GENUS 2 CLOSED HYPERBOLIC 3-MANIFOLDS OF ARBITRARILY LARGE VOLUME

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ABSTRACT. We describe a class of genus 2 closed hyperbolic 3-manifolds of arbitrarily large volume.

The purpose of this note is to advertise the existence of a class of genus 2 closed hyperbolic 3-manifolds of arbitrarily large volume. The class described here consists merely of appropriate Dehn fillings on 2-bridge knots. That this class has the properties claimed follows directly from [4], [6], and the Gromov-Thurston 2π-Theorem. The existence of such a class of hyperbolic 3-manifolds is known, as pointed out by Daryl Cooper [3], who mentions that branched covers of the figure 8 knot provide another such class. We believe that the existence of such a class deserves to be more widely known. For general definitions and properties concerning knot theory, see [7] or [8].

The following definition and theorem are due to M. Lackenby.

Definition 1. *Given a link diagram D, we call a complimentary region having two crossings in its boundary a <u>bigon region</u>. A twist is a sequence $v_1, \dots, v_l$ of vertices such that v_i and v_{i+1} are the vertices of a common bigon region, and that is maximal in the sense that it is not part of a longer such sequence. A single crossing adjacent to no bigon regions is also a twist. The twist number $t(D)$ of a diagram D its number of twists.*

Theorem 1. *(Lackenby) Let D be a prime alternating diagram of a hyperbolic link K in S^3. Then $v_3(t(D)-2)/2 \leq Volume(S^3 - K) < v_3(16t(D) - 16)$, where $v_3(\approx 1.01494)$ is the volume of a regular hyperbolic ideal 3-simplex.*

A particularly nice class of alternating diagrams is given by 2-bridge knots that are not torus knots. The following lemma is a well known consequence of work of Hatcher and Thurston.

Lemma 1. *There are 2-bridge knots whose complements support complete hyperbolic structures of arbitrarily large volume.*

Proof. It follows from [5] that 2-bridge knots are simple and from [12] that the complement of a 2-bridge knot that is not a torus knot supports a complete finite volume hyperbolic structure.

Claim: There are 2-bridge knots that are not torus knots with diagrams of arbitrarily high twist number.

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A 2-bridge knot is determined by a sequence of integers $[c_1, \dots, c_n]$ denoting the number of crossings in its twists, read from top to bottom, and with c_i being the number of positive crossings if i is odd and the number of negative crossings if i is even. Such a sequence gives rise to a rational number

$$\frac{p}{q} = 1 + \frac{1}{c_2 + \frac{1}{c_3 + \dots}}.$$

For instance, the Figure 8 Knot, with sequence $[2, 2]$ corresponds to $\frac{5}{2} = 2 + \frac{1}{2}$.

Two 2-bridge knots, with corresponding rational numbers $\frac{p}{q}$ and $\frac{p'}{q'}$, are equivalent if and only if $p = p'$ and $q - q'$ is divisible by p. It follows from [10] (for a shorter proof see [11]) that the bridge number of a (p, q)-torus knot is $min(p, q)$. Thus a 2-bridge knot that is also a torus knot must be a $(2, n)$-torus knot. The rational number corresponding to the $(2, n)$-torus knot is n, an integer. Examples of 2-bridge knots that are not torus knots with diagrams of arbitrarily high twist number can thus be easily constructed, e.g.: $[2, 2]$, $[2, 2, 2]$, $[2, 2, 2, 2]$, ... These have corresponding nonintegral rational numbers $\frac{5}{2} = 2 + \frac{1}{2}$, $\frac{12}{5} = 2 + \frac{2}{5}$, $\frac{29}{12} = 2 + \frac{5}{12}, \dots$.

Since there are 2-bridge knots that are not torus knots with diagrams of arbitrarily high twist number, Lackenby's Theorem [6] implies that there are 2-bridge knots of arbitrarily large volume. □

Definition 2. *A tunnel system for a knot K is a collection of disjoint arcs $\mathcal{T} = t_1 \cup \dots \cup t_n$, properly embedded in $C(K) = S^3 - \eta(K)$ such that $C(K) - \eta(\mathcal{T})$ is a handlebody. The tunnel number of K, denoted by $t(K)$, is the least number of arcs required in a tunnel system for K.*

A Heegaard splitting of a closed 3-manifold M is a decomposition $M = V \cup_S W$ in which V, W are handlebodies with $\partial V = \partial W$ is the surface S, called the splitting surface. The genus of M is the minimal genus required for a splitting surface of M.

The following Lemma is well known (see for instance [9]).

Lemma 2. *2-bridge knots have tunnel number 1.*

Recall the 2π-Theorem (for a proof, see for instance [2, Theorem 9]): (Here $X(s_1, \dots, s_n)$ is the 3-manifold obtained by Dehn filling X along $s_1 \cup \dots \cup s_n$.)

Theorem 2. *(Gromov-Thurston) Let X be a compact orientable hyperbolic 3-manifold. Let $s_1, \dots, s_n$ be a collection of slopes on distinct components $T_1, \dots, T_n$ of ∂X. Suppose that there is a horoball neighborhood of $T_1 \cup \dots \cup T_n$ on which each s_i has length greather than 2π. Then $X(s_1, \dots, s_n)$ has a complete finite volume Riemannian metric with all sectional curvatures negative.*

More recently, in their investigation of Dehn surgery, D. Cooper and M. Lackenby established the following relationship between the Thurston norm of a compact hyperbolic 3-manifold and that of its Dehn fillings ([4, Proposition 3.3]):

Theorem 3. *(Cooper-Lackenby) There is a non-increasing function $\beta : (2\pi, \infty) \to (1, \infty)$, which has the following property. Let X be a compact hyperbolic 3-manifold and let $s_1, \dots, s_n$ be slopes on distinct components $T_1, \dots, T_n$ of ∂X. Suppose that there is a maximal horoball neighborhood of $T_1 \cup \dots \cup T_n$ on which $l(s_i) > 2\pi$ for each i. Then*

$$|X(s_1, \dots, s_n)| \leq |X| < |X(s_1, \dots, s_n)| \beta(min_{1 \geq i \geq n} l(s_i))$$

.

Recall that the that the volume and the Thurston norm of a compact hyperbolic 3-manifold M satisfy $vol(M) = v_3|M|$, where v_3 is the volume of a regular ideal 3-simplex in hyperbolic 3-space.

Theorem 4. *There exist genus 2 closed hyperbolic 3-manifolds of arbitrarily large volume.*

Proof. Let $N \epsilon \mathbf{R}^+$. Choose $\varepsilon > 0$ and choose a 2-bridge knot $K_{p/q}$ that is not a torus knot such that its complement $X = S^3 - \eta(K)$ has $|X| = \frac{vol(X)}{v_3} \geq \beta(2\pi + \varepsilon)N$, for β as provided by Cooper-Lackenby's Theorem. Let r be a slope satisfying the hypotheses of Thurston's Hyperbolic Surgery Theorem (see [13, Theorem 5.8.2] or [1, Section E.5]), then $X(r)$ is hyperbolic.

Let α be an arc in X that is a tunnel system for $K_{p/q}$. Let $\tilde{V} = \eta(\partial X \cup \alpha)$ and let $W = closure(X - V)$. By abusing notation slightly, we may consider W to be lying in $X(r)$. Set $V = closure(X(r) - W)$ and $S = V \cap W$. Then $X(r) = V \cup_S W$ is a genus 2 Heegaard splitting of $X(r)$.

Suppose that $V \cup_S W$ is reducible. Then either $X(r)$ is reducible, or $V \cup_S W$ is stabilized. In case of the former, $X(r)$ would be the connected sum of two lens spaces. In case of the latter, $X(r)$ would have genus 1, i.e., be a Lens space, or genus 0, i.e., be S^3, but all of these outcomes would contradict the fact that $X(r)$ is hyperbolic. Thus $X(r)$ has genus 2.

By the theorem of Cooper-Lackenby,

$$vol(X(r)) = v_3|X(r)| > \frac{1}{\beta(l(r))}|X| \geq \frac{\beta(2\pi + \varepsilon)}{\beta(l(r))}N \geq N.$$

□

Corollary 1. *There are closed manifolds with fundamental group of rank ≤ 2 of arbitrarily large hyperbolic volume.*

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