Pergamon

Topology Vol. 37, No. 5, pp. 1089–1112, 1998

0040-9383/98 $19.00 + 0.00

PII: S0040-9383(97)00072-4

IRREDUCIBLE HEEGAARD SPLITTINGS OF SEIFERT FIBERED SPACES ARE EITHER VERTICAL OR HORIZONTAL

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(*Received* 12 *June* 1995; *in revised form* 3 *June* 1997)

0. INTRODUCTION

Irreducible 3-manifolds are divided into Haken manifolds and non-Haken manifolds. Much is known about the Haken manifolds and this knowledge has been obtained by using the fact that they contain incompressible surfaces. On the other hand, little is known about non-Haken manifolds. As we cannot make use of incompressible surfaces we are forced to consider other methods for studying these manifolds. For example, exploiting the structure of their Heegaard splittings. This approach is enhanced by the result of Casson and Gordon [6] that irreducible Heegaard splittings are either strongly irreducible (see Definition 1.2) or the manifold is Haken. Hence, the study of Heegaard splittings as a mean of understanding 3-manifolds, whether they are Haken or not, takes on a new significance.

Let M be an orientable Seifert fibered space with m exceptional fibres and an orientable base space of genus g_0. These manifolds were known to have "vertical" (see Definition 2.1) Heegaard splittings of genus $2g_0 + m - 1$. These Heegaard splittings were classified by Lustig and Moriah in [12] and [25], unless $g_0 = 0$ and $0 < m \leqslant 4$. Heegaard splittings of manifolds of genus 2 (i.e., $g_0 = 0$ and $m = 3$) in this class were classified by Boileau *et al.* [1] and separately by Moriah [14] using the work of Boileau and Otal in [3]. In this case there are manifolds which have "horizontal" Heegaard splittings (see Definition 3.1). Schultens [17] classified Heegaard splittings of manifolds which are (orientable surfaces) $\times S^1$ and showed that these are all vertical. More recently, she showed [20] that all irreducible Heegaard splittings of orientable Seifert fibered spaces over an orientable base space with nonempty boundary are vertical. It should be mentioned that Waldhausen [24] classified Heegaard splittings for S^3, Bonahon and Otal [5] for Lens spaces and Boileau and Otal [2] did so for T^3.

The main result of this paper is the following theorem.

THEOREM 0.1. *Let M be an orientable Seifert fibered space over an orientable base space S. Then every irreducible Heegaard splitting of M is either vertical or horizontal.*

As a consequence of the proof we also have:

[†] Partially supported by a grant from the Toronto Academic Lectureship at the Technion.
[‡] Partially supported by a grant from the University Research Council at Emory University.

THEOREM 0.2. *Let M be an orientable Seifert fibered space with an orientable base space and let Σ be a Heegaard splitting surface for M. Then there is an isotopy of M taking a fiber onto the surface.*

Let M be an orientable Seifert fibered space with an orientable based space S of genus g_0, m exceptional fibers and Euler number e_0, i.e., $M = \{g_0, e_0 \,|\, (\alpha_1, \beta_1), \ldots, (\alpha_m, \beta_m)\}$, where g.c.d. $(\alpha_j, \beta_j) = 1$ and β_j is normalized so that $0 < \beta_j < \alpha_j$. The numbers (α_j, β_j) are the Seifert gluing invariants of the jth exceptional fiber and e_0 is the rational Euler number. For further details see [22]. Note that if $g_0 = 0$ and $m \leqslant 2$ then M is a Lens space. Set

$$\alpha_0 = 1 \quad \text{and} \quad \beta_0 = b = -e_0 - \sum_{j=1}^{m} \beta_j/\alpha_j$$

Let $\alpha^i = \text{l.c.m.}\{\alpha_j\}$, $j = 0, \ldots, m$, $j \neq i$. Let s_i, t_i be two integers such that $s_i(\sum_{j=0, j\neq i}^{m} \beta_j \alpha^i/\alpha_j) + t_i \alpha^i = 0$ and $|s_i|$ is minimal.

Horizontal Heegaard splittings arise in a very special way, described in Section 3. In particular, not every Seifert fibered space possesses horizontal Heegaard splittings. Each horizontal splitting corresponds either to one of the singular fibers f_i ($i = 1, \ldots, m$) or to a regular fiber which we denote by f_0. We associate the invariants (α_0, β_0) with f_0. Whether a Seifert fibered space possesses a horizontal Heegaard splitting can be determined from its Seifert invariants. The precise conditions are given in the following theorem:

THEOREM 0.3. *Let $M = \{g_0, e_0 \,|\, (\alpha_1, \beta_1), \ldots, (\alpha_m, \beta_m)\}$ be an orientable Seifert fibered space with an orientable base space S. The manifold M has a horizontal Heegaard splitting corresponding to the fiber f_i if and only if f_i is null-homologous in M. In particular if and only if*

(a) $s_i = \alpha^i$ *and*

(b) *There are a pair of integers u_i, v_i such that $s_i v_i - t_i u_i = 1$ and the equation $\{\alpha_i, \beta_i\} = \{ns_i + u_i, nt_i + v_i\}$ (where $nt_i + v_i$ is considered* $\text{mod}(ns_i + u_i)$) *holds for some $n \in \mathbb{Z}$.*

Theorem 0.1 tells us that given an irreducible Heegaard splitting of one of the Seifert fibered spaces under consideration, one of two situations occurs: Either the handlebodies of the Heegaard splitting contain the singular fibers as cores or there is a fiber f which is isotopic into the splitting surface Σ and $\Sigma - N(f)$ is incompressible in $M - N(f)$, where $N(f)$ is a regular neighborhood of the fiber. Recent work of Moriah and Rubinstein [15] shows that irreducible Heegaard splittings of hyperbolic manifolds have similar structural features.

We will call a Seifert fibered space exceptional if it has S^2 as base space, three exceptional fibers and rational Euler number 0. The following two results are consequences of Theorems 0.1 and 0.3:

COROLLARY 0.4. *Let M be an orientable Seifert fibered space over an orientable base space S. Assume that M has rational Euler number 0. Then every irreducible Heegaard splitting of M is vertical.*

COROLLARY 0.5. *An orientable circle bundle over an orientable surface has, up to homeomorphism, a unique irreducible Heegaard splitting. This Heegaard splitting is horizontal if and only if the Euler number is ± 1. If the Euler number is not ± 1 the Heegaard splitting is unique up to isotopy.*

A priori it is possible for horizontal and vertical Heegaard splittings to be isotopic and, in fact there are some cases in which this is known to happen. For example, when $g_0 = 0$ and $m = 3$ (see [3]). However, this is not a common phenomena as can be seen from Proposition 5.1 and Theorem 5.2. If either $g_0 > 0$ or $m > 3$, then the vertical Heegaard splittings contain disjoint compressing disks on both sides of the surface and hence are weakly reducible (see Definition 1.2). So in order to show that horizontal and vertical Heegaard splittings are not isotopic it would be sufficient to show that the horizontal Heegaard splittings are strongly irreducible. Theorem 5.2 establishes the strong irreducibility of most horizontal Heegaard splittings using a result of Casson and Gordon which is proven in the appendix. For more background about Seifert fibered spaces see [16, 22, 23].

Remark 0.6. Theorems 0.1 and 0.3 generalize Theorem 1.1(i) of [4] and resolve the undecided cases there. The manifolds $M = \{0, e_0 | (2, 1), \dots, (2, 1), (\alpha_m, \beta_m)\}$ with $\alpha_m = 2\lambda + 1$, $m \geqslant 6$ and even, have horizontal Heegaard splittings of genus $m - 2$ if and only if $\beta_m = \pm(\lambda + 1) \bmod \alpha_m$. This is a minimal genus Heegaard splitting as the rank of $\pi_1(M)$ is $m - 2$. Otherwise $g(M) = m - 1$.

1. PUSHING FIBERS ONTO HEEGAARD SURFACES

In this section we prove a generalization of Proposition 1.1 in [3].

Definition 1.1. A compression body W is a 3-manifold obtained by adding 2-handles to a (surface)$\times I$ along simple closed curves on (surface)$\times\{0\}$ and capping off resulting 2-shperes. The component (surface)$\times\{1\}$ is denoted by $\partial_+ W$, and $\partial W - \partial_+ W$, which might be disconnected, is denoted by $\partial_- W$. Note that if $\partial_- W = \emptyset$, then W is a handlebody. Recall that a Heegaard splitting for a 3-manifold M with boundary is a decomposition of M into two compression bodies so that $M = W_1 \cup W_2$, and $\Sigma = W_1 \cap W_2 = \partial_+ W_1 = \partial_+ W_2$. We call Σ the splitting surface.

Definition 1.2. A Heegaard splitting surface Σ is reducible (weakly reducible) if there is a compressing disk D_1 for Σ in W_1 and a compressing disk D_2 for Σ in W_2 so that $|D_1 \cap D_2| = 1$ ($|D_1 \cap D_2| = 0$). If the manifold is not weakly reducible then we say it is strongly irreducible.

PROPOSITION 1.3. *Either a Heegaard splitting surface Σ is weakly reducible or there is an isotopy of M pushing some fiber onto Σ.*

We prove this proposition at the end of the section.

Let M be a Seifert fibered space with base space S an orientable surface of genus g_0, with m exceptional fibers and Euler number e_0 i.e., $M = \{g_0, e_0 | (\alpha_1, \beta_1), \dots, (\alpha_m, \beta_m)\}$, where g.c.d.$(\alpha_j, \beta_j) = 1$ and β_j is normalized so that $0 < \beta_j < \alpha_j$. Remove small open disk neighborhoods $\mathscr{D}_1, \dots, \mathscr{D}_m$ of the points $x_1, \dots, x_m$ on S corresponding to the exceptional fibers, to get a surface S^*. Choose a point p on S^* corresponding to a regular fiber and a cutting system of curves $a_1, b_1, \dots, a_{g_0}, b_{g_0}$ for S^* based at p as indicated in Fig. 1.

In addition, choose a system of simple closed curves $c_1, \dots, c_m$ also based at p which are pairwise disjoint and so that each c_j goes once around the disk $\mathscr{D}_j$ (see Fig. 1). There is an embedding of S^* in M and a projection of M-{regular neighborhood of exceptional fibers}

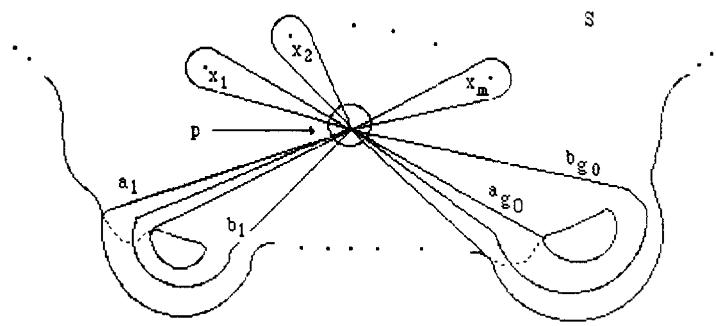

Fig. 1.

onto S^*. The preimage of the curves $a_1, b_1, \ldots, a_{g_0}, b_{g_0}, c_1, \ldots, c_m$ under this projection is a collection of annuli $A_1, B_1, \ldots, A_{g_0}, B_{g_0}, C_1, \ldots, C_m$ in M.

Now, $M - N((\bigcup A_i)\cup(\bigcup B_i)\cup(\bigcup C_j)) = V_{f_1} \cup \cdots \cup V_{f_m} \cup V$ where V_{f_j} is a regular neighborhood of the singular fiber f_j, $1 \leqslant j \leqslant m$, and V is a regularly fibered solid torus. Notice that $\partial V_{f_j} = C_j$, $1 \leqslant j \leqslant m$, and $\partial V = (\bigcup C_j)\cup(\bigcup A_i^+)\cup(\bigcup A_i^-)\cup(\bigcup B_i^+)\cup(\bigcup B_i^-)$, where A_i^+ and A_i^- (B_i^+ and B_i^-) are parallel copies of A_i (B_i).

Let Σ be a Heegaard splitting surface for M. A Heegaard surface determines a Morse function h on M so that its splitting surface Σ is a level surface which lies between the critical levels of index 0, 1 and those of index 2, 3 (for details see [17, section 3]). Let $\mathscr{L}$ denote the link $f_1 \cup \cdots \cup f_m \cup f_p$ in M. By general position we can push the link $\mathscr{L}$ into a collar $\Sigma \times I \subset M$ and after a small isotopy we can arrange that $h|\mathscr{L}$ is a Morse function (see [13]).

Let h be a Morse function on M such that $h|\mathscr{L}$ has critical levels $u_0, \ldots, u_n$ on $\mathscr{L}$ distinct from the critical levels on M. Let $r_1, \ldots, r_n$ be regular values for h so that $u_{i-1} < r_i < u_i$. Then $h^{-1}(r_i)$ is a level surface F_i. Let $|F_i \cap \mathscr{L}|$ denote the number of intersection points of $F_i \cap \mathscr{L}$.

Definition 1.4. A link $\mathscr{L}$ in thin position within its isotopy class if it minimizes the sum over all i of $|F_i \cap \mathscr{L}|$.

In what follows, we shall assume that $\mathscr{L}$ is in thin position with respect to the Morse function h induced by the Heegaard splitting with Heegaard surface Σ. For the proof of Proposition 1.3 we require the following two lemmas.

LEMMA 1.5. *If no fiber in M can be isotoped onto the surface Σ, then after an isotopy the transverse intersection of $\Sigma \cap A_i \subset A_i$, $\Sigma \cap B_i \subset B_i$, $\Sigma \cap C_j \subset C_j$, contains, for each annulus A_i, B_i, C_j, $1 \leqslant i \leqslant g_0$, $1 \leqslant j \leqslant m$, at least one essential arc, no non-essential arcs, and perhaps some null-homotopic curves.*

Proof. This follows from the proof of Lemma 3.3 in [17]. □

It follows from Lemma 1.5 that if we cannot isotope any fiber onto the surface Σ, then ∂V_{f_j}, ∂V_{f_p} contain simple closed curves that are either null homotopic curves in the annuli A_i^+, A_i^-, B_i^+, B_i^-, C_j or are simple closed curves that are the union of essential arcs on these annuli.

LEMMA 1.6. *If no fiber in M can be isotoped onto the surface* Σ. *Then the essential simple closed curves on* ∂V *and* ∂V_{f_j} *(i.e., those comprised of the essential arcs in the annuli above) are meridians bounding disks in the solid tori* V, V_{f_j}, $1 \leqslant j \leqslant m$.

Proof. Let γ denote such a simple closed curve. If γ is not a meridian of, say, V then it must follow around the core of some torus V at least once. There is a singular annulus between the core of the torus V and the curve γ. A singular annulus is the image of a map $\sigma(A) \rightarrow V$ of a regular annulus A. We can choose a level surface isotopic to Σ (also denoted by Σ) whose intersection with σ(int A) is not empty. When we consider the intersection pattern of Σ and $\sigma(A)$ on A i.e., $\sigma^{-1}(\sigma(A) \cap \Sigma)$, we see a collection of level arcs with end points on exactly one of the boundary components of A, namely the one which s mapped to the fiber. These arcs must intersect in a configuration as indicated in Fig. 2.

Consider the arcs (y_0, y_1) and (y_2, y_3) on ∂A, there are two possibilities. If the images, $\sigma(y_0, y_1)$, and $\sigma(y_2, y_3)$ of the two arcs are distinct, then the disks D_1 and D_2 or D_2 and D_3 are an upper and lower disk pair. They are disjoint from the other components of the link $\mathscr{L}$ as they are contained inside a regular neighborhood of one of the fibers away from the boundary. Hence, we can reduce the number of intersections with the level curves. This contradicts the fact that the link $\mathscr{L}$ is in thin position (see also [3, 9]). If $\sigma(y_0, y_1)$ and $\sigma(y_2, y_3)$ coincide, then $\sigma(y_1, y_3)$ is a copy of the fiber. As $\sigma(A)$ is an embedding on the interior of A, the union of the disks D_1 and D_2 or D_2 and D_3 is an embedded disk which describes an isotopy of the core of V onto Σ, contrary to the assumption that this is impossible. Hence, these simple closed curves must be meridians of each of the solid tori V, V_{f_j}, $1 \leqslant j \leqslant m$. □

Proof of Proposition 1.3. Let us assume that there is no isotopy pushing a fiber onto the Heegaard surface Σ. Then by Lemmas 1.5 and 1.6 the intersection curves of Σ and the tori ∂V_{f_j}, ∂V, $1 \leqslant j \leqslant m$ are either null homotopic or meridians for the solid tori V, V_{f_j}, $1 \leqslant j \leqslant m$. Hence, there is a set of compressing disks for the surface Σ so that Σ compresses along them to a surface which is the union of the meridians disks in $V_{f_1} \cup \cdots \cup V_{f_m} \cup V$. It is thus transverse to all fibers and hence is a horizontal incompressible surface. Since handlebodies do not contain closed incompressible surfaces the compressions cannot have been only to one side of the surface Σ. This implies that Σ is weakly reducible. □

Note that in particular an orientable Seifert fibered space M contains a horizontal incompressible surface if and only if $e_0 = 0$ or, equivalently, if and only if M fibers as a circle bundle over S^1.

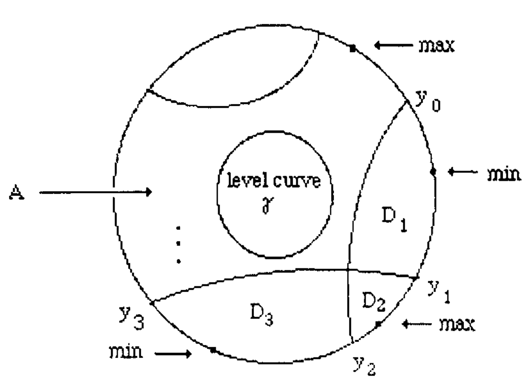

Fig. 2.

2. VERTICAL HEEGAARD SPLITTINGS

It follows from [17] that irreducible Heegaard splittings which are weakly reducible are obtained by a process called amalgamation. We will use this fact to prove that irreducible but weakly reducible Heegaard splittings of M are vertical whenever M is an orientable Seifert fibered space over an orientable base space, but is not exceptional.

Definition 2.1. We call a Heegaard splitting vertical if it is isotopic to one obtained by the following construction: Let M be an orientable Seifert fibered space with an orientable base space S, m exceptional fibers $f_1, \dots, f_m$ and d boundary components $\partial_1, \dots, \partial_d$ (where d could be 0). Let p, a_1, $b_1, \dots, a_{g_0}$, b_{g_0}, $\mathcal{D}_1, \dots, \mathcal{D}_m$, $c_1, \dots, c_m$ be as in Section 1. Furthermore, choose a collection of arcs σ_j, with one end point at p and the other at x_j, $j = 1, \dots, m$, a system of simple arcs μ_i connecting p to ∂_i, $i = 1, \dots, d$, and a system of simple closed curves $d_1, \dots, d_d$ based at p, so that d_i goes once around the i-th boundary component ∂_i. These curves can be chosen so that they are all disjoint and so that S, cut along $a_1, b_1, \dots, a_{g_0}, b_{g_0}, d_1, \dots, d_d, c_1, \dots, c_m$ is a disk. Now in the case where $d \neq 0$ (case 1), choose two subsets of indices $\{j_1, \dots, j_r\} \subset \{1, \dots, m\}$ and $\{i_1, \dots, i_s\} \subset \{2, \dots, d\}$ at least one of which is not empty. In the case where $d = 0$ and $m > 1$ (case 2) choose one nonempty subset of indices $(j_1, \dots, j_r\} \subset \{2, \dots, m\}$. In the case where $d = 0$ and $m \leqslant 1$ (case 3), we denote by f either the unique singular fiber or, if it does not exist, a regular fiber. In case 1 let $\{k_1, \dots, k_{m-r}\}$ and $\{l_1, \dots, l_{d-s-1}\}$ be the complementary sets and in case 2, let $\{k_1, \dots, k_{m-r-1}\}$ be the complementary set. In case 1 denote by $\Gamma(j_1, \dots, j_r, i_1, \dots, i_s)$ the graph embedded in M which is the union of the curves:

$$a_1, b_1, \dots, a_{g_0}, b_{g_0}, \sigma_{j_1}, f_{j_1}, \dots, \sigma_{j_r}, f_{j_r}, c_{k_1}, \dots, c_{k_{m-r}}, \mu_{i_1}, \dots, \mu_{i_s}, d_{l_1}, \dots, d_{l_{d-s-1}}.$$

In case 2 denote by $\Gamma(j, \dots, j_r, i_l, \dots, i_s)$ $(s = 0)$ the graph embedded in M which is the union of the curves:

$$a_1, b_1, \dots, a_{g_0}, b_{g_0}, \sigma_{j_1}, f_{j_1}, \dots, \sigma_{j_r}, f_{j_r}, c_{k_1}, \dots, c_{k_{m-r-1}}$$

and in case 3 denote by Γ $(j_1, \dots, j_r, i_1, \dots, i_s)$ $(r \leqslant 1,\ s = 0)$ the graph embedded in M which is the union of the curves:

$$a_1, b_1, \dots, a_{g_0}, b_{g_0}, f.$$

Set $W_1 = N(\Gamma(j_1, \dots, j_r, i_1, \dots, i_s) \cup (\partial_{i_1} \times S^1) \cup \dots \cup (\partial_{i_s} \times S^1))$ and $W_2 = \text{closure}(M - W_1)$ (see Fig. 3). Clearly W_1 is a compression body. For the proof that W_2 is a compression body

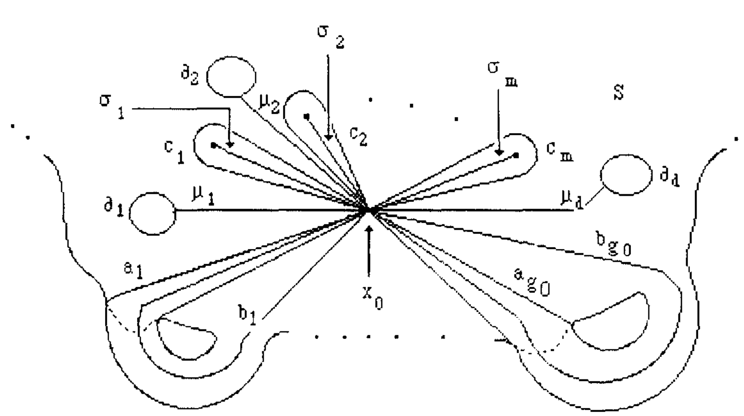

Fig. 3.

and that (W_1, W_2) is well defined, see [4, 12, 20]. Note that if $d = 0$ then W_1, W_2 are handlebodies.

The following defines the process of amalgamation of Heegaard splittings. This process produces a Heegaard splittings for M from Heegaard splittings of submanifolds of M.

Definition 2.2. Let R be a closed surface contained in the boundary of a 3-manifold M. Let U_1, U_2 be a pair of compression bodies defining a Heegaard splitting for M, and assume that $R \subset \partial U_1$. Note that there is some component $R' \subset \partial U_1$ (R' can be empty) so that $U_1 = N(R \cup R') \cup \{1\text{-handles}\}$. Let h be a homeomorphism $N(R) \to R \times I$ and $p: R \times I \to R$ the projection onto the first factor.

Let M_1, M_2 be two manifolds each with non-empty boundary and with Heegaard splittings (U_1, U_2), (V_1, V_2) respectively. Let R_1, R_2 be two homeomorphic surfaces such that $R_1 \subset \partial U_1 \subset \partial M_1$ and $R_2 \subset \partial V_1 \subset \partial M_2$ and let h_i, p_i $i = 1, 2$, be the corresponding functions, respectively.

Define an equivalence relation $\sim$ on $M_1 \cup M_2$ as follows:

(1) If x_i, y_i are points such that $x_i, y_i \in N(R_i)$ and $p_i h_i(x_i) = p_i h_i(y_i)$ then $x_i \sim y_i$.
(2) If $x \in R_1$, $y \in R_2$ and $g(x) = y$, where $g: R_1 \to R_2$ is the homeomorphism between the surfaces, then $x \sim y$.

Furthermore, we can arrange that the attaching disks on $R_1 \times I$ ($R_2 \times 1$) for the one handles in U_1 (V_1) respectively, have disjoint images in R_1 (R_2) and hence they do not get identified to each other. Now set

$$M = (M_1 \cup M_2)/\sim, \qquad W_1 = (U_1 \cup V_2)/\sim, \qquad W_2 = (U_2 \cup V_1)/\sim.$$

Note that $W_1 = V_2 \cup N(R_1') \cup (1\text{-handles})$ and $W_2 = U_2 \cup N(R_2') \cup (1\text{-handles})$ (The 1-handles connect $\partial_+ V_2$ to $\partial N(R_1')$ ($\partial_+ U_2$ to $\partial N(R_2')$, respectively)) so that W_1, W_2 are compression bodies defining a Heegaard splitting (W_1, W_2) for M (see also [17]).

The Heegaard splitting (W_1, W_2) of M is called the amalgamation of the Heegaard splittings (U_1, U_2) of M_1 and (V_1, V_2) of M_2 along R_1, R_2.

A weakly reducible Heegaard splitting surface Σ in M compresses to both sides along a maximal system of disjoint non-parallel compressing disks Δ. The result is a possibly disconnected surface. We denote by $\Sigma^* = \sigma(\Sigma, \Delta)$ the surface obtained from Σ by doing 2-surgery along the curves $\partial\Delta$ and deleting the 2-sphere components. If Σ is irreducible then $\Sigma^* \neq \emptyset$ (see [6]). We will assume that Δ minimizes the geometric intersection of Σ^* with Σ.

The next two lemmas are proved in [17]. We include the proof of Lemma 2.3, because it illustrates how the Heegaard splitting of M naturally yields a Heegaard splitting for certain submanifolds of M. In particular, it defines the induced Heegaard splitting for N as in the lemma.

LEMMA 2.3. *Let (W_1, W_2) be a Heegaard splitting of M with splitting surface Σ. Assume that Σ is weakly reducible and let Δ be as above. Let N denote the closure of a component of $M - \Sigma^*$. Then the Heegaard splitting (W_1, W_2) induces a Heegaard splitting (U_1, U_2) of N. Moreover, $\partial N - \partial M$ is contained either entirely in $\partial_- U_1$ or entirely in $\partial_- U_2$.*

Proof. We can assume that $N \subset W_1 \cup N(\Delta_2)$ where $\Delta = \Delta_1 \cup \Delta_2$ and Δ_i is the subcollection of Δ consisting of compression disks for Σ in W_i. Set $U_1 = W_1 \cap N$. We can obtain N from U_1 by attaching 2-handles and hence one can obtain U_1 from N by removing 2-handles (i.e., by drilling out tunnels), thus U_1 is connected. So U_1 is a single component of

$W_1 - N(\Delta_1)$ and hence is a compression body. Now, $U_2 = N - U_1$ is obtained from a collar of $N \cap \Sigma^*$ by attaching 1-handles. It is connected because $\partial_+ U_1 = \partial_+ U_2$ and therefore is also compression body. Thus, (U_1, U_2) is a Heegaard splitting for N. It is called the induced Heegaard splitting on N. Note that $\partial N - \partial M$ is contained either entirely in $\partial_- U_1$ or entirely in $\partial_- U_2$. □

LEMMA 2.4. *Let (W_1, W_2) be a Heegaard splitting of M with splitting surface Σ. Assume that Σ is weakly reducible and denote by Δ the pairwise disjoint collection of compressing disks on both sides of Σ. Let $N_1, \ldots, N_n$ be the closure of the components of $M - \Sigma^*$ and let $(U_1, U_2)_1, \ldots, (U_1, U_2)_n$ be the induced Heegaard splittings on $N_1, \ldots, N_n$. Then (W_1, W_2) is the amalgamation of $(U_1, U_2)_1, \ldots, (U_1, U_2)_n$ along $\Sigma^\# = (\bigcup \partial N_i) - \partial M$.*

Proof. See proof of Proposition 2.8 in [17]. □

The following theorem is due to the second author. For the excluded case, the exceptional manifolds, the question remains as to whether or not a Heegaard splitting which is obtained as the amalgamation of two Heegaard splittings of (closed orientable surface) $\times I$ is isotopic to a vertical Heegaard splitting.

Remark 2.5. Recall that a connected incompressible surface S in an orientable Seifert fibered space over an orientable base space is either a vertical annulus or torus, or is a horizontal surface which is also a fiber in fibrations over S^1. If S is the boundary of a twisted I-bundle over a surface F then S is a connected 2-fold cover of F. Thus F must be non-orientable. This is a contradiction as F intersects every fiber transversally and hence is a non-orientable cover of the orientable base space. This argument also holds if F has boundary and S is the boundary of a twisted I-bundle over F less the annuli which are the restriction of the bundle to the boundary components (see [11, VI.34]). Hence S cannot be the boundary of a twisted I-bundle over a surface F.

THEOREM 2.6. *A weakly reducible Heegaard splitting of an orientable Seifert fibered space M with orientable base space is either reducible or isotopic to a vertical Heegaard splitting.*

Proof. Assume first that M is not an exceptional space and that the splitting is irreducible. Let Δ be a maximal set of compressing disks for Σ as above. Compressing Σ along Δ, suppose we obtain an incompressible horizontal surface Σ^*. Note that if M is to contain a horizontal incompressible surface of positive genus, then either the base space of M has positive genus or M has at least three exceptional fibers. This fact together with the assumption that M is not exceptional guarantees that M has saturated essential tori. Let Σ_i^* be a component of Σ^*, hence M is a Σ_i^* fiber bundle over S^1 as in Remark 2.5. If we cut M along Σ_i^* we obtain a manifold homeomorphic to $\Sigma_i^* \times I$. By case 1 of Theorem 10.3 of [10] all components Σ_j^* of Σ^* are isotopic. The surface Σ^* is homologous to Σ; hence, it must be separating and so has an even number of components. Let T be a saturated incompressible torus in M. Consider a component c of $\Sigma^* \cap T$ and note that c is essential in both Σ^* and T as both surfaces are incompressible. Let N_1, N_2 be the components of $M - \Sigma^*$ whose boundary contains c. It follows that $N_i = \Sigma_i^* \times I$, $i = 1, 2$. As in Lemma 2.3, Σ induces a Heegaard splitting on $N_i = \Sigma_i^* \times I$, $i = 1, 2$.

Heegaard splittings of $\Sigma_i^* \times I$ are standard by a result of Scharlemann and Thompson (see [21]). It follows that the induced Heegaard splitting is defined by two copies of the

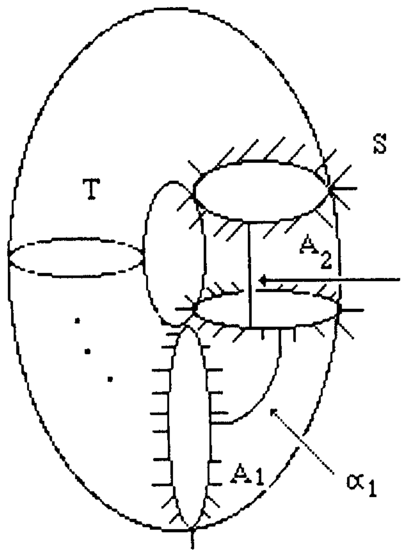

Fig. 4.

surface Σ_i^* together with the boundary of a regular neighborhood of a spanning arc (in the terminology of [21] it is standard of type II). Note that the Heegaard splitting is independent of the choice of the arc.

Therefore, we can choose the spanning arcs α_1, α_2 to be straight arcs on the annular components A_1, A_2 of $N_1 \cap T$, $N_2 \cap T$ (see Fig. 4).

By slightly pushing the disks $A_1 - \alpha_1$ and $A_2 - \alpha_2$ to opposite sides of T we obtain two disjoint disks, one in each handlebody, such that when we compress along these disks, we obtain a surface which intersects T two fewer times than did Σ^*. If this new surface is compressible, we may compress it further to obtain an incompressible surface of lower genus than Σ^*. Thus, it is possible to choose a collection of compressing disks Δ' satisfying all the conditions that Δ does, but such that either $\sigma(\Sigma;\Delta')$ has lower genus than Σ^* or $|\sigma(\Sigma;\Delta') \cap T| \leqslant |\sigma(\Sigma;\Delta) \cap T| - 2$. When we choose a collection Δ that minimizes (genus($\sigma(\Sigma;\Delta')$), $|\sigma(\Sigma;\Delta) \cap T|$), the intersection must be empty. Hence, Σ^* is a collection of vertical tori, contradicting our assumption that Σ^* is a horizontal surface.

If on the other hand Σ compresses to a vertical incompressible surface then it must be a collection of saturated incompressible tori. In other words the Heegaard splitting (W_1, W_2) determined by Σ is an amalgamation of Heegaard splittings of Seifert fibered spaces with boundary. Theorem 4.2 of [20] states that all irreducible Heegaard splittings of fiberwise orientable Seifert manifolds with non-empty boundary are vertical. Proposition 1.3 of [20] states that a Heegaard splitting of Seifert fibered manifolds which is the amalgamation of vertical Heegaard splittings along vertical tori is itself vertical.

If M is an exceptional Seifert fiber space then any weakly reducible Heegaard splitting is reducible by Theorem 1 of [18]. Hence the claim follows. □

3. HORIZONTAL HEEGAARD SPLITTINGS

Not all Seifert fibered spaces have horizontal Heegaard splittings. We begin by describing a method to construct horizontal Heegaard splittings in the Seifert fibered spaces which admit them. Consider a Seifert fibered space M^* with one torus boundary component. Such manifolds are surface fiber bundles over S^1 (see [11, VI.32]). Consider a surface fiber S in such a fibration of M^* over S^1. It is a once punctured surface and hence a regular neighborhood of S is a handlebody H_1 whose genus is $2 \times (\text{genus } S)$. The manifold $M^* - N(S)$ is homeomorphic to $S \times I$ and is also a handlebody H_2. The two handlebodies H_1, H_2 are glued to each other along their boundaries less two annuli $A_1 \subset H_1$, $A_2 \subset H_2$. The two annuli A_1, A_2 are glued to each other along their boundaries to form the boundary torus (see Fig. 5).

Any Dehn filling on ∂M^* produces a closed Seifert fibered space. However, only surgery corresponding to n-Dehn twists along one of the annuli, say A_1, produces a manifold for which the surface ∂H_1 is a splitting surface. This can be seen as follows: The solid torus V in the Dehn filling is glued to A_1 along an annulus $A'_1 \subset \partial V$. A necessary and sufficient condition for the resulting manifold to be a handlebody is that the generator of $\pi_1(A'_1)$ is also a generator in $\pi_1(V)$. Thus the 2×2-matrix in $GL_2(\mathbb{Z})$ with entries a, b, c, d determining the Dehn filling must have $a = \pm 1$. Hence, the meridian of V is glued to a $1/n$ curve. Note that the surgery coefficients are computed with respect to the framing determined by ∂A_1.

Definition 3.1. Let M be a Seifert fibered space and let f_i be a fiber (regular or exceptional) in M. Let S be a surface in a fibration of $M^* = M - N(f_i)$ over S^1. Suppose that M is obtained from M^* by $1/n$ -Dehn filling with respect to the framing determined by ∂S. Then the Heegaard splitting for M constructed as above (using M^* and S) is called a horizontal Heegaard splitting corresponding to the fiber f_i.

Remark 3.2. It should be pointed out that the Heegaard surface of a horizontal Heegaard splitting is *not* a horizontal surface in the standard sense. More specifically it is transverse to the Seifert fibration everywhere except on an annulus in the splitting surface.

Proof of Theorem 0.3. Let M be an orientable Seifert fibered space with an orientable base space. If M has a horizontal Heegaard splitting corresponding to the fiber f_i then from the construction f_i and ∂S cobound an annulus and hence f_i is homologous to ∂S and is null-homologous. If on the other hand f_i is null-homolougous then it bounds a surface S' in M. We may assume that this surface is incompressible as if not we compress it as much as

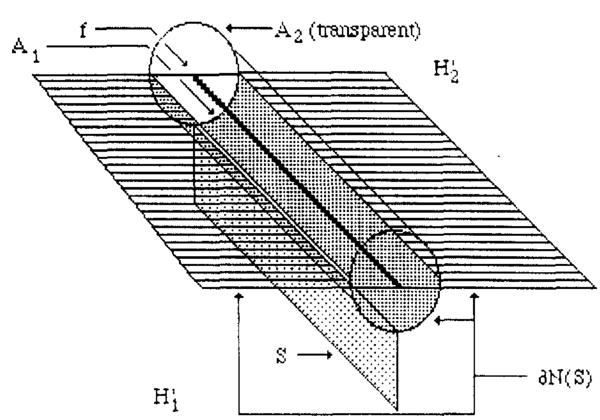

Fig. 5.

possible. Remove int($N(f_i)$) from M to obtain a Seifert fibered space M^* with one torus boundary component. The surface $S = S' \cap M^*$ is a fiber in a fibration of M^* over S^1 (By [11, VI.34 and Remark 2.5]) and hence determines a horizontal Heegaard splitting.

The construction above shows that M has a horizontal Heegaard splitting corresponding to f_i if and only if we obtain M from M^* by a $1/n$-Dehn filling with respect to the framing determined by the boundary component ∂S. In order to check this condition is fulfilled we need to determine the coordinates of ∂S with respect to the basis {cross curve, regular fiber}.

Let T be an incompressible torus in M^* separating the exceptional fibers and the boundary torus from the rest of the manifold. We can assume that M^* has k exceptional fibers with invariants $(\alpha_1, \beta_1), \ldots, (\alpha_k, \beta_k)$. That is $k = m$ if the fiber on the surface is a regular one and otherwise $k = m - 1$. If we cut M^* along T we obtain two components, one, M^{0*} is a Seifert fibered space over S^2 with k exceptional fibers and two boundary components and the other, M^1, is a (once punctured surface $S^*) \times S^1$. If we cap off the boundary component in M^{0*} corresponding to T by a trivially fibered torus we get a Seifert fibered space M^0 over S^2 with k exceptional fibers and one boundary component. If S^0 is a horizontal surface in M^0 then we can obtain a horizontal surface in M^* by the following process. Remove the interior of a regular neighborhood V of a regular fiber in M^0, ($\partial V = T$). For each boundary component of $S^0 \cap T$ select a copy of S^* in M^1 and glue M^1 to M^0 so that the boundary curves of the copies are glued to the curves of $S^0 \cap T$ and also so that regular fibers in both spaces match up. This can always be done as the surfaces intersect the regular fibers transversely. In fact, any horizontal surface in M^* can be cut up by T into a horizontal surface S^0 in M^0 and some copies of S^* in M^1. The number of copies needed is exactly the number of intersection points of a regular fiber and the horizontal surface in M^0.

Any fibration of M^* is determined by a homomorphism $\pi_1(M^*) \to \mathbb{Z}$. Hence, to understand the fibrations of M^*, it is sufficient to consider homomorphisms $\pi_1(M^0) \to \mathbb{Z}$ (see [8], pp. 90–91]). We give an argument using the fundamental group but as the referee pointed out a homological argument would suffice here.

Denote the regular fiber f_p by f_0 and set $\alpha_0 = 1$, $\beta_0 = b$. Assume that we have removed the fiber f_i, for some i in $\{0, \ldots, m\}$. The group $\pi_1(M^0)$ has a presentation:

$$\pi_1(M^0) = \langle q_0, \ldots, q_m, h \mid [q_j, h], j = 0, \ldots, m; q_j^{\alpha_j} h^{\beta_j}, j = 0, \ldots, m, j \neq i; q_0, \ldots, q_m \rangle.$$

We get a homomorphism $\varphi: \pi_1(M^0) \to \mathbb{Z}$ as follows. Set $\alpha^i = \text{l.c.m.}\{\alpha_j\}$, $j = 0, \ldots, m, j \neq i$ and set $\varphi(h) = \alpha^i$, $\varphi(q_j) = -\beta_j \alpha^i / \alpha_j$. It is immediate that the relators $q_j^{\alpha_j} h^{\beta_j}, j = 0, \ldots, m$, are satisfied, so we have a homomorphism. It is also clear that any homomorphism $\pi_1(M^0) \to \mathbb{Z}$ must satisfy these relators and hence is a "multiple" of φ. As a consequence of the last relator we get $q_i \to \sum_{j=0, j \neq i}^{m} \beta_j \alpha^i / \alpha_j \in \mathbb{Z}$. The boundary curve $\partial S \subset \partial M^0$ must be mapped to $0 \in \mathbb{Z}$. So we are looking for a pair of integers s_i, t_i such that $s_i \cdot \varphi(q_i) + t_i \cdot \varphi(h) = 0$, $|s_i|$ is minimal. If $\varphi(q_i) \neq 0$, a curve on ∂M which intersects ∂S once is given by a $\{u_i, v_i\}$ curve in the $\{q_i, h\}$ basis so that $s_i v_i - u_i t_i = 1$. In the case $\varphi(q_i) = 0$ this curve is just the regular fiber h. The $1/n$-Dehn surgery coefficients with respect to a framing determined by ∂S, given in $\{q_i, h\}$ coordinates are $n(\{s_i, t_i\}) + \{u_i, v_i\}$. Thus a necessary condition for the existence of a horizontal Heegaard splitting is that the Seifert invariants must be: $\alpha_i = ns_i + u_i$, $\beta_i = nt_i + v_i$ at the ith exceptional fiber. (Or $1 = ns_0 + u_0$, $b = nt_0 + v_0$ when we remove a regular fiber.)

The horizontal surface, if it exists, is a branched cover of the base space branched over $m - 1$ points (or m points if the removed fiber is regular) with branching indices $\alpha_1, \ldots, \hat{\alpha}_i, \ldots, \alpha_k$ (i.e., α_i excluded). Hence, the degree of the covering must divide by each α_j, $j \neq i$, in fact it must be equal to $\alpha^i = \text{l.c.m.}(\alpha_1, \ldots, \hat{\alpha}_i, \ldots, \alpha_k)$. The surface must also have a unique boundary component. The degree of the cover is equal to the number of intersection points between h and ∂S which is exactly s_i. Hence, $s_i = \alpha^i$. Note that if $s_i \mid \alpha^i$ but $s_i \neq \alpha^i$, then we have more than one boundary component for the surface fiber and in this case we do not get a horizontal Heegaard splitting.

If $\varphi(q_i) = 0$, then $[\partial S] = \alpha^i[q_i]$, that is q_i is only one of the components of ∂S. So in this case the construction will not yield a horizontal Heegaard splitting. Applying the above considerations to the fibers $f_0, \ldots, f_m$ one by one proves the "only if" part of the theorem.

On the other hand, if $s_i = \alpha^i$ and $\alpha_i = ns_i + u_i$, $\beta_i = nt_i + v_i$, for some n, then we can define a homomorphism φ as above. This homomorphism induces a fibration of $M - N(f_i)$ as a fiber bundle over S^1. Let S be a surface so that $[S] = \varphi^{-1}(0)$. The conditions above enable us to complete this surface S in $M - N(f_i)$ to a horizontal Heegaard splitting of M. □

Remark 3.3. It is an easy exercise using the formula for the Euler characteristic $\mathscr{X}(S)$ to show that indeed $\mathscr{X}(S)$ is always an odd integer as it should be for a surface with one boundary component.

Proof of Corollary 0.4. Since M has $e_0 = 0$ it fibers over S^1. Thus a regular fiber and hence any fiber cannot be null-homologus in M i.e., a fiber cannot bound a surface in M. This implies that M does not have a horizontal Heegaard splitting. Hence, by Theorem 0.1 all Heegaard splittings of M are vertical. □

Proof of Corollary 0.5. By Theorem 0.3 a necessary and sufficient condition for M to have a horizontal Heegaard splitting is that a fiber be null-homologous i.e., $e_0 = \pm 1$. If $e_0 = \pm 1$, Dehn twists along vertical tori act transversally on the collection of horizontal Heegaard splittings. By Corollary 12 of [19] the vertical Heegaard splittings are stabilizations of the horizontal Heegaard splitting. In the case $e_0 \neq \pm 1$ the irreducible Heegaard splitting is unique up to isotopy by Theorems 0.1 and 0.3. □

Examples. 3.4. The first two manifolds have horizontal Heegaard splittings by [3]. We corroborate their result using our computations. In our third example we provide a manifold that does not have a horizontal Heegaard splitting.

(1) Let $M = S(0; -1/42 \mid \{2,1\}, \{3,2\}, \{7,6\})$. Remove the singular fiber $\{7,6\}$. We compute $\alpha^3 =$ l.c.m. $(2,3) = 6$, so $\varphi(h) = 6$, $\varphi(q_1) = -\beta_1\alpha^3/\alpha_1 = -3$, $\varphi(q_2) = -\beta_2\alpha^3/\alpha_2 = -4$, and $b = -2$, therefore $\varphi(q_3) = (-2)\ 6 + 3 + 4 = -5$. Thus, $s_3\ (-5) + t_3\ 6 = 0$ implies that $s_3 = 6$ and $t_3 = 5$ and, consequently, $u_3 = 1$, $v_3 = 1$. Hence, in order to get a horizontal Heegaard splitting we must have $\alpha_3 = 6n + 1$, $\beta_3 = 5n + 1$ for some n, and indeed for $n = 1$ we have $\alpha_3 = 7$, $\beta_3 = 6$, so M has a horizontal Heegaard splitting.

Note that if we remove the fiber $\{3,2\}$ we get $\alpha^2 =$ l.c.m. $(2,7) = 14$, so $\varphi(h) = 14$, $\varphi(q_1) = -\beta_1\alpha^2/\alpha_1 = -7$, $\varphi(q_2) = -\beta_2\alpha^2/\alpha_2 = -12$ and $b = -2$, therefore $\varphi(q_2) = (-2)14 + 7 + 12 = -9$. Thus, $s_2(-9) + t_2\ 14 = 0$ implies that $s_2 = 14$ and $t_2 = 9$ and, consequently, $u_2 = 3$, $v_2 = 2$. Hence, in order to get a horizontal Heegaard splitting we must have $\alpha_2 = 14n + 3$, $\beta_2 = 9n + 2$ for some n and indeed for $n = 2$ we have $\alpha_2 = 3$, $\beta_2 = 2$, so M has a horizontal Heegaard splitting coming from this fiber. (They are distinguished in 3.5.)

(2) Let $M = S(0;\text{-}1/21 \mid \{3,2\}, \{3,2\}, \{7,5\})$. Remove the singular fiber $\{7,5\}$. We compute $\alpha^3 =$ l.c.m.$(3,3) = 3$, so $\varphi(h) = 3$, $\varphi(q_1) = -\beta_1\alpha^3/\alpha_1 = -2$, $\varphi(q_2) = -\beta_2\alpha^3/\alpha_2 = -2$ and $b = -2$, therefore $\varphi(q_3) = -6 + 2 + 2 = -2$. Thus $s_3(-2) + t_3 3 = 0$, implies that $s_3 = 3$ and $t_3 = 2$ and, consequently, $u_3 = 1$, $v_3 = 1$. Hence, in order to get a horizontal

Heegaard splitting we must have $\alpha_3 = 3n + 1$, $\beta_3 = 2n + 1$ for some n and indeed for $n = 2$ we have $\alpha_3 = 7$, $\beta_3 = 5$ so M has a horizontal Heegaard splitting.

If we remove the fiber $\{3,2\}$ we get $\alpha^2 = \text{l.c.m.}\ (3,7) = 21$ so $\varphi(h) = 21$, and $\varphi(q_1) = -\beta_1\alpha^2/\alpha_1 = -14$, $\varphi(q_2) = -\beta_2\alpha^2/\alpha_2 = -15$ and $b = -2$, therefore $\varphi(q_3) = (-2)21 + 14 + 15 = -13$. Thus, $s_2(-13) + t_2\, 21 = 0$ implies that $s_2 = 21$ and $t_2 = 13$ and, consequently, $u_2 = 8$, $v_2 = 5$. Hence, in order to get a horizontal Heegaard splitting we must have $\alpha_2 = 3 = 21n + 8$, $\beta_2 = 2 = 13n + 5$ for some n and so M has no horizontal Heegaard splitting corresponding to this fiber.

(3) Let $M = S(5; -21/40 \mid \{3,1\}, \{6,1\}, \{8,5\}, \{5,2\})$. Remove the singular fiber $\{5,2\}$. We compute $\alpha^4 = \text{l.c.m.}\ \{3,6,8\} = 24$, so $\varphi(h) = 24$, $\varphi(q_1) = -\beta_1\alpha^4/\alpha_1 = -8$, $\varphi(q_2) = -\beta_2\alpha^4/\alpha_2 = -4$, $\varphi(q_3) = -\beta_3\alpha^4/\alpha_3 = -15$ and $b = -1$, therefore, $\varphi(q_4) = (-1)24 + 8 + 4 + 15 = 3$. Thus, $s_4(3) + t_4 24 = 0$ implies that $s_4 = 8 \neq 24 = \alpha^4$ and, consequently, M has no horizontal Heegaard splittings corresponding to this fiber.

The genus of the horizontal Heegaard splitting can be computed. In the generic case it tends to be high, as we see below. The horizontal surface contains α^i copies of S^* each of genus g_0. The horizontal surface S^0 in M^0 also contributes to the genus. It is a α^i-fold branched cover of the disk branched over either $m - 1$ or m points depending on whether we removed a singular fiber or a regular one. The branching indices are $\alpha_1, \ldots, \hat{\alpha}_i, \ldots, \alpha_k$. The formula of the Euler characteristic of $\mathscr{X}(S^0)$ is given by

$$\mathscr{X}(S^0) = \mathscr{X}(D)\alpha^i - \sum_{j \neq i}^{m} (1 - 1/\alpha_j)\, \alpha^i = \alpha^i \left(1 - \sum_{j \neq i}^{m} (1 - 1/\alpha_j)\right)$$

where D is the base space of M^0. We need to remove α^i disks and attach α^i copies of S^* each with Euler characteristic $1 - 2g_0$. So the Euler characteristic of the horizontal surface S in M^* is

$$\mathscr{X}(S) = \alpha^i \left(1 - \sum_{j \neq i}^{m} (1 - 1/\alpha_j)\right) - \alpha^i + \alpha^i(1 - 2g_0) = \alpha^i \left(1 - 2g_0 - \sum_{j \neq 1}^{m} (1 - 1/\alpha_j)\right).$$

Recall that the Heegaard surface Σ is the boundary of a regular neighborhood of S. Hence, the genus of the horizontal Heegaard splitting Σ is given by

$$g(\Sigma) = 1 - (\mathscr{X}(S)).$$

Examples 3.5. We compute the genus of horizontal Heegaard splittings of the manifold M in Example (1) of 3.4.

(1) Let $M = S(0; -1/42 \mid \{2,1\}, \{3,2\}, \{7,6\})$ and remove the $\{7,6\}$ fiber as in Example (1) of 3.4. We have $g_0 = 0$ and $s_3 = 6$ so $g(\Sigma_3) = 1 - 6\,(1 - (1 - 1/2) - (1 - 1/3)) = 2$ as we know by [3].

(2) Let $M = S(0; -1/42 \mid \{2,1\}, \{3,2\}, \{7,6\})$ and remove the $\{3,2\}$ fiber as in example (1) of 3.4. We have $g_0 = 0$ and $s_2 = 14$ so $g(\Sigma_2) = 1 - 14\,(1 - (1 - 1/2) - (1 - 1/7)) = 6$.

Hence, the two horizontal Heegaard splittings of M are different. A more general result is the following:

PROPOSITION 3.6. *Let $M = \{g_0; e_0 \mid (\alpha_1\, \beta_1), \ldots, (\alpha_m, \beta_m)\}$ be a Seifert fibered space so that the invariants α_j are pairwise relatively prime. Then horizontal surfaces corresponding to different fibers are non-homeomorphic.*

Proof. For each k, $1 \leqslant k \leqslant m$, $k \neq i$ abuse notation and write $q_i = (\sum_{j=0, j\neq i,k}^{m} \beta_j \alpha^i/\alpha_j) + \beta_k \alpha^i/\alpha_k$. Note that α_k divides the first factor but does not divide the second factor as both β_k and α^i/α_k are relatively prime to α_k. Hence, q_i is relatively prime to α^i. This implies that $s_i = \alpha^i$. Now consider α_i and α_k, for some fixed k, $k \neq i$ and the Euler characteristic of the corresponding surfaces S_i and S_k.

$$\mathscr{X}(S_i) = \alpha^i\left(1 - 2g_0 - \sum_{j \neq i}^{m}(1 - 1/\alpha_j)\right) = \alpha^i\left(N + \sum_{j \neq i}^{m} 1/\alpha_j\right)$$

$$\mathscr{X}(S_k) = \alpha^k\left(N + \sum_{j \neq i}^{m} 1/\alpha_j\right), \qquad N = 2 - 2g_0 - m.$$

Assume that $\mathscr{X}(S_i) = \mathscr{X}(S_i)$ to derive a contradiction. We can assume that $i = 1$, $k = 2$ and note that $\alpha^1/\alpha_2 = \alpha^2/\alpha_1$. Hence,

$$\alpha^1 N + \alpha^1/\alpha_3 + \cdots + \alpha^1/\alpha_m = \alpha^2 N + \alpha^2/\alpha_3 + \cdots + \alpha^2/\alpha_m$$

thus

$$-(\alpha_2 - \alpha_1)N\alpha_3 \cdots \alpha_m = \alpha^2/\alpha_3 - \alpha^1/\alpha_3 + \alpha^2/\alpha_4 - \alpha^1/\alpha_4 + \cdots + \alpha^2/\alpha_m - \alpha^1/\alpha_m.$$

Therefore, we can divide both sides by $(\alpha_2 - \alpha_1)$ and after rearranging obtain

$$-N\alpha_3 \cdots \alpha_m - \alpha_3\alpha_5 \cdots \alpha_m - \alpha_3\alpha_4\alpha_6 \cdots \alpha_m - \cdots - \alpha_3\alpha_4 \cdots \alpha_{m-1} = \alpha_4\alpha_5 \cdots \alpha_m$$

but the left-hand side divides by α_3 while the right-hand side does not. So the genus of the horizontal Heegaard splitting surfaces S_i and S_k is not equal and the surfaces are not homeomorphic. □

4. THE MAIN THEOREM

We prove Theorem 0.1.

Proof. Let M be an orientable Seifert fibered space over an orientable base space S. Let Σ be the splitting surface of the irreducible Heegaard splitting (H_1, H_2) of M. If M is a Lens space then all its Heegaard splittings are vertical by [5]. Furthermore, if Σ is weakly reducible, then it is vertical by Theorem 2.6. Thus we can assume, in what follows, that M is not a Lens space and Σ is strongly irreducible.

By Proposition 1.3 we can isotope a fiber f (either regular or singular) into the surface Σ. Let $M^* = M - N(f)$ and let $\Sigma^* = \Sigma - N(f)$. Since $\Sigma \cap N(f)$ is an annulus, Σ^* has two boundary components and since Σ is separating Σ^* is also separating. There are two possible cases, either Σ^* is incompressible in M^* or Σ^* is compressible in M^*.

Case 1. The surface Σ^* is incompressible in M^*.

Since Σ^* is an orientable, separating, incompressible surface with two boundary components in M^* it is either a vertical annulus (boundary parallel or saturated) or consists of two fibers in a fibration of M^* as a surface bundle over S^1 (see [11, VI. 34]). The surface Σ^* cannot be the boundary of a twisted I-bundle over a compact surface by Remark 2.5.

If Σ^* is a vertical annulus then Σ is a torus and is a genus one Heegaard splitting of M. This is impossible when M is not a Lens space.

If the separating surface Σ^* is a fiber in a fibration of M^* as a surface bundle over S^1 then it must consist of two components Σ_1^*, Σ_2^*. The components Σ_1^*, Σ_2^*, must be parallel. For if we cut M^* along Σ_1^* then $\Sigma_2^* \subset \Sigma_1^* \times I$ and parallelity follows from [10], 10.3 Case 1.

Now, the handlebody H_1, say, is obtained from the handlebody $\Sigma_1^* \times I$ by gluing on a solid torus along an annulus. As in the first paragraph of Section 3, H_1 is a handlebody if and only if M is obtained from M^* by $1/n$-Dehn surgery with respect to the framing determined by $\partial\Sigma^*$. In these cases, Σ will be a horizontal Heegaard splitting of M. In order to determine whether or not this case occurs in M we need to calculate the Seifert invariants of the fiber f with respect to the basis of the homology of ∂M^* determined by $\partial\Sigma^*$ and a curve intersecting it once. Note that as genus (Σ_i^*) = genus $(\Sigma)/2$, this can occur only when genus (Σ) is even.

Case 2. The surface Σ^* is compressible in M^*.

Let Δ be a collection of disjoint compressing disks for Σ^* minimizing intersection with Σ^*. If Δ is on both sides of $\Sigma^* \subset M^*$ then in particular Δ would be on both sides of $\Sigma \subset M$ contraducting the facts that Σ is strongly irreducible. Thus Δ is either entirely in H_1 or entirely in H_2. Say $\Delta \subset H_2$. Denote by Σ^{**} the incompressible surface obtained from Σ^* by ambient surgery along the components of Δ.

As in Case 1, if Σ^{**} is connected then it is an annulus and if it is not connected then it consists of exactly two parallel fibers in a fibration of M^* as a surface bundle over S^1. If Σ^{**} is an annulus then one of three things may happen depending on how many singular fibers are cut off by Σ^{**}:

(a) The annulus Σ^{**} is boundary parallel i.e., it cuts off a solid torus not containing a singular fiber; or
(b) The torus $\Sigma^{**} \cup \{\Sigma \cap N(f)\} \subset H_2$ bounds a solid torus in the handlebody H_2 containing a singular fiber f in its interior: or
(c) The torus $\Sigma^{**} \cup \{\Sigma \cap N(f)\} \subset H_2$ bounds an incompressible torus in the handlebody H_2.

In the first and second cases f and f', respectively, are cores of H_2, since they intersect a meridian disk cut off of H_2 by Σ^{**} exactly once. The third case clearly is impossible.

In cases (a) and (b) we may, after a small isotopy of Σ, remove a small regular neighborhood of f (or f' resp.) from M to obtain a manifold homeomorphic to M^* (or $M - N(f')$ resp.) such that Σ is also the splitting surface of a Heegaard splitting of M^* ($M - N(f')$ resp.). By Theorem 4.2 of [20], Σ is a vertical Heegaard splitting of M^* ($M - N(f')$ resp.), hence it follows from the construction that Σ is a vertical Heegaard splitting of M.

We show that if the surface Σ^{**} consists of two parallel fibers Σ_1^*, Σ_2^{**} in a fibration of M as a surface bundle over S^1, then Σ is reducible. The two surfaces Σ_1^{**}, Σ_2^{**} separate M^* into two handlebodies H_1^*, H_2^*. (We obtain H_1 from H_1^* by drilling out tunnels as indicated in Fig. 6(a).) After an isotopy which moves the boundary parallel annulus $A(f) = \Sigma \cap N(f)$ across $\partial N(f) = \partial M^*$ we obtain a Heegaard splitting (H_1', H_2') of H_1^* by setting:

$$H_1' = (H_2 \cap H_1^*) \cup (\text{collar of } \partial H_1^* \text{ in } H_1^*) \text{ and } H_2' = (H_1^* - H_1') \text{ (see Fig. 6(b))}.$$

The Heegaard splitting (H_1', H_2') is a Heegaard splitting of a handlebody. Recall that Heegaard splittings of handlebodies are all standard (see [6]). As $\Delta \neq \emptyset$ the Heegaard splitting is reducible. Notice that a pair of reducing disks for the splitting (H_1', H_2') is also a pair of reducing disks for the splitting (H_1, H_2), so the claim is proved.

It follows that all irreducible Heegaard splittings of orientable Seifert fibered spaces M over an orientable base space S are either vertical or horizontal. □

Proof of Theorem 0.2. Let Σ be a Heegaard splitting for M. If it is a stabilization of a vertical Heegaard splitting then every singular fiber can be pushed onto Σ, as the singular

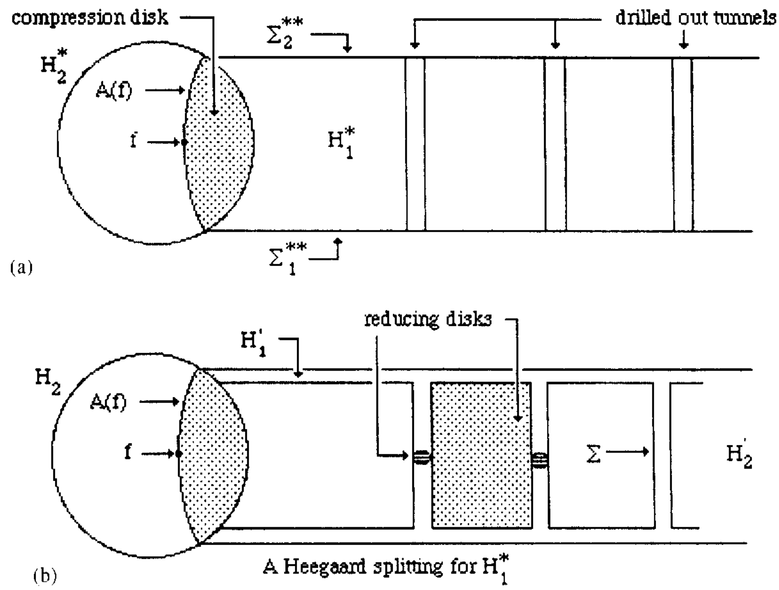

Fig. 6.

fibers are cores of the handlebodies. If Σ is a stabilization of a horizontal Heegaard splitting it is obtained as above; hence there is some fiber which can be pushed onto Σ. □

5. HORIZONTAL AND VERTICAL HEEGAARD SPLITTINGS

In this section we will show that in some sense almost all irreducible horizontal Heegaard splitting surfaces cannot be isotopic to vertical Heegaard splittings. The most elementary invariant which distinguishes between irreducible Heegaard splittings is the genus of the splitting surface. In partial answer to the question at hand, we can say the following:

PROPOSITION 5.1. *Let* $M = \{g_0, e_0 | (\alpha_1, \beta_1), \dots, (\alpha_m, \beta_m)\}$, $m \geqslant 2$, *be a Seifert fibered space with an orientable base surface of genus* g_0. *If* $g_0 > 0$ *or if* $g_0 = 0$ *and one of the following possibilities holds*: (1) $\alpha^i \geqslant 5$, (2) $\alpha^i \geqslant 4$ *and* $m > 4$, (3) $\alpha^i \geqslant 3$ *and* $m > 5$, *then all irreducible horizontal Heegaard splittings are not isotopic to vertical Heegaard splittings.*

Proof. We can assume that if $g_0 = 0$ then $m > 3$. The other cases, Lens spaces and small Seifert fibered manifolds were treated in [3, 5]. Let Σ be a vertical Heegaard splitting, then the genus of Σ is $2g_0 + m - 1$. Assume that Σ is also a horizontal Heegaard splitting for M. By the formula for the genus (as in Section 3) we have

$$2 - m - 2g_0 = \mathscr{X}(S) = \alpha^i(1 - 2g_0 - \sum_{j \neq i}^{m} (1 - 1/\alpha_j)) = \alpha^i(2 - 2g_0 - m) + \alpha^i \sum_{j \neq i}^{m} 1/\alpha_j.$$

As $\alpha_i > 1$ we have

$$(2g_0 - 1)/2 + (m - 1)/2 \leqslant (2g_0 + m - 2)(\alpha^i - 1)/\alpha^i = \sum_{j \neq i}^{m} 1/\alpha_j \leqslant (m - 1)/2$$

which is a contradiction if $g_0 > 0$. *If* $g_0 = 0$ and $\alpha^i \geqslant 5$ then $(m-2)4/5 \leqslant (m-2)(\alpha^i - 1)/\alpha^i \leqslant (m-1)/2$, hence $3m \leqslant 11$, which contradicts $m > 3$. Similarly, if $\alpha^i \geqslant 4$, then $m \leqslant 4$, contradicting $m > 4$, and if $\alpha^i \geqslant 3$, then $m \leqslant 5$, contradicting $m > 5$. □

This theorem does not answer the questions of whether or not a given horizontal Heegaard splitting is actually irreducible and the related question of whether or not a given horizontal Heegaard splitting is a stabilization of a vertical one.

We mentioned in the introduction that for $M = \{g_0, e_0 \mid (\alpha_1, \beta_1),, \ldots, (\alpha_m, \beta_m)\}$ with $g_0 > 0$ or $m > 3$, all vertical Heegaard splittings are weakly reducible. To see this, in the case $g_0 > 0$, consider a cocore disk D_1 of a regular neighborhood of a_1 in H_1 and a disk $D_2 = (b_1 \times S^1) - b_1$. Note that D_2 is an essential disk in H_2. In the case where $g_0 = 0$ and $r < m - 1$, we may construct D_1 and D_2 by using a cocore of a regular neighborhood of f_{j_1} and the curve c_{j_k}, where $k < r$ and k not equal 1. Finally, if $g_0 = 0$ and $r = m - 1$, we can construct D_1 and D_2 by using a cocore of a regular neighborhood of f_{j_1} and replacing b_1 by an arc which is the union of $\sigma_{j_{r-1}}$ and σ_{j_r}.

It is of independent interest whether any Heegaard splitting is strongly irreducible. The method by which we show that a Heegaard splitting is strongly irreducible is due to Casson and Gordon (unpublished work, see [7]). Note that their theorem, quoted below, is a theorem about Heegaard splittings of a sequence of manifolds and it only gives us specific information if we know that some manifold in that sequence has a weakly reducible Heegaard splitting. We give a proof, in the appendix, of this theorem based on notes taken during Casson's presentation of the result.

THEOREM 5.2. *Let* $M = \{g_0;\, e_0 \mid (\alpha_1, \beta_1), \ldots, (\alpha_m, \beta_m)\}$ *be a Seifert fibered space with an orientable base space. Let S be a horizontal Heegaard splitting corresponding to a fiber* (α_i, β_i), $1 \leqslant i \leqslant m$, *that is* $\alpha_i = s_i n_0 + u_i$, $\beta_i = t_i n_0 + v_i$. *Then either S is strongly irreducible or there are at most five manifolds* $M = \{g_0;\, e_0 \mid (\alpha'_1, \beta'_1), \ldots, (\alpha'_m, \beta'_m)\}$ *so that* $(\alpha'_j, \beta'_j) = (\alpha_j, \beta_j)$ *for* $1 \leqslant j \leqslant m, j \neq i$, *and* $\alpha'_i = s_i n' + u_i$, $\beta'_i = t_i n' + v_i$, $|n' - n_0| \leqslant 2$ *which have weakly reducible horizontal Heegaard splittings corresponding to the fiber* (α'_i, β'_i).

Proof. Let $M = H_1 \cup H_2$ where $\partial H_i = \Sigma$ is the Heegaard surface. Let $k \subset \Sigma$ be an essential separating simple closed curve. Let $T: \Sigma \to \Sigma$ be a Dehn twist in k then $M(1/n) = H_1 \cup_{T^n} H_2$ is the manifold obtained by a $1/n$-Dehn surgery on k (as in Section 3). The new Heegaard splitting surface of $M(1/n) = H_1 \cup_{T^n} H_2$ is $\Sigma' = \Sigma$ (with a n-Dehn twist). In [7] the following theorem is proved.

THEOREM A (Casson-Gordon). *Suppose* $M = H_1 \cup H_2$ *is a weakly reducible Heegaard splitting for M and* $\Sigma - N(k)$ *is incompressible in both* H_1 *and* H_2. *Then* $M(1/n) = H_1 \cup_{T^n} H_2$ *is a strongly irreducible Heegaard splitting for* $M(1/n)$, *for* $|n| \geqslant 6$.

Recall that in the case of a horizontal Heegaard splitting the surface $\Sigma - N(f_i)$ is incompressible. Also the boundary of $\Sigma - N(f_i)$ is a (s_i, t_i) curve in terms of the chosen basis (q_i, h) for the homology of $\partial N(f_i)$. Hence, $1/n$-Dehn surgery on M-$N(f_i)$ corresponds to Seifert invariants $\alpha_i = s_i n + u_i$, $\beta_i = t_i n + v_i$. If we assume a weakly reducible Heegaard splitting for $\alpha_i = s_i n_0 + u_i$, $\beta_i = t_i n_0 + v_i$ Theorem 5.2 follows from Theorem A of Casson and Gordon. □

Acknowledgements—The authors wish to thank the Department of Mathematics at the Technion, where part of this work was done, for its generous hospitality. Thanks also to Marty Scharlemann, Bronek Wajnryb and Michah Sageev for helpful discussions. We also wish to thank the referee for helpful comments.

REFERENCES

1. Boileau, M., Collins, D. J. and Zieschang, H., *Genus 2 Heegaard decompositions of small Seifert manifolds.* Preprint.
2. Boileau, M. and Otal, J. P., Sur les scindements de Heegaard du tore T^3. *Journal of Differential Geometry*, 1990, **32**, 209–233.
3. Boileau, M. and Otal, J. P., Scindements de Heegaard et groupe des homeotopies des petites varietes de Seifert. *Inventory Mathematics*, 1991, **106**, 85–107.
4. Boileau, M. and Zieschang, H., Heegaard genus of closed orientable Seifert 3-manifolds. *Inventory Mathematics*, 1984, **76**, 455–468.
5. Bonahon, F. and Otal, J. P., Scindements de Heegaard des espaces lenticulairs. *Annuals Scientifiques del'Ecole Norm Sup.*, 1983, **16**, 451–467.
6. Casson, A. and Gordon, C., Reducing Heegaard splittings. *Topology and its Applications*, 1987, **27**, 275–283.
7. Casson, A. and Gordon, C., Manifolds with irreducible Heegaard splittings of arbitrary large genus. Unpublished.
8. Eisenbud, D. and Neumann, W., Three dimensional link theory and invariants of plane curve singularities. *Annuals of Mathematics Studies*, **100**, 1985, Princeton University Press, Princeton.
9. Gabai, D., Foliations and the topology of 3-manifolds III. *Journal of Differential Geometry*, 1987, **26**, 479–536.
10. Hempel, J., 3-Manifolds. *Annals of Mathematics Studies*, **86**, 1976, Princeton University Press, Princeton.
11. Jaco, W., Lectures on three-manifold topology. CBMS Regional Conference Series in Mathematics **43**, 1980, American, Mathematics Society.
12. Lustig, M. and Moriah, Y., Nielsen equivalence in Fuchsian groups and Seifert fibered spaces. *Topology*, 1991, **30**, 191–204.
13. Milnor, J., Lectures on the *h*-cobordism theorem. Notes by L. Siebenmann and J. Sondow, Princeton Mathematics Notes, Princeton University Press, 1965.
14. Moriah, Y., Heegaard splittings of Seifert fibered spaces. *Inventory Mathematics* 1998, **91**, 465–481.
15. Moriah, Y. and Rubinstein, H., Heegaard structure of negatively curved 3-manifolds. *Com. in Geom and Ann.*
16. Orlik, P., Seifert manifolds. *Lecture Notes in Mathematics*, **291**, Springer, Berlin, 1972.
17. Schultens, J., The classification of Heegaard splittings for (closed orientable surfaces) $\times S^1$. *Proceedings of the London Mathematical Society*, 1993, **67**(15), 401–487.
18. Schultens, J., Weakly reducible Heegaard splittings of Seifert fibered spaces. Preprint.
19. Schultens, J., The stabilization problem for Heegaard splittings of Seifert fibered spaces. *Topology and its Applications*, 1996, **73**, 133–139.
20. Schultens, J., The Heegaard splittings of Seifert fibered spaces with boundary. *Transactions of the American Mathematical Society*, to appear.
21. Scharlemann, M. and Thompson, A., Heegaard splittings of (surface) $\times I$ are standard. *Mathematische Annalen*, 1993, **295**, 549–564.
22. Scott, P., The eight geometries of 3-manifolds. *Bulletin of London Mathematical Society*, 1993, **67**(3), 425–448.
23. Seifert, Topologie dreidimensionaler gefaserter Raume. *Acta Mathematica*, 1993, **60**, 147–238. Translation by W. Heil, memo notes Florida State University, 1976.
24. Waldhausen, F., Heegaard-Zerlegungen der 3-sphare. *Topology*, 1968, **7**, 195–203.
25. Lustig, M., Nielsen equivalence and simple homotopy type. *Proc. London Math. Soc.* 1991, **62**, 537–562.

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APPENDIX

Here we prove the theorem due to Casson and Gordon that we used in Section 5. Casson and Gordon used this theorem to establish the irreducibility of Heegaard splitting of arbitrarily high genus of manifolds obtained by surgery on certain pretzel knots. The proof given here, due to Casson, is not the original proof. We would like to thank Martin Lustig for his remarks concerning Definition A.3.

Let M be a closed orientable 3-manifold and let $M = H_1 \cup H_2$ be a Heegaard splitting for M with $\Sigma = \partial H_i$ as the splitting surface. Fix some hyperbolic metric on Σ once and for all. Let $K \subset \Sigma$ be an essential separating simple closed curve and let $T: \Sigma \to \Sigma$ be a Dehn twist in K. Denote the manifold obtained by a $1/n$-Dehn surgery on K by $M(1/n)$ (as in Section 3). Then Σ defines a Heegaard splittng surface for $M(1/n)$ which we denote by Σ^n.

THEOREM A (Casson-Gordon [7]). *Suppose $M = H_1 \cup H_2$ is a weakly reducible Heegaard splitting for the closed manifold M. Let K be a simple closed curve in Σ such that $\Sigma - N(K)$ is incompressible in both H_1 and H_2. Then Σ^n, for all $|n| \geqslant 6$, is a strongly irreducible Heegaard splitting for $M(1/n)$.*

Definition A1. A basis $\mathbb{B}$ for a genus g handlebody H is a collection of g simple closed curves $B_1, \ldots, B_g$ in $\partial H = \Sigma$ bounding disks $D_1, \ldots, D_g$ such that H-int$(N(\cup(D_i))$ is a 3-ball.

Definition A2. If $\mathbb{B}$ is a basis for a handlebody H, a wave for $\mathbb{B}$ is an arc $\omega \subset \Sigma$ such that int$(\omega) \cap \mathbb{B} = \emptyset$, the two points of $\partial\omega$ lie in the same component B of $\mathbb{B}$ and ω approaches B from the same side. We furthermore require that $(\omega, \partial\omega)$ is not homotopic, in Σ, into a component of $\mathbb{B}$ (see Fig. A1).

Note that a wave together with a subarc of the disk bounded by B bound a disk in the handlebody.

LEMMA A1. *Let C be a simple closed curve in Σ bounding a disk D in H that is not parallel to an element of $\mathbb{B}$. Assume that the intersection of $\mathbb{B}$ and C is minimal (for instance by taking the components of $\mathbb{B}$ and C to be geodesics) and non empty. Then C contains a wave for $\mathbb{B}$.*

Proof. Consider the intersection between D and the D_i's. We can eliminate simple closed curves in the intersection by an innermost disk argument (since handlebodies are irreducible). Hence, we may assume that the intersection is a collection of arcs. Let α be an outermost arc in D and $\beta \subset \partial D$ be an arc cut off by α for which int$(\beta) \cap B = \emptyset$. Then $\alpha \cup \beta$ bound a sub-disk of D. Since H is orientable β is on one side of B, the component of $\mathbb{B}$ containing $\partial\alpha$. Hence, β is a wave for $\mathbb{B}$. □

LEMMA A2. *Let $K \subset \Sigma$ be a simple closed curve so that $\Sigma - N(K)$ is incompressible in H. Let $\mathbb{B}$ be a basis for H chosen so that the intersection $|K \cap \mathbb{B}|$ is minimal. Then every wave for $\mathbb{B}$ must intersect K essentially (i.e., the wave cannot be homotoped to reduce its intersection with K).*

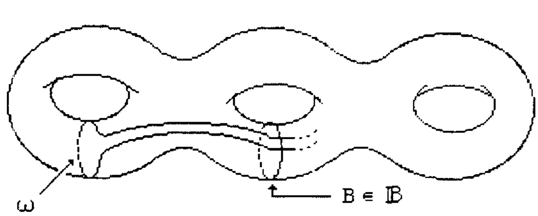


Fig. A1.

Proof. Suppose ω is a wave for $\mathbb{B}$. As $\Sigma - N(K)$ is incompressible in H, K must intersect every component $B \in \mathbb{B}$, in particular, the component B for which $\partial\omega \in B$. The two points in $\partial\omega$ separate B into two arcs which we denote by α and β. Since ω is not homotopic, in Σ, to a subarc of B both simple closed curves $\alpha\cup\omega$ and $\beta\cup\omega$ are essential in Σ. Let D_1, D_2 be the disks bounded by $\alpha\cup\omega$, $\beta\cup\omega$ (respectively) and let D_ω be the disk bounded by ω and a subarc of D. Since $\Sigma - N(K)$ is incompressible in H, K intersects both $\alpha\cup\omega$ and $\beta\cup\omega$. Now, cut H along $\mathbb{B} - B$ to obtain a solid torus V. Note that D is a meridian disk for V. If neither D_1 nor D_2 is a meridian disk for V then neither is D. Hence, we may assume that D_1, say, is a meridian disk for V. We may now replace B by $\alpha\cup\omega$ to obtain a new basis $\mathbb{B}'$ for H. If $\omega\cap K = \emptyset$ then $|K\cap\mathbb{B}'| < |K\cap\mathbb{B}|$ contradicting our assumption on minimality. □

LEMMA A3. *Suppose C is a geodesic simple closed curve in Σ bounding a disk D in H. Then, perhaps after isotopy, there is a lift $\tilde{C}$ of C in the universal cover $\mathbb{H}^2$ of Σ meeting lift $\tilde{K}$ of K so that if $\tilde{B}$ is a lift of any component B for which $\tilde{B}\cap\tilde{K} \neq \emptyset$ then $\tilde{C}\cap\tilde{B} = \emptyset$.*

Proof. If C is parallel to a component B of $\mathbb{B}$ we are done, so suppose that $C\cap B \neq \emptyset$ for some B in $\mathbb{B}$. Now assume to the contrary that every lift $\tilde{C}$ of C which meets a lift $\tilde{K}$ of K intersects some lift $\tilde{B}$ of B which also intersects $\tilde{K}$. By Lemma A1, the curve C contains a wave ω for some component B. Let p* be a lift of a point p in $\omega\cap$B. The lift $\tilde{\omega}$ of ω emanating from p^* is contained in $\tilde{C}$ between two lifts $\tilde{B}$ and $\tilde{B}'$ of B. On $\tilde{\omega}$ there are points of intersection with copies of $\tilde{K}$, since K inersects ω. These copies of $\tilde{K}$ must intersect either $\tilde{B}$ or $\tilde{B}'$ otherwise the geometry of $\mathbb{H}^2$ would not allow $\tilde{K}$ to intersect any other lifts of (disjoint) components of $\mathbb{B}$ which intersect $\tilde{C}$, contrary to our assumption. Thus every copy of $\tilde{K}$ meeting $\tilde{\omega}$ must intersect a copy of $\tilde{B}$ (see Fig. A2).

Consider an innermost triangle T between $\tilde{C}$, $\tilde{K}$ and $\tilde{B}$. The covering projection must map the triangle T injectively into the surface Σ since neither the arc between p^* and $\tilde{K}\cap\tilde{B}$ nor the arc between p^* and $\tilde{K}\cap\tilde{C}$ project to a closed loop. We now use the projection of the triangle T to isotope K off the wave. The assumption above a ensures that a triangle T always exists and we can repeat the process until the intersection of ω and K is empty, contradicting Lemma A2. □

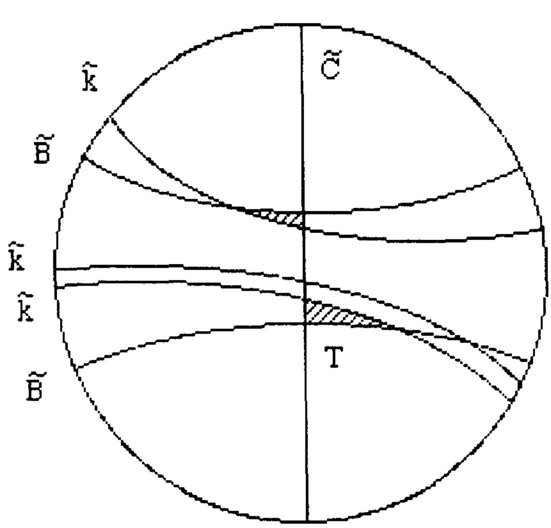


Fig. A2.

Definition A3. Consider $\mathbb{H}^2$, the universal cover of Σ. Let $\tilde{K}$ be a lift of K and let $\tilde{C}$ be a lift of a simple closed curve C which intersects $\tilde{K}$. We may assume that $\tilde{K} \cap \tilde{C} = 0 \in \mathbb{H}^2$. Draw perpendiculars from $\tilde{C} \cap S^1_\infty$ onto $\tilde{K}$ and let p_1 and p_2 be the points where the perpendiculars meet $\tilde{K}$. Define $\pi(\tilde{C}) = d(p_1, p_2)$ if the angle between $\tilde{K}$ and $\tilde{C}$ in the direction in which the Dehn twist is to take place is bigger than $\pi/2$ and $\pi(\tilde{C}) = -d(p_1, p_2)$ if it is less than $\pi/2$ (see Fig. A3. Where the angle α is acute as the Dehn twist is always to the right and hence $\pi(\tilde{C}) < 0$). Denote the length of K on Σ by k.

LEMMA A4. *If C, C' are disjoint (or coincident) geodesics and $\tilde{C}$, $\tilde{C}'$ are lifts of C, C' to $\mathbb{H}^2$ both meeting a lift $\tilde{K}$ of K, then:*

$$|\pi(\tilde{C}) - \pi(\tilde{C}')| \leqslant k.$$

Proof. Let $m = \pi(\tilde{C})$ and $1 = \pi(\tilde{C})$. Let $p = \tilde{K} \cap \tilde{C}$ and let $p' = \tilde{K} \cap \tilde{C}'$. Since the length of K is k we may rechoose $\tilde{C}'$ so that $d = d(p, p') \leqslant k/2$ (this choice does not affect $|\pi(\tilde{C}) - \pi(\tilde{C}')|$). Apply an isometry to $\mathbb{H}^2$ translating along $\tilde{K}$ so that the intersection point x of the perpendicular from the end point of $\tilde{C}$ to $\tilde{K}$ farthest away from p' is mapped to 0 (as in Fig. A4). The interval $(0, p + m/2)$ is the projection of $\tilde{C}$ onto $\tilde{K}$ and the interval $(p' - 1/2, p' + 1/2)$ is the porjection of $\tilde{C}'$ onto $\tilde{K}$. As $\tilde{C}$ and $\tilde{C}'$ are disjoint, we may assume that $\tilde{C}'$ is "above" $\tilde{C}$ (as in Fig. A4). Hence $p' > p$, $p' - 1/2 \geqslant p - m/2 = 0$ and $p' + 1/2 \geqslant p + m/2$. Since the geodesics are distinct we cannot have two equalities at the same time. In the case where $\pi(\tilde{C}') > 0$ we have:

(a) $p' - p \geqslant 1/2 - m/2$ and
(b) $p' - p \geqslant m/2 - 1/2$ and thus,

$$|\pi(\tilde{C}) - \pi(\tilde{C}')| = |m - 1| \leqslant 2(p' - p) \leqslant k.$$

If both $\pi(\tilde{C})$ and $\pi(\tilde{C}')$ are less then zero the argument is similar. If $\pi(\tilde{C}') < 0$ and $\pi(\tilde{C}) > 0$ as the angle between the $\tilde{C}'$ and $\tilde{K}$ is acute and the geodesics $\tilde{C}$, $\tilde{C}'$ are disjoint we must have $k/2 > m/2 + 1/2$ (see Fig. A4(c)) and hence,

$$|\pi(\tilde{C}) - \pi(\tilde{C}')| = |m + 1| \leqslant k. \qquad \square$$

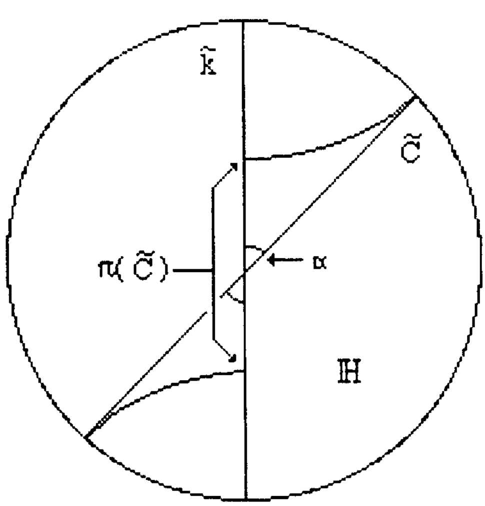
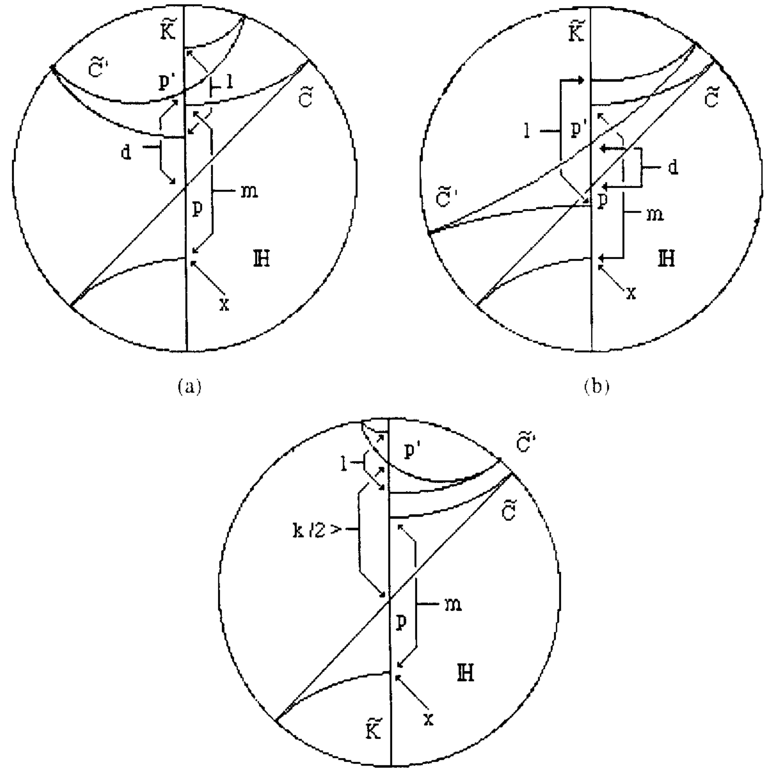


Fig. A3.



(a) (b)

(c)

Fig. A4.

LEMMA A5. *If C is a geodesic bounding a disk in a handlebody H, then there is a lift $\tilde{C}$ of C meeting $\tilde{K}$ so that $|\pi(\tilde{C}) - \pi(\tilde{B})| = |m-1| \leqslant 2(p'-p) \leqslant k$ for all lifts $\tilde{B}$ of components B of $\mathbb{B}$ which intersect $\tilde{K}$.*

Proof. The claim follows from Leemas A3 and A4. Lemma A3 ensures that there is a lift $\tilde{C}$ of C which is disjoint from all lifts of components of $\mathbb{B}$ which intersect $\tilde{K}$. Now Lemma A4 establishes the result. □

LEMMA 6. *Let $M = H_1 \cup H_2$ be a Heegaard splitting for M and let $\mathbb{B}, \mathbb{B}'$ be basis systems for the handlebodies H_1, H_2, respectively. Let K and k be as above. If $M = H_1 \cup H_2$ is a weakly reducible Heegaard splitting then*

$$|\pi(\tilde{B}) - \pi(\tilde{B}')| = \leqslant 3k.$$

Proof. Since $M = H_1 \cup H_2$ is a weakly reducible Heegaard splitting there disjoint distinct curves C on H_1 and C' on H_2 both bounding disks $D \subset H_1$ and $D' \subset H_2$, respectively. By Lemmas A4 and A5 we have

$$|\pi(\tilde{B}) - \pi(\tilde{B}')| \leqslant |\pi(\tilde{B}) - \pi(\tilde{C})| + |\pi(\tilde{C}) - \pi(\tilde{C}')| + |\pi(\tilde{C}') - \pi(\tilde{B}')| \leqslant 3k. \quad \square$$

Consider the two systems of basis curves $\mathbb{B}, \mathbb{B}'$ on the surface $\Sigma = \partial H_1 = \partial H_2$. Let $T: \Sigma \to \Sigma$ be a Dehn twist in K. Composing the gluing map of the two handlebodies with T changes the way the two handlebodies are glued to each other. This amounts to changing one of the basis systems, say $\mathbb{B}'$, by the map T. Consider the system $T(\mathbb{B}')$ and choose a system of geodesics representing the components of $T(\mathbb{B}')$. Denote the new system by $\mathbb{B}^*$ and their lifts to the universal cover by $\tilde{B}^*$.

LEMMA A7. *If* $\mathbb{B}$, $\mathbb{B}'$, B^* *and* T *are as above and* $\tilde{B}^*$, $\tilde{B}'$ *are lifts of any component of* B^* *and* $\mathbb{B}$; *respectively, then*:

$$|\pi(\tilde{B}^*) - \pi(\tilde{B}')| > k.$$

Proof. Let $N(K)$ be a neighborhood of K in Σ in which the Dehn twist takes place. We can lift $N(K)$ to the universal cover $\mathbb{H}^2$ of Σ to obtain a collection of strips $\tilde{\mathbb{N}}(K)$. The map T lifts to a homeomorphism $\tilde{T}$ of $\mathbb{H}^2$. Consider the collection A of arcs consisting of all the lifts of $\tilde{B}'$ minus their intersection with $\tilde{N}(K)$. The effect of $\tilde{T}$ on a copy of $\tilde{B}'$ in $\tilde{B}' - \tilde{\mathbb{N}}(K)$ is to shift all but one of the arcs of $\tilde{B}' - \tilde{\mathbb{N}}(K)$ to other arcs in A; inside $\tilde{\mathbb{N}}(K)$, $\tilde{T}(\tilde{B}')$ make a right turn every time it intersects $\tilde{K}$ traveling along $\tilde{K}$ until it reaches the next intersection point with $\tilde{B}'$ (see Fig. A5).

Now, consider the geodesic $\tilde{B}^*$ in $\mathbb{H}^2$ with the same end points as $T(\tilde{B}')$ and the image $\tilde{B}'_{\#}$ of $\tilde{B}'$ under an isometry which is a translation of length k along $\tilde{K}$. We may assume that $\tilde{K}$ and $\tilde{B}$ intersect in the point 0 of $\mathbb{H}^2$. Let s be the distance between 0 and the outermost intersection point of a perpendicular from an end point of $\tilde{B}'_{\#}$ to $\tilde{K}$ (see Fig. A6). We refer to the relevant end point as the point "above" $\tilde{K}$ (as indicated in Fig. A6).

Note that as a Dehn twist is always to the right, the end points of $T(\tilde{B}')$ must be between the end points of the $\tilde{B}'_{\#}$ and the end points of $\tilde{K}$ so that $\pi(\tilde{B}^*) > s + \pi(\tilde{B}')/2$, for s as in Fig. A6, if $\pi(\tilde{B}') > 0$ and $\pi(\tilde{B}^*) > s - \pi(\tilde{B}')/2$ if $\pi(\tilde{B}') < 0$. If $\pi(\tilde{B}') < 0$ it follows immediately that

$$|\pi(\tilde{B}^*) - \pi(\tilde{B}')| > k.$$

If $\pi(\tilde{B}') > 0$ denote the distance between the intersection point of perpendicular to $\tilde{K}$ from $\tilde{B}'$ and 0 by $r = \pi(\tilde{B}')/2$. The triangles Δ_1, Δ_2, in Fig. A6, are isometric hence $|r - s|$ is equal to the distance between 0 and $\tilde{K} \cap \tilde{B}'_{\#}$ which is k by choice of $\tilde{B}'_{\#}$. Hence, if

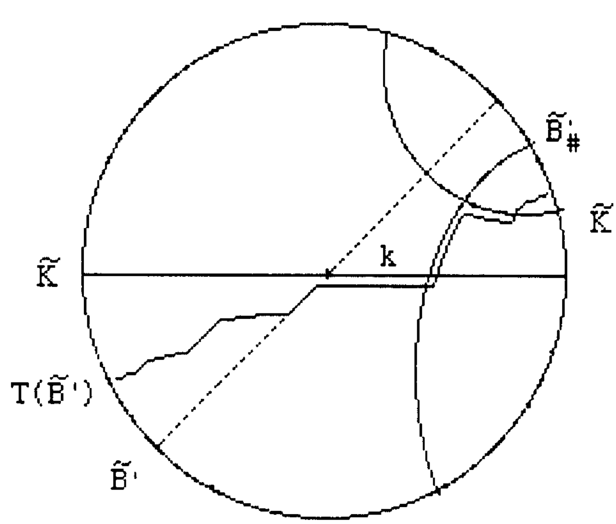


Fig. A5.

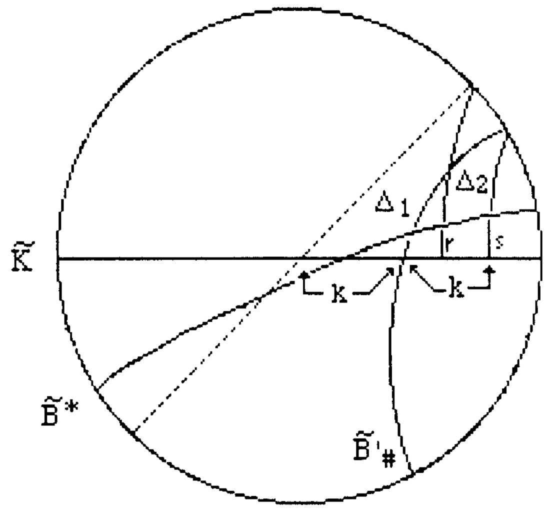


Fig. A6.

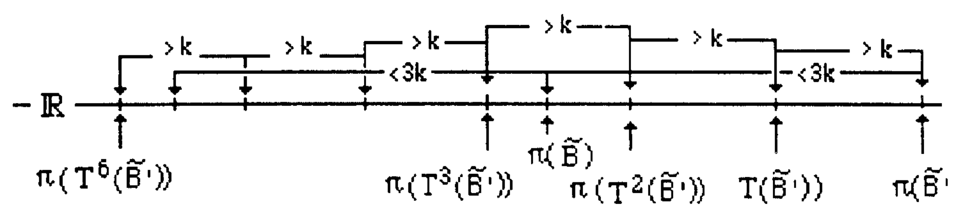


Fig. A7.

$\pi(\tilde{B}') > 0$ it follows that

$$|\pi(\tilde{B}^*) - \pi(\tilde{B}')| > |s + r - 2r| = k.$$

Proof (*Theorem A*). Since (H_1, H_2) is a weakly reducible Heegaard splitting for M we have two disjoint curves C and C' on the splitting surface $\Sigma \subset M$ bounding disks in H_1, H_2, respectively. Hence, by Leema A6 there are two basis systems $\mathbb{B}$ and $\mathbb{B}'$ on H_1, H_2, respectively, minimizing the intersection with K and components $B \in \mathbb{B}$ and $B' \in \mathbb{B}'$ so that

$$(*) \qquad |\pi(\tilde{B}) - \pi(\tilde{B}')| = \leqslant 3k.$$

In $M(1/n)$ the system $\tilde{B}'$ would be changed to a system $T^n(\tilde{B}')$. If $M(1/n) = H_1 \cup_{T^n} H_2$ is a weakly reducible Heegaard splitting for $M(1/n)$ then (using a different set of disjoint curves C and C' on $\Sigma^n \subset M(1/n)$) we get

$$(*) \qquad |\pi(\tilde{B}) - \pi(T^n(\tilde{B}'))| = \leqslant 3k.$$

However, generalizing the argument in Lemma A7 we have

$$|\pi(T^n(\tilde{B}')) - \pi(T(\tilde{B}'))| = > nk.$$

So

$$|\pi(\tilde{B}) - \pi(T^n(\tilde{B}'))| = |\pi(\tilde{B}) - \pi(\tilde{B}') + \pi(\tilde{B}') - \pi(T^n(\tilde{B}'))|$$

$$\geqslant |\pi(\tilde{B}') - \pi(T^n(\tilde{B}'))| - |\pi(\tilde{B}) - \pi(\tilde{B}')| > nk - 3k.$$

Hence, $(*)$ is violated for $|n| \geqslant 6$ (see Fig. A7). Hence, $M(1/n)$ is strongly irreducible for $|n| \geqslant 6$. □