THE CLASSIFICATION OF HEEGAARD SPLITTINGS FOR (COMPACT ORIENTABLE SURFACE) $\times S^1$

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1. *Introduction*

This paper gives the classification of Heegaard splittings for (compact orientable surface) $\times S^1$. In particular, I prove that the irreducible Heegaard splittings for (closed orientable surface) $\times S^1$ are unique. Michel Boileau and Jean-Pierre Otal have proven this for the case (torus) $\times S^1 = T^3$. Martin Scharlemann and Abigail Thompson have classified the Heegaard splittings for (closed surface) $\times I$; this includes the case (torus) $\times I =$ (annulus) $\times S^1$. I generalize methods in these papers and use the results.

Section 2 describes the irreducible Heegaard splittings of (compact orientable surface) $\times S^1$. Section 3 shows that any Heegaard splitting of (compact orientable surface) $\times S^1$ is obtained by amalgamating Heegaard splittings of (annulus) $\times S^1$, (thrice punctured sphere) $\times S^1$ and (closed orientable surface) $\times I$ along incompressible surfaces. Section 4 classifies the Heegaard splittings for (thrice punctured sphere) $\times S^1$. Finally, § 5 shows that the amalgamation process yields the Heegaard splittings described in § 2. J. Pitts and H. Rubinstein have announced a proof of a similar theorem using different methods. I would like to thank Martin Scharlemann for his helpful comments.

2. *The standard Heegaard splittings of* (compact orientable surface) $\times S^1$

A *compression body* is a 3-manifold W for which there is a closed connected orientable surface F with $W = (F \times I) \cup$ (2-handles) $\cup$ (3-handles), where the 2-handles are attached along $F \times \{0\}$ and the 3-handles are attached so as to cap off any resulting 2-sphere boundary components. Define $\partial_+ W = F \times \{1\}$ and $\partial_- W = \partial W - \partial_+ W$. A compression body with $\partial_- W = \varnothing$ is called a *handlebody*. A *spine* of a compression body W is a properly embedded 1-complex X such that W collapses to $X \cup \partial_- W$. A *Heegaard splitting* of a 3-manifold M is a pair (W_1, W_2) of compression bodies such that $W_1 \cup W_2 = M$ and $W_1 \cap W_2 = \partial_+ W_1 = \partial_+ W_2$. The surface $F = \partial_+ W_1 = \partial_+ W_2$ is called the *splitting surface*. Two Heegaard splittings are equivalent if their splitting surfaces are isotopic in M. The *genus* of a Heegaard splitting is the genus of its splitting surface.

An *elementary stabilization* of F is the splitting surface obtained by taking the connected sum of pairs $(M, F) \# (S^3, T)$, where T is the standard unknotted torus in S^3. If the splitting surface of (V_1, V_2) is the result of a finite sequence of elementary stabilizations of the splitting surface of (W_1, W_2) then (V_1, V_2) is a *stabilization* of (W_1, W_2); in this case (V_1, V_2) is called a *stabilized* Heegaard splitting. Observe that a Heegaard splitting is stabilized if and only if there

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are properly embedded disks D_1, D_2 such that $(D_i, \partial D_i) \subset (W_i, \partial_+ W_i)$ and $|\partial D_1 \cap \partial D_2| = 1$. For details on the definitions above, see **[9]**.

Let Q be a compact orientable surface with boundary components $B_1, \ldots, B_n$. Let X be a properly embedded 1-complex in Q and $X_+ = X \cup (B_{k+1} \cup \ldots \cup B_n)$. For $k > 0$, X is a *spine of* Q *rel* $\{B_1, \ldots, B_k\}$ if $Q - X_+$ is an open regular neighbourhood of $B_1 \cup \ldots \cup B_k$. For $k = 0$, X is a *spine of* Q *rel* $\emptyset$ if $Q - X_+$ is an open disk.

Spines of Q rel $\{B_1, \ldots, B_k\}$ give rise to Heegaard splittings of $Q \times S^1$. Let $Q \times \{\text{point}\}$ be a copy of Q in $Q \times S^1$. For the case where $k = n$, suppose X is a spine of Q rel $\{B_1, \ldots, B_k\}$ and p is a point in X. Then

$$(X \times \{\text{point}\}) \cup (\{p\} \times S^1)$$

is the spine of a handlebody W_1. Now $W_2 = (Q \times S^1) - \text{interior}(W_1)$ is a compression body with $\partial_- W_2 = \partial Q \times S^1$. (Remark 2.2 below explains why W_2 is a compression body.) Hence (W_1, W_2) is a Heegaard splitting. For $k < n$, suppose X is a spine of Q rel $\{B_1, \ldots, B_k\}$; then $(X \times \{\text{point}\}) \cup ((X_+ - X) \times S^1)$ is the spine of a compression body W_1 in $Q \times S^1$. In addition, $W_2 = (Q \times S^1) - \text{interior}(W_1)$ is a compression body with $\partial_- W_2 = \{B_1 \times S^1, \ldots, B_k \times S^1\}$. Hence (W_1, W_2) is a Heegaard splitting. In both cases, the Heegaard splittings obtained in this way are called *Heegaard splittings induced by* X.

Figure 1 depicts a thrice punctured torus with boundary components B_1, B_2, B_3; X is a spine rel $\{B_3\}$.

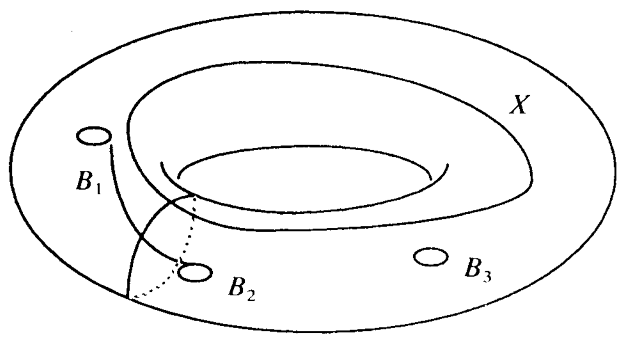

FIG. 1

LEMMA 2.1. *For a given collection* $\{B_1, \ldots, B_k\}$ *of components of* ∂Q, *all Heegaard splittings induced by spines of* Q *rel* $\{B_1, \ldots, B_k\}$ *are equivalent.*

The Heegaard splitting of $Q \times S^1$ induced by a spine of Q rel $\{B_1, \ldots, B_k\}$ and its stabilizations are called the *standard Heegaard splittings of* $Q \times S^1$ *rel* $\{B_1, \ldots, B_k\}$. A Heegaard splitting of $Q \times S^1$ is *standard* if it is a standard Heegaard splitting of $Q \times S^1$ rel $\{B_1, \ldots, B_k\}$ for some $0 \leq k \leq n$.

Proof of Lemma 2.1. Let X be a spine of Q rel $\{B_1, \ldots, B_k\}$, and let $\eta(X_+)$ be a closed regular neighbourhood of X_+ in Q. Then for $k > 0$, $Q - \text{interior}(\eta(X_+))$ is a closed regular neighbourhood of $B_1 \cup \ldots \cup B_k$. Hence it follows from the isotopy uniqueness of regular neighbourhoods that, for X and

X' two spines of Q rel $\{B_1, \ldots, B_k\}$, $\eta(X_+)$ and $\eta(X'_+)$ are isotopic in Q. For $k=0$, $Q - \text{interior}(\eta(X_+))$ and $Q - \text{interior}(\eta(X'_+))$ are closed regular neighbourhoods of points q and q' in Q. Since Q is path connected, q can be isotoped to coincide with q'. It then follows from the isotopy uniqueness of regular neighbourhoods of q' that $\eta(X_+)$ and $\eta(X'_+)$ are isotopic in Q.

A product of $\eta(X_+)$ and an ε-neighbourhood of {point} in S^1 is a regular neighbourhood of X_+ in $Q \times S^1$. For $k<n$ the lemma then follows from the isotopy uniqueness of regular neighbourhoods of $X \cup ((X_+ - X) \times S^1)$. For $k=n$, let X and X' be spines rel ∂Q, and let $p \in X$, $p' \in X'$ be such that $(X \times \{\text{point}\}) \cup (\{p\} \times S^1)$ and $(X \times \{\text{point}\}) \cup (\{p'\} \times S^1)$ are spines of W_1, W'_1. Since $\eta(X_+)$ and $\eta(X'_+)$ are isotopic and path connected, p can be isotoped to coincide with p'. The lemma then follows from the isotopy uniqueness of regular neighbourhoods of $X \cup (\{p'\} \times S^1)$.

REMARK 2.2. If (W_1, W_2) is the standard irreducible Heegaard splitting rel $\{B_1, \ldots, B_k\}$ of $Q \times S^1$, then (W_2, W_1) is the standard irreducible Heegaard splitting rel $\{B_{k+1}, \ldots, B_n\}$ of $Q \times S^1$. This is obvious in the case where $n=0$, since if (W_1, W_2) is the standard irreducible Heegaard splitting rel $\varnothing$, then (W_2, W_1), which is equivalent to (W_1, W_2), is also the standard irreducible Heegaard splitting rel $\varnothing$.

In the case where $0<k \leqslant n$, let X be a spine of Q rel $\{B_1, \ldots, B_k\}$ and let Y be a spine of Q rel $\{B_{k+1}, \ldots, B_n\}$. Let I_1, I_2 be intervals in S^1 such that $S^1 = I_1 \cup I_2$ and $\partial I_1 = \partial I_2$. Then

$$\begin{aligned} W_1 &= (\text{closure}(\eta(X_+)) \times I_1) \cup (\text{closure}(\eta(X_+ - X)) \times I_2) \\ &= ((Q - \eta(B_1 \cup \ldots \cup B_k)) \times I_1) \cup (\eta(B_{k+1} \cup \ldots \cup B_n) \times I_2) \\ &= ((Q - \eta(Y_+ - Y)) \times I_1) \cup ((Q - \eta(Y)) \times I_2) \end{aligned}$$

and hence $W_2 = (\text{closure}(\eta(Y_+ - Y)) \times I_1) \cup (\text{closure}(\eta(Y)) \times I_2)$. The other case follows similarly. In this sense the standard irreducible Heegaard splitting rel $\{B_1, \ldots, B_k\}$ is *dual* to the standard irreducible Heegaard splitting rel $\{B_{k+1}, \ldots, B_n\}$.

The main theorem in this paper, Theorem 5.7, is that all Heegaard splittings of $Q \times S^1$ are standard. The following definitions lay the foundation for an inductive proof of the main theorem by enabling cutting and pasting of manifolds with Heegaard splittings.

An *essential disk* in (M, F) is a disk D in M such that $D \cap F = \partial D$ and ∂D is essential in F. A Heegaard splitting (W_1, W_2) is *reducible* if there exist essential disks $D_1 \subset W_1$ and $D_2 \subset W_2$ such that $\partial D_1 = \partial D_2$. It is *weakly reducible* if there exists a disjoint collection of disk $\Delta = \Delta_1 \cup \Delta_2$, such that both Δ_1 and Δ_2 are non-empty, and each disk $D_i \subset \Delta_i$ is an essential disk in W_i. In particular, $|\partial\Delta_1 \cap \partial\Delta_2| = 0$. The collection Δ is called a *weakly reducing collection of disks for* (W_1, W_2). (*Note*: in the terminology of **[2]** a weakly reducible Heegaard splitting is *not strongly irreducible*; in the terminology of **[1]** it is *fortement réductible*.) It is well-known that a stabilized Heegaard splitting whose genus is greater than 1 is reducible, and further, that a reducible Heegaard splitting of an irreducible 3-manifold is stabilized. Hence in the present context, a Heegaard splitting is stabilized if and only if it is reducible.

Let S be a surface in a 3-manifold M, and let Δ be a disjoint union of disks in M such that $\Delta \cap S = \partial\Delta$. Denote by $\sigma(S\,;\partial\Delta)$ the surface obtained from S by performing ambient 2-surgeries along the components of Δ. Let F be the splitting surface for a given Heegaard splitting and Δ a weakly reducing collection of disks. Set $F^* = \sigma(F\,;\partial\Delta) - (\text{2-sphere components of } \sigma(F\,;\partial\Delta))$. Let C be the closure of a component of $M - F^*$. Then ∂C is the union of $\partial M \cap C$ and some components of F^*. Denote the latter by $\partial_F C$. Let C^* be the union of C with a collar $\partial_F C \times I$ of $\partial_F C$ lying in M − interior(C).

REMARK 2.3. If $F^* = \emptyset$, then the $\sigma(F\,;\partial\Delta)$ are 2-spheres. In this case the Heegaard splitting defined by F is reducible (see **[2**, 3.1] or, for more detail, **[9**, 5.1]). It follows that for an irreducible but weakly reducible Heegaard splitting, Δ can be chosen so that F^* is a non-empty collection of incompressible surfaces.

Figure 2a depicts a weakly reducing collection of disks $D_1 \cup D_2$ for a Heegaard splitting of T^3. Figure 2b depicts F^*.

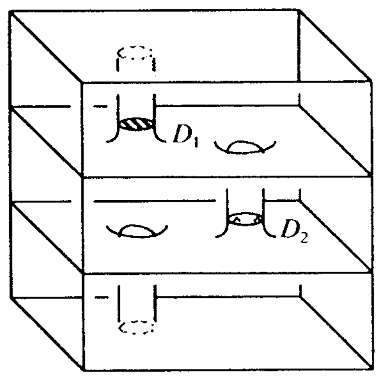

FIG. 2a

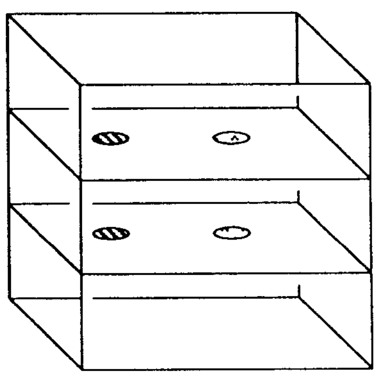

FIG. 2b

LEMMA 2.4. *The Heegaard splitting* (W_1, W_2) *of* M *induces a Heegaard splitting* (W_1^*, W_2^*) *of* C^*.

The splitting (W_1^*, W_2^*) is called *the induced Heegaard splitting of* C^*.

Proof. By construction, $C \subset W_i \cup \eta(\Delta_j)$, with $i \neq j$; say $C \subset W_1 \cup \eta(\Delta_2)$. Set $W_1^* = W_1 \cap C$. Now C is obtained from W_1^* by attaching 2-handles, so, dually, W_1^* is obtained from C by drilling out tunnels with ends on $\partial_F C$. Thus W_1^* is connected. Since W_1^* is a single component of $W_1 - \eta(\Delta_1)$, it is a compression body.

Now $W_2^* = C^* - W_1^*$ is obtained from the collar $\partial_F C \times I$ by attaching 1-handles which are the interiors of the tunnels. Hence W_2^* is a union of compression bodies. Since $\partial_+ W_1^* = \partial_+ W_2^*$, W_2^* is connected and is a single compression body. Hence (W_1^*, W_2^*) is a Heegaard splitting.

REMARK 2.5. Induced Heegaard splittings have the property that $\partial_F C \times I$ is contained in one compression body.

The following defines a notion of *complexity* for the weakly reducing collection of disks Δ. For a closed surface S, set $\bar{c}(S) = \sum (1 - \chi(S_i))$, where the sum is taken over all components of S which are not 2-spheres. Define the complexity c of Δ by $c(\Delta) = \bar{c}(F) - \bar{c}(F^*)$.

REMARK 2.6. Note that $c(\Delta) \leq \bar{c}(F)$, so Δ can be chosen so as to maximize $c(\Delta)$. If $c(\Delta)$ is maximal, then the Heegaard splittings induced on the components of $M - F^*$ are not weakly reducible. Indeed, let C be the closure of a component of $(Q \times S^1) - F^*$. If the induced Heegaard splitting (U_1, U_2) on C^* is weakly reducible, let Δ' be a weakly reducing collection of disks for (U_1, U_2). Then $c(\Delta \cup \Delta') > c(\Delta)$.

A weakly reducing collection of disks enables the cutting of a manifold with a Heegaard splitting into components with Heegaard splittings. The following notion of amalgamation enables the pasting of manifolds with Heegaard splittings so as to obtain a manifold with a Heegaard splitting.

Let N, L be 3-manifolds with R a closed subsurface of ∂N, and S a closed subsurface of ∂L, such that R is homeomorphic to S via a homeomorphism h. Further, let (U_1, U_2), (V_1, V_2) be Heegaard splittings of N, L such that $\eta(R) \subset U_1$, $\eta(S) \subset V_1$. Then, for some $R' \subset \partial N - R$ and $S' \subset \partial L - S$, $U_1 = \eta(R \cup R') \cup (\text{1-handles})$ and $V_1 = \eta(S \cup S') \cup (\text{1-handles})$. Thus $\eta(R)$ is homeomorphic to $R \times I$ via a homeomorphism f and $\eta(S)$ is homeomorphic to $S \times I$ via a homeomorphism g. Let $\sim$ be the equivalence relation on $N \cup L$ generated by

(1) $x \sim y$ if $x, y \in \eta(R)$ and $p_1 \circ f(x) = p_1 \circ f(y)$,

(2) $x \sim y$ if $x, y \in \eta(S)$ and $p_1 \circ g(x) = p_1 \circ g(y)$,

(3) $x \sim y$ if $x \in R$, $y \in S$ and $h(x) = y$,

where p_1 is projection onto the first coordinate. Perform isotopies so that for D an attaching disk for a 1-handle in U_1, D' an attaching disk for a 1-handle in V_1, $[D] \cap [D'] = \varnothing$. Set $M = (N \cup L)/\sim$, $W_1 = (U_1 \cup V_2)/\sim$, and $W_2 = (U_2 \cup V_1)/\sim$. In particular, $\eta(R) \cup \eta(S)/\sim \cong R, S$. Then $W_1 = V_2 \cup \eta(R') \cup (\text{1-handles})$, where the 1-handles are attached to $\partial_+ V_2$ and connect $\partial\eta(R')$ to $\partial_+ V_2$, and hence W_1 is a compression body. Similarly, W_2 is a compression body. So (W_1, W_2) is a Heegaard splitting of M. The splitting (W_1, W_2) is called *the amalgamation of* (U_1, U_2) *and* (V_1, V_2) *along* $\{R, S\}$ *via* h.

REMARK 2.7. The amalgamation of a Heegaard splitting of genus n of a manifold N and a genus l Heegaard splitting of a manifold L along boundary components $R \subset \partial N$ and $S \subset \partial L$ of genus k has genus $n + l - k$.

Figure 3a partially depicts two Heegaard splittings of 3-manifolds with boundary. Figure 3b partially depicts an amalgamation.

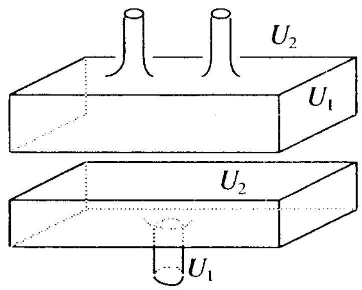

FIG. 3a

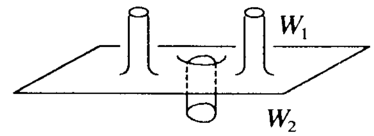

FIG. 3b

In the following proposition, $\bar{F} = (\bigcup \partial C^*) - \partial M$, where the union is taken over the components of $M - F^*$. Note that $\bar{F}$ is homeomorphic to two parallel copies of F^*. There is a natural map between the two homeomorphic copies of F^* in $\bar{F}$ defined by the collar structure of the C^* and the identity map on F^*. This natural map will be denoted v.

PROPOSITION 2.8. *Let (W_1, W_2) be a Heegaard splitting of M with a weakly reducing collection of disks Δ. Let $C_1, \ldots, C_n$ be the closures of the connected components of $M - F^*$ and let $(W_1^1, W_2^1), \ldots, (W_1^n, W_2^n)$ be the induced Heegaard splittings on $C_1^*, \ldots, C_n^*$. Then (W_1, W_2) is the amalgamation of $(W_1^1, W_2^1), \ldots, (W_1^n, W_2^n)$ along $\bar{F}$ via v.*

Proof. The proof proceeds by induction on n.

Step 1. If $M - F^*$ consists of two components C_1, C_2, let (W_1^1, W_2^1), (W_1^2, W_2^2) be the induced Heegaard splittings on C_1^*, C_2^*. Then by definition (W_1, W_2) is the amalgamation of (W_1^1, W_2^1) and (W_1^2, W_2^2) along $\bar{F}$.

Step 2. Suppose it is true that whenever $\#|M - F^*| < n$, then (W_1, W_2) is the amalgamation along $\bar{F}$ of the induced Heegaard splittings on the components of $M - F^*$. Let $\Delta' = \Delta \cap C_1$ and $\Delta'' = \Delta - \Delta'$ and define $(F')^*$, $(F'')^*$, $\bar{F}'$, $\bar{F}''$ analogously to F^*, $\bar{F}$. Then $M - (F')^*$ has two components, the closures of which are C_1 and C, where $C = C_2 \cup \ldots \cup C_n$. Moreover, the closures of the components of $C - (F'')^*$ are $C_2, \ldots, C_n$.

The induced Heegaard splitting (U_1, U_2) on C^* induces Heegaard splittings on $C_2^*, \ldots, C_n^*$ equivalent to $(W_1^2, W_2^2), \ldots, (W_1^n, W_2^n)$. By the inductive assumption, (U_1, U_2) is the amalgamation of $(W_1^2, W_2^2), \ldots, (W_1^n, W_2^n)$ along $\bar{F}''$. By Step 1, (W_1, W_2) is the amalgamation of (W_1^1, W_2^1) and (U_1, U_2) along $\bar{F}'$. But $\bar{F} = \bar{F}' \cup \bar{F}''$, and hence (W_1, W_2) is the amalgamation along $\bar{F}$ of $(W_1^1, W_2^1), \ldots, (W_1^n, W_2^n)$.

Let Q_1, Q_2 be compact orientable surfaces with boundary components $\{A_1, \ldots, A_n\}$, $\{B_1, \ldots, B_m\}$, and let $\tau_1 = A_1 \cup \ldots \cup A_k$ and $\tau_2 = B_1 \cup \ldots \cup B_k$, for some $0 < k \leq n, m$. Let Q be the surface obtained from $Q_1 \cup Q_2$ by identifying τ_1

and τ_2. Let X be a spine of Q_1 rel A, where $\tau_1 \subset A \subset \partial Q_1$, and let Y be a spine of Q_2 rel $\partial Q_2 - B$, where $\tau_2 \subset B \subset \partial Q_2$. Then $Q_1 - \eta(X)$ is a collar of A and hence contains a collar of τ_1, and $\eta(Y)$ contains a collar of ∂B and hence contains a collar of τ_2. Isotope Y so that it intersects the collar of τ_2 in fibres. Together, the collars of τ_1 and τ_2 form a bicollar of $\tau_1 = \tau_2$ in Q. Let $\sim$ be the equivalence relation on Q which identifies all points in each interval fibre of the bicollar. Let $q\colon Q \to Q/\sim$ be the quotient map. Since q merely collapses an imbedded annulus to its core circle, $Q \cong Q/\sim$.

Lemma 2.9. *The image of* $\eta(X_+) \cup \eta(Y_+)$ *under the quotient map* $q\colon Q \to Q/\sim$ *is a regular neighbourhood of a spine of* Q *rel* $((A - \tau_1) \cup (\partial Q_2 - B))$.

Proof. Each endpoint y of Y in τ_2 lies on one end of an interval fibre of the bicollar; the other end is a point y' on $\partial\eta(X)$. The track of y' under a retraction $\eta(X) \to X$ is, generically, an interval with one end at y' and the other end on X. Let Z be the proper graph in Q obtained from $X \cup Y$ by attaching to each end of Y incident to τ_2 the fibre connecting y to y' and the interval with one end at y' and the other end on X. Then the only effect of q on Z is to collapse subintervals of some of its edges. So $q(\eta(X_+) \cup \eta(Y_+))$ can be regarded as just $\eta(Z)$ in Q. In particular, it is the neighbourhood of some graph in Q. To see that Z is a spine rel $((A - \tau_1) \cup (\partial Q_2 - B))$, observe that $Q - \eta(Z)$ is

$$Q - (\eta(X_+) \cup \eta(Y_+) \cup \text{bicollar}) = \eta(A - \tau_1) \cup \eta(\partial Q_2 - B).$$

Proposition 2.10. *The amalgamation of standard Heegaard splittings of* (compact orientable surface) $\times S^1$, *via a homeomorphism respecting the product structure induced on the toral boundary components along which the amalgamation occurs, is a standard Heegaard splitting of* (compact orientable surface) $\times S^1$.

Proof. It suffices to prove the lemma for standard irreducible Heegaard splittings. Let (U_1, U_2), (V_1, V_2) be standard irreducible Heegaard splittings of Q_1 rel A, Q_2 rel B with $\tau_1 \subset A$, $\tau_2 \subset B$ for $\tau_1 \times S^1$, $\tau_2 \times S^1$ the toral boundary components along which the amalgamation occurs. Let X be a spine of Q_1 rel A, and Y a spine of Q_2 rel $\partial Q_2 - B$. In particular, for I_1, I_2 as in Remark 2.2,

$$U_1 = (\text{closure}(\eta(X_+)) \times I_1) \cup (\text{closure}(\eta(X_+ - X)) \times I_2),$$
$$V_2 = (\text{closure}(\eta(Y_+)) \times I_1) \cup (\text{closure}(\eta(Y_+ - Y)) \times I_2).$$

So

$$\begin{aligned} W_1 &= U_1 \cup V_2/\sim \\ &= ((\text{closure}(\eta(X_+) \cup \eta(Y_+)) \times I_1) \cup (\text{closure}(\eta(X_+ - X) \cup \eta(Y_+ - Y)) \times I_2))/\sim \\ &= (\text{closure}(\eta(Z_+)) \times I_1) \cup (\text{closure}(\eta(Z_+ - Z)) \times I_2), \end{aligned}$$

for Z a spine of Q rel $(A - \tau_1) \cup (\partial Q_2 - B)$, by Lemma 2.9. Note that here it is essential that the homeomorphism via which the amalgamation occurs respects the product structure of $\tau_1 \times S^1$ and $\tau_2 \times S^1$. Hence (W_1, W_2) is standard.

3. *Weak reducibility and incompressible surfaces*

In this section it is shown that for compact orientable surfaces of positive genus or with sufficiently small Euler characteristic all Heegaard splittings of (compact orientable surface) $\times S^1$ are weakly reducible.

Heegaard splittings correspond to Morse functions. Let (W_1, W_2) be a Heegaard splitting of the 3-manifold M. Then $W_1 = ((\partial_- W_1 \times I) \cup (0\text{-handles})) \cup (1\text{-handles})$ and $W_2 = (\partial_+ W_2 \times I) \cup (2\text{-handles}) \cup (3\text{-handles})$, so

$$W_1 \cup W_2 = ((\partial_- W_1 \times I) \cup (0\text{-handles})) \cup (1\text{-handles}) \cup (2\text{-handles}) \cup (3\text{-handles})$$

gives a handlebody description of M. Excess 0-handles can be cancelled with 1-handles and (dually) excess 3-handles with 2-handles, after which there is at most one 0-handle (precisely when W_1 is a handlebody), and at most one 3-handle (precisely when W_2 is a handlebody). This handlebody description can be used to define a Morse function h on M (apply [**6**, Theorem 3.12] repeatedly). Call h a *Morse function induced by* (W_1, W_2). Then h can be taken to have singular values $0 < a_1 < \ldots < a_m < b_0 < \ldots < b_n < 1$, with the critical point at level a_i $(i = 1, \ldots, m)$ the centre of a 1-handle, and the critical point at level b_j $(j = 1, \ldots, n)$ the centre of a 2-handle. For the converse, constructing a Heegaard splitting from a Morse function, see [**7**, 1.3].

Let γ be a properly embedded arc or circle in $Q \times S^1$. Assume that $h|_\gamma$ is a Morse function on γ, and that the critical values of $h|_\gamma$ are distinct from the critical values of h on $Q \times S^1$. Let $l_0, \ldots, l_n$ be the critical values of γ, where $0 = l_0 < l_1 < \ldots < l_n = 1$. Let $x_1, \ldots, x_n$ be regular values of h such that $l_{n-1} < x_n < l_n$. Then each $h^{-1}(x_i)$ is a level surface S_i. Define the *width* of γ to be $|S_0 \cap \gamma| + \ldots + |S_n \cap \gamma|$. Alternatively, $\text{width}(\gamma) = \sum_i |(h|_\gamma)^{-1}(x_i)|$. An isotopic copy of γ is in *thin position* if it minimizes the width of γ. An isotopic copy of γ is in *thin position rel* $\partial\gamma$ if it minimizes the width of γ among the isotopy class of γ rel $\partial\gamma$.

PROPOSITION 3.1. *If* $\text{genus}(Q) > 0$ *or* $\chi(Q) < -1$, *then any Heegaard splitting* (W_1, W_2) *of* $Q \times S^1$ *is weakly reducible.*

That is, as long as the compact orientable surface Q is not a sphere, a disk, an annulus, or a thrice punctured sphere, then any Heegaard splitting of $Q \times S^1$ is weakly reducible.

The cases where $\partial Q = \emptyset$ and $\partial Q \neq \emptyset$ are proved separately. In both cases the idea is to find a level surface of h which can be compressed so as to yield incompressible surfaces, which, in the case where $\partial Q \neq \emptyset$, are not boundary parallel. Combined with Lemma 3.2, this shows that (W_1, W_2) is weakly reducible.

LEMMA 3.2. *Let F be the splitting surface of a Heegaard splitting (W_1, W_2) of M. Suppose $T \subset F$ is a compact subsurface so that every component of ∂T is essential in F. Suppose further that each component of ∂T bounds a disk in M disjoint from* interior(T). *Either ∂T bounds a collection of disks in a single compression body W_i, or F is weakly reducible.*

Proof. See [**9**, 2.6].

Suppose Q is a closed surface of genus $g > 0$. Then Q can be obtained by identifying opposite sides of a $4g$-gon Υ. In Q the boundary of Υ can be viewed as a bouquet of $2g$ circles with base point p. Then $\partial\Upsilon \times S^1$ is a collection $T_1, \ldots, T_{2g}$ of essential tori in $Q \times S^1$ intersecting in the curve $\gamma = \{p\} \times S^1$. Alternatively, let G be the solid torus $\Upsilon \times S^1$. The boundary of G is the union of

$4g$ annuli $A_1, \dots, A_{4g}$, one for each side of Y. Call these annuli the *sides* of G. Then $Q \times S^1$ is obtained from G by identifying opposite sides A_i and A_{i+2g} of G. An arc in a side of G is *essential* if it spans the side; otherwise it is inessential. A circle c in ∂G is *strictly essential* if it intersects each side only in essential arcs.

A properly embedded surface S in $Q \times S^1$ is vertical at $q \in S$ if p_1^*: $\tau(S) \to \tau(Q)$ is singular at q. The surface S is *vertical* if it is vertical for all q in S. It follows that a vertical surface is a union of tori and annuli, each projecting to a circle or a proper arc in Q. Say that S is *horizontal* if it is nowhere vertical. In particular, p_1^* is everywhere non-singular, so $p_1|_S$: $S \to Q$ is a covering map. In addition, S is strictly horizontal if p_2^*: $\tau(S) \to \tau(S^1)$ is always trivial. That is, S is *strictly horizontal* if $S = Q \times \{t_1, \dots, t_n\}$ for some $\{t_1, \dots, t_n\} \subset S^1$.

Suppose there is an inessential arc α of $h^{-1}(r) \cap A_i$, for some i. Then α, together with a piece γ' of γ, bounds a disk D in A_i. If $\operatorname{int}(D) \cap h^{-1}(r)$ is empty or consists of simple closed curves, then γ' is *above* $h^{-1}(r)$ if it lies on the side of $h^{-1}(r)$ containing $h^{-1}(1)$; otherwise it is *below*. If γ' is above $h^{-1}(r)$, α is called a *low arc*; if γ' is below $h^{-1}(r)$, α is called a *high arc*. Note that r could be a critical value.

LEMMA 3.3. *Let h: $Q \times S^1 \to [0, 1]$ be a Morse function such that $h|_{Q \times S^1}$, $h|_{T_i}$, for $i = 1, \dots, 2g$, and $h|_\gamma$ are Morse functions with distinct critical values. If γ is in thin position with respect to h, then there is a regular value r of $h|_{Q \times S^1}$, $h|_{T_i}$, for $i = 1, \dots, 2g$, or $h|_\gamma$ such that either*

(1) *there is an essential vertical torus in $Q \times S^1$ which contains a level curve*
1 *$l \subset h^{-1}(r)$ parallel to γ, or*
4 (2) *$h^{-1}(r) \cap$ (sides of G) $\neq \varnothing$ and all components of $h^{-1}(r) \cap$ (sides of G) are strictly essential.*

Proof. Since γ is compact, $h|_\gamma$ takes on a maximum and a minimum value. Let m be the largest minimum value of $h|_\gamma$ and M a maximum value of $h|_\gamma$ such that $h|_\gamma$ has no singular values in (m, M). Set

$$R^+ = \{r \in [m, M] \mid h^{-1}(r) \text{ contains a high arc in some } A_i\},$$

$$R^- = \{r \in [m, M] \mid h^{-1}(r) \text{ contains a low arc in some } A_i\}.$$

Since m is not a critical value of $h|_{T_i}$, for $i = 1, \dots, 2g$, R^+ contains a neighbourhood of m in $[m, M]$ and is hence non-empty. Similarly, $R^- \neq \varnothing$, since it contains a neighbourhood of M in $[m, M]$. Note that R^-, R^+ are closed, by [**4**, § 4].

Case 1: $R^+ \cup R^- \neq [m, M]$. Here $[m, M] - (R^+ \cup R^-)$ contains an open interval; hence there is a value $r \in [m, M] - (R^+ \cup R^-)$ such that r is a regular value for $h|_{Q \times S^1}$, $h|_{T_i}$ $(i = 1, \dots, 2g)$, $h|_\gamma$. By definition, all components of $h^{-1}(r) \cap$ (sides of G) are strictly essential.

Case 2: $R^+ \cup R^- = [m, M]$. We claim that there is a regular value $r \in R^+ \cap R^-$.

Suppose there is a singular value $r' \in R^+ \cap R^-$. Let α_1 be a high arc and α_2 a low arc in $h^{-1}(r')$ with $\alpha_1 \subset A_i$, $\alpha_2 \subset A_j$. If both α_1 and α_2 are singular leaves, their singularities must coincide, since no two singularities occur at the same level. Hence α_1, α_2 are both on A_i, for some i. Furthermore, α_1 and α_2 must intersect in an arc, due to gradient considerations. But all possible such cases contradict the thin position of γ. Thus only α_1 or α_2, say α_1, is a singular leaf.

Then there is an ε' such that for all $r \in (r' - \varepsilon', r')$, r is a regular value of $h|_{Q\times S^1}$, $h|_{T_i}$ $(i = 1, \ldots, 2g)$, $h|_\gamma$, and $h^{-1}(r) \cap A_i$ is a high arc; and there is an ε'' such that for all $r \in (r' - \varepsilon'', r' + \varepsilon'')$, r is a regular value of $h|_{Q\times S^1}$, $h|_{T_i}$ $(i = 1, \ldots, 2g)$, $h|_\gamma$, and $h^{-1}(r) \cap A_j$ is a low arc. Set $\varepsilon = \min\{\varepsilon', \varepsilon''\}$. Then there is a regular value r of $h|_{Q\times S^1}$, $h|_{T_i}$ $(i = 1, \ldots, 2g)$, $h|_\gamma$ in $(r' - \varepsilon, r') \subset R^+ \cap R^-$. This proves the claim.

Now $R^+ \cap R^-$ is non-empty, and hence by the claim it contains a regular value r of $h|_{Q\times S^1}$, $h|_{T_i}$ $(i = 1, \ldots, 2g)$, $h|_\gamma$. By thin position, the high arc $\alpha_1 \in h^{-1}(r)$ and the low arc $\alpha_2 \in h^{-1}(r)$ have the same endpoints in γ, although they may lie on different annuli A_i and A_j. If $i = j$, $\alpha_1 \cup \alpha_2$ is a level curve parallel to γ on the essential vertical torus T_i. If $i \neq j$, $l = \alpha_1 \cup \alpha_2$ is a level curve l parallel to γ on the essential vertical torus $(T_i) +_d (T_j)$ (for the definition of the double curve sum $+_d$ see [**8**, p. 560]).

Definition. Let h: $Q \to R$ be a Morse function on a surface Q and let Φ be the singular foliation by level sets of h on Q. Suppose a singular leaf γ is contained in the interior of Q. By Morse general position, γ contains a single critical point, so γ is a wedge of two circles. If both circles are essential in Q, then γ is an *essential saddle of* Φ. If all three components of $\partial\eta(\gamma)$ are essential in Q then γ is *completely essential.*

Lemma 3.4. *Suppose Q is a compact surface and h: $Q \to R$ is a Morse function which is constant on each component of ∂Q. Let $\varepsilon(Q) = \max\{2 - |\partial Q|, 0\}$. Let Φ be the singular foliation by level sets of h on Q.*

(a) *At least $\varepsilon(Q) - \chi(Q)$ leaves of Φ are essential saddles.*

(b) *At least $-\chi(Q)$ are completely essential.*

In particular, if genus$(Q) > 0$ *or* $\chi(Q) < 0$, *Φ contains an essential saddle. If, in addition, $\partial Q \neq \emptyset$, then Φ contains a completely essential saddle.*

In Fig. 4 all saddles are essential, but only σ_1 and σ_2 are completely essential.

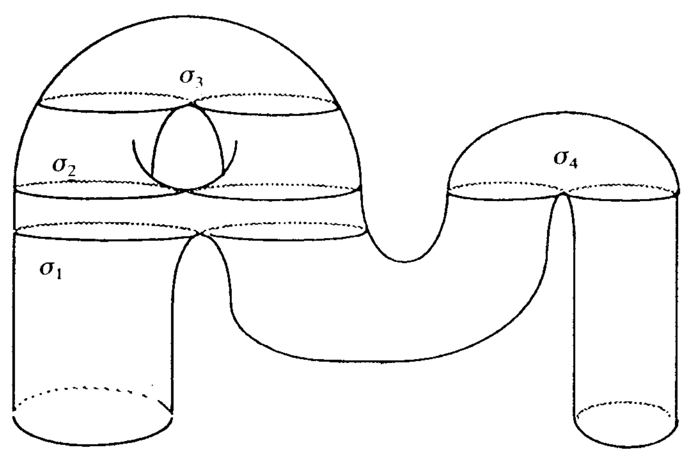

Fig. 4

Proof. The gradient of h defines a vector field v on Q. The vector field v has zeros at critical points of h. If the critical point is a maximum or minimum, the index is $+1$; at a saddle it is -1. By the Poincaré–Hopf index theorem, the sum of the indices is $\chi(Q)$.

The proof of the lemma is by induction on the number n of interior maxima and minima. Since Q is compact, h has at least one minimum and one maximum on Q, and hence at least $\varepsilon(Q)$ interior extrema, so $n \geq \varepsilon(Q)$. There are $n - \chi(Q)$ saddles. If all are essential, as must be the case when $n=0$, then (a) follows. Otherwise, at least one of the imbedded circles contained in a singular leaf bounds a disk D in Q. It is easy to redefine h near D (by first making it constant across D, then tilting slightly) so it has no singularities at all near D. This removes the inessential saddle and reduces n by at least one (an extremum in interior(D) is removed). This completes the inductive step.

For (b) consider the surface $\tilde{Q}$ obtained from Q by removing each singular leaf γ which is an essential saddle, but not a completely essential saddle, together with the disk bounded by a component of $\partial\eta(\gamma)$. If there are k such leaves, then $\chi(\tilde{Q}) = \chi(Q)$ and $|\partial\tilde{Q}| = |\partial Q| + 2k$. If $k=0$, then (b) follows from (a). If $k>0$, then $\varepsilon(\tilde{Q}) = 0$. The argument used in (a) still applies to the vector field $\tilde{v}$ on $\tilde{Q}$ obtained from restricting v to $\tilde{Q}$. So the number of saddles in $\tilde{Q}$ is at least $\tilde{\varepsilon}(Q) - \chi(\tilde{Q}) = -\chi(Q)$. But, by construction, all essential saddles in $\tilde{Q}$ are completely essential saddles in Q.

LEMMA 3.5. *Let h be a Morse function on a torus T and Φ be the singular foliation of T given by the level sets of h. Suppose α is a regular essential leaf of Φ and β is a curve in T which intersects α in a single point. After an isotopy which fixes a neighbourhood of $\alpha \cap \beta$, one can assume that β passes through every essential saddle in Φ and that $h|_\beta$ is monotonic between two essential saddles. In particular, each regular leaf of Φ which intersects β in a regular point of $h|_\beta$ is isotopic to α. Furthermore, this isotopy can be chosen so that it does not increase the width of β.*

Proof. See [**1**, Lemma 2.2].

REMARK 3.6. The proof of [**1**, Lemma 2.2] also shows that Lemma 3.4 is true in a slightly different setting. Replace T by an annulus A, and β by a spanning arc on A with $\partial\beta \subset \partial A$, and suppose that h is constant on each component of ∂A. Then after an isotopy which fixes a neighbourhood of $\alpha \cap \beta$ and of $\partial\beta$, one can assume that β passes through every essential saddle in Φ and that $h|_\beta$ is monotonic between two essential saddles. In particular, each regular leaf of Φ which intersects β in a regular point of $h|_\beta$ is isotopic to α. Furthermore, this isotopy can be chosen so that it does not increase the width of β rel $\partial\beta$. This fact is used in § 4.

In Fig. 5, β satisfies the above conditions.

LEMMA 3.7. *Let Q be a closed surface with* genus$(Q) > 0$. *Then any Heegaard splitting (W_1, W_2) of $Q \times S^1$ is weakly reducible.*

In the proof of Lemma 3.7 two cases need to be considered. In one case, the proof relies heavily on the following result.

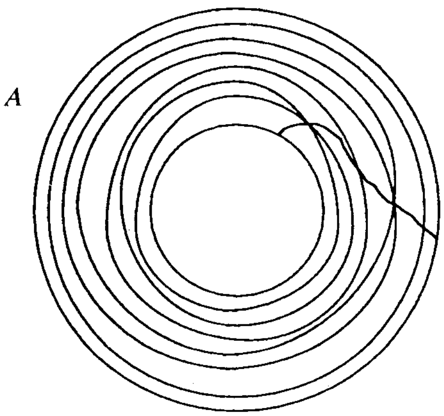

FIG. 5

SUBLEMMA 3.8. *Let Q be a surface containing an essential simple closed curve β. Suppose h: $Q \to R$ and $h|_\beta$ are Morse functions and β is in thin position with respect to h. Let Φ' be the singular foliation of Q given by the level sets of h. Then there is an essential regular leaf which intersects β transversely.*

Proof. Consider $S \subset Q \times \{\text{point}\}$, where S is the smallest union of leaves satisfying:

(a) if $\gamma \cap \beta \neq \varnothing$ for some leaf $\gamma \in \Phi'$, then $\gamma \subset S$;

(b) if $D \subset Q \times \{\text{point}\}$ is a disk such that $\partial D \subset S$, then $D \subset S$.

It suffices to prove the following claim.

Claim. The surface S has the following properties:

(1) $\beta \subset S$;

(2) S is a subsurface of $Q \times \{\text{point}\}$, possibly with boundary;

(3) S contains an essential regular leaf γ' intersecting β.

Proof of Claim. (1) This is obvious from the definitions.

(2) Let $p \in S$. Either p lies on a leaf γ of Φ' for which $\gamma \cap \beta \neq \varnothing$, or p lies in the interior of a disk D such that $\partial D \subset S$. If p lies on a regular leaf $\gamma \in \Phi'$ which intersects β transversely, then for some bicollar Γ of γ consisting of leaves in Φ', all leaves in Γ also intersect β, so $\Gamma \subset \Phi'$. If p lies on the regular leaf $\gamma \in \Phi'$ which is tangent to β, then on one side of γ a collar $\gamma \times I$ of regular leaves also intersects β, all but γ transversely. If p lies on a singular leaf $\gamma \in \Phi'$, consider a bicollar neighbourhood $P = \eta(\gamma)$ consisting of leaves in Φ'. Note that P is a thrice punctured sphere. Since the singular leaf γ intersects β transversely by Morse general position, γ must intersect P in at least one arc with endpoints on different boundary components for P small. Set $\beta' = \beta \cap P$. Now γ cuts P into three annuli A_1, A_2, A_3, and β' intersects at least two of the annuli, say A_1, A_2. So A_1, $A_2 \subset S$. Possibly $A_3 \subset S$. In all the above cases, and certainly in the case where $p \in \text{int}(D) \subset S$, p lies in a region in S homeomorphic to the plane or the upper half plane.

(3) Since $\beta \subset S$ is essential in $Q \times \{\text{point}\}$, S is not simply connected. Hence $\chi(S) \leq 0$. If S is an annulus, then a regular leaf in the interior of S near ∂S is an essential regular leaf of Φ' which intersects β transversely. Since this leaf is essential in S, it is also, by construction of S, essential in $Q \times \{\text{point}\}$.

If S is not an annulus, then $\varepsilon(S) - \chi(S) > 0$. By Lemma 3.4, $\Phi'|_S$ contains an essential (possibly singular) leaf. If this essential leaf is regular, the result follows. If this essential leaf is singular, then for P, A_1, A_2, A_3 as above, either A_1 or A_2 contains an essential regular leaf intersecting β.

Proof of Lemma 3.7. Let h: $Q \times S^1 \rightarrow [0, 1]$ be a Morse function induced by (W_1, W_2), which satisfies the hypotheses of Lemma 3.3. Then only two cases need to be considered.

Case 1. There is a regular value r of $h|_{Q \times S^1}$, $h|_{T_i}$ $(i = 1, \ldots, 2g)$, $h|_\gamma$ such that there is an essential vertical torus T in $Q \times S^1$ which contains a level curve $\alpha \in h^{-1}(r)$ parallel to γ.

Let $\beta = (Q \times \{\text{point}\}) \cap T$ and let Φ be the singular foliation of T by level sets of h. By Lemma 3.5, one can assume that β passes through every essential saddle of Φ and that $h|_\beta$ is monotonic between two essential saddles.

By Sublemma 3.8 there is an essential regular leaf γ_1 in $Q \times \{\text{point}\}$ which intersects β transversely. Perhaps by choosing a nearby leaf one may assume that $r = h(\gamma_1)$ is a regular value of $h|_{Q \times S^1} h|_T$. By Lemma 3.5, a leaf γ_2 in $h^{-1}(r) \cap T$ is a regular leaf isotopic to α on T, whence parallel to γ. Thus in $\pi_1(Q \times S^1) = \pi_1(Q) \times \pi_1(S^1)$, $[\gamma_1][\gamma_2^{-1}][\gamma_1^{-1}][\gamma_2] = 1$. Let $\eta(\gamma_1 \cup \gamma_2)$ be a closed regular neighbourhood of $\gamma_1 \cup \gamma_2$ in $h^{-1}(r)$. Then $\partial\eta(\gamma_1 \cup \gamma_2)$ bounds a disk D in $Q \times S^1$. Since $h^{-1}(r)$ is contained in a compression body and hence cannot be an essential torus, $\partial\eta(\gamma_1 \cup \gamma_2)$ must be essential in $h^{-1}(r)$. Without loss of generality, assume that D is disjoint from the interior of $\eta(\gamma_1 \cup \gamma_2)$. (For γ_2 is parallel to γ and hence can be isotoped to lie in the essential torus $\gamma_1 \times S^1$. Then $\eta(\gamma_1 \cup \gamma_2)$ is isotopic to a closed regular neighbourhood $\hat{\eta}(\gamma_1 \cup \gamma_2)$ of $\gamma_1 \cup \gamma_2$ in $\gamma_1 \times S^1$ and D can be chosen to be the inverse image of $(\gamma_1 \times S^1) - \eta(\gamma_1 \cup \gamma_2)$ under this isotopy.)

The surface F is constructed from $h^{-1}(r)$ by performing ambient 1-surgeries along arcs lying in $h^{-1}([r, 1])$. These arcs can be chosen so that their endpoints are disjoint from a closed regular neighbourhood $\eta(\gamma_1 \cup \gamma_2)$ of $\gamma_1 \cup \gamma_2$ in F. Then $\partial D = \partial\eta(\gamma_1 \cup \gamma_2)$ is essential in F. Let $T = \eta(\gamma_1 \cup \gamma_2)$. Then $\sigma(T\,; \partial D)$ is an essential vertical and hence incompressible torus. The handlebodies W_1, W_2, cannot contain incompressible tori. So D is not contained in a single handlebody. Hence (W_1, W_2) is weakly reducible by Lemma 3.2.

Case 2. There is a regular value r of $h|_{Q \times S^1}$, $h|_{T_i}$ $(i = 1, \ldots, 2g)$, $h|_\gamma$ such that $h^{-1}(r) \cap (\text{sides of } G) \neq \varnothing$ and all components of $h^{-1}(r) \cap (\text{sides of } G)$ are strictly essential.

Claim. Each component l of $h^{-1}(r) \cap \partial G$ is a meridian μ of ∂G.

Let $p_1, \ldots, p_k$ be the points in $l \cap \gamma$ chosen in order around γ. For $i = 1, \ldots, k$ choose a point q_i so that $p_1, q_1, \ldots, p_k, q_k$ are also chosen in order around γ. This can be done even if $k = 1$. Set $E_i = Q \times \{q_i\}$ and let $\mu_i = E_i \cap \partial G$, a meridian. Subarcs of μ_i are identified pairwise in $Q \times S^1$ along with the annuli they are contained in. Endow μ_i with the orientation induced by that of $Q \times S^1$. Set $\mu^* = \bigcup_i \mu_i$.

On A_i, define the slope of an arc α in $l \cap A_i$ to be the algebraic intersection number of α with μ^*. Since all arcs in $h^{-1}(r) \cap A_i$ are spanning arcs, adjacent arcs in $l \cap A_i$ have the same slope up to sign, which is determined by the orientation of the arc. Further, since $l \cap \partial G$ is strictly essential, adjacent arcs have the same orientation. Thus on a given component l, the slope m_i is well-defined on A_i. Opposite annuli A_i and A_{i+2g} are identified in $Q \times S^1$, whence $s_i = -s_{i+2g}$. Now as l winds around ∂G once, l traverses opposite annuli, so $\sum_i s_i = 0$. Hence l is a closed curve which is a meridian μ. Note that the product structure of $Q \times S^1$ is crucial in this argument. This completes the proof of the claim.

Let $l_1, \ldots, l_n$ be the components of $(h^{-1}(r)) \cap \partial G$, each a 1-complex in $h^{-1}(r)$. Let T be a closed regular neighbourhood of $l_1 \cup \ldots \cup l_n$ in $h^{-1}(r)$. Consider $A = (Q \times S^1) - \partial(\eta(l_1 \cup \ldots \cup l_k))$, where $\eta(l_1 \cup \ldots \cup l_k)$ is an open regular neighbourhood of $l_1 \cup \ldots \cup l_k$ and $T \cap A = \partial T$. By the claim, each component of ∂T bounds a disk in A. Call these disks $D_1, \ldots, D_n$ and set $\Delta = (D_1 \cup \ldots \cup D_n)$. In G, $T \cup \Delta$ is a collection of disks. In $Q \times S^1$ this collection of disks is identified with an incompressible surface. Since neither W_1 nor W_2 contains an incompressible surface, (W_1, W_2) is weakly reducible, by Lemma 3.2.

The proof of the analogous result in the case where $\partial Q \neq \emptyset$ uses the existence of completely essential saddles in a singular foliation of Q by level sets of h when $\chi(Q) \leq -1$.

LEMMA 3.9. *Suppose $\partial Q \neq \emptyset$ and either* genus$(Q) > 1$ *or* $\chi(Q) < -1$. *Then any Heegaard splitting (W_1, W_2) of $Q \times S^1$ is weakly reducible.*

Proof. Let h be a Morse function induced by (W_1, W_2). And let β_1, β_2 be non-parallel essential arcs in Q with $\partial\beta_1, \partial\beta_2 \subset B$, for some component B of ∂Q. Set $A_i = \beta_i \times S^1$. Identify β_i with $A_i \cap (Q \times \{p\})$ for some $p \in S^1$. Without loss of generality, assume that h, $h|_{Q\times\{p\}}$, $h|_{A_i}$, $h|_{\beta_i}$ are Morse functions with distinct critical values and $h(B) = 0$. By Remark 3.5, one can assume that β_i passes through every essential saddle in the singular foliation Φ_i of $h|_{A_i}$ by level sets of h and that β_i is monotonic between two essential saddles of $h|_{A_i}$.

Figure 6 illustrates an essential arc (β_1) and an inessential arc (β_2).

Claim. Let Φ be the singular foliation of $Q \times \{p\}$ with respect to h. An arc $\beta \subset Q \times \{p\}$ with $\partial\beta \subset B$ which does not intersect a completely essential saddle in Φ is inessential.

Observe that in an essential saddle which is not completely essential, the two circles whose wedge is the saddle are parallel. Hence if one is boundary parallel, then so is the other. An arc β, with $\partial\beta \subset B$, which does not intersect a completely essential saddle, lies in an annulus which is a regular neighbourhood of B. Thus β is inessential. This proves the claim.

Let r^* be a singular value of $h|_{Q\times\{p\}}$ for which $(h|_{Q\times\{p\}})^{-1}(r^*)$ contains a completely essential saddle σ and such that the subarcs of β_1 and β_2 connecting B to σ do not intersect any other completely essential saddles. Then $(h|_{Q\times\{p\}})^{-1}(r^*)$ contains a figure eight; nearby regular values of $h|_{Q\times\{\text{point}\}}$ compose a thrice

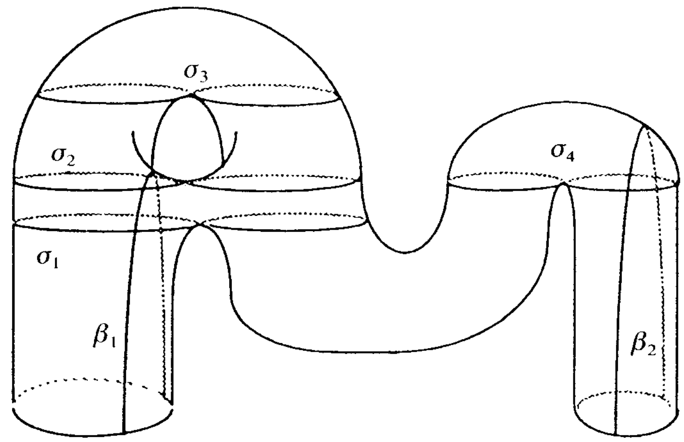

FIG. 6

punctured sphere P which is a regular neighbourhood of this figure eight. The boundary ∂P has three components, B_1, B_2, B_3, all essential in Q. By choice of r^*, one of them, say B_1, is parallel to B in $Q - P$. If $\max\{h|_{\beta_i}\} < r^*$, β_i must lie in the annulus between B and B_1. Since β_i is essential, this is impossible. Thus $\max\{h|_{\beta_i}\} > r^*$ and β_i must intersect either B_2 or B_3.

Suppose B_2 is parallel in Q to a component B' of ∂Q. If β_1 and β_2 are both disjoint from B_3, then they both lie in the thrice punctured sphere bounded by B, B' and B_3. This contradicts the fact that in a thrice punctured sphere, two essential arcs with both endpoints in the same boundary component are parallel. Hence one of the β_i must intersect B_3. Since Q is not itself a thrice punctured sphere, B_2 and B_3 cannot both be boundary parallel in Q. Thus either β_1 or β_2, say β_1, intersects an essential, non-boundary parallel circle γ_1 (B_2 or B_3) which lies in $h^{-1}(r) \cap Q$, for some regular value r of h, $h|_{Q\times\{p\}}$, $h|_{A_1}$. By Remark 3.6, the component γ_2 of $h^{-1}(r) \cap A_1$ which intersects γ_1 is an essential vertical level curve. By the argument used in Lemma 3.7, Case 1, (W_1, W_2) is weakly reducible.

Proposition 3.1 follows from Lemmas 3.6 and 3.9.

REMARK 3.10. Since any Heegaard splitting (W_1, W_2) of $Q \times S^1$ is weakly reducible, except in the case where Q is a torus, a disk, an annulus, a sphere, or a thrice punctured sphere, there is a weakly reducing collection of disks for (W_1, W_2). By the proof of Proposition 3.1, $F^* \neq \emptyset$, that is, F^* contains at least one incompressible surface which is not boundary parallel. Further, since F is separating, F^* is separating. These facts, together with the classification of incompressible surfaces in $Q \times S^1$ (Lemma 3.11), enable the inductive proof of the main theorem.

LEMMA 3.11. *Let p_1: $Q \times S^1 \to Q$ be the projection map. Let S be a closed incompressible surface in $Q \times S^1$. Then S can be isotoped so that it is either vertical or horizontal. That is, after isotopy, either S is a collection of vertical tori, or $p_1|_S$: $S \to Q$ is a covering map.*

There is an immediate corollary.

COROLLARY 3.12. *Let Q be a compact surface which is not a sphere, a disk, an annulus, or a thrice punctured sphere. Any Heegaard splitting of $Q \times S^1$ can be constructed by a series of amalgamations, either out of Heegaard splittings of* (annulus) $\times S^1$ *and* (thrice punctured sphere) $\times S^1$ *or out of Heegaard splittings of* (closed orientable surface) $\times I$.

Proof of Corollary 3.12. It suffices to prove this for irreducible Heegaard splittings (W_1, W_2) of $Q \times S^1$. Let Δ be a weakly reducing collection of disks for (W_1, W_2) of maximal complexity. Since F^* is properly embedded, it is either a disjoint collection of vertical incompressible surfaces or a disjoint collection of horizontal incompressible surfaces. Furthermore, the induced Heegaard splittings on the components of $Q \times S^1 - F^*$ are not weakly reducible. In the case that F^* is vertical, this means that the closures of the components of $Q \times S^1 - F^*$ must be (annulus) $\times S^1$ and (thrice punctured sphere) $\times S^1$. In the case that F^* is horizontal, the closures of the components of $Q \times S^1 - F^*$ must be (closed orientable surface) $\times I$.

Proof of Lemma 3.11. This is classical 3-manifold topology. If Q is closed, consider $S \cap A_i$, for Υ, $A_1, \dots, A_{4g}$, γ, G as in the remarks preceding Lemma 3.3. The collection consists of four types of components:

(1) inessential circles;

(2) inessential arcs;

(3) essential circles;

(4) essential arcs.

Standard innermost-disk-outermost-arc arguments allow removal of components of Type (1) and of Type (2).

If $S \cap A_i$ contains only components of Type (3) and of Type (4), for $i = 1, \dots, 4g$, then in fact it contains either only components of Type (3), or only components of Type (4); because if $S \cap A_i$ contains a component of Type (3), for some i, then $S \cap A_i$ cannot contain a component of Type (4), whence $|S \cap \gamma| = 0$, and thus no A_i contains a component of Type (4). In this case $S \cap \partial G$ consists of longitudes of ∂G. Since S is incompressible, they bound a collection of annuli in G; hence $S \cap (Q \times S^1)$ is a collection of vertical tori in $Q \times S^1$. If $S \cap A_i$ contains only components of Type (4), for $i = 1, \dots, 4g$, then by the proof of Lemma 3.6, $S \cap \partial G$ consists of $\#|S \cap \gamma|$ meridians of ∂G. Since S is incompressible, they bound disks in G; hence $S \cap (Q \times S^1)$ consists of $\#|S \cap \gamma|$ disks, which, after isotopy, project onto Υ under p_1. The sides of these disks are identified to yield an incompressible surface for which $p_1|_S \colon S \to Q$ is a covering map.

The proof is similar in the general case. Note that if $\partial Q \neq \emptyset$, then after isotopy, S is vertical.

4. *Heegaard splittings of* (thrice punctured sphere) $\times S^1$

In the following, $P =$ (thrice punctured sphere), (W_1, W_2) is a Heegaard splitting of $P \times S^1$ and h is a Morse function on $P \times S^1$ induced by (W_1, W_2).

Theorem 4.5 gives the classification of Heegaard splittings for $P \times S^1$. The proof breaks down into two cases. If an irreducible Heegaard splitting of $P \times S^1$

is not weakly reducible, then there is an essential horizontal arc in the spine of one of the compression bodies. This determines the structure of the Heegaard splitting. If a Heegaard splitting of $P \times S^1$ is weakly reducible, then an inductive argument shows that it is the amalgamation of standard Heegaard splittings and hence standard by Proposition 2.10.

Let Υ be an octagon with sides, in order, $\gamma_1, \ldots, \gamma_8$. Set $G = \Upsilon \times I$ and $E_i = \gamma_i \times I$. Then $P \times I$ may be obtained from G by identifying E_1 with E_3 and E_5 with E_7, Let $P_0 = P \times \{0\}$ and $P_1 = P \times \{1\}$ in $P \times I$. Call $E_1, E_3, E_5, E_7, P_0, P_1$ the *sides* of G. Call E_2, E_4, E_6, E_8 the *boundary sides* of G. Note that $P \times S^1$ may be obtained from $P \times I$ by identifying P_0 with P_1. The boundary sides of G become the boundary components B_0, B_1 and B_2 of $P \times S^1$.

It follows from Lemmas 4.1 and 4.2 that if a Heegaard splitting of $P \times S^1$ is not weakly reducible, then, after arc slides and isotopy, the spine of one of its compression bodies contains an essential horizontal arc.

Figure 7 depicts the (octagon) $\times I$ suggestively.

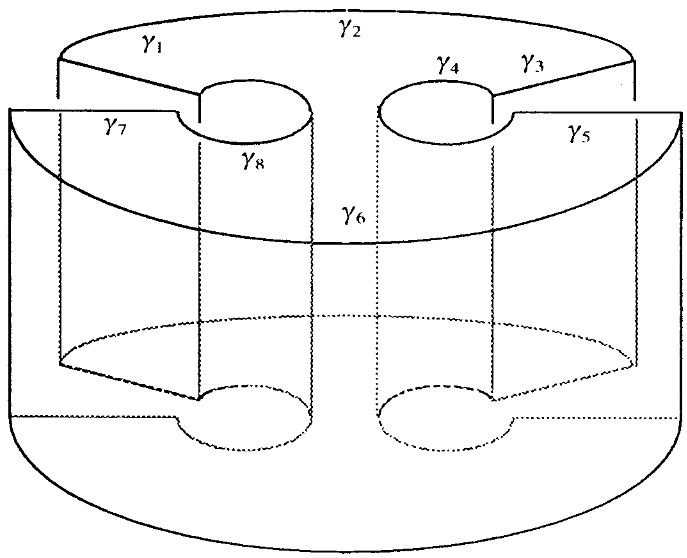

FIG. 7

LEMMA 4.1. *Suppose that γ_1, γ_5, as above, are in thin position with respect to h. Suppose that each of $h|_{\gamma_1}$, $h|_{\gamma_5}$ has at most one critical point. Then, after arc slides and isotopy, the spine of either W_1 or W_2 contains an essential horizontal arc $\alpha \times \{\text{point}\}$.*

Proof. Let W_1 be the compression body containing B_0. By the proof of [**9**, 4.1] there is a spine X for W_1 and there is a complete set of disks Δ for W_2 such that $X \cap \gamma_1 = \emptyset$, $X \cap \gamma_5 = \emptyset$ and $\Delta \cap \gamma_1 = \emptyset$, $\Delta \cap \gamma_5 = \emptyset$. Let $\bar{X}$ be a small closed regular neighbourhood of X. Consider $\Delta \cap$ (sides of G). These intersections consist of a collection of six different types of components, all disjoint from γ_1, γ_5, and hence contained in a single side K:

(1) simple closed curves;

(2) arcs with endpoints on distinct components of $\bar{X} \cap K$;

Let $q \in S^1$. Then F^* induces a homomorphism $f\colon H_1(Q \times \{q\}) \to \mathbb{Z}$ by setting $f(\alpha) = [F^*] \cdot \alpha$. Thus f defines a class $[f]$ in $\mathrm{Hom}(H_1(Q \times \{q\}), \mathbb{Z})$. The sequence

$$0 \to \mathrm{Hom}(H_1(Q \times \{q\}), \mathbb{Z}) \to H^1(Q \times \{q\}\,;\mathbb{Z}) \to \mathrm{Ext}(H_0(Q \times \{q\}), \mathbb{Z}) \to 0$$

is exact by the universal coefficient theorem. Here $\mathrm{Ext}(H_0(Q \times \{q\}), \mathbb{Z}) = 0$, since $H_0(Q \times \{q\})$ is free. Hence $[f]$ defines a 1-dimensional cohomology class $[\bar{f}]$. Let c be the Poincaré dual of $[\bar{f}]$. Then $c = n\bar{c}$ for some primitive element $\bar{c}$ of $H_1(Q \times \{q\})$. Represent $\bar{c}$ by an embedded simple closed curve also called $\bar{c}$ as in [5]. Then for $[\alpha] \in H_1(Q \times \{q\}), f([\alpha]) = n(\bar{c} \cdot \alpha)$.

Let c' be a circle intersecting $\bar{c}$ once. Set $T = \eta(\bar{c} \cup c')$, $Q' = Q - T$. It now suffices to show that each component of F^* can be isotoped to be strictly horizontal over Q'. Hence in what follows, F^* can be taken to be connected.

Let X be a spine of Q'. Let β be a simple closed curve in X and $T_\beta = \beta \times S^1$. Then $F^* \cap T_\beta = \gamma_1 \cup \ldots \cup \gamma_n$, a collection of simple closed curves on T_β, any two of which are either parallel or anti-parallel, since they are disjoint.

Claim. The curve γ_i is parallel to γ_j, for $i, j = 1, \ldots, n$.

A Riemannian metric on Q and the standard Riemannian metric on S^1 endow $Q \times S^1$ with a Riemannian metric via the natural isomorphism $\tau(Q \times S^1) \cong \tau(Q) \times \tau(S^1)$. This Riemannian metric has the property that a tangent plane to $Q \times S^1$ is vertical if and only if its normal vector is strictly horizontal. Let v be the normal bundle to F^* in $Q \times S^1$. Since $P_1\colon F^* \to Q$ is a covering map, $(p_1)^*$ is never singular, and hence $(p_2)^*(v)$ is never zero. Now F^* is connected, so $(p_2)^*(v)$ has the same orientation in $\tau(S^1)$ on all of F^*, and, in particular, on $T_\beta \cap F^* = \gamma_1 \cup \ldots \cup \gamma_n$. Therefore all the γ_i (oriented via the cross-product of v and a normal orientation of T_β) are oriented so that their normals project to the same orientation in $\tau(S^1)$ and are thus parallel. This proves the claim.

Now $0 = [F^*] \cdot \beta = \sum \gamma_i \cdot \beta$. But since $\gamma_1, \ldots, \gamma_n$ are isotopic, $\gamma_i \cdot \beta = \gamma_j \cdot \beta$, for $i, j = 1, \ldots, n$. Thus $\gamma_j \cdot \beta = 0$. By the classification of simple closed curves on the torus, $\gamma_1, \ldots, \gamma_n$ are therefore all parallel or all anti-parallel to β. In particular, $\gamma_1, \ldots, \gamma_n$ can be isotoped to be strictly horizontal over β and to be disjoint from β. Hence F^* can be isotoped to be disjoint from X and to be strictly horizontal over X. These properties can be extended to $\eta(X) \cong Q'$. This makes F^* strictly horizontal over Q'.

When $p_1|_{F^*}\colon F^* \to Q$ is a finite covering map, F^* can be viewed as a k-fold covering space of Q.

REMARK 5.3. *For Q, Q' as above, let q be a point in Q' and set $\gamma = \{q\} \times S^1$. Then $|\gamma \cap F^*| = k$, but $\gamma \cdot [F^*] = 0$, since $[F^*] = [F] = 0$. Hence k must be even. In particular, $k > 1$. Furthermore, the closure of each component of $Q \times S^1 - F^*$ is of the form* (genus-$\bar{g}$ closed orientable surface) $\times I$.

For $\bar{q}$ a closed orientable surface of genus $\bar{g}$, $\bar{Q} \times \{\frac{1}{2}\}$ splits $\bar{Q} \times I$ into two compression bodies, and hence defines a Heegaard splitting (W_1, W_2). The splitting (W_1, W_2) and its stabilizations are called the *standard Heegaard splittings* *of* $\tilde{Q} \times I$ *rel* $\tilde{Q} \times \{0\}$. Let γ be a vertical arc (that is, $\{\text{point}\} \times I$) in $\tilde{Q} \times I$ joining $\tilde{Q} \times \{0\}$ and $\tilde{Q} = \{1\}$. Set $W_1 = \text{closure}(\eta(\gamma \cup \tilde{Q} \times \{0\} \cup \tilde{Q} \times \{1\}))$, a compression body, and $W_2 = \tilde{Q} \times S^1 - \text{interior}(W_1)$. Since $W_2 \cong (\tilde{Q} - \{\text{point}\}) \times I$ is a handlebody, (W_1, W_2) is a Heegaard splitting. Here (W_1, W_2) and its stabilizations are called the *standard Heegaard splittings of* $\tilde{Q} \times I$ *rel* $\partial(\tilde{Q} \times I)$.

THEOREM 5.4. *All Heegaard splittings of* $Q \times I$ *are stabilizations of a standard Heegaard splitting.*

Proof. See [**9**].

For Q a torus, Lemmas 5.5 and 5.6 follow from [**1**, 1.6]. Hence in the following, genus$(Q) \geq 2$, so that genus$(Q') \geq 1$, for Q' satisfying the conclusions of Lemma 5.2.

LEMMA 5.5. *If* F^* *is a k-fold covering space of* Q *with* $k > 2$, *then* (W_1, W_2) *is reducible.*

Proof. By Remark 2.5, the induced Heegaard splittings on the closures of the components $C_1 = \tilde{Q} \times I$ and $C_2 = \tilde{Q} \times J$ of $(Q \times S^1) - F^*$ are standard rel ∂C_1, ∂C_2. Thus C_1, C_2 can be chosen so that after isotopy the proper arcs defining the induced Heegaard splittings are $\{p_1\} \times I$ and $\{p_2\} \times J$ for two distinct points p_1, p_2 in Q', and $I = [t_{i-1}, t_i]$, $J = [t_i, t_{i+1}]$, for some $i \in \mathbb{Z}_k$. Since genus$(Q') > 0$, there are essential circles c_1, c_2 in Q' running through p_1, p_2 respectively, which intersect in exactly one point in $Q' - \{p_1, p_2\}$. Set $D_1 = (c_1 - \eta(p_1)) \times I$ and $D_2 = (c_2 - \eta(p_2)) \times J$. By construction, $(D_i, \partial D_i) \subset (W_i, \partial_+ W_i)$ with ∂D_i essential in F and $\#|\partial D_1 \cap \partial D_2| = 1$. Hence (W_1, W_2) is reducible.

LEMMA 5.6. *If* F^* *is a 2-fold covering space of* Q, *there is a weakly reducing collection of disks* $\tilde{\Delta}$ *for* (W_1, W_2), *such that* $\tilde{F}^* = \sigma(F ; \partial\tilde{\Delta}) = \{\text{vertical tori}\}$.

Proof. Again, the induced Heegaard splittings on the closures C_1, C_2 of the components of $(Q \times S^1) - F^*$ are standard rel ∂C_1, ∂C_2; and after isotopy the proper arcs defining the induced Heegaard splittings are $\{p_1\} \times I$, $\{p_2\} \times J$ for two distinct points p_1, p_2 in Q' and $I = [t_1, t_2]$, $J = [t_2, t_1]$ (here $I \cup J = S^1$). Let c_1, c_2 be non-intersecting parallel essential curves in Q' through p_1, p_2. Set $D_1 = (c_1 - \eta(p_1)) \times I$ and $D_2 = (c_2 - \eta(p_2)) \times I$. By construction, $(D_i, \partial D_i) \subset (W_i, \partial_+ W_i)$ with ∂D_i essential in F and $\#|\partial D_1 \cap \partial D_2| = 0$. Hence $D_1 \cup D_2$ is a weakly reducing collection of disks for (W_1, W_2), such that $\sigma(F ; \partial(D_1 \cup D_2))$ contains at least one vertical torus. Thus any weakly reducing collection of disks $\tilde{\Delta}$, with $D_1 \cup D_2 \subset \tilde{\Delta}$ has $\tilde{F}^* = \{\text{vertical tori}\}$.

THEOREM 5.7. *All Heegaard splittings of* (compact orientable surface) $\times S^1$ *are standard.*

Proof. It suffices to show this for an irreducible Heegaard splitting (W_1, W_2) of $Q \times S^1$. Then Lemma 5.3 to 5.6 show that (Δ_1, Δ_2) can be chosen so that F^* is a collection of vertical tori. So by Lemma 5.1, (W_1, W_2) is standard.

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