Thin position for knots and 3-manifolds

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April 7, 2022

Abstract

We prove that for 2-bridge knots and 3-bridge knots in thin position the double branched cover inherits a manifold decomposition in thin position. We also argue that one should not expect this sort of correspondance to hold in general.

1 Preliminaries

Thin position for knots was first defined by D. Gabai in [2]. He used this notion to prove Property R for knots. A few years later M. Scharlemann and A. Thomspon developed a notion of thin position for 3-manifolds. Both of these notions have become vital tools in many geometric arguments. We here investigate the similarities and differences between the two notions.

Definition 1.1. *Suppose L is a submanifold of M. We will denote an open regular neighborhood of L in M by $\eta(L, M)$ or simply by $\eta(L)$, if the ambient manifold is clear from the context. Similarly, we will denote a closed regular neighborhood of L in M by $N(L, M)$ or simply by $N(L)$.*

We define thin position for knots as in [15]:

Definition 1.2. *Let k be a knot in the 3-sphere. The complement of k is $S^3 - \eta(k)$.*

Definition 1.3. *A* <u>*meridional planar surface*</u> *in the complement of k is a planar surface properly embedded in the knot complement whose boundary components are meridians.*

Definition 1.4. *A boundary parallel annulus with meridional boundary components is a* <u>*trivial*</u> *meridional planar surface.*

Definition 1.5. *Let $h : \{S^3 - (\text{two points})\} \to [0, 1]$ be a height function on S^3 that restricts to a Morse function on k. Choose a regular value t_i between each pair of adjacent critical values of $h|_k$. The* <u>*width of k with respect to h*</u> *is $\sum_i \#|k \cap h^{-1}(t_i)|$.*

Define the <u>*width of k*</u> *to be the minimum width of k with respect to h over all h. A thin position of k is the presentation of k with respect to a height function that realizes the width of k.*

For any regular value t_0 of $h|_k$ we can find a neighborhood $\eta(k)$ such that $P = h^{-1}(t_0) - \eta(k)$ is a meridional planar surface in the complement of k.

Definition 1.6. *An* <u>*upper disk*</u> *for a meridional planar level surface P is a disk D such that $\partial D' = \alpha \cup \beta$, where α is an arc properly embedded in P, β is an arc embedded on the boundary of $\eta(k)$, parallel to an arc of k, $\partial\alpha = \partial\beta$, $int(D)$ intersects P in simple closed curves, and a small product neighborhood of α in D lies on the side of P containing $h^{-1}(1)$, i.e., it lies above P. A* <u>*strict upper disk*</u> *for P is is an upper disk D for P whose interior is disjoint from P.*

<u>*Lower disks*</u> *for P and* <u>*strict lower disks*</u> *for P are defined similarly.*

If for some meridional planar level surface P there is a disjoint pair of upper and lower disks for k, then these disks describe a width reducing isotopy. In other words, if k is in thin position, then there is no such level surface.

Definition 1.7. *A* <u>*thin level*</u> *for k is a 2-sphere S such that the following hold:*

1) $S = h^{-1}(t_0)$ for some regular value t_0;

2) t_0 lies between adjacent critical values x and y of h, where x is a minimum of k lying above t_0 and y is a maximum of k lying below t_0.

A <u>thick level</u> is a 2-sphere S such that the following hold:

1) $S = h^{-1}(t_0)$ for some regular value t_0;

2) t_0 lies between adjacent critical values x and y of h, where x is a maximum of k lying above t_0 and y is a minimum of k lying below t_0.

More recently, the width of k has been defined via a lexicographically minimal non-increasing sequence of numbers involving the numbers of intersection of thick and thin levels with k. This is analogous to the notion of width of a 3-manifold, as defined below. Whether a thin position of a knot with respect to one definition is also a thin position of the knot with respect to the other is an open question. In most applications the precise definition does not make a difference. What matters is that both definitions imply that if k is in thin position, then there is no level surface at which there is a disjoint pair of upper and lower disks for k. For the knots under consideration here, the two notions of thin position resulting from the two notions of width coincide.

We borrow definitions of a thin manifold decomposition fairly directly from from [10]:

> One way to construct a 3-manifold is to take a 0-handle, add some 1-handles, then some 2-handles, and finally capping off with some 3-handles. An alternative is to start with 0-handles, add some 1-handles, then some 2-handles, then alternately more 1-handles and more 2-handles before finally capping off with some 3-handles. The former gives a traditional Heegaard spliting of a 3-manifold, with one hadlebody comprised of the 0-handles and the 1-handles, and the other handlebody made up of the remaining handles (the 2 and 3-handles). The latter gives a generalized Heegaard splitting.
>
> If M is an orientable, closed, connected 3-manifold, with $M = b_0 \cup N_1 \cup T_1 \cup N_2 \cup T_2 \cup \cdots \cup N_k \cup T_k \cup b_3$, where b_0 is made up of of 0-handles, b_3 is made up of 1-handles, and for each i, each N_i consists of 1-handles, and each T_i consists of 2-handles. M can now be built in stages, starting with b_0 then adding N_1,

then T_1, and so on. Let $S_i 1 \leq i \leq k$ be the surface obtained from $\partial[b_0 \cup N_1 \cup T_1 \cup N_2 \cup T_2 \cup \cdots \cup N_i]$ by deleting all spheres bounding 0- or 3- handles in the decomposition. Let $F_i 1 \leq i \leq k-1$ be the surface obtained from $\partial[b_0 \cup N_1 \cup T_1 \cup N_2 \cup T_2 \cup \cdots \cup T_i]$ by similarly deleting all such spheres.

Let $W_i =$ (collar of F_{i-1}) $\cup N_i \cup T_i$ together with every 0- and 3-handle incident to N_i or T_i. W_i is divided by a copy of S_i into two compression bodies: $\overline{N_i} = (0-\text{handles}) \cup (\text{collar of } F_{i-1}) \cup N_i$ and $\overline{T_i} = (\text{collar of } S_i) \cup T_i \cup 3\text{-handles}$. Thus S_i describes a Heegaard splitting of W_i into compression bodies $\overline{N_i}$ and $\overline{T_i}$.

Definition 1.8. *Let the complexity of a connected surface S be $c(S) = 1 - \chi(S) = 2genus(S) - 1$ for S of positive genus. Define $c(S^2) = 0$. For S not necessarily connected define $c(S) = \Sigma\{c(S')|S' a connected component of S\}$*

Definition 1.9. *Let the width of the decomposition of M be the set of integers $\{c(S_i)|1 \leq i \leq k\}$*

Order these integers in monotonically non-increasing order. Compare the ordered multi-sets lexicographically.

Definition 1.10. *Define the width $w(M)$ of M to be the minimal width over all decompositions using the above ordering of the sets of integers.*

Finally we arrive at

Definition 1.11. *A given decomposition of M is thin if the width of the decomposition is the width of M.*

The investigation here is motivated by the question as to how the two notions of thin position are related. The double branched cover construction provides a natural link between the two notions. We prove that for 2-bridge knots and 3-bridge knots in thin position the double branched cover inherits a manifold decomposition in thin position.

Theorem 1.12. *If k is either a 2-bridge knot or a 3-bridge knot, then the manifold decomposition that the double branched cover of (S^3, k) inherits from a thin position of k is thin.*

However, one can't expect this sort of correspondance to hold in general. In [5] Tsuyoshi Kobayashi and Yoav Rieck point out the following example: torus knots can have arbitrarily high bridge number. (For a proof of this fact, see [8] or for a more modern and self-contained proof, see [12].) Furthermore, their complements contain no meridional incompressible surfaces. (They are Seifert fibered spaces. Incompressible surfaces in Seifert fibered spaces are completely characterized.) Thus thin position is bridge position for torus knots. (By the main theorem of [15].) For a torus knot of bridge number b, the manifold decomposition that the double branched cover of (S^3, k) inherits from a thin position of k is then a genus $b - 1$ Heegaard splitting. On the other hand, the double branched cover of (S^3, k) is a small Seifert fibered space and such a manifold possesses a genus 2 Heegaard splitting. Thus for $b > 3$, the analog of Theorem 1.12 is false. In other words, Theorem 1.12 is sharp.

We wish to thank Marty Scharlemann and others for asking questions providing the motivation for this paper. This work is supported in part by the grant NSF-DMS 0203680.

2 Thin position versus bridge position

The notion of bridge number for a knot was invented by H. Schubert in "Über eine numerische Knoteninvariante". There he proved, among other things, that bridge number is subadditive under connected sum of knots. (For a short modern proof see [11].) The notion of bridge number has once again proven useful in our investigation below.

Definition 2.1. *A knot k is in bridge position if all its maxima occur above its minima. A level sphere that lies above all minima and below all maxima of k is called a* <u>*bridge sphere*</u> *of k. The* <u>*bridge number*</u> *of a knot k is half of the smallest possible number of times a bridge sphere of k intersects k. We denote it by $b(k)$.*

For knots with low bridge number thin position can be characterized, see below. To do so, we employ a calculation developed by M. Scharlemann and the second author in [9]. This calculation derives the width of k from a thin position of k.

Theorem 2.2. *Let k be a knot in thin position. Denote the thick levels of k by $S_1, \ldots, S_n$, where S_1 is the lowest thick level, S_2 the next lowest,* *and so forth and S_n the highest. Similarly, denote the thin levels of k by $L_1, \ldots, L_{n-1}$. Denote the number of intersections of k with S_i by s_i and the number of intersections of k with L_i by l_i. Then the following holds:*

$$width(k) = 2(\sum_{i=1}^{n}(\frac{s_i}{2})^2 - \sum_{i=1}^{n-1}(\frac{l_i}{2})^2)$$

Lemma 2.3. *If the thick sphere for a knot k in bridge position intersects k $2n$ times (as it would for an $n-$bridge knot) any presentation that is thinner cannot have a level sphere intersecting k in $2n$ points or more.*

Proof. Suppose that k is in thin position and suppose there is a level sphere that intersects k in $2n$ points. Then this level sphere is parallel to a thick or thin level. In particular, it intersects k in the same number of points as this thick or thin level. Note that if a thin level intersects k in $2n$ points, then both adjacent thick levels intersect k in strictly more than $2n$ points. I.e., $s_i > l_{i-1}$ and $s_i > l_i$. Thus we may assume that our level sphere is parallel to a thick level, say S_l. Then,

$$width(k) = 2(\sum_{i=1}^{n}(\frac{s_i}{2})^2 - \sum_{i=1}^{n-1}(\frac{l_i}{2})^2)$$

$$2((\frac{s_l}{2})^2 + \sum_{i=1}^{l-1}((\frac{s_i}{2})^2 - (\frac{l_i}{2}^2)) + \sum_{l}^{n}((\frac{s_{i+1}}{2})^2 - (\frac{l_i}{2})^2) \geq 2(\frac{s_l}{2})^2$$

Moreover, equality holds if and only if $n = 1$. I.e., if thin position is bridge position. □

Corollary 2.4. *Thin position equals bridge position for two bridge knots.*

Proof. The only knot that has a presentation with no level sphere intersecting the knot in more than two points is the unknot. □

Corollary 2.5. *Thin position equals bridge position for a three bridge knot k if and only if k is prime. If the knot is not prime every thin position has a thick layer intersecting the knot in four points, followed by a thin layer intersecting the knot in two points, followed by a thick layer intersecting the knot in four points, reflecting the fact that k is the connect sum of two two bridge knots.*

Proof. The theorem of H. Schubert mentioned above states that $b(K_1\#K_2) = b(K_1) + b(K_2) - 1$. Thus if a three bridge knot is not prime, it must be the connected sum of two two bridge knots. By a theorem of A. Hatcher and W. Thurston [3], 2-bridge knots are small. By a theorem of Y. Rieck and E. Sedgwick (see [7]), thin position for the sum of two small knots is obtained by stacking a thin position of one of the knots on top of a thin position of the other, as in Figure 1.

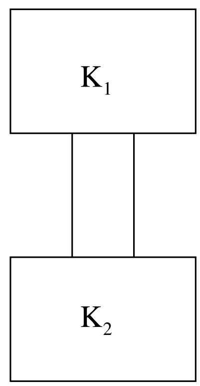

Figure 1: Schematic diagram for the sum of two small knots in thin position

Now if k is a prime 3-bridge knot, then the following hold:

- k can't have a thick level meeting k in 2 points, for it is not the unknot.
- k can't have a thick level meeting k in 4 points. If it does, then the adjacent thin levels meet k in 2 points and k is a connected sum. If there are not adjacent thin levels, then k is a 2-bridge knot.

Thus k has a thick level meeting k in 6 points. By Lemma 2.3 thin position must be bridge position for k. □

3 Interconnections

The question motivating this paper is the following:

Question 3.1. *How is thin position in knots related to thin position in 3-manifolds?*

The definitions exhibit much similarity, but are not totally analogous. A natural way in which the two definitions can be related is via the construction of double branched cover . Let k be a knot and denote the double branched cover of (S^3, k) by M.

If k is in thin position, then M inherits a manifold decomposition as follows: Denote the thick levels of k by $S_1, \ldots, S_n$ and the thin levels by $L_1, \ldots, L_{n-1}$. Each S_i and each L_i is a sphere that meets the knot some (even) number of times. More specifically, each S_i meets k at least 4 times and each L_i meets k at least 2 times. Denote the surface in M corresponding to S_i by $\tilde{S}_i$ and the surface in M corresponding to L_i by $\tilde{L}_i$. Each $\tilde{S}_i$ is a closed orientable surface of genus at least 1 and each $\tilde{L}_i$ is a closed orientable surface. More specifically, if S_i meets k exactly $2l$ times, then $\tilde{S}_i$ is a closed orientable surface of genus $l-1$. And if L_i meets k exactly $2l$ times, then $\tilde{L}_i$ is a closed orientable surface of genus $l-1$.

Now consider the 3-ball bounded by S_1. It contains l subarcs of k each with exactly one minimum. Moreover, it contains l disjoint strict lower disks . Each strict lower disk corresponds to a disk in M whose boundary is an essential curve on $\tilde{S}_1$. Moreover, these l compressing disks for $\tilde{S}_1$ cut the 3-manifold bounded by $\tilde{S}_1$ into two 3-balls. In particular, $\tilde{S}_1$ bounds a handlebody in M. An analogous argument shows that $\tilde{S}_n$ bounds a handlebody in M.

For $2 \leq i \leq n$, consider the $(disk) \times I$ cobounded by L_{i-1} and S_i. It meets k in some number, say m, of vertical arcs, and some number, say l, of arcs each with exactly one minimum. Then it also contains l disjoint strict lower disks . These strict lower disks again correspond to compressing disks. This time for the compact submanifold of M cobounded by $\tilde{L}_{i-1}$ and $\tilde{S}_i$. They have their boundaries on $\tilde{S}_i$ and cut the 3-manifold cobounded by $\tilde{S}_i$ and $\tilde{L}_{i-1}$ into $\tilde{L}_{i-1} \times I$. In particular, $\tilde{L}_{i-1}$ and $\tilde{S}_i$ cobound a compression body in M. An analogous argument shows that $\tilde{L}_i$ and $\tilde{S}_i$ cobound a compression body. Note that in both cases it is $\tilde{S}_i$ that compresses, in one case towards $\tilde{L}_{i-1}$ and in the other case towards $\tilde{L}_i$.

Now a natural question to ask is the following:

Question 3.2. *If a knot k is in thin position, then is the manifold decomposition that the double branched cover inherits also thin?*

As mentioned in the introduction, the answer to this question is no. Torus knots provide counterexamples. On a more philosophical level, there may be other reasons for this negative answer. For in a thin manifold decomposition the thin levels are incompressible. This was proven by M. Scharlemann and A. Thompson [10, Rule 5]. On the other hand, given a knot in thin position, it is likely that there are compressing disks for a thin level in the complement of the knot. See the schematic knot diagram in Figure 2 due to T. Kobayashi [4]. Though we can't prove that a knot of this form really must be in thin position, there are reasons to believe that sufficiently complicated braids would force it to be so.

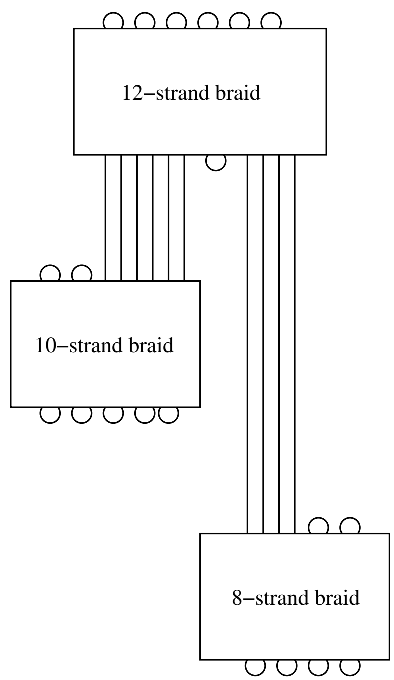

Figure 2: A schematic knot diagram, probably representing knots in thin position

But for knots with bridge number 2 or 3, the answer to the question is in fact yes.

The following three propositions establish Theorem 1.12.

Proposition 3.3. *If k is a 2-bridge knot, then the manifold decomposition that the double branched cover of (S^3, k) inherits from a thin position of k is thin.*

Proof. Suppose that k is a 2-bridge knot. In this case the double branched cover is a lens space that is not S^3 or $S^2 \times S^1$. In particular, M is irreducible.

By Corollary 2.4, thin position is bridge position for k. Thus the manifold decomposition that M, the double branched cover of (S^3, k), inherits from a thin position of k is defined by a genus 1 surface bounding solid tori on either side. The width of this manifold decomposition is $\{1\}$. Thinner manifold decompositions must include 0's. A 3-manifold with a width that includes 0's is reducible. Since M is irreducible, its width is exactly $\{1\}$. □

Proposition 3.4. *If k is a 3-bridge knot for which thin position is not bridge position, then the manifold decomposition that the double branched cover of (S^3, k) inherits from a thin position of k is thin.*

Proof. Suppose that k is a 3-bridge knot for which thin position is not bridge position. In this case the double branched cover M of (S^3, k) is not S^3 and is not itself a lens space, though as we shall see, it contains lens spaces.

By Corollary 2.5, thin position of k is achieved by stacking a thin position of one of the 2-bridge summands on top of a thin position for the other 2-bridge summand, as in Figure 1.

Thus the manifold decomposition that M inherits corresponds to a connected sum decomposition of M into two lens spaces (not equal to S^3 or $S^2 \times S^1$). The width of this decomposition is $\{1, 1, 0\}$, the smallest possible width for a 3-manifold that is not prime. Thus the manifold decomposition is thin. □

Proposition 3.5. *If k is a 3-bridge knot for which thin position is bridge position, then the manifold decomposition that the double branched cover of (S^3, k) inherits from a thin position of k is thin.*

Proof. Suppose that k is a 3-bridge knot for which thin position is bridge position. In this case the double branched cover M of (S^3, k) is again not S^3 and not a lens space. Here the manifold decomposition that M inherits corresponds to a genus 2 Heegaard splitting and thus has width $\{3\}$.

The only manifold decompositions thinner than that are those whose width begins with a 1. A 3-manifold whose width contains a 1 is equal to or contains a lens space. If M is a lens space, then it has a genus 1 Heegaard splitting that can be isotoped to be invariant under the involution. The quotient of the splitting torus would be a 2-sphere defining a 2-bridge presentation for k. But this is impossible.

If M contains a lens space summand, then Corollary 3 of [6] states that for a manifold that is a connected sum, an involution factors into involutions on the summands. This implies that if M is a connected sum, then it is the double branched cover of (S^3, k) for a composite knot k. If this is the case, then the reasoning in the proof of Proposition 3.4 shows that thin position is not bridge position for k. Thus here M must be prime. In particular, it can't have a manifold decomposition beginning with 0. Thus the width of M is $\{3\}$. $\square$

4 3-bridge knots revisited

In this section we present a combinatorial proof of Proposition 3.4. In endeavoring to extend Theorem 1.12 to knots with higher bridge number, the short proof above would fail, as there would certainly be prime knots for which bridge position does not minimize width. For this reason we include the more direct combinatorial argument below.

4.1 Cut vertices and a theorem of R. Stong

Cut vertices play a role in graph theory and geometric group theory. Below, we define a graph related to a Heegaard splitting. In this context cut vertices help identify certain properties.

Definition 4.1. *Given a connected graph Γ, a vertex v_i of Γ is said to be a* *cut vertex* *if $\Gamma - v_i$ is not connected.*

Definition 4.2. *Given a handlebody H and a curve $\alpha \subset \partial H$, α is said to be* *disk busting* *if it intersects the boundary of every essential disk in H.*

Let H be a genus n handlebody. Let $\{D_1, \ldots D_n\}$ be a set of defining disks for H. Let $N(D_i) = D_i \times [0,1]$ be a closed regular neighborhood of D_i in H. Denote $D_i \times 1$ by V_i and $D_i \times 0$ by V_i'. When we remove $D_i \times (0,1)$ from H we obtain a ball with $2n$ (fat) vertices, $V_1, V_1', \ldots, V_n, V_n'$, on its boundary. If there is a curve on the boundary of H, then the subarcs of this curve form the edges of graph with $2n$ (fat) vertices.

The following theorem is due to Richard Stong. See [14]. For completeness, we include a proof.

Theorem 4.3. *Let α be a simple closed curve on the boundary of the genus n handlebody H. The above construction yields a graph Γ. If the D_i can be chosen so that Γ is connected and has no cut vertices, then α is disk busting in H.*

Proof. Let $\{D_1, \dots D_n\}$ be a set of defining disks for H. Suppose that this set is chosen so that Γ is connected and has no cut vertices. Let D be a disk in H that does not meet α. Closed components of intersection of D with the $D_i's$ can be removed via an innermost disk argument.

Now consider an outermost arc β of $D \cap \cup_i D_i$ in D. Then β cuts a disk D' off of D whose interior is disjoint from $\cup_i D_i$. The boundary of D' is partitioned into two subarcs: β and an arc γ. Here γ is an arc on the ball that meets Γ only in its endpoints. Both of these lie on a single (fat) vertex. Since Γ contains no cut vertices, γ together with a subarc of the boundary of this (fat) vertex must bound a disk whose interior does not meet Γ. But this means that we may isotope D' through $\cup_i D_i$ to reduce the number of components of $D \cap \cup_i D_i$.

It follows that we may isotope D to be disjoint from $\cup_i D_i$. Thus D lies in the ball whose boundary contains Γ. Since ∂D separates the boundary of this ball into two disks and since Γ is connected, one of these disks is disjoint from Γ. Therefore D is parallel into the boundary of this ball and thus parallel into ∂H. I.e., D is inessential in H. We conclude that every essential disk in H meets α. $\square$

This theorem is related to work of J. Berge, J.R. Stallings (see [13]) and probably the thoughts of J.H.C. Whitehead.

4.2 Explicit combinatorial argument proving Proposition 3.4

We here provide a combinatorial argument to prove that in this case the manifold decomposition the double branched cover of (S^3, k) inherits from thin position of k corresponds to a strongly irreducible genus 2 Heegaard splitting. A 3-manifold that has a strongly irreducible genus 2 Heegaard splitting is irreducible. Furthermore, it can be neither a 3-sphere, by a theorem of F. Waldhausen (see [16]), nor a lens space, by a theorem of F. Bonahon and J.P. Otal (see [1]). It follows that its width is realized by this manifold decomposition and equals $\{3\}$.

Let k be an n bridge knot for which thin position equals bridge position. Let S be a bridge sphere of k (intersecting k in $2n$ points). Since thin position equals bridge position, there will be at least n strict upper disks for k based at S and n strict lower disks for k based at S. We denote the arcs in which the strict upper disks meet S by $\{a^+_1, \dots a^+_n\}$ and the arcs in which the strict lower disks meet S by $\{a_1^-, \dots a_n^-\}$. Note that if a_i^+ can be chosen disjoint from a_j^{-1} for any i, j then these disks yield an isotopy showing that k was not in thin position.

We denote the double branched cover of (S^3, k) by M. The double branched cover of S is a separating genus $n-1$ surface $\hat{S}$ in M. The double branched cover of a strict upper (lower) disk is a disk whose boundary lies in $\hat{S}$ and that lies to one side of $\hat{S}$. It follows that the double branched cover of the upper (lower) hemisphere is a genus $n-1$ handlebody H^+ (H^-). Thus we obtain a Heegaard splitting $M = H^+ \cup_{\hat{S}} H^-$. We denote the compressing disks for H^+ coming from the strict upper disks by $\{D_1^+, \dots D_n^+\}$ and the compressing disks for H^- coming from the strict lower disks by $\{D_1^-, \dots D_n^-\}$. We further denote the boundary of the disk D_i^* by c_i^*.

Lemma 4.4. *For each i, c_i^+ is disk busting in H^- and c_i^- is disk busting in H^+.*

The proof of this lemma relies heavily on the methods in [14].

Proof. In this setting the bridge sphere S meets k six times, $\hat{S}$ is a genus 2 surface, $\{D_1^+, D_2^+, D_3^+\}$ are compressing disks for H^+ with boundaries c_1^+, c_2^+, c_3^+, respectively, and $\{D_1^-, D_2^-, D_3^-\}$ are compressing disks for H^- with boundaries c_1^-, c_2^-, c_3^-, respectively.

By symmetry, it suffices to show that c_1^+ is disk busting in H^-. To do so, we work with $\{D_1^+, D_2^+, D_3^+\}$, but choose the strict lower disks leading to $\{D_1^-, D_2^-, D_3^-\}$ subject to the following condition:

- $c_1^- \cup c_2^- \cup c_3^-$ have the smallest total number of intersections with c_1^+.
Call this Minimality Property 1.

Subject to Minimality Property 1 we choose the strict lower disk leading to D_3^- subject to the following condition:

- c_3^- intersects c_1^+ the <u>most</u> times (of all of the minimal ways to choose the strict lower disks leading to $\{D_1^-, D_2^-, D_3^-\}$).

Call this Maximality property 2.

Split H^- along D_1^- and D_2^- as in [14]. Let Γ_1^+ be the graph that results from c_1^+. Let V_i^+ and V_i^- be the (fat) vertices of Γ_1^+ that result from splitting along D_i^- $(i \in 1, 2)$. Recall that k is in thin position. Hence each strict upper disk meets each strict lower disk in points on S. Therefore each c_i^+ intersects each c_j^- (at least twice). Futher note that there are no isotopies removing intersections. Thus for any given vertex of Γ_1^+ there must be an edge of Γ_1^+ meeting that vertex. And there must also be edges of Γ_1^+ meeting the waistband, c_3^+. (See Figure 3.)

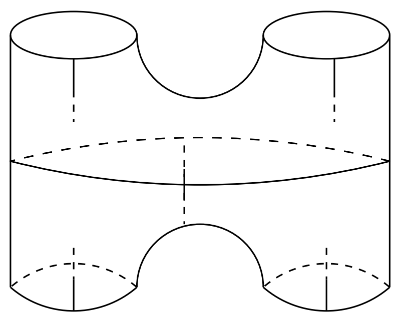

Figure 3: Edges everywhere

Sublemma 4.5. Γ_1^+ *must have a Hamiltonian cycle.*

Proof. Let d_1^+ be the valence of V_1^+, d_2^+ of V_2^+, d_1^- of V_1^- and d_2^- of V_2^-. Note that $d_i^+ = d_i^-$ as a result of the way the graph is formed. Let d_3 be the number of times Γ_1^+ intersects c_3^+. Note that $d_1^+ + d_2^+ + d_3$ was minimized in the selection leading to the D_i's.

Γ_1^+ can have up to six possible types of edges

- Class 1 connects $V_1^+ to V_1^-$.
- Class -1 connects $V_2^+ to V_2^-$.
- Class 2 connects $V_1^+ to V_2^+$.
- Class -2 connects $V_1^- to V_2^-$.
- Class 3 connects $V_1^+ to V_2^-$.
- Class -3 connects $V_1^- to V_2^+$.

We will show that Γ_1^+ has at least one edge each of class $i, -i, j, -j$ where $i, j \in \{1, 2, 3\}$ and $i \neq j$. This yields the desired Hamiltonian cycle.

Since $d_1^+, d_2^+, d_1^-, d_2^-$ are all positive there must be at least two types of edges. We note the following:

- Classes 1 and -1.

  All the edges cannot be from classes 1 and -1 because c_1^- is connected.

- Classes 2 and -2.

  All the edges cannot be from just Classes 2 and -2. If there are no other classes of arcs, then there is a properly embedded disk Δ in the complement of the $V_i's$ separating the two sets of arcs. Indeed, edges of this form can be thought of as rational tangles. Full twists, around the essential disk that meets D_3^- and D_3^- itself in turn, reduce the situtation to that in Figure 4 where the disk Δ is as pictured.

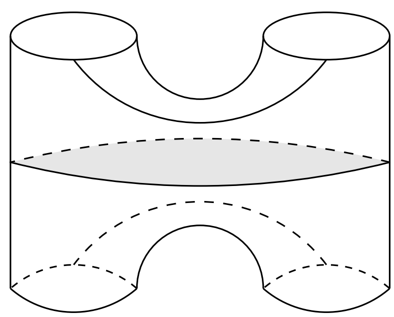

Figure 4: The disk Δ disjoint from edges of Class 2 and -2

  To see Δ in the original picture, reverse the twists. This procedure also makes clear that Δ is symmetric with respect to the involution. Hence Δ maps to a strict lower disk in (S^3, k) that misses the strict upper disk that gave rise to $c_1^+ = \partial D_1^+$. This, however, contradicts the fact that k was in thin position. Thus we cannot have only edges of Classes 2 and -2.

- Classes 3 and -3.

  All edges cannot be from only Classes 3 and -3 for the same reasons as they cannot be from only Classes 2 and -2. See Figure 5.

Claim: Γ_1^+ contains the same number of edges of Class 2 as -2. Γ_1^+ also contains the same number of edges of Class 3 as -3.

Note that $d_1^+ + d_2^+ = d_1^- + d_2^-$. An edge of Class 2 contributes 2 to the left side of the equation, but 0 to the right side. Edges of Class $1, -1, 3$, and -3 affect both sides equally, contributing 1 to each side, so the only way to

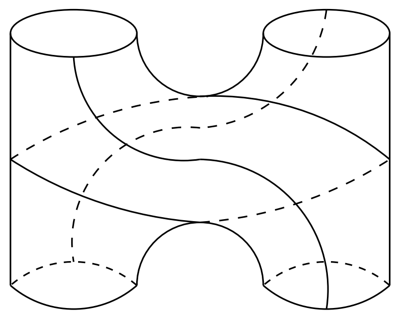

Figure 5: The boundary of the disk Δ disjoint from edges of Class 3 and -3

balance the equation is to add an edge of Class -2, which contributes 2 to the right side of the equation and 0 to the left side. Thus for the equation to hold each edge of Class 2 must be accompanied by exactly one edge of Class -2 and vice versa by a symmetric argument.

Since $d_1^+ = d_1^-$ and $d_2^+ = d_2^-$, we also have that $d_1^+ + d_2^- = d_1^- + d_2^+$. In the same manner as above, this allows us to argue that edges of Class 3 are always accompanied by the same number of edges of Class -3. This proves the claim.

Summarizing our observations so far, we see that since $d_1^+, d_2^+, d_1^-, d_2^-$ are all positive we must have at least 2 classes of edges. Then as all edges cannot be of Class j and $-j$ for a fixed j, we can assume without loss of generality that we have one of the following cases:

- Case a: An edge of Class 2 and an edge of Class 3.
- Case b: An edge of Class 1 and an edge of Class 2.
- Case a: An edge of Class 1 and an edge of Class 3.

In Case a) the claim above dictates that we have edges of Class -2 and -3, too. This yields the desired result: a Hamiltonian cycle.

In Case b), assume that we are not in Case a) and also not in Case c). Then the claim above dictates that we have an edge of Class -2 also. If we have an edge of Class -1 we have a Hamiltonian cycle and we are done. If not we have edges of Class 1,2, -2 and no others.

As in the argument above, if D_3^- is not already disjoint from the edges of Class 2 and -2, a disk Δ can be chosen that is disjoint from them. This disk, as before, could replace D_3^- since again it can be assumed to be symmetric with respect to the involution.

This is a contradiction, though, since it shows that D_3^- could have been chosen differently to decrease $d_1^+ + d_2^+ + d_3$ contradicting Minimality Property 1. Thus the edges of Class 2 and -2 are disjoint from D_3^-.

Thus we have a picture like Figure 4. Switching the roles of D_1^- and D_3^- contradicts Maximality Property 2 since c_1^- intersects Γ_1^+ in more points that c_3^+. This eliminates the possibility of being in Case b) without a Hamiltonian cycle.

In Case c) the argument is essentially the same as in Case b).

Thus in all possible cases there is a Hamiltonian cycle. $\square$(Sublemma 4.5)

To complete the proof of Lemma 4.4 note that if Γ_1^+ has a Hamiltonian cycle, then it has no cut vertices. This implies by Theorem 4.3 that c_1^+ is disk busting in H^-. The exact same argument can be made for c_2^+ and c_3^+, so they are disk busting, too. Note that no special assumption was made about D_1^+, D_2^+, or D_3^+ (unlike D_1^-, D_2^-, and D_3^- which had minimality assumptions placed on them) so all possible choices of D_1^+, D_2^+, and D_3^+ result in disk busting curves c_1^+, c_2^+ and c_3^+.

A symmetric argument shows that all possible choices of strict lower disks yielding D_1^-, D_2^-, D_3^- result in disk busting curves c_1^-, c_2^-, c_3^-. This completes the proof of Lemma 4.4. $\square$

Lemma 4.6. *The boundary of every essential disk in H^+ is disk busting in H^- and the boundary of every essential disk in H^- is disk busting in H^+.*

Proof. Let D_4^+ be a random essential disk properly embedded in H^+. Let c_4^+ be the boundary of D_4^+. We can assume without loss of generality that D_4^+ is chosen in such a way that the total number of intersections of c_4^+ with $\{D_1^-, D_2^-, D_3^-\}$ is minimal over all disks isotopic to D_4^+.

The proof that c_4^+ is disk busting largely mimics the proof of Theorem 4.4. As before form a graph Γ_4^+ for H^-. As before set d_1^+ equal to the number of times Γ_4^+ intersects V_1^+ and so on. Define the edge classes as before, too.

Since Lemma 4.4 shows that c_1^-, c_2^- and c_3^- are all disk busting in H^+, c_4^+ must intersect V_1^+, V_2^+, V_1^- and V_2^- as well as c_3^- so $d_1^+, d_2^+, d_1^-, d_2^-$, and d_3 are all positive. Thus as in the proof of Lemma 4.4 we have at least two edge types in Γ_4^+. As before, we note the following:

- Classes 1 and -1.

  As before, all the edges cannot be from classes 1 and -1 because c_4^+ is connected.

- Classes 2 and -2.

  As before, all the edges cannot be from just Classes 2 and -2 or there is a properly embedded disk Δ in the complement of the $V_i's$ separating the two sets of arcs. Indeed the edges can be thought of as a rational tangle and there is always a properly embedded disk Δ in the complement of any rational tangle. As above, we see that Δ is invariant under the involution. Thus alternative choices could have given us Δ instead of D_3^-. Recall that by Lemma 4.4 any choices of three disjoint strict lower disks yielded three disks in H^- whose boundaries were disk busting in H^+. I.e., $\partial\Delta$ must meet c_4^+, a contradiction. Thus we cannot have only edges of Class 2 and -2.

- Classes 3 and -3.

  A similar argument works to show that we cannot have just edges of Class 3 and -3.

The rest of the proof follows exactly as it did before, showing c_4^+ is disk busting, but D_4^+ was chosen at random and thus the boundary of every disk in H^+ is disk busting in H^-. A symmetric argument works for H^- completing the proof of Lemma 4.6. □

Our alternative proof of 3.4 is completed as follows:

Proof. As seen in the introduction to this section, the manifold decomposition that the double branched cover M of (S^3, k) inherits has width $\{3\}$ is defined by a genus 2 Heegaard splitting. By Lemma 4.6, this genus 2 Heegaard splitting is strongly irreducible. Since M has a strongly irreducible genus 2 Heegaard splitting, M can't be reducible. By a theorem of F. Waldhausen, every genus 2 Heegaard splitting of S^3 is stabilized, thus M can't be S^3. By a theorem of F. Bonahon and J.P. Otal, every genus 2 Heegaard splitting of a lens space is stabilized, thus M can't be a lens space. Thus the manifold decomposition defined by the strongly irreducible Heegaard splitting is thin. □

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