tubes added in T_0 to create S^j is $p-2$ if T_0 abuts $\mathcal{F} \cup \partial M$ and is p or $p-1$ if it does not. The required inequality then follows from the inequalities $p - 2 \geq \frac{p-1}{2}$ when $p > 2$ and T_0 abuts $\mathcal{F} \cup \partial M$ and $p - 1 \geq \frac{p}{2}$ when $p > 1$ and T_0 does not abut $\mathcal{F} \cup \partial M$.

The previous argument can be extended to include the case in which one or both ends of B abut a dipping annulus A on its root side (when the exact construction would need to handle a tube running through B; cf. Figure 8): Just slide the end of B out (away from T) just beyond the end of A (so the end of B cuts off a small collar of a boundary component of $\mathcal{S} \cap \mathcal{A}$) before attaching plumbline annuli in T_A. The argument then goes through as above. □

7. Counting Annuli in the general case

In this section we use cut and paste techniques to bound the number of dipping annuli in a strongly irreducible generalized Heegaard splitting.

Definition 7.1. *Suppose $\mathcal{A} \subset M$ is a properly embedded collection of annuli. Then a component of $M - \mathcal{A}$ that is a solid torus whose boundary intersects $\mathcal{A}$ in a single longitudinal annulus is a* <u>*parallelism*</u>.

Put another way, an annulus that is ∂-parallel cuts off a parallelism, unless there are other components of $\mathcal{A}$ lying between it and ∂M.

Lemma 7.2. *Suppose that $(\mathcal{A}, \partial\mathcal{A}) \subset (W, \partial_+ W)$ is a collection of essential annuli in a compression body W and suppose $J(W) > 0$ (i. e. W is not a product). Then there is an essential disk D in W such that D is disjoint from $\mathcal{A}$. Moreover, if not all components of ∂A are spanning, then D may be chosen so that at least one side of D in $W - \eta(D)$ lies on a parallelism of $\mathcal{A}$ in $W - \eta(D)$.*

Proof: Let $\mathcal{D} \neq \emptyset$ be a set of defining disks for W. We argue by induction on $|\mathcal{A} \cap \mathcal{D}|$. If $\mathcal{A} \cap \mathcal{D} = \emptyset$ then all non-spanning components of $\mathcal{A}$ are ∂-parallel in the product $W - \eta(\mathcal{D})$. One of them cuts off a parallelism U which, because $\mathcal{A}$ is essential in W, is adjacent to some of the disks in $\mathcal{D}$. Then a disk in the boundary annulus $\partial U - \mathcal{A}$ that contains all such disks corresponds, back in W, to the required disk D.

If $\mathcal{A} \cap \mathcal{D} \neq \emptyset$, consider $\mathcal{D} \cap \mathcal{A}$. Since the annuli are incompressible, we can easily remove any circle of intersection. If there is an arc in $\mathcal{D} \cap \mathcal{A}$ that is inessential in $\mathcal{A}$, then let α be an outermost such arc in $\mathcal{A}$, and we may cut the disk D in $\mathcal{D}$ containing α along α and paste on two copies of the disk cut off by α in $\mathcal{A}$ to obtain a new disk D'. Replacing D by D' in $\mathcal{D}$ produces a new set of defining disks $\mathcal{D}'$ with $|\mathcal{A} \cap \mathcal{D}'| < |\mathcal{A} \cap \mathcal{D}|$. The result follows by induction.

On the other hand, if all arcs in $\mathcal{A} \cap \mathcal{D}$ are essential in $\mathcal{A}$, let β be an arc in $\mathcal{A} \cap \mathcal{D}$ that is outermost in $\mathcal{D}$. Let A be the annulus in $\mathcal{A}$ that gives rise to β. Cutting and pasting A along β and the outermost disk cut off in $\mathcal{D}$ yields a disk D' disjoint from $\mathcal{A}$. Since A is essential, it follows that D' is essential, and is as required. □

Lemma 7.3. *Suppose there is a collection $\mathcal{B}$ of essential annuli in a compression body W and $\mathcal{A} \subset \mathcal{B}$ is the subcollection of non-spanning annuli. Let $W_- \subset W - \eta(\mathcal{B})$ consist of those components which are incident to $\partial_- W$. Let $S = \partial_+ W - \eta(\mathcal{B})$, let $S_- \subset S = S \cap W_-$ and let $S_+ = S - S_-$. Let a denote the number of annulus components of S_+.*

Then $\mathcal{A}$ has at most $J(W) + \frac{a}{2}$ components.

Proof: With no loss of generality we can ignore spanning annuli and assume $\mathcal{B} = \mathcal{A}$.

Let $c(W, A) = 2J(W) + a - 2|\mathcal{A}|$; it suffices to show that $c(W, A) \geq 0$. The proof is by induction on $J(W)$. When $J(W) = 0$ there are no annuli, and there is nothing to prove. When $J(W) > 0$ it follows from Lemma 7.2 that there is a ∂-reducing disk D for W that is disjoint from $\mathcal{A}$. The result of cutting W along D is either one or two compression bodies W' with $J(W') = J(W) - 2$. In particular, J is lower in (each component of) W', so we can assume the Lemma is true in W'. Remove all inessential annuli from $\mathcal{A} \cap W'$ and call the result $\mathcal{A}'$. Define $c' = c(W', \mathcal{A}')$. By induction, it suffices to show that $c' \leq c = c(W, A)$.

The first step, cutting W along D, decreases $2J(W)$ by 4 and raises a by the number of inessential annuli that result, no more than 2. So this step decreases c by at least two. It may create, however, one or two components of $W' - \mathcal{A}'$ that are parallelisms.

Next examine what happens when an inessential annulus is removed: $-2|\mathcal{A}|$ goes up by two. Also a drops by at least one and will drop by two if A is adjacent to an annulus component of $\partial_+ W - \mathcal{A}$. The latter will happen, for example, if the inessential annulus that's removed is parallel to another one. Thus this step either reduces the number of components of $W' - \mathcal{A}'$ that are parallelisms and simultaneously increases c by at most one, or leaves both c and the number of parallelisms unchanged. Continue the process until all parallelisms (which, at the beginning, are no more than two) are eliminated. The result is to increase c by at most two. Combining both steps, c has not increased. □

Return now to the original context: M is a compact orientable 3-manifold and $\mathcal{A}$ a complete properly imbedded collection of essential annuli in M. Suppose

$$(V_1 \cup_{S_1} W_1) \cup_{F_1} \cdots \cup_{F_{m-1}} (V_m \cup_{S_m} W_m)$$

is a strongly irreducible generalized Heegaard splitting of M chosen so that $F_0 = \partial_- V_1$ contains $\partial\mathcal{A}$ and isotoped so that $(\mathcal{F} \cup \mathcal{S}) \cap A$ consists only of curves essential in both $\mathcal{F} \cup \mathcal{S}$ and $\mathcal{A}$, and such that this number is minimal. Let $M^1, ..., M^n$ be the components into which M is divided by the family of annuli $\mathcal{A}$. We apply Lemma 7.3 to the annuli $\mathcal{A} \cap V_i$.

Theorem 7.4. *For M and $\mathcal{A}$ as described above, let k be the number of dipping annuli in $\mathcal{A} - \mathcal{S}$, and w be the number of wide annuli in $\mathcal{S} - \mathcal{A}$. Then*

$$k \leq (\Sigma_{i=1}^{m} J(V_i)) + \frac{w}{2}.$$

Proof: Lemma 7.3 says that for each $1 \leq i \leq m$ the number of dipping annuli in $\mathcal{A} \cap V_i$ is at most $J(V_i) + \frac{a}{2}$, where a (defined in Lemma 7.3) counts the number of annuli components of $S_i - \mathcal{A}$ with a property that assures that they are adjacent to dipping annuli and therefore wide. That is, when Lemma 7.3 is applied to V_i the inequality remains true if a is replaced by the number of wide annuli in S_i. Summing over all V_i we get $k \leq (\Sigma_{i=1}^m J(V_i)) + \frac{w}{2}$ as required. □

In the appendix, Andrew Casson constructs an example that the inequality of Theorem 7.4 is in some sense best possible. That is, given any $c < 1$ there is an example for which $w = 0$ and $k > c(\Sigma_{i=1}^m J(V_i))$.

Corollary 7.5. *Let M and $\mathcal{A}$ be as described above. Then there is a Heegaard splitting $V_-^j \cup_{S_-^j} W_-^j$ for each M^j so that*

$$3(\Sigma_{i=1}^m J(V_i)) \geq \sum_j J(V_-^j).$$

Proof: Let k denote the number of dipping annuli in $\mathcal{A} \cap (\cup_i V_i)$, and let w denote the number of wide annuli in $\mathcal{S} - \mathcal{A}$. The construction in Theorem 5.2 yields generalized Heegaard splittings for $C(K^j)$ such that $2k + \sum J(V_i) \geq \sum_{i,j} J(V_i^j)$.

Now substitute for k from Theorem 7.4 to get

$$3(\Sigma_{i=1}^m J(V_i)) + w \geq \sum_{i,j} J(V_i^j).$$

According to Lemma 6.8 the induced Heegaard splittings for the $C(K^j)$ can be destabilized at least $\frac{w}{2}$ times, yielding Heegaard splittings $V_-^j \cup_{S_-^j} W_-^j$ for each $C(K^j)$ with $\sum_j J(V_-^j) \leq \sum_{i,j} J(V_i^j) - w$. It follows that

$$3(\Sigma_{i=1}^m J(V_i)) \geq \sum_j J(V_-^j).$$

□

Corollary 7.6. *If $K^1, \dots, K^n \subset S^3$ are prime knots knots then*

$$t(K^1 \# \dots \# K^n) \geq \frac{1}{3}(t(K^1) + \dots + t(K^n)).$$

Proof: By Lemma 2.10 there is a generalized Heegaard splitting

$$C(K^1 \# \dots \# K^n) \cong (V_1 \cup_{S_1} W_1) \cup_{F_1} \cdots \cup_{F_{m-1}} (V_m \cup_{S_m} W_m)$$

for which $\sum_{i=1}^m J(V_i) = 2t(K^1 \# \dots \# K^n)$. Let $\mathcal{A}$ be the $n-1$ annuli that decompose $C(K^1 \# \dots \# K^n)$ into the complements of the constituent prime knots. We may assume that $\mathcal{F} \cup \mathcal{S}$ and $\mathcal{A}$ have been chosen so that $(\mathcal{F} \cup \mathcal{S}) \cap \mathcal{A}$ consists only of essential curves and then also minimizes the number of such intersections. The family $\mathcal{A}$ is complete since each K^j is prime. Apply Corollary 7.5, substitute from Lemma 2.10, and divide by two to get $3t(K^1 \# \dots \# K^n) \geq t(K^1) + \dots + t(K^q)$ as required. □

Following Theorem 5.6, the same sort of argument can be applied to essential tori in M. For M a knot complement, the application is to satellite knots.

Definition 7.7. [BZ] *Let $\tilde{K}$ be a knot in a 3-sphere S^3 and V an unknotted solid torus in S^3 with $\tilde{K} \subset V \subset S^3$. Assume that $\tilde{K}$ is not contained in a 3-ball of V. A homeomorphism $h : V \to \hat{V}$ onto a tubular neighborhood $\hat{V}$ of a nontrivial knot $\hat{K} \subset S^3$ which maps a meridian of $S^3 - V$ onto a longitude of $\hat{K}$ maps $\tilde{K}$ onto a knot $K = h(\tilde{K}) \subset S^3$. The knot K is called a satellite of $\hat{K}$, and $\hat{K}$ is called its companion. The pair $(V, \tilde{K})$ is called a pattern of K.*

Theorem 7.8. *Let K be a satellite knot, then $t(K) \geq \frac{1}{3}(t(\tilde{K}) + t(\hat{K}))$.*

Proof: We merely sketch the proof. Let $(V_1 \cup_{S_1} W_1) \cup_{F_1} \cdots \cup_{F_{m-1}} (V_m \cup_{S_m} W_m)$ be a generalized Heeggard splitting for $C(K)$ for which $\sum_{i=1}^{m} J(V_i) = 2t(K)$. Isotope ∂V so it intersects $\mathcal{F} \cup \mathcal{S}$ only in curves essential in both, and in a minimal number of these curves. Attach copies of $torus \times I$ to $S^3 - \hat{V}$ and to $\hat{V}$ and complete the surfaces $(\mathcal{F} \cup \mathcal{S}) - \hat{V}$ and $(\mathcal{F} \cup \mathcal{S}) \cap \hat{V}$ to give Heegaard splittings of $C(\hat{K})$ and $V - \eta(\tilde{K})$ respectively, following Theorem 5.6. The latter can be made a Heegaard splitting of $C(\tilde{K})$ by just filling in a solid torus along ∂V. Destabilizations can be found for these Heeegaard splittings just as in Lemma 6.8. □

8. Counting annuli - specialized case

We can improve the count of dipping annuli in Section 7 if we add the one further assumption that each component of $\mathcal{S} - \mathcal{A}$ has an even number of boundary components. This condition will certainly be satisfied when the annuli are the decomposing annuli from a knot summation. More generally,

Definition 8.1. *For C a collection of circles, the parity class $\alpha \in H^1(C, \mathbb{Z}_2)$ is the class that is non-trivial on each fundamental class of a component of C.*

Informally, the parity class just counts the parity of the number of components, even or odd.

Lemma 8.2. *Suppose the properly embedded essential annuli $\mathcal{A} \subset M$ have the property that the parity class of $\partial\mathcal{A} \subset \partial M$ is the restriction of a class in*

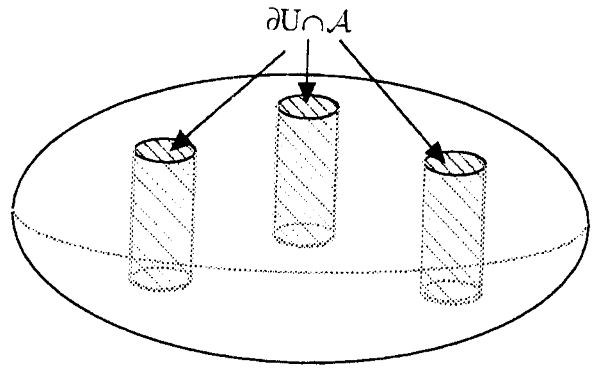

FIGURE 9. A special component with $r = 3, I = 0$.

$H^1(M, \mathbb{Z}_2)$. As usual, let $M^1, \ldots, M^j$ be the closed complementary components of $\mathcal{A}$ in M, so each annulus in $\mathcal{A}$ becomes two annuli in $\partial(\cup_j M^j)$. If $(S, \partial S) \subset (M^j, \mathcal{A} \cap M^j)$ is an orientable, properly embedded surface in any M^j, then $|\partial S|$ is even.

Proof: We may as well assume S is connected. Let $[S]$ denote the non-trivial class of $H_2(S, \partial S; \mathbb{Z}_2)$, let α be the parity class of $\partial\mathcal{A} \subset \partial M$ and let $\tilde{\alpha} \in H^1(M)$ be a class such that $i^*(\tilde{\alpha}) = \alpha$. Then, after homotoping ∂S to ∂A we see that the evaluation $[\alpha, \partial[S]] = [\delta(\alpha), [S]] = [\delta(i^*(\tilde{\alpha})), [S]] = 0$ by the exactness of the cohomology sequence for the pair $(M, \partial A)$. □

Definition 8.3. *A collection of annuli $\mathcal{A} \subset M$ as in Lemma 8.2 satisfies the parity condition.*

This section will be a repeat of Section 7 in the special case in which the annuli $\mathcal{A}$ satisfy the parity condition in M, so every component of $\mathcal{S} - \mathcal{A}$ has an even number of boundary components. The more delicate analysis will require a new notion:

Definition 8.4. *Suppose that $(\mathcal{A}, \partial\mathcal{A}) \subset (W, \partial_+ W)$ is a collection of essential annuli in a compression body W. A component U of $W - \eta(\mathcal{A})$ is called special if there is a planar surface P with boundary components $p_0, \ldots, p_r$ and a homeomorphism $(U, \partial U \cap \mathcal{A}) \cong (P \times I, \cup_{i=1}^r p_i \times I)$. See Figure 9.*

The index $I(U)$ of the special component U is defined to be $3 - r$.

For example, a boundary parallel annulus cuts off from W a parallelism (see Definition 7.1). This is a special component of index 2, for in this case P is an annulus and $r = 1$. More generally, a special component is obtained from a collection of r such examples by connecting them with $r - 1$ 1-handles. An exceptional component (see Definition 6.6) is special in the compression body in which it lies. Its index is 1. Any ∂-reducing disk $(D, \partial D) \subset (U, \partial U \cap \partial_+ W)$ for W that lies in a special component U divides U into two special components in the boundary-reduced manifold W', and the sum of their indices is $index(U) + 3$.

We now embark on improving the bound of Lemma 7.3 under the parity assumption: every component of $\partial W - \mathcal{B}$ has an even number of boundary components. The argument is in three stages, incorporated in the following lemmas that culminate in Lemma 8.7:

Lemma 8.5. *Suppose there is a collection $\mathcal{B}$ of essential annuli in a compression body W and $\mathcal{A} \subset \mathcal{B}$ is the subcollection of non-spanning annuli. Let $W_- \subset W - \eta(\mathcal{B})$ consist of those components which are incident to $\partial_- W$. Let $S = \partial_+ W - \eta(\mathcal{B})$ and suppose every component of S has an even number of boundary components. Let $S_- \subset S = S \cap W_-$ and let $S_+ = S - S_-$. Let m be the number of annuli in $\mathcal{A}$ that are adjacent to W_- (on either side or both sides), n be the number of components of $\mathcal{A}$ that are not adjacent to W_-, and p be the number of spanning annuli. Let s be the number of special components of $W - \mathcal{B}$ (see Definition 8.4).*

Then

$$J(W) + |S_+| - |S_-| + genus(S_-) - m + p + \frac{\chi(\partial_- W)}{2} - 2n + s \geq 0.$$

Proof: If a spanning component of $\mathcal{B}$ is removed, the only effect is to lower p by one and to raise $[-|S_-| + genus(S_-)]$ by one. So we may as well assume that there are no spanning annuli, so $\mathcal{B} = \mathcal{A}$ and $p = 0$.

The proof will be by induction on $J = J(W)$. When $J = 0$ then W is a product, and there are no annuli. In particular, $S_- \cong \partial_- W$ so the inequality follows from

$$|\partial_- W| - genus(\partial_- W) = \frac{\chi(\partial_- W)}{2}.$$

Adding 1-handles on $\partial_+ W$ only increases $J(W) + genus(S_-)$ and has no other effect, so the Lemma follows from the case $J = 0$ whenever $\mathcal{A} = \emptyset$. So we henceforth assume $\mathcal{A} \neq \emptyset$.

For the collection $(\mathcal{A}, \partial \mathcal{A}) \subset (W, \partial_+ W)$ define

$$b(W, \mathcal{A}) = J(W) + |S_+| - |S_-| + genus(S_-) - m + p - 2n + s.$$

Following Lemma 7.2, there is an essential disk D in W that is disjoint from $\mathcal{A}$ and cuts off a parallelism from the component of $W - \mathcal{A}$ in which it lies. The result of cutting W along D is either one or two compression bodies W' with $J(W') = J(W) - 2$. In particular, J is lower in (each component of) W', so we can assume the Lemma is true in W'. Recall that D has been chosen so that at least one of the adjacent components of $W' - \mathcal{A}$ is a parallelism of $\mathcal{A}$ in W'. Remove all inessential annuli from $\mathcal{A} \cap W'$ and call the result $\mathcal{A}'$.

Abbreviate $b(W, \mathcal{A})$ to simply b and set $b' = b(W', \mathcal{A}')$. We will show that $b \geq b'$. Since it follows from the inductive hypothesis that

$$b' \geq -\frac{\chi(\partial_- W')}{2} = -\frac{\chi(\partial_- W)}{2},$$

we will then be able to conclude that

$$b \geq -\frac{\chi(\partial_- W)}{2}$$

as required.

The process by which b' can be calculated from b consists of two steps: First ∂-reduce W along D, then remove all resulting inessential annuli. We examine the effect of each move on b (in the interim extending the definition of b also to the case in which $\mathcal{A}$ may contain boundary parallel annuli). There are a number of possible cases that arise in each step. We will examine each in turn and determine the effect on each of the constituents of b, denoting, for example, that $|S_+|$ goes up by at most two with the notation $|S_+|\uparrow \leq 2$.

Step A: ∂-reducing W along D.

1. *D is contained in a special component.* Then $J\downarrow 2$, $|S_+|\uparrow 1$, $s\uparrow 1$ so b is unchanged.
2. *D is contained in a non-special component not in W_-.* Then $J\downarrow 2$, $|S_+|\uparrow 1$, $s\uparrow 1$ so b is unchanged.
3. *D lies in W_-.* Then $J\downarrow 2$, $S_-\uparrow 1$ and $s\uparrow 1$. The annulus abutting the parallelism created either abuts W_- on the other side (in which case $-m$ and n are both unchanged) or does not (in which case $-m\uparrow 1$ and $-2n\downarrow 2$). Hence in any case b does not increase.

Step B: Inessential annuli A removed. There are various cases, depending on what sort of component C of $W - \mathcal{A}$ lies on the other side of the annulus A (we will say that A lies *on* the component C) and whether or not the two boundary components of A lie on the same or different components of $C \cap \partial_+ W$ (we will say that A is respectively non-separating or separating). For example, it may be that C becomes special when A is removed. In this case A is necessarily separating, since, by definition of "special", $\partial C \cap \partial_+ W'$ is planar.

1. *A is separating, doesn't lie on W_-, and isn't on a component that becomes special.* Then $|S_+|\downarrow 2$, $-2n\uparrow 2$, $s\downarrow 1$ (the last since the product component cut off by A is special by definition). Hence b drops by 1.
2. *A lies on a component that becomes special.* Then $|S_+|\downarrow 2$, $-2n\uparrow 2$ so b is unchanged.

3. *$\mathcal{A}$ is non-separating and doesn't lie on W_-.* Then $|S_+|\downarrow 1$, $-2n\uparrow 2$, $s\downarrow 1$. Hence b is unchanged.
4. *$\mathcal{A}$ is separating and lies on W_-.* Then $|S_+|\downarrow 1$, $-|S_-|\uparrow 1$, $-m\uparrow 1$, $s\downarrow 1$. Hence b is unchanged.
5. *$\mathcal{A}$ is non-separating and lies on W_-.* Then $|S_+|\downarrow 1$, $genus(S_-)\uparrow 1$, $-m\uparrow 1$, $s\downarrow 1$. Hence b is unchanged.

Thus we conclude that $b' \leq b$ as required. □

Lemma 8.6. *With the hypotheses and notation of Lemma 8.5, let also a denote the number of annuli components of S_+, as in Lemma 7.3. Then $\mathcal{A}$ has at most $\frac{3}{4}J(W) + \frac{1}{2}(a+s)$ components.*

Proof: By the special assumption that any orientable properly embedded surface in M^j has an even number of ∂-components, each component of S_+ is either an annulus or has Euler characteristic no greater than -2. Hence we have

$$|S_+| \leq a - \frac{\chi(S_+)}{2} = a - \frac{\chi(\partial_+ W)}{2} + \frac{\chi(S_-)}{2} \tag{1}$$

On the other hand,

$$-|S_-| + genus(S_-) + m + p \leq -\frac{\chi(S_-)}{2}$$

so

$$-|S_-| + genus(S_-) - m + p + \frac{\chi(S_-)}{2} \leq -2m. \tag{2}$$

(Equality fails when an annulus abuts W_- on both sides, thereby contributing 4 components to ∂S_-.) Adding inequalities (1) and (2) to the equality

$$\frac{\chi(\partial_- W)}{2} - \frac{\chi(S_-)}{2} = \frac{J}{2} + \frac{\chi(\partial_+ W)}{2} - \frac{\chi(S_-)}{2}$$

we get

$$|S_+| - |S_-| + genus(S_-) - m + p + \frac{\chi(\partial_- W)}{2} \leq a + \frac{J}{2} - 2m$$

so from Lemma 8.5 we get

$$\frac{3J}{2} + a - 2m - 2n + s \geq 0$$

as required. □

Lemma 8.7. *With the hypotheses and notation of Lemma 8.6, let also I be the sum of the indices of the special components of $W - \mathcal{A}$ (See Definition 8.4). Then $\mathcal{A}$ has at most $\frac{3}{4}J(W) + \frac{1}{2}(a+I)$ components.*

Proof: Let $c = \frac{3}{2}J(W) + a + I - 2|\mathcal{A}|$; it suffices to show that $c \geq 0$. The proof is by induction on J. If there are no special components (e. g. when $J = 0$), the result follows immediately from Lemma 8.6. Otherwise, let D be a ∂-reducing disk for W that lies in a special component U of $W - \mathcal{A}$. We will show that ∂-reduction of W along D and the removal of any resulting inessential annuli cannot increase c in the compression body (or compression bodies) that arise.

The first step, cutting W along D, decreases $\frac{3}{2}J(W)$ by 3, raises I by 3 and raises a by the number $\pi = 1$ or 2 of parallelisms in the two components $U - D$. Thus $c - \pi$ is unchanged.

Now examine the result of removing an inessential annulus $A \in \mathcal{A}$ that cuts off a parallelism, an annulus such as is created when W is cut along D. Let U' be the component of $W - \mathcal{A}$ that abuts A on the opposite side from U. U' may become special when A is removed. There are several cases:

1. *A was not adjacent to an annulus component of $S_+ \subset \partial W_+$* Then even if U' becomes special, $I(U') \leq 0$ so $-2|\mathcal{A}|\uparrow 2$, $a\downarrow 1$, $I\downarrow \geq 2$ (since the index of a parallelism is 2), $-\pi\uparrow 1$. Hence $c - \pi\downarrow \geq 0$.
2. *A is adjacent to an annulus component of $S_+ \subset \partial W_+$, U' is not special or $I(U') \leq 0$.* Then $-2|\mathcal{A}|\uparrow 2$, $a\downarrow 2$, $I\downarrow \geq 2$, $-\pi\uparrow 1$. Hence $c - \pi\downarrow \geq 1$.
3. *A is adjacent to an annulus component of $S_+ \subset \partial W_+$, U' is special and $I(U') = 1$.* Then $-2|\mathcal{A}|\uparrow 2$, $a\downarrow 2$, $I\downarrow 1$, $-\pi\uparrow 1$. Hence $c - \pi$ is unchanged.
4. *A is adjacent to an annulus component of $S_+ \subset \partial W_+$, U' is special and $I(U') = 2$.* Then U' becomes a parallelism and $-2|\mathcal{A}|\uparrow 2$, $a\downarrow 2$, both I and $-\pi$ unchanged. Hence $c - \pi$ is unchanged.

So $c - \pi$ never increases and may decrease. Continue removing annuli cutting off parallelisms until $\pi = 0$, as it was before the disk D was removed. In the end, c also will not have increased. □

Return now to the original context: M is a compact orientable 3-manifold and $\mathcal{A}$ is a complete properly imbedded collection of essential annuli in M satisfying the parity condition (see Definition 8.3). Suppose

$$(V_1 \cup_{S_1} W_1) \cup_{F_1} \cdots \cup_{F_{m-1}} (V_m \cup_{S_m} W_m)$$

is a strongly irreducible generalized Heegaard splitting of M chosen so that $F_0 = \partial_- V_1$ contains $\partial\mathcal{A}$ and isotoped so that $(\mathcal{F} \cup \mathcal{S}) \cap A$ consists only of curves essential in both $\mathcal{F} \cup \mathcal{S}$ and $\mathcal{A}$, and such that this number is minimal. Let $M^1, \ldots, M^n$ be the components into which M is divided by the family of annuli $\mathcal{A}$. We will soon apply Lemma 8.7 to the annuli $\mathcal{A} \cap V_i$.

Notice that the index of a special component of $V_i - \mathcal{A}$ is non-positive unless the component U is of index 1, when $\partial_S U = \partial U \cap S_i$ is a 4-punctured sphere. Since $\partial_S U$ visibly compresses in U, it follows from strong irreducibility that $\partial_S U$ also compresses in W_i. It then follows from a standard outermost arc argument, that $\partial_S U \cap S_i$ compresses in $W_i - \mathcal{A}$. The result is two annuli. If the two annuli are parallel, this implies that U is exceptional (see Definition 6.6). If they are not, then the union of the two annuli, together with the annuli $\partial U - \partial_S U$, cuts off a 2-bridge knot or link complement from $M - \mathcal{A}$, necessarily one of the M^j since $\mathcal{A}$ is complete.

Definition 8.8. *A component of M^j which is a 2-bridge knot or link complement containing, as above, an index 1 special component of some $V_i - \mathcal{A}$ is called an* <u>*interlaced*</u> *2-bridge complement.*

The following theorem now improves Theorem 7.4, when $\mathcal{A} \subset M$ satisfies the parity condition:

Theorem 8.9. *For M and $\mathcal{A}$ as described above, let k be the number of dipping annuli in $\mathcal{A} - \mathcal{S}$, w be the number of wide annuli in $\mathcal{S} - \mathcal{A}$, and e_V be the number of exceptional components cut off by $\mathcal{A}$ from all the V_i (see Definitions 5.1, 6.6). Let $c \leq m, n$ be the number of interlaced 2-bridge complements among the M^j. Then*

$$k \leq \frac{3}{4}(\Sigma_{i=1}^m J(V_i)) + \frac{w + e_V + c}{2}.$$

Proof: We will apply Lemma 8.7 to the annuli $\mathcal{A} \cap V_i$.

Suppose, as a first simplification, that each special component of each $V_i - \mathcal{A}$ has non-positive index, so in particular $e = c = 0$. Then Lemma 8.7 says that the number of dipping annuli in $\mathcal{A} \cap V_i$ is at most $\frac{3}{4}J(V_i) + \frac{a}{2}$, where a counts the number of annuli components of $S_i - \mathcal{A}$ with a property (defined in Lemma 8.6) that assures that they are adjacent to dipping annuli and therefore wide. That is, when Lemma 8.7 is applied to V_i the inequality remains true if a is replaced by the number of wide annuli in S_i. Summing over all V_i we get $k \leq \frac{3}{4}(\Sigma_{i=1}^m J(V_i)) + \frac{w}{2}$ as required.

Now examine what happens if $V_i - \mathcal{A}$ has a special component U of positive index. Then, as explained above, U is of index 1 and so is either an exceptional component or part of an interlaced 2-bridge complement. In either case, the positive index contributed by U to I in Lemma 8.7 is incorporated into $e_V + c$.

□

Lemma 8.10. *For M and $\mathcal{A}$ as described above, let M^- be the manifold obtained from M by replacing each interlaced 2-bridge complement M^j with an unknot or unlink complement (matching longitudes). Then M^- has a (perhaps weakly reducible) generalized Heegaard splitting*

$$(V_1^- \cup_{S_1} W_1^-) \cup_{F_1} \cdots \cup_{F_{m-1}} (V_m^- \cup_{S_m} W_m^-)$$

so that each $V_i^- \cong V_i$ and $W_i^- \cong W_i$.

Proof: One way of constructing M^- would be, on each index 1 special component $U \subset (V_i - \mathcal{A})$ that comes from an interlaced 2-bridge complement, reglue V_i to W_i differently along $\partial_S U = \partial U \cap S_i$. Depending on how the ∂-reducing disk D_W for W_i separates pairs of boundary components of the 4-punctured sphere $\partial_S U$, this regluing can be done so that the ∂-reducing disk D_U of U intersects ∂D_W in either two points or none. In the former case, the replacement is by an unknot complement and U becomes exceptional (see Definition 6.6). In the latter, the replacement is by the complement of an unlink of two components, and M^- is reducible. □

Corollary 8.11. *Let M and $\mathcal{A}$ be as described above, with $\mathcal{A} \subset M$ a complete collection of annuli satisfying the parity condition. Order the summands so that the interlaced 2-bridge complements among the M^j, if any, are $M^{q+1}, \dots, M^n$. Then there is a Heegaard splitting $V_-^j \cup_{S_-^j} W_-^j$ for each M^j so that*

$$\frac{5}{2}(\Sigma_{i=1}^m J(V_i)) + (n-q) \geq \sum_j J(V_-^j).$$

Proof: We may assume that the surfaces $\mathcal{F} \cup \mathcal{S}$ from the generalized Heegaard splitting and the annuli $\mathcal{A}$ have been isotoped so that $(\mathcal{F} \cup \mathcal{S}) \cap \mathcal{A}$ consists only of essential curves and a minimal number of them. Let k denote the number of dipping annuli in $\mathcal{A} \cap (\cup_i V_i)$, w denote the number of wide annuli in $\mathcal{S} - \mathcal{A}$, e denote the total number of exceptional components in the $V_i - \mathcal{A}$ and the $W_i - \mathcal{A}$, and e_V denote only the number that lie in the $V_i - \mathcal{A}$. The construction in Theorem 5.2 yields generalized Heegaard splittings for $C(K^j)$ such that $2k + \sum J(V_i) \geq \sum_{i,j} J(V_i^j)$.

Now substitute for k from Theorem 8.9 to get

$$\frac{5}{2}(\Sigma_{i=1}^m J(V_i)) + w + e_V + (n-q) \geq \sum_{i,j} J(V_i^j).$$

According to Lemma 6.8 the induced Heegaard splittings for the $C(K^j)$ can be destabilized at least $\frac{w}{2} + e$ times, yielding Heegaard splittings $V_-^j \cup_{S_-^j} W_-^j$ for each $C(K^j)$ with

$$\sum_j J(V_-^j) \leq \sum_{i,j} J(V_i^j) - w - 2e \leq \sum_{i,j} J(V_i^j) - w - e_V.$$

It follows that

$$\frac{5}{2}(\Sigma_{i=1}^m J(V_i)) + (n-q) \geq \sum_j J(V_-^j)$$

as required. □

9. Tunnel Numbers

In this section we apply the results above to tunnel numbers of composite knots. The results can also be formulated for composite links, though the statements are sometimes a bit more cumbersome.

Theorem 9.1. *Let $K^1, \dots, K^n$ be prime knots, and suppose $K^{p+1}, \dots, K^n$ are the 2-bridge knots among the K^j. Then there is some q such that $p \leq q \leq n$ and $t(K^1\# \dots \# K^n)$ is no smaller than any of:*

- $\frac{2}{5}(t(K^1) + \dots + t(K^q)) + \frac{n-q}{5}$
- $(n-q)$
- $t(K^1\# \dots \# K^q)$

Proof: By Lemma 2.10 there is a generalized Heegaard splitting

$$C(K^1\# \dots \# K^n) \cong (V_1 \cup_{S_1} W_1) \cup_{F_1} \dots \cup_{F_{m-1}} (V_m \cup_{S_m} W_m)$$

for which $\sum_{i=1}^m J(V_i) = 2t(K^1\# \dots \# K^n)$. Let $\mathcal{A}$ be the $n-1$ annuli that decompose $C(K^1\# \dots \# K^n)$ into the complements of the constituent prime knots. Clearly the non-trivial element of $H^1(C(K^1\# \dots \# K^n), \mathbb{Z}_2)$ restricts to the parity class on meridians, so the parity condition holds for $\mathcal{A} \subset C(K^1\# \dots \# K^n)$.

Let $q \geq p$ be the number of components of the $C(K^j)$ that are not interlaced 2-bridge complements. The family $\mathcal{A}$ is complete since each K^j is prime. It then follows from Corollary 8.11 that

$$\frac{5}{2}(\Sigma_{i=1}^m J(V_i)) + (n-q) \geq \sum_j J(V_-^j).$$

Since 2-bridge knots have tunnel number one, we further know that, for $j \geq p$, hence $j \geq q$, that $J(V_-^j) \geq 2$. Hence another way to write the inequality is

$$\frac{5}{2}(\Sigma_{i=1}^m J(V_i)) \geq \sum_{j=1}^q J(V_-^j) + (n-q)$$

or, substituting from Lemma 2.10 and dividing by two,

$$\frac{5}{2}t(K^1\# \dots \# K^n) \geq t(K^1) + \dots + t(K^q) + \frac{n-q}{2}.$$

This gives the first inequality.

The second inequality follows from the fact that each V_i contributes at least 2 to $\Sigma_{i=1}^m J(V_i)$ and the number $n-q$ of interlaced 2-bridge knot complements can be no bigger than m.

The last inequality follows from Lemma 8.10. Here M^- is the manifold obtained by replacing each $C(K^j)$, $q+1 \leq j \leq n$ with the complement of the unknot. Hence $M^- = C(K^1\#K^2\# \dots K^q)$. It has a generalized Heegaard splitting whose constituent compression bodies are the same as that of the untelescoped minimal genus Heegaard splitting for M itself, so when the splitting of M^- is amalgamated, the result is a Heegaard splitting of genus no higher than that of M. □

Corollary 9.2. *Let $K^1, \ldots, K^n$ be prime knots.*

1. $t(K^1 \# \ldots \# K^n) \geq \frac{1}{3}(t(K^1) + \ldots + t(K^n))$.
2. *If none of the K^j are 2-bridge knots then* $t(K^1 \# \ldots \# K^n) \geq \frac{2}{5}(t(K^1) + \ldots + t(K^n))$.
3. $t(K^1 \# K^2) \geq \frac{2}{5}(t(K^1) + t(K^2))$.

Proof: The first statement of course is just Corollary 7.6 but here it can be seen (somewhat intriguingly) to follow also from the first and second inequalities of Theorem 9.1 and the fact that, for any $a, b \geq 0$, $max\{(\frac{2}{5}a + \frac{1}{5}b), b\} \geq \frac{1}{3}(a+b)$.

The second statement follows immediately from Theorem 9.1 by setting $p = n$.

The last is immediate if both or neither of the K^j are 2-bridge. If exactly K^2 is 2-bridge, the result follows from the last inequality of Theorem 9.1. □

Appendix A. Examples with many dipping annuli

ANDREW CASSON

Theorem A.1. *For any number $c < 2$, there is a 3-manifold*

$$M = X_1 \cup Y_1 \cup X_2 \cup Y_2 \cup \cdots \cup X_k \cup Y_k$$

where X_i and Y_i are compression bodies, and an essential annulus A properly embedded in M, such that the following conditions are satisfied.

1. *$\partial M = \partial_- X_1$ is an incompressible torus and $\partial_- Y_k = \emptyset$.*
2. *$X_i \cap Y_i = \partial_+ X_i = \partial_+ Y_i$, and $X_i \cup Y_i$ is strongly irreducible.*
3. *$Y_i \cap X_{i+1} = \partial_- Y_i = \partial_- X_{i+1}$ is incompressible in M.*
4. *The components of $A \cap X_i$, $A \cap Y_i$ are essential sub-annuli of A.*
5. *If X_i has n_i 2-handles, and $A \cap X_i$ contains d_i components which are disjoint from $\partial_- X_i$, then $\sum_{i=1}^{k} d_i \geq c \sum_{i=1}^{k} n_i$.*

Most of the pairs $(X_i, A \cap X_i)$ and $(Y_i, A \cap Y_i)$ are homeomorphic to a standard model (V_n, F_n), constructed in the following lemma.

Lemma A.2. *For each integer $n > 0$ there is a compression body V_n, a homeomorphism $h : \partial_+ V_n \to \partial_+ V_n$, and a surface F_n properly embedded in V_n, satisfying the following conditions.*

1. *V_n has n 2-handles and $\partial_- V_n$ is connected.*
2. *$V_n \cup_h V_n$ is strongly irreducible.*
3. *F_n consists of $2n-1$ essential, non-parallel annuli disjoint from $\partial_- V_n$ together with $4n-2$ essential, non-parallel annuli meeting both $\partial_+ V_n$ and $\partial_- V_n$.*
4. *$h(F_n \cap \partial_+ V_n) = F_n \cap \partial_+ V_n$.*
5. *$F_n \cup_h F_n$ consists of $4n-2$ annuli, each meeting both boundary components of $V_n \cup_h V_n$.*

Proof: Let S^2 be the unit sphere $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$. Let r_x, r_y, r_+ and r_- be the reflections of S^2 in the planes $y = 0$, $x = 0$, $x = y$ and $x = -y$ respectively, generating a group Γ of order 8. The fixed point

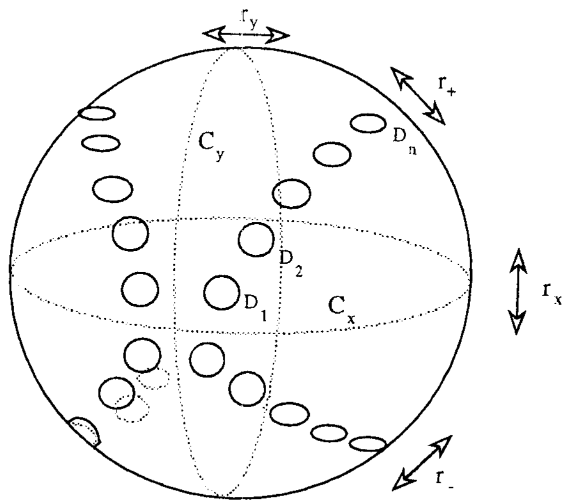

FIGURE 10. The $4n$ disks D, with $n = 5$.

sets of r_x and r_y are the circles C_x and C_y where S^2 intersects the xz and yz planes respectively.

Let $Q^+ = \{(x, y, z) \in S^2 : x \geq 0, y \geq 0\}$. Let $D_1, D_2, \ldots D_n$ be disjoint closed disks in the interior of Q^+ such that $r_+(D_i) = D_i$, for example, small round disks in S^2 with centers evenly spaced on the semicircle where Q^+ meets the plane $x = y$. Let $D^+ = D_1 \cup D_2 \cup \cdots \cup D_n$ and $D = D^+ \cup r_x(D^+) \cup r_y(D^+) \cup r_-(D^+)$; then D is a Γ-invariant disjoint union of $4n$ disks in S^2. (See Figure 10.) Set $P^+ = \overline{Q^+ \setminus D^+}$ and $P = \overline{Q \setminus D}$, so P is a $4n$-punctured sphere on which Γ acts. Let $S = P \cup_\partial P'$, where P' is a connected surface with $\partial P' = \partial P$ such that the action of Γ extends over P'. For example, one could choose $P' = P$; then S is the double of P, a surface of genus $4n - 1$.

Let $C_x^+ = P^+ \cap C_x$, and let $R = (P^+ \times 0) \cup (C_x^+ \times I) \cup (P^+ \times 1)$ as a subset of of $\partial(P^+ \times I)$. There is a homeomorphism $g : R \to P^+ \cup r_x(P^+)$ such that $g(p, 0) = p$ and $g(p, 1) = r_x(p)$ for all $p \in P^+$ outside a small neighborhood N of C_x^+, and $r_x g(p, t) = g(p, 1 - t)$ for every point $(p, t) \in R$. Let V_n be the 3-manifold obtained from the disjoint union of $S \times I$ and $P^+ \times I$ by identifying each point $(p, t) \in R \subset P^+ \times I$ with $(g(p, t), 1) \in S \times 1 \subset S \times I$. By using a collar neighborhood of R in $P^+ \times I$, construct an embedding $e : P^+ \times I \to V_n$ such that $e(p, t) = (g(p, t), 0) \in S \times 0$ for all $(p, t) \in R$, and $e(P^+ \times I)$ is disjoint from the subset $(S \setminus (P^+ \cup r_x(P^+)) \times I$ of V_n.

Let $I_1, I_2, \ldots I_n$ be disjoint arcs properly embedded in P^+, with end-points on C_x^+ and such that I_j separates D_j from each D_i with $i \neq j$. Then $e(I_j \times I)$ is an embedded disk in V_n with boundary in $S \times 0$. This exhibits V_n as a compression body with $\partial_+ V_n = S \times 0$ and with $e(I_1 \times I), e(I_2 \times I), \ldots e(I_n \times I)$

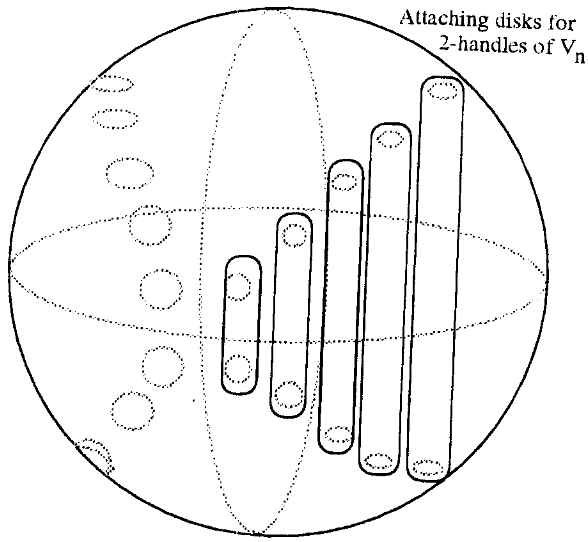

FIGURE 11. The attaching disks $e(\partial(I_j \times I))$

as the cores of the 2-handles. (See Figure 11.) Since P' was chosen to be connected, $\partial_- V_n$ is connected.

For $1 \leq i \leq n$ let C_i be a simple loop in P^+ that is parallel to the boundary component ∂D_i and invariant under r_+. Enlarge $C_1, C_2, \ldots C_n$ to a maximal collection $C_1, C_2, \ldots C_{2n-1}$ of disjoint essential non-parallel simple loops in P^+ in such a way that $r_+(C_i) = C_i$ for all i. (See Figure 12.) Set $A_i = e(C_i \times I)$; then $A_1, A_2, \ldots A_{2n-1}$ are disjoint properly embedded annuli in V_n. If C_i is chosen disjoint from the neighborhood N of C_x^+, then $\partial A_i = C_i \cup r_x(C_i)$. V_n also contains disjoint properly embedded annuli $A'_1, A'_2, \ldots A'_{2n-1}$ and $A''_1, A''_2, \ldots A''_{2n-1}$, where $A'_i = r_y(C_i) \times I$ and $A''_i = r_-(C_i) \times I$. Let F_n be the union of the annuli A_i, A'_i and A''_i for $1 \leq i \leq 2n-1$.

Let $h = t_x t_y r_+ : S \to S$, where t_x and t_y are Dehn twists in C_x and C_y respectively. By Corollary A.4 below, $V_n \cup_h V_n$ is strongly irreducible. The intersection of F_n with $\partial_+ V_n$ is the union of the curves C_i, $r_x(C_i)$, $r_y(C_i)$ and $r_-(C_i)$ for $1 \leq i \leq 2n-1$, which is invariant under h. In $V_n \cup_h V_n$, each of $(A_i \cup A'_i) \cup_h (A_i \cup A'_i)$ and $A''_i \cup_h A''_i$ is a single properly embedded annulus.

□

Lemma A.3. *Let N be a regular neighborhood of $C_x \cup C_y$ in S. Suppose L is an essential simple loop in S which bounds a disk in V_n and intersects $\partial N \cup C_y$ minimally. Then some component of $L \cap N$ intersects C_x and is disjoint from C_y.*

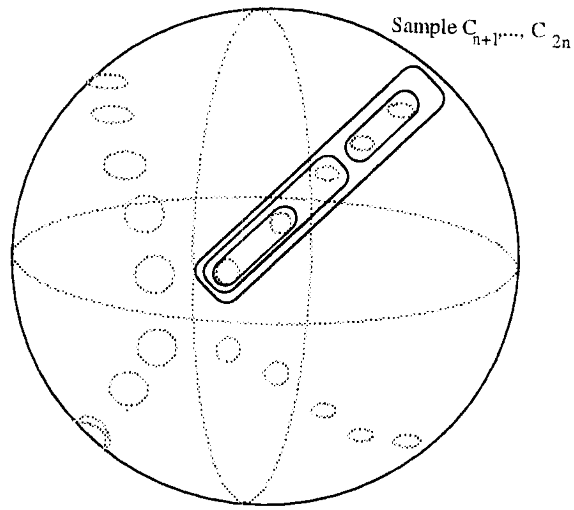

FIGURE 12. A maximal collection of of disjoint essential non-parallel loops

Proof: A similar argument is given in [MS, Appendix]. It follows from the construction of V_n as $S \times I \cup_g P^+ \times I$ that $\pi_1(V_n)$ is an HNN extension of $\pi_1(S \setminus C_x)$, so $\pi_1(S \setminus C_x)$ injects into $\pi_1(V_n)$. Moreover, every loop in S which bounds a disk in V_n has zero homological intersection number with C_x. Therefore every essential loop in S which bounds a disk in V_n intersects C_x at least twice. Observe that C_y bounds a disk in V_n and intersects C_x in exactly two points; the regular neighborhood N of $C_x \cup C_y$ is a 4-punctured sphere.

Suppose that L is an essential simple loop in S which bounds a disk in V_n and intersects ∂N minimally, but every component of $L \cap N$ either intersects C_y or is disjoint from C_x. Since some component of $L \cap N$ intersects C_x, L must intersect C_y also.

Therefore L contains an arc α (a wave) with $\alpha \cap C_y = \partial\alpha$ and such that if β is either arc of $C_y \setminus \partial\alpha$ then $\alpha \cup \beta$ bounds a disk in V_n. Each component of $\alpha \cap N$ not containing an end-point of α is a component of $L \cap N$ which is disjoint from C_y, and is therefore disjoint from C_x also. Therefore every component of $\alpha \cap N$ contains an end-point of α.

It follows that $(\alpha, \partial\alpha)$ is isotopic in (S, C_y) to an arc α' not meeting C_x; let β', β'' be the arcs of $C_y \setminus \partial\alpha'$. Then $\alpha' \cup \beta'$, $\alpha' \cup \beta''$ are essential simple closed curves in S bounding disks in V_n, and at least one intersects C_x in less than two points, a contradiction. □

Corollary A.4. *With V_n and h as in Lemma A.2, $V_n \cup_h V_n$ is strongly irreducible.*

Lemma A.5. *Let S be a closed orientable surface, possibly disconnected, and let $C_1, C'_1, \dots C_k, C'_k$ be disjoint essential simple closed curves on S. Then there is a compact 3-manifold N with incompressible boundary $\partial N = S$, containing disjoint annuli $A_1, A_2, \dots A_k$ such that $\partial A_i = C_i \cup C'_i$.*

Proof: Let $A_1, A_2, \dots A_k$ be annuli, and construct

$$N' = (S \times I) \cup (A_1 \times I) \cup (A_2 \times I) \cup \dots \cup (A_k \times I)$$

where $(A_i \times I) \cap (S \times I) = (\partial A_i) \times I$ is a regular neighborhood of $C_i \cup C'_i$ in $S \times 1$. The boundary of N' consists of S together with a closed surface S'; since C_i and C'_i are essential, S and S' are incompressible in N'. Choose a compact 3-manifold N'' with incompressible boundary $\partial N'' = S'$, and set $N = N' \cup_{S'} N''$. □

Proof of Theorem A.1 Choose $n > (1 - c/2)^{-1}$ and let V_n, h and F_n be as in Lemma A.2. Let C, C' be parallel essential simple closed curves on a torus T. Let S be the disjoint union of T and $\partial_- V_n$, and let $C_1, C'_1, \dots C_{2n}, C'_{2n}$ be the components of $(F_n \cap \partial_- V_n) \cup C \cup C'$, in any order. By Lemma A.5, there is a 3-manifold N with incompressible boundary $T \cup \partial_- V_n$, containing disjoint annuli $A_1, A_2, \dots A_{2n}$ such that

$$(F_n \cap \partial_- V_n) \cup C \cup C' = \partial A_1 \cup \partial A_2 \cup \dots \cup \partial A_{2n}.$$

If instead $S = \partial_- V_n$ and $C_1, C'_1, \dots C_{2n-1}, C'_{2n-1}$ are the components of $F_n \cap \partial_- V_n$ in any order then Lemma A.5 gives a 3-manifold N' with incompressible boundary $\partial_- V_n$ and containing disjoint annuli $A'_1, A'_2, \dots A'_{2n-1}$ such that

$$F_n \cap \partial_- V_n = \partial A'_1 \cup \partial A'_2 \cup \dots \cup \partial A'_{2n-1}.$$

Choose the orderings in such a way that if $F = A_1 \cup A_2 \cdots \cup A_{2n}$ and $F' = A'_1 \cup A'_2 \cup \dots \cup A'_{2n-1}$, then $F \cup F' \subset N \cup_S N'$ is a single annulus.

By [SS1, Lemma 3], $(N; T, \partial_- V_n)$ has a strongly irreducible untelescoped Heegaard splitting $N = X_1 \cup Y_1 \cup \dots \cup X_k \cup Y_k$ such that the components of $A_i \cap X_j$ and $A_i \cap Y_j$ are essential sub-annuli of A_i. Similarly, $(N'; \partial_- V_n, \emptyset)$ has an untelescoped Heegaard splitting $N' = X'_1 \cup Y'_1 \cup \dots \cup X'_{k'} \cup Y'_{k'}$ with similar properties.

Let q be the total number of 1-handles in these generalized Heegaard splittings for N and N'. Choose $m > 2q$ and let $m(V_n \cup_h V_n)$ denote the union $(V_n \cup_h V_n) \cup (V_n \cup_h V_n) \cup \dots \cup (V_n \cup_h V_n)$ of m copies of $V_n \cup_h V_n$. Set $M = N \cup m(V_n \cup V_n) \cup N'$ with the generalized Heegaard splitting

$$X_1 \cup Y_1 \cup \dots \cup X_k \cup Y_k \cup m(V_n \cup_h V_n) \cup X'_1 \cup Y'_1 \cup \dots \cup X'_{k'} \cup Y'_{k'}$$

and set $A = F \cup m(F_n \cup F_n) \cup F'$, a single annulus properly embedded in M.

The total number of 1-handles is $mn + q$, and there are at least $(2n - 1)m$ "dipping" annuli. Since $n \geq (1 - c/2)^{-1}$ and $m \geq 2q$, $(2 - c)mn \geq 2m \geq m + 2q \geq m + cq$, so $(2n - 1)m \geq c(mn + q)$, as required. □

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