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TUNNEL NUMBERS OF SMALL KNOTS DO NOT GO DOWN UNDER CONNECTED SUM

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Abstract. Let K_1 and K_2 be two knots in S^3 and $t(K_1)$, $t(K_2)$ the tunnel numbers of them. In this paper, we show that if both K_1 and K_2 are small, then $t(K_1\#K_2) \geq t(K_1)+t(K_2)$. Moreover we show that $t(K_1\#K_2\#\cdots\#K_n) \geq t(K_1)+t(K_2)+\cdots+t(K_n)$ for any small knots $K_1, K_2, \cdots, K_n$.

1. Introduction

Let K be a knot in the 3-sphere S^3 and $t(K)$ the tunnel number of K. Here, $t(K)$ is the minimum number of mutually disjoint arcs properly embedded in $E(K)$ whose exterior is a handlebody, where $E(K) = cl(S^3 - N(K))$ and $N(K)$ is a regular neighborhood of K in S^3. For two knots K_1 and K_2, we denote the connected sum of them by $K_1\#K_2$.

Concerning the problem if tunnel numbers of knots go down or not under connected sum, in 1992 the first author showed the existence of those knots whose tunnel numbers go down. In fact he got:

Theorem 1 ([Mo1, Theorem]). *There are infinitely many knots K such that $t(K) = 2$ and $t(K\#K') = 2$ for any 2-bridge knot K'.*

After then, by taking the connected sum of knots obtained by modifying those knots in Theorem 1, Kobayashi showed:

Theorem 2 ([Ko, Theorem]). *For any positive integer n, there are infinitely many pairs of knots K_1 and K_2 such that $t(K_1\#K_2) < t(K_1) + t(K_2) - n$.*

Theorem 2 says that tunnel numbers of knots can arbitrarily highly degenerate. Moreover, we see that those knots in Theorem 1 and Theorem 2 have the property that the exteriors contain closed essential surfaces.

Now, we say that a knot K is small if $E(K)$ contains no closed essential surfaces. Then the second author showed:

Theorem 3 ([St2, Corollary 13]). *If both K_1 and K_2 are small, then*

$$t(K_1\#K_2) \geq t(K_1) + t(K_2) - 1.$$

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This theorem says that tunnel numbers of small knots either do not go down or go down by one under connected sum. In this paper, we show that we can get rid of the term " $-$ 1". Namely we prove:

Theorem 4. *If both* K_1 *and* K_2 *are small, then*

$$t(K_1 \# K_2) \geq t(K_1) + t(K_2).$$

More generally, we will prove the following. We note that $K \# K'$ is no longer small even if both K and K' are small.

Theorem 5. *For any small knots* $K_1, K_2, \cdots, K_n$*,*

$$t(K_1 \# K_2 \# \cdots \# K_n) \geq t(K_1) + t(K_2) + \cdots + t(K_n).$$

Throughout this paper, for an m-manifold M ($m = 2$ or 3 resp.) and an n-manifold N ($n = 1$ or 2 resp.) properly embedded in M, a component of $M - N$ means the closure of a component of $M - N$. And for a manifold X and a subcomplex Y of X, $N(Y)$ denotes a regular neighborhood of Y in X.

2. Preliminaries

Let K_1 and K_2 be two knots in S^3, and put $K = K_1 \# K_2$ in S^3. Let $N(K)$ be a regular neighborhood of K in S^3, and put $E(K) = cl(S^3 - N(K))$. Let $\Gamma = \{\gamma_1, \gamma_2, \cdots, \gamma_t\}$ be an unknotting tunnel sysytem for K, i.e. γ_i $(i = 1, 2, \cdots, t)$ is an arc properly embedded in $E(K)$ and $cl(E(K) - N(\gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_t))$ is a genus $t + 1$ handlebody, where $N(\gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_t)$ is a regular neighborhood of $\gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_t$ in $E(K)$.

Put $V_1 = N(\partial E(K) \cup \gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_t)$ in $E(K)$, $V_2 = cl(E(K) - V_1)$ and $V_1 \cap V_2 = F$. Then V_1 is a genus $t + 1$ compressionbody with $\partial V_1 - F = \partial E(K)$ and V_2 is a genus $t + 1$ handlebody with $\partial V_2 = F$. Hence (V_1, V_2) is a genus $t + 1$ Heegaard splitting of $E(K)$.

Let Δ_i $(i = 1, 2)$ be a disk properly embedded in V_i with $\partial \Delta_i \subset F$. Then we say that Δ_i is essential if $\partial \Delta_i$ is an essential loop in F. We say that the Heegaard splitting (V_1, V_2) is reducible if there is an essential disk Δ_i in V_i $(i = 1, 2)$ with $\partial \Delta_1 = \partial \Delta_2$, and that the Heegaard splitting (V_1, V_2) is irreducible if it is not reducible. Moreover according to [CG], we say that (V_1, V_2) is weakly reducible if there is an essential disk Δ_i in V_i $(i = 1, 2)$ with $\partial \Delta_1 \cap \partial \Delta_2 = \emptyset$, that (V_1, V_2) is strongly irreducible if it is not weakly reducible.

Now, suppose that the unknotting tunnel system Γ for $K = K_1 \# K_2$ realizes the tunnel number of K. Then the corresponding Heegaard splitting (V_1, V_2) of $E(K)$ is irreducible. Thus hereafter we assume that the Heegaard splitting is irreducible. Then the second author showed:

Lemma 6 ([St2, Theorem 9])**.** *Let both* K_1 *and* K_2 *be small, and suppose the corresponding Heegaard splitting* (V_1, V_2) *is weakly reducible. Then* $t(K_1 \# K_2) \geq t(K_1) + t(K_2)$.

Let S be the 2-sphere giving the connected sum of $K = K_1 \# K_2$. Then we can put $S \cap N(K) = D_1^* \cup D_2^*$, where $D_i^* (i = 1, 2)$ is a meridian disk of $N(K)$. Put $A = cl(S - (D_1^* \cup D_2^*))$; then A is a separating essential annulus properly embedded in $E(K)$. Then the second author showed:

Lemma 7 ([St2, Lemma 6])**.** *If the Heegaard splitting* (V_1, V_2) *is strongly irreducible, then after some ambient isotopy for* A*,* $A \cap F$ *consists of essential loops in both* A *and* F*.*

Moreover, she showed:

Lemma 8 ([St2, Lemma 11])**.** *If both* K_1 *and* K_2 *are small, then* (V_1, V_2) *is weakly reducible, or we can choose the Heegaard splitting* (V_1, V_2) *so that* $A \cap F$ *consists of two essential loops or of four essential loops in both* A *and* F*.*

Remark 9. The Heegaard splitting (V_1, V_2) in Lemma 8 may not be isotopic to the Heegaard splitting we take first because the argument in the proof of [St2, Lemma 11] exchanges the Heegaard splittings. In fact, the argument in the proof of [St2, Lemma 11] has been done under the two assumptions stated in [St2, section 4], but we have no those assumptions in this paper. Therefore the statement in the above lemma is somewhat different from that in [St2, Lemma 11].

3. Proof of theorem 4

In this section, we show the following, which is a refinement of Lemma 8.

Lemma 10. *If both* K_1 *and* K_2 *are small, then* (V_1, V_2) *is weakly reducible, or we can choose the Heegaard splitting* (V_1, V_2) *so that* $A \cap F$ *consists of two essential loops in both* A *and* F*.*

Remark 11. If $A \cap F$ consists of two essential loops in both A and F, then by a more detailed argument we can show that (V_1, V_2) is weakly reducible. Hence this lemma says that we can always take the Heegaard splitting of $E(K_1 \# K_2)$ corresponding to the tunnel number $t(K_1 \# K_2)$ to be weakly reducible.

Proof. We denote the number of components of $A \cap F$ by $|A \cap F|$. Then by Lemma 8, we may assume that $|A \cap F| = 4$ and each component of $A \cap F$ is an essential loop in both A and F. Then since A is a separating essential annulus in $E(K)$, we can put $A \cap V_1 = E_1 \cup E_2 \cup E_0$ and $A \cap V_2 = G_1 \cup G_2$, where E_i $(i = 1, 2)$ is an annulus in V_1 connecting F and $\partial E(K)$, E_0 is an essential annulus in V_1 with $\partial E_0 \subset F$ and G_i $(i = 1, 2)$ is an essential annulus in V_2. Then we can regard E_0 as a union of an essential disk D_0 in V_1 and a band b_0, G_i $(i = 1, 2)$ as a union of an essential disk D_i in V_2 and a band b_i. Since the annulus E_i $(i = 1, 2)$ extends to a non-separating disk $E_i \cup D_i^*$ in the handlebody $V_1 \cup N(K)$, E_i is a non-separating annulus in V_1. Moreover, since A is a separating annulus in $E(K)$, $E_1 \cup E_2 \cup E_0$ is a separating 2-manifold in V_1. Hence according to whether $E_1 \cup E_2$ is a separating 2-manifold or not in V_1, we have the following two cases.

Case I: $E_1 \cup E_2$ splits V_1 into two components and E_0 is a separating annulus in one of the two components (Figure 1(I)).

Case II: $E_1 \cup E_2$ does not split V_1 and E_0 is a non-separating annulus in V_1 such that $E_1 \cup E_2 \cup D_0$ splits V_1 into two components (Figure 1(II)).

Suppose we are in Case I.

In this case, since $E_1 \cup E_2$ splits V_1 into two components and D_0 splits one of the two components into two components, b_0 is contained in one of the two components. Then we have the following claim. Recall the annulus A consists of five annuli $E_1 \cup G_1 \cup E_0 \cup G_2 \cup E_2$.

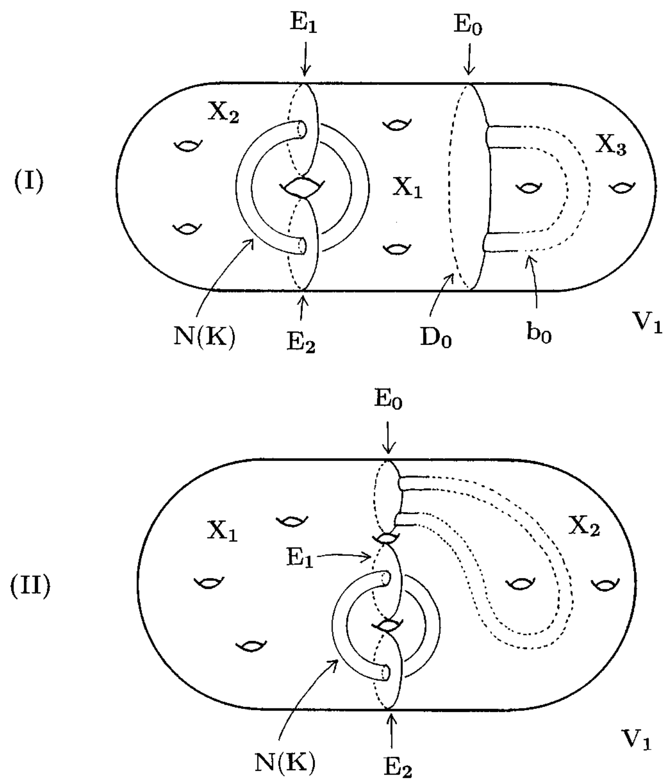

FIGURE 1

Claim 12. *We may assume that the band b_0 is contained in the component of $V_1 - (E_1 \cup E_2 \cup D_0)$ not meeting $E_1 \cup E_2$ as illustrated in Figure* 1(I).

Proof. Suppose b_0 is contained in the component of $V_1 - (E_1 \cup E_2 \cup D_0)$ meeting $E_1 \cup E_2$. Push out the band b_1 from V_2 into V_1; then a band in V_1, say b_1', is produced. Since b_1 is a part of G_1, b_1' is a band connecting E_1 and E_0. Then, since b_0 is contained in the above component, b_1' does not run over b_0. Hence we can push out b_0 from V_1 into V_2 leaving b_1' in V_1. Then after these ambient isotopies, $A \cap F$ consists of two loops, and this completes the proof. □

Suppose one of G_1 and G_2 is a separating annulus in V_2; then since A is a separating annulus in $E(K)$, the other is a separating annulus too. Then both D_1 and D_2 are separating disks in V_2, and $D_1 \cup D_2$ splits V_2 into three components R_0, R_1, R_2, where $D_1 \subset \partial R_1$, $D_2 \subset \partial R_2$ and $D_1 \cup D_2 \subset \partial R_0$.

Suppose the band b_1 is not contained in R_1, and let H_1 be the component of $V_2 - G_1$ containing R_1. Then $H_1 \cap F$ is a connected surface with two boundary components, and $H_1 \cap F$ is identified with $X_2 \cap F$ or $X_3 \cap F$. Hence G_1 connects E_1 and E_2 or $\partial G_1 = \partial E_0$, a contradiction. If the band b_2 is not contained in R_2, then we have the same contradiction. Thus b_i is contained in R_i $(i = 1, 2)$. Then $G_1 \cup G_2$ splits V_2 into three components H_1, H_2, H_0, where $G_1 \subset \partial H_1$, $G_2 \subset \partial H_2$ and $G_1 \cup G_2 \subset \partial H_0$. If one of $H_1 \cap F$ and $H_2 \cap F$ is connected, then we have a contradiction as above. Hence each of $H_1 \cap F$ and $H_2 \cap F$ has two components.

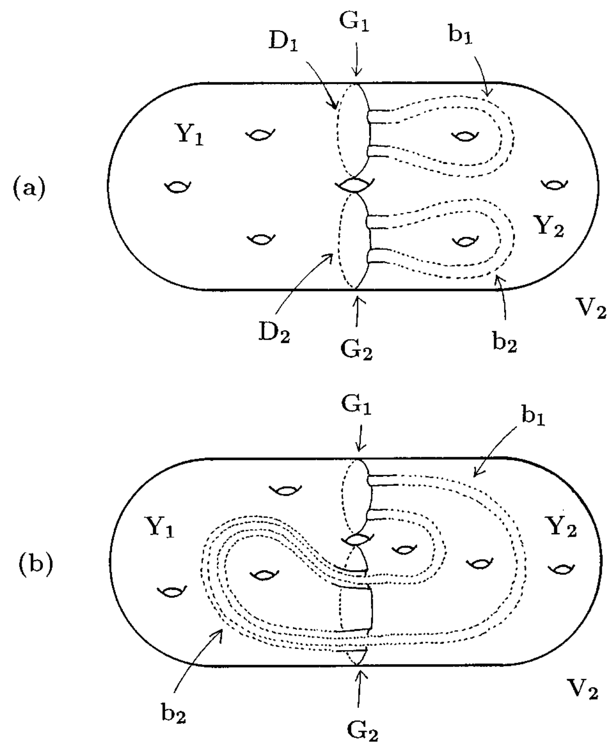

FIGURE 2

Then there is no component in $H_1 \cap F$, $H_2 \cap F$ and $H_0 \cap F$ which has two boundary components. But $X_2 \cap F$ has two boundary componects, a contradiction.

Thus both G_1 and G_2 are non-separating annuli in V_2 and $G_1 \cup G_2$ splits V_2 into two components (Figure 2). Figure 2(a) is the case when the bands b_1 and b_2 are contained in one of the two components $V_2 - (D_1 \cup D_2)$. Figure 2(b) is the case when b_2 is contained in one of the two components $V_2 - (D_1 \cup D_2)$ and b_1 or a part of b_1 is contained in the other component, and in the latter case b_1 runs over the band b_2. We note that in the case of Figure 2(b) the band b_2 does not run over the band b_1.

Let X_1, X_2 and X_3 be the three components of $V_1 - (E_1 \cup E_2 \cup E_0)$ indicated in Figure 1(I), and Y_1 and Y_2 the two components of $V_2 - (G_1 \cup G_2)$ indicated in Figure 2.

Claim 13. *$X_1 \cap F$ is identified with $Y_1 \cap F$.*

Proof. Suppose $X_1 \cap F$ is identified with a part of $Y_2 \cap F$. Push out the band b_1 from V_2 into V_1, and let b_1' be the band in V_1 produced by the ambient isotopy. Then b_1' is contained in X_2 or X_3. This means that ∂G_1 is contained in $\partial(E_1 \cup E_2)$ or that $\partial G_1 = \partial E_0$. This is a contradiction because ∂G_1 consists of a component of ∂E_1 and a component of ∂E_0, and completes the proof. □

By the above claim, $(X_2 \cup X_3) \cap F$ is identified with $Y_2 \cap F$. We denote the images of E_1, E_2, E_0, G_1 and G_2 in ∂X_1, ∂X_2, ∂X_3, ∂Y_1 and ∂Y_2 by the same

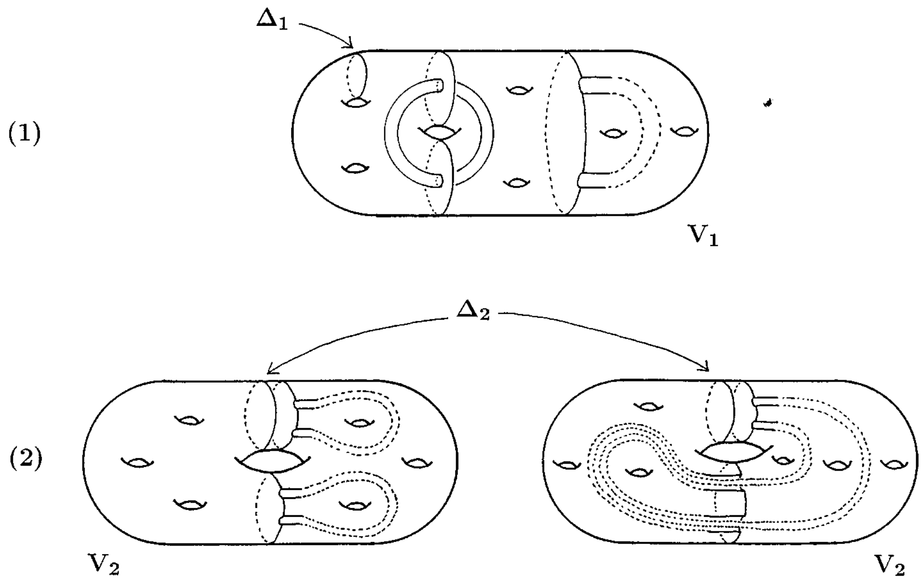

FIGURE 3

notations. According as $X_2 \cap F$ is an annulus or not, we have the following two subcases.

Case I-(1): $X_2 \cap F$ is not an annulus.

In this case, since $X_2 \cap F$ has a positive genus, we can take an essential disk, say Δ_1, properly embedded in $X_2 \subset V_1$ indicated in Figure 3(1). Let Δ_2 be an essential disk properly embedded in $Y_1 \subset V_2$ parallel to D_1 indicated in Figure 3(2).

Then, since $X_1 \cap F$ is identified with $Y_1 \cap F$, $\partial\Delta_1 \cap \partial\Delta_2 = \emptyset$. This shows that (V_1, V_2) is weakly reducible.

Case I-(2): $X_2 \cap F$ is an annulus.

Let N_1 and N_2 be the two components of $N(K) - (D_1^* \cup D_2^*)$ so that $N_i \cap X_i$ $(i = 1, 2)$ is an annulus. Put $B_1 = X_1 \cup N_1 \cup Y_1$ and $B_2 = X_2 \cup N_2$. Then B_2 is a 3-ball. And since B_1 is a component of $S^3 - S$, B_1 is a 3-ball too. Moreover, since $B_1 \cap X_3 = X_1 \cap X_3 = E_0$ is an annulus, B_1 is a 2-handle for the handlebody X_3 along E_0. And since $B_2 \cap Y_2 = X_2 \cap F$ is an annulus, B_2 is a 2-handle for the handlebody Y_2.

Put $W_1 = B_1 \cup X_3$ and $W_2 = B_2 \cup Y_2$. Then $W_1 \cup W_2 = S^3$ and $W_1 \cap W_1 = \partial W_1 = \partial W_2$. Hence at least one of W_1 and W_2 has a compressible boundary. Then we have the following two subcases.

(i): ∂W_1 is compressible in W_1.

Suppose W_1 is a 3-ball. Then X_3 is a solid torus. If the annulus E_0 winds around the handle of X_3 more than once, then by [Mo2, Proposition 1.3] S^3 has a lens space summand, a contradiction. Hence E_0 winds around the handle of X_3 exactly once, and we can push out E_0 from V_1 into V_2. This makes $|A \cap F|$ to be 2. Hence we may asuume that W_1 is not a 3-ball. Then W_1 has a compressible boundary with a positive genus. Then by [Ja, Theorem 2] (cf. [Sr1, Lemma 1.1]), $\partial X_3 - E_0$ is compressible in X_3. Then there is a compressing disk, say Δ_1, for $\partial X_3 - E_0$ in X_3. And, since E_0 is an incompressible annulus in ∂X_3, Δ_1 is an essential disk in

X_3 and in V_1. Let Δ_2 be an essential disk properly embedded in $Y_1 \subset V_2$ parallel to D_1 indicated in Figure 3(2). Then $\partial\Delta_1 \cap \partial\Delta_2 = \emptyset$. This shows that (V_1, V_2) is weakly reducible.

(ii): ∂W_2 is compressible in W_2.

Put $B_2 \cap Y_2 = G_3$. Then G_3 is an annulus in ∂Y_2, and ∂G_3 consists of a component of ∂G_1 and a component of ∂G_2. In this case, the four components $\partial G_1 \cup \partial G_2$ are all mutually parallel to each other in ∂Y_2.

If W_2 is a 3-ball, then Y_2 is a solid torus. Then $\partial Y_2 - (G_1 \cup G_2 \cup G_3)$ is an annulus, and it is identified with $\partial X_3 - E_0$. Hence X_3 is a solid torus, and by the argument in the case (i), we may assume that W_2 is not a 3-ball. Then by the argument in the case (i) ([Ja, Theorem 2]), we have an essential disk, say Δ_2, properly embedded in $Y_2 \subset V_2$ with $\partial\Delta_2 \subset \partial Y_2 - (G_1 \cup G_2 \cup G_3)$. Let Δ_1 be an essential disk properly embedded in $X_1 \subset V_1$ parallel to D_0 (then Δ_1 is a separating disk in $X_1 \subset V_1$). Then $\partial\Delta_1 \cap \partial\Delta_2 = \emptyset$, and this shows that (V_1, V_2) is weakly reducible. This completes the proof of Case I.

Suppose we are in Case II.

Let X_1 and X_2 be the two components of $V_1 - (E_1 \cup E_2 \cup E_0)$ indicated in Figure 1(II). Suppose one of G_1 and G_2 is a separating annulus in V_2; then by the same reason as in Case I, the other is a separating annulus too. Then both D_1 and D_2 are separating disks in V_2, and $D_1 \cup D_2$ splits V_2 into three components R_0, R_1, R_2, where $D_1 \subset \partial R_1$, $D_2 \subset \partial R_2$ and $D_1 \cup D_2 \subset \partial R_0$.

Suppose the band b_1 is not contained in R_1, and let H_1 be the component of $V_2 - G_1$ containing R_1. Then $H_1 \cap F$ is a connected surface with two boundary components, and $H_1 \cap F$ is not identified with $X_1 \cap F$. Then by the argument in the proof of Claim 12, we can change the bands b_0 and b_1 and reduce the number of $|A \cap F|$. Hence b_i is contained in R_i ($i = 1, 2$). Then $G_1 \cup G_2$ splits V_2 into three components H_1, H_2 and H_0, where $G_1 \subset \partial H_1$, $G_2 \subset \partial H_2$ and $G_1 \cup G_2 \subset \partial H_0$. Since $H_0 \cap F$ is a connected surface with four boundary components, $H_0 \cap F$ is identified with $X_1 \cap F$ and $(H_1 \cap F) \cup (H_2 \cap F)$ is identified with $X_2 \cap F$.

Consider the incompressibility of $H_1 \cap F$. If $H_1 \cap F$ is compressible in $H_1 \subset V_2$, then we have a compressing disk, say Δ_2, for $H_1 \cap F$ properly embedded in $H_1 \subset V_2$, and let Δ_1 be the essential disk properly embedded in $X_1 \subset V_1$ parallel to D_0. Then $\partial\Delta_1 \cap \partial\Delta_2 = \emptyset$. This shows that (V_1, V_2) is weakly reducible, a contradiction. If $H_1 \cap F$ is compressible in $X_2 \subset V_1$, then we have a compressing disk, say Δ_1, for $H_1 \cap F$ properly embedded in $X_2 \subset V_1$, and let Δ_2 be the essential disk properly embedded in $H_0 \subset V_2$ parallel to D_1. Then $\partial\Delta_1 \cap \partial\Delta_2 = \emptyset$, a contradiction. Hence $H_1 \cap F$ is incompressible in both H_1 and X_2.

Glue $H_1 \cup H_2$ and X_2 along $(H_1 \cap F) \cup (H_2 \cap F) = X_2 \cap F$, and get $E(K_2) = H_1 \cup H_2 \cup X_2$. By this identification, $X_2 \cap F$ has two components each of which has two boundary components. Hence $H_1 \cap F$ is a connected surface with two boundary components. Put $Q = H_1 \cap F$. Then Q is an incompressible surface properly embedded in $E(K_2)$, and ∂Q consists of two meridian loops of $N(K_2)$. If Q is not an annulus, then Q is essential and by [CGLS, Theorem 2.0.3(iii)], $E(K_2)$ contains a closed essential surface, a contradiction. Thus Q is an annulus and hence H_1 is a solid torus. Then by [Mo2, Proposition 1.3], G_1 is parallel to an annulus in ∂V_2. Then we can reduce the number of $|A \cap F|$, a contradiction. Thus both G_1 and G_2 are non-separating annuli.

Let Y_1 and Y_2 be the two components of $V_2 - (G_1 \cup G_2)$ indicated in Figure 2. Then by the argument similar to the proofs of Claims 12 and 13, $X_i \cap F$ ($i = 1, 2$

resp.) is identified with $Y_i \cap F$ ($i = 1, 2$ resp.) We denote the images of E_1, E_2, E_0, G_1 and G_2 in ∂X_1, ∂X_2, ∂Y_1 and ∂Y_2 by the same notations.

If $X_2 \cap F$ is compressible in X_2, then we have a compressing disk, say Δ_1, for $X_2 \cap F$ properly embedded in $X_2 \subset V_1$. Let Δ_2 be an essential disk in $Y_1 \subset V_2$ indicated in Figure 3(2); then $\partial\Delta_1 \cap \partial\Delta_2 = \emptyset$. This shows that (V_1, V_2) is weakly reducible. If $Y_2 \cap F$ is compressible in Y_2, then we have a compressing disk, say Δ_2, for $Y_2 \cap F$ properly embedded in $Y_2 \subset V_2$. Let Δ_1 be an essential disk in $X_1 \subset V_1$ parallel to D_0; then $\partial\Delta_1 \cap \partial\Delta_2 = \emptyset$. This shows that (V_1, V_2) is weakly reducible. Hence we assume that $X_2 \cap F$ ($Y_2 \cap F$ resp.) is incompressible in X_2 (Y_2 resp.).

Glue X_2 and Y_2 along $X_2 \cap F = Y_2 \cap F$. Then we get $E(K_2) = X_2 \cup Y_2$. Put $Q = X_2 \cap F = Y_2 \cap F$. Then Q is an incompressible 2-manifold properly embedded in $E(K_2)$, and ∂Q consists of four meridian loops of $N(K_2)$. If Q has a component which is not an annulus, then the component is an essential surface in $E(K_2)$ and by [CGLS, Theorem 2.0.3(iii)], $E(K_2)$ contains a closed essential surface, a contradiction.

Suppose Q consists of two annuli. Then X_2 is a solid torus homeomorphic to (an annulus, say R) $\times[1, 4]$ so that $R \times \{1\} = E_1$, $R \times \{4\} = E_2$ and (a component of ∂R) $\times[2, 3] = E_0$. Put $E_3 = R \times \{\frac{5}{2}\}$. Then, since a component of ∂E_3 is contained in $E_0 \subset A$, the component splits A into two annuli A_1 and A_2. And, since small knots are prime, one of $A_1 \cup E_3$ and $A_2 \cup E_3$, say $A_1 \cup E_3$, is a decomposing annulus of $E(K_1\#K_2)$ and A_2 is isotopic rel. $\partial E(K_1\#K_2)$ to E_3. This reduces the number of $|A \cap F|$ and completes the proof of Case II and Lemma 10. □

Proof of Theorem 4. Let (V_1, V_2) be a Heegaard splitting corresponding to an unknotting tunnel system for $K = K_1\#K_2$ which realizes the tunnel number of K. Then (V_1, V_2) is irreducible, and by Lemma 10 we may assume that (V_1, V_2) is weakly reducible or the decomposing 2-sphere of $K_1\#K_2$ intersects the Heegaard surface in two essential loops. In the former case, by Lemma 6 we have $t(K_1\#K_2) \geq t(K_1) + t(K_2)$. In the latter case, by the argument in the proof of Case 1 of [St2, Theorem 12], we have $t(K_1\#K_2) \geq t(K_1) + t(K_2)$. This completes the proof of Theorem 4. □

4. Proof of Theorem 5

Put $K = K_1\#K_2\#\cdots\#K_n$, and let (V_1, V_2) be the Heegaard splitting of $E(K)$ corresponding to an unknotting tunnel system for K which realizes the tunnel number of K.

Lemma 14. *If* (V_1, V_2) *is weakly reducible, then*

$$t(K_1\#K_2\#\cdots\#K_n) \geq t(K_{i_1}\#\cdots\#K_{i_j}) + t(K_{i_{j+1}}\#\cdots\#K_{i_n})$$

for some j.

Proof. Since (V_1, V_2) is weakly reducible, by considering the untelescoping of (V_1, V_2) and by [CG, Theorem 3.1] and [St2, Remark 7] (cf. [LM, Theorem 1.3]), there is a closed incompressible surface S in $E(K)$ such that S splits (V_1, V_2) into two Heegaard splittings (V_1^1, V_2^1) and (V_1^2, V_2^2). Then (V_1, V_2) is an amalgamation of (V_1^1, V_2^1) and (V_1^2, V_2^2) along the surface S, and $g(V_1, V_2) = g(V_1^1, V_2^1) + g(V_1^2, V_2^2) - g(S)$, where $g(\cdot)$ is the genus of the Heegaard splitting or is the genus of the surface. For the definition of amalgamation, see [St1]. And for the definition of the untelescoping, see [St2, Definition 12], [Sr2] or [ST].

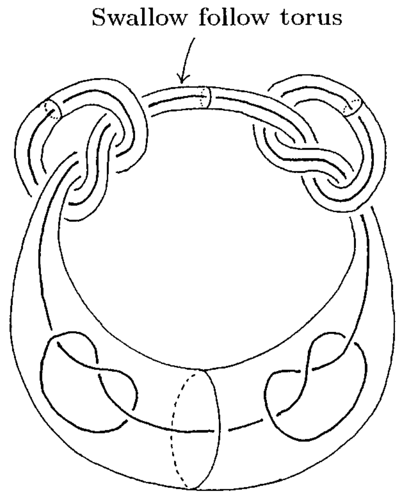

FIGURE 4

If S is boundary parallel in $E(K)$, then $E(K)$ has a lower genus Heegaard splitting than $g(V_1, V_2)$, a contradiction. Hence S is essential. Then by [St2, Lemma 14], S is a swallow follow torus as illustrated in Figure 4, and S splits $E(K)$ into $E(K_{i_1}\#\cdots\#K_{i_j})-$(a solid torus) and $E(K_{i_{j+1}}\#\cdots\#K_{i_n})$. Then, since (V_1^1, V_2^1) ((V_1^2, V_2^2) resp.) is a Heegaard splitting of $E(K_{i_1}\#\cdots\#K_{i_j})-$(a solid torus) ($E(K_{i_{j+1}}\#\cdots\#K_{i_n})$ resp.), $t(K_{i_1}\#\cdots\#K_{i_j}) \leq g(V_1^1, V_2^1) - 1$ and $t(K_{i_{j+1}}\#\cdots\ \#K_{i_n}) \leq g(V_1^2, V_2^2) - 1$. Thus by noting $g(S) = 1$, we have $t(K_{i_1}\#\cdots\#K_{i_j}) + t(K_{i_{j+1}}\#\cdots\#K_{i_n}) \leq g(V_1^1, V_2^1) - 1 + g(V_1^2, V_2^2) - 1 = g(V_1, V_2)$ $-1 = t(K_1\#K_2\#\cdots\ \#K_n)$. This completes the proof. □

Let $S_1, S_2, \cdots, S_{n-1}$ be the decomposing 2-spheres giving the connected sum of $K = K_1\#K_2\#\cdots\#K_n$. Put $A_i = S_i \cap E(K)$ $(i = 1, 2, \cdots, n-1)$. Then A_i is a separating essential annulus properly embedded in $E(K)$. By Claim 2 in the proof of [St2, Theorem 15], we have:

Lemma 15. (V_1, V_2) *is weakly reducible, or we can choose the Heegaard splitting* (V_1, V_2) *and the decomposing* 2*-spheres so that* F *intersects at most one of the decomposing annuli in* 2 *or* 4 *essential loops and intersects the others in* 2 *essential loops in both* F *and the annuli.*

By Lemmas 14 and 15, and by exchanging the decomposing 2-spheres if necessary, we can put $|F \cap A_{i_1}| = 2$ or 4 and $|F \cap A_{i_j}| = 2$ $(j = 2, \cdots, n-1)$. Then by repeating the argument in the proof of Case 1 of [St2, Theorem 12], we have $t(K_1\#K_2\#\cdots\#K_n) \geq t(K_{i_1}\#K_{i_2}) + t(K_{i_3}) + \cdots + t(K_{i_n})$. Moreover by Theorem 4, $t(K_{i_1}\#K_{i_2}) \geq t(K_{i_1}) + t(K_{i_2})$. This completes the proof of Theorem 5. □

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