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Weakly reducible Heegaard splittings of Seifert fibered spaces ☆

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Abstract

We prove that for exceptional Seifert manifolds all weakly reducible Heegaard splittings are reducible. This provides the missing case for the Main Theorem in (Moriah and Schultens, to appear). It follows that for all orientable Seifert fibered spaces which fiber over an orientable base space, irreducible Heegaard splittings are either horizontal or vertical. © 2000 Published by Elsevier Science B.V. All rights reserved.

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1. Introduction

Weakly reducible Heegaard splittings are either reducible, or they are the amalgamation of Heegaard splittings of submanifolds along incompressible surfaces. For the manifolds here under consideration, orientable Seifert fibered spaces which fiber over an orientable base space, we can use our knowledge of the incompressible surfaces they contain to show that their weakly reducible Heegaard splittings are in fact reducible. This completes the missing case in the Main Theorem of [4]. I wish to thank Martin Scharlemann for helpful discussions.

Definitions. An orientable Seifert fibered space over S^2 with three exceptional fibers and rational Euler number 0 is called an exceptional Seifert fibered space.

A Heegaard splitting (H_1, H_2) is reducible (respectively weakly reducible), if there are embedded disks $(D_i, \partial D_i) \subset (H_i, \partial H_i)$ such that $|\partial D_1 \cap \partial D_2| = 1$ (respectively $= 0$).

A Heegaard splitting which is not weakly reducible is said to be strongly irreducible.

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For the definition of amalgamation and fundamental lemmas pertaining to amalgamation, see [7, Section 2].

For an exceptional Seifert fibered space a vertical Heegaard splitting is a Heegaard splitting in which one of the handlebodies consists of a regular neighborhood of two of the exceptional fibers together with a lift of a simple arc in the base space connecting the two corresponding exceptional points.

Theorem 1. *Let* (H_1, H_2) *be a weakly reducible Heegaard splitting of an exceptional Seifert fibered space. Then* (H_1, H_2) *is reducible.*

Corollary 2. *Let* (H_1, H_2) *be a weakly reducible Heegaard splitting of an orientable Seifert fibered space over an orientable base space. Then* (H_1, H_2) *is either vertical or reducible.*

Corollary 3. *Let* (H_1, H_2) *be an irreducible Heegaard splitting of an orientable Seifert fibered space over an orientable base space. Then* (H_1, H_2) *is either horizontal or vertical.*

2. Proofs

Proof of Theorem 1. Let (H_1, H_2) be a weakly reducible Heegaard splitting of the exceptional Seifert fibered space M, and let F be the splitting surface of (H_1, H_2). Let S be a connected incompressible surface in M. It follows from [3, Theorem VI.34] that M fibers as an S-bundle over S^1. We may think of M as $S \times I/\sim_\phi$, where $\sim_\phi$ is the equivalence relation derived from a periodic automorphism $\phi : S \to S$ by setting $\{x\} \times \{1\} \sim_\phi \phi(x) \times \{0\}$.

Step 1. There is an essential curve γ in S such that $|\phi^i(\gamma) \cap \gamma| = 1$ for some i. Recall that M has base space S^2 with three exceptional fibers. We consider the orbifold P which has underlying space S^2 and three cone points (with indices corresponding to the indices of the exceptional fibers of M). Then S orbifold covers P. This implies that P is a good orbifold and has either a Euclidean or hyperbolic structure depending on whether S has a Euclidean or hyperbolic structure [10, Theorem 5.5.3]. The pullback via the orbifold covering map of the Riemannian metric on P defines a Riemannian metric on S which is ϕ-invariant. In particular, with respect to this Riemannian metric, ϕ is an isometry. This Riemannian metric is either Euclidean or hyperbolic. (All that is really needed here is a Riemannian metric which is invariant under ϕ. The argument below does not rely on constant curvature. We could also have constructed such a metric by averaging the pullbacks under powers of ϕ of a given Riemannian structure on S. This is possible by [5, Main Theorem].)

Let γ be a shortest essential geodesic in S. Then the length of $\phi(\gamma)$ is the length of γ (i.e., $\phi(\gamma)$ is also a shortest essential geodesic), so $|\phi(\gamma) \cap \gamma| \leqslant 1$ (for if $|\phi(\gamma) \cap \gamma| \geqslant 2$ we could construct a shorter essential geodesic by exchanging arcs and rounding corners). Suppose the order of ϕ is n and let $\gamma, \phi(\gamma), \ldots, \phi^n(\gamma)$ be the translates of γ under ϕ. If

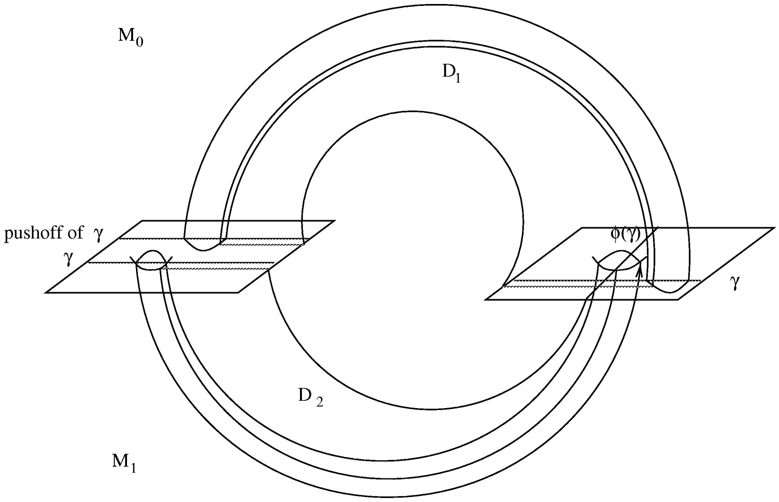

Fig. 1.

$\gamma, \phi(\gamma), \ldots, \phi^n(\gamma)$ are pairwise disjoint, then these curves describe the intersection in M of S with a properly embedded essential vertical torus. Since exceptional Seifert fibered spaces do not contain essential vertical tori, this scenario is impossible, and a pair of these curves must intersect (exactly once). We may assume that γ intersects $\phi^i(\gamma)$.

Step 2. Constructing reducing disks. Suppose F is weakly reducible. Following [2, Theorem 3.1] either F is reducible, or we may compress F along disks both into H_1 and into H_2. The result is an incompressible surface S^*. Since S^* is separating, it must have an even number of components. In M, this incompressible surface S^* must consist of copies of a fiber in a fibration of M as a surface bundle over S^1 (again by [3, Theorem VI.34]). We think of this fiber as the surface S in the above M as $S \times I/\!\sim_\phi$. Then [7, Proposition 2.8] shows that F is the amalgamation along S^* of Heegaard splittings of $S \times I$. By [7, Remark 2.5] these Heegaard splittings of $S \times I$ are standard of type II (in the terminology of [6]); i.e., their splitting surfaces are obtained by taking the component of $\partial(N(S \times 0, 1) \cup (\mathit{point} \times I))$ which lies in the interior of $S \times I$. Fig. 1 illustrates F in the case where S^* has two components.

Let γ be as in step 1. We may choose i to be the smallest number for which $\phi^i(\gamma)$ intersects γ in exactly one point. The components of S^* separate M into components homeomorphic to $S \times I$. Choose two adjacent such components M^{i-1} and M^i, where we assume that $S \times \{1\} \subset M^{i-1}$ coincides with $S \times \{0\} \subset M^i$ and that the glueing of these two copies of S is via ϕ. The glueings of the other copies of S is via the identity map. Define disks $D_1 = (\phi^{i-1}(\gamma) - \{\mathit{point}\}) \times I \subset M^{i-1}$ (where $\phi^{i-1}(\gamma)$ is considered in $S \times \{0\} \subset M^{i-1}$) and $D_2 = ((\gamma) - \{\mathit{point}\}) \times I \subset M^i$ (where γ is considered in $S \times \{0\} \subset M^i$). The points should be chosen away from $\phi^i(\gamma) \cap \gamma$. In the case where S^* has more than two components, it is clear that $|\partial D_1 \cap \partial D_2| = 1$. In the case where S^* has exactly two components, note that M^{i-1} and M^i are glued together via ϕ along $S \times \{1\} \subset M^{i-1}$ and $S \times \{0\} \subset M^i$ but via the identity along $S \times \{0\} \subset M^{i-1}$ and

$S \times \{1\} \subset M^i$. Here we may need to isotope a copy of γ off itself in the construction of D_2. Then again, $|\partial D_1 \cap \partial D_2| = 1$. (See Fig. 1.) □

Proof of Corollary 2. This follows from Theorem 1 and [4, Theorem 2.6]. □

Proof of Corollary 3. This follows from Theorem 1 and [4, Theorem 0.1]. □

There are some circumstances under which a horizontal Heegaard splitting coincides with a vertical Heegaard splitting. Boileau and Otal show in [1, Lemma 2.4] that in certain small Seifert manifolds such as the Brieskorn manifolds $V(2, 3, a)$, $a = 4, 5$, the (in this case unique) horizontal Heegaard splitting coincides with a vertical Heegaard splitting. The fact that these are the only examples of this phenomenon follows from [8, Theorem 13] and from the Main Theorem in [9].

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