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STABILITY INEQUALITIES AND UNIVERSAL SCHUBERT CALCULUS OF RANK 2

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Abstract. The goal of the paper is to introduce a version of Schubert calculus for each dihedral reflection group W. That is, to each "sufficiently rich" spherical building Y of type W we associate a certain cohomology theory $H^*_{BK}(Y)$ and verify that, first, it depends only on W (i.e., all such buildings are "homotopy equivalent"), and second, $H^*_{BK}(Y)$ is the associated graded of the coinvariant algebra of W under certain filtration. We also construct the dual homology "pre-ring" on Y. The convex "stability" cones in $(\mathbb{R}^2)^m$ defined via these (co)homology theories of Y are then shown to solve the problem of classifying weighted semistable m-tuples on Y in the sense of [KLM1]; equivalently, they are cut out by the *generalized triangle inequalities* for thick Euclidean buildings with the Tits boundary Y. The independence of the (co)homology theory of Y refines the result of [KLM2], which asserted that the Stability Cone depends on W rather than on Y. Quite remarkably, the cohomology ring $H^*_{BK}(Y)$ is obtained from a certain universal algebra A_t by a kind of "crystal limit" that has been previously introduced by Belkale–Kumar for the cohomology of flag varieties and Grassmannians. Another degeneration of A_t leads to the homology theory $H_*(Y)$.

1. Introduction

Alexander Klyachko in [K] solved the old problem on eigenvalues of sums of hermitian matrices. His solution was to interpret the eigenvalue problem as an existence problem for certain parabolically stable bundles over $\mathbb{CP}^1$, so that the inequalities on the eigenvalues are stated in terms of the Schubert calculus on Grassmannians. Klyachko's work was later generalized by various authors to cover general semisimple groups; see, e.g., [BS], [KLM1]. The stable bundles were replaced in [KLM1] with *semistable weighted configurations* on certain spherical buildings and the eigenvalue problem was interpreted as a *triangle inequalities problem* for the vector-valued distance function on nonpositively curved symmetric spaces and Euclidean buildings. Still, the solution depended heavily on (and was formulated in terms of) Schubert calculus on generalized Grassmannians G/P, where G is a (complex or real) semisimple Lie group and P's are maximal parabolic subgroups of G.

The present work is a part of our attempt to generalize Lie theory to the case

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of nonexistent Lie groups having noncrystallographic dihedral groups $W = I_2(n)$ (of order $2n$) as their Weyl groups. For such Weyl groups, one cannot define G and P, but there are spherical (Tits) buildings Y, whose vertex sets serve as generalized Grassmannians G/P. Moreover, we also have thick discrete and nondiscrete Euclidean buildings for the groups $I_2(n)$ (see [BK]), so both problems of existence of semistable weighted configurations and of computation of triangle inequalities for vector-valued distance functions on Euclidean buildings certainly make sense. The goal of this paper is to compute these inequalities (by analogy with [BS], [KLM1]) in terms of the Borel model for $H^*(G/P)$ and to verify that they solve the problem of existence of semistable weighted configurations and the equivalent problem of computation of triangle inequalities in the associated affine buildings.

Our main results can be summarized as follows.

Let $\mathfrak{Y}$ be a rank 2 affine building with the Weyl group $W = I_2(n)$ and let Δ denote the positive Weyl chamber for W. We then obtain a Δ-valued distance function $d_\Delta(x, y)$ between points $x, y \in \mathfrak{Y}$; see [KLM1] or [KLM3]. Then

Theorem 1. *There exists a geodesic m-gon $x_1 \cdots x_m$ in $\mathfrak{Y}$ with the Δ-side-lengths $\lambda_1, \dots, \lambda_m$ if and only if the vectors $\lambda_1, \dots, \lambda_m$ satisfy the* Weak Triangle Inequalities (*the* stability inequalities):

$$w(\lambda_i - \lambda_j^*) \leq_{\Delta^*} \sum_{k \neq i, k \neq j} \lambda_k^*, \qquad w \in W, \tag{1}$$

taken over all distinct $i, j \in \{1, \dots, m\}$.

Here $\lambda^* = -w_\circ(\lambda)$ is the vector contragredient to λ ($w_\circ \in W$ is the longest element). The order $\leq_{\Delta^*}$ is defined with respect to the obtuse cone Δ^* dual to Δ:

$$\Delta^* = \{\nu \mid \nu \cdot \lambda \geq 0, \ \forall \lambda \in \Delta\}.$$

(Recall that $\lambda \leq_{\Delta^*} \nu \iff \nu - \lambda \in \Delta^*$.)

The key idea behind the proof is that although we do not have smooth homogeneous manifolds G/P, we still can define some kind of Schubert calculus on the sets of "points" Y_1 and "lines" Y_2 in appropriately chosen Tits buildings Y (replacing G/P's). We define certain "homology pre-rings" $H_*(Y_l, \widehat{\mathbf{k}})$, $l = 1, 2$ ("Schubert precalculus") which reflect the intersection properties of "Schubert cycles" in Y_l. We then show that this calculus is robust enough to solve the existence problem for weighted semistable configurations.

We next promote the cohomology pre-rings to rings. To this end, we introduce the *universal Schubert calculus*, i.e., we define a *cohomology ring* $H^*(Y, \mathbf{k}) = A_t$ for each reflection group of rank 2, based on a generalization of the Borel model for the computation of cohomology rings of flag varieties. One novelty here is that in the definition of A_t we allow $t \in \mathbb{C}^\times$, thereby providing an interpolation between cohomology rings of complex flag manifolds; for t a primitive nth root of unity, A_t defines $H^*(Y, \mathbf{k})$, the cohomology rings of the buildings Y with the Weyl group $W = I_2(n)$. We therefore think of the family of rings A_t as the "universal Schubert calculus" in rank 2. An odd feature of the rings A_t is that even for the values of t which are roots of unity, the structure constants of A_t are typically irrational (t-binomials), so we do not have a natural geometric model for these A_t. In order to link A_t to geometry, we define a (trivial) deformation $A_{t,\tau}, \tau \in \mathbb{R}_+$, of A_t. Sending τ to 0 we obtain an analogue of the Belkale–Kumar degeneration $H^*_{BK}(Y, \mathbf{k}) = \mathrm{gr}(A_t)$ of A_t. On the other hand, by sending τ to ∞, we recover the pre-ring $H_*(Y, \widehat{\mathbf{k}})$ given by the Schubert precalculus. Therefore, A_t interpolates between $H^*_{BK}(Y, \mathbf{k})$ and $H_*(Y, \widehat{\mathbf{k}})$. The same relation holds for the cohomology rings of "Grassmannians," $B_t^{(l)} = H^*(Y_l, \mathbf{k}) \subset A_t$, their Belkale–Kumar degenerations $H^*_{BK}(Y_l, \mathbf{k}) = \mathrm{gr}(B_t^{(l)})$ and pre-rings $H_*(Y_l, \widehat{\mathbf{k}})$. We then observe (Section 16) that the system of strong triangle inequalities defined by $H_*(Y, \widehat{\mathbf{k}})$ also determines the Stability Cone $\mathcal{K}_m(Y)$ for the building Y. In Section 17 we introduce systems of linear inequalities determined by certain based rings A, generalizing A_t. Specializing these inequalities to the case $A = A_t$, using the results of Section 16, we recover the Stability Cones $\mathcal{K}_m(Y)$. Therefore, the systems of inequalities defined by $A_t, B_t^{(l)}, \mathrm{gr}(A_t), \mathrm{gr}(B_t^{(l)})$ and $H_*(X)$ are all equivalent. In this section, we also prove that the system of Weak Triangle Inequalities, determined by $H^*_{BK}(Y_l, \mathbf{k})$, equivalently, $H_*(Y_l, \widehat{\mathbf{k}})$, $(l = 1, 2)$ is irredundant. This is reminiscent of the result by Ressayre who proved irredundancy of the Beklale–Kumar inequalities in the context of complex reductive groups.

After this paper was submitted, we received a preprint [C] by Carlos Ramos Cuevas, in which he gives an alternative proof of Theorem 1. His proof does not rely upon development of Schubert precalculus, but rather on direct geometric arguments.

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2. Coxeter complexes

Let A, the *apartment*, be either the Euclidean space $E = E^N$ or the unit $N-1$-sphere $S = S^{N-1} \subset E^N$ (we will be primarily interested in the case of the Euclidean plane and the circle). If $A = S$, a *Coxeter group* acting on A is a finite group W generated by isometric reflections. If $A = E$, a *Coxeter group* acting on A is a group W_{af} generated by isometric reflections in hyperplanes in A, so that the linear part of W_{af} is a Coxeter group acting on S. Thus, $W_{\mathrm{af}} = \Lambda \rtimes W$, where Λ is a certain (countable or uncountable) group of translations in E. We will use the notation $\mathbf{1}$ for the identity in W and $w_\circ$ for the longest element of W with respect to the word-length function $\ell : W \to \mathbb{Z}$ with respect to the standard Coxeter generators s_i.

Definition 1. A spherical or Euclidean[1] *Coxeter complex* is a pair (A, G), of the

[1] Also called *affine*.

form (S, W) or (E, W_{af}). The number N is called the *rank* of the Coxeter complex.

A *wall* in the Coxeter complex (A, G) is the fixed-point set of a reflection in G. A *half-apartment* in A is a closed half-space bounded by a wall. A *regular point* in a Coxeter complex is a point which does not belong to any wall. A *singular point* is a point which is not regular.

Remark 1. Note that in the spherical case, there is a natural cell complex in S associated with W. However, in the affine case, when W_{af} is nondiscrete, there will be no natural cell complex attached to W_{af}.

Chambers in (S, W) are the fundamental domains for the action $W \curvearrowright S$, i.e., the closures of the connected components of the complement to the union of walls. We will use the notation Δ_{sph} for a fixed (positive) fundamental domain.

An *affine* Weyl chamber in (A, W_{af}) is a fundamental domain $\Delta = \Delta_{\mathrm{af}}$ for a conjugate W' of W in W_{af}, i.e., it is a cone over Δ_{sph} with the tip at a point o fixed by W'.

A *vertex* in (A, G) is a (component of, in the spherical case) 0-dimensional intersection of walls. We will consider almost exclusively only *essential* Coxeter complexes, i.e., complexes that have at least one vertex. Equivalently, these are spherical complexes where the group G does not have a global fixed point and those Euclidean Coxeter complexes where W does not have a fixed point in S.

In the spherical case, the notion of *type* is given by the projection

$$\theta : S \to S/W = \Delta_{\mathrm{sph}},$$

where the quotient is the spherical Weyl chamber.

Let $s_i \in W$ be one of the Coxeter generators. We define the *relative length functions* ℓ_i on W as follows: $\ell_i(w)$ is the length of the shortest element of the coset $w\langle s_i\rangle \subset W/\langle s_i\rangle$. In the case when W is a finite dihedral group, $\ell_i(w)$ equals the combinatorial distance from the vertex $w(\zeta_i)$ to the positive chamber Δ_{sph} in the spherical Coxeter complex (S^1, W). Here, ζ_i is the vertex of Δ_{sph} fixed by s_i.

3. Metric concepts

Notation 2. *Let Y, Z be subsets in a metric space X. Define the (lower) distance $d(Y, Z)$ as*

$$\inf_{y \in Y, z \in Z} d(y, z).$$

If Z is a singleton $\{z\}$, we abbreviate $d(\{z\}, Y)$ to $d(z, Y)$. In the examples we are interested in, the above infimum is always realized.

For a subset $Y \subset X$, we let $B_r(Y)$ denote the closed r-neighborhood *of Y in X, i.e.,*

$$B_r(Y) := \{x \in X \mid d(x, Y) \leq r\}.$$

For instance, if $Y = \{y\}$ is a single point, then $B_r(Y) = B_r(y)$ is the closed r-ball centered at y. Similarly, we define "spheres centered at Y"

$$S_r(Y) := \{x \in X \mid d(x, Y) = r\}.$$

A metric space X is called *geodesic* if every two points in X are connected by a (globally distance-minimizing) geodesic. Most metric spaces considered in this paper will be geodesic. Occasionally, we will have to deal with metrics on disconnected graphs. In this case we declare the distance between points in distinct connected components to be infinite.

For a pair of points x, y in a metric space X we let $\overline{xy}$ denote a closed geodesic segment (if it exists) in X connecting x and y. Because, most of the time, we will deal with spaces where every pair of points is connected by the unique geodesic, this is a reasonable notation.

We refer to [B] or [BH] for the definition of a CAT(k) metric space. We will think of the distances in CAT(1) spaces as *angles* and, in many cases, denote these distances $\angle(xy)$.

The following characterization of 1-dimensional CAT(1) spaces will be important:

A 1-dimensional metric space (a metric graph) is a CAT(1) space if and only if the length of the shortest embedded circle in X is $\geq 2\pi$.

If G is a metric graph, where each edge is given the length π/n, then the CAT(1) condition is equivalent to the assumption that the girth of G is $\geq 2n$.

Fix an integer $n \geq 2$. Similarly to [BK], a type-preserving map of bipartite graphs $f : G \to G'$ is said to be $(n-1)$-isometric if:

1. $\forall x, y \in V(G), d(x, y) < n-1 \Rightarrow d(f(x), f(y)) = d(x, y)$.
2. $\forall x, y \in X, d(x, y) \geq n-1 \Rightarrow d(f(x), f(y)) \geq n-1$.

Here d is the combinatorial path-metric on G, which is allowed to take infinite values on points which belong to distinct connected components. One can easily verify that the concept of an $(n-1)$-isometric map is equivalent to the notion of a type-preserving map graphs which preserves the *bounded distance* on the graphs defined in [Te].

4. Buildings

Spaces modeled on Coxeter complexes

Let (A, G) be a Coxeter complex (Euclidean or spherical).

Definition 2. A space *modeled on the Coxeter complex* (A, G) is a metric space X together with an atlas where charts are isometric embeddings $A \to X$ and the transition maps are restrictions of the elements of G. The maps $A \to X$ and their images are called *apartments* in X. Note that (unlike in the definition of an atlas in a manifold) we do not require the apartments to be open in X.

Therefore, all G-invariant notions defined in A extend to X. In particular, we will talk about vertices, walls, chambers, etc.

Notation 3. *We will use the notation Δ_i for chambers in spherical buildings.*

The *rank* of X is the rank of the corresponding Coxeter complex.

A space X modeled on (A, G) is called *discrete* if the group G is discrete. This is automatic in the case of spherical Coxeter complexes since G is finite in this case.

A *spherical building* modeled on (S, W) is a CAT(1) space Y modeled on (S, W) which satisfies the following condition:

Axiom ("Connectedness"). Every two points $y_1, y_2 \in Y$ are contained in a common apartment.

The group W is called the *Weyl group* of the spherical building Y.

Spherical buildings of rank 2 (with the Weyl group of order ≥ 4) are called *generalized polygons.* They can be described combinatorially as follows:

A building Y is a bipartite graph of girth $2n$ and valence ≥ 2 at every vertex, so that every two vertices are connected by a path of the combinatorial length $\leq n$. To define a metric on Y, we identify each edge of the graph with the segment of length π/n.

A *Euclidean* (*or affine*) *building* modeled on (A, W_{af}) is a CAT(0) space X modeled on (A, W_{af}) which satisfies the following conditions:

Axiom 1 ("Connectedness"). Every two points $x_1, x_2 \in X$ belong to a common apartment.

Axiom 2. There is an extra axiom (comparing to the spherical buildings) of "Angle rigidity", which will be irrelevant for the purposes of this paper. It says that for every $x \in X$, the space of directions $Y = \Sigma_x(X)$ satisfies the following:

$$\forall \xi, \eta \in Y, \quad \angle(\xi, \eta) \in W \cdot \angle(\theta(\xi), \theta(\eta)).$$

Here $\theta : Y \to \Delta_{\mathrm{sph}}$ is the *type projection.* We refer to [BK], [KL], [P] for the details. Note that Axiom 2 is redundant in the case of discrete Euclidean buildings.

The finite Coxeter group W (the linear part of W_{af}) is called the *Weyl group* of the Euclidean building X.

A building X is called *thick* if every wall in X is the intersection of (at least) three apartments.

We now specialize our discussion of buildings to the case of rank 2 (equivalently, 1-dimensional) spherical buildings.

Chambers Δ_1, Δ_2 in a spherical building Y are called *antipodal* if the following holds. Let $A \subset Y$ be an apartment containing both Δ_1, Δ_2 (it exists by the Connectedness Axiom). Then $\Delta_1 = -\Delta_2$ inside A. More generally, if Y is a bipartite graph of diameter n, then two edges e_1, e_2 of Y are called *antipodal* if the minimal distance between vertices of these edges is exactly $n - 1$.

Let $W = I_2(n)$ be the dihedral group of order $2n$. We regard the *type* of a vertex x (denoted type(x)) of a bipartite graph (in particular, of a spherical building with the Weyl group W) to be an element of $\mathbb{Z}/2$. We let $W_l, l = 1, 2$ denote the stabilizer of the vertex of type l in the positive (spherical) chamber Δ_+ of W.

Let Y be a rank 2 spherical building with the Weyl group W; we will use two metrics on Y:

1. The combinatorial path-metric $d = d_Y$ between the vertices of Y, where each edge is given the unit length. This metric extends naturally to the rest of Y: we will occasionally use this fact.

2. The (angular) path metric $\angle$ on Y where every edge has the length π/n.

Given a subset $Z \subset Y$ we let $B_r(Z)$ and $S_r(Z)$ denote the closed r-ball and r-sphere in Y with respect to the combinatorial metric.

The building Y has two vertex types identified with $l \in \mathbb{Z}/2$; accordingly, the vertex set of Y is the disjoint union $Y_1 \cup Y_2$ of the *Grassmannians* of type $l = 1, 2$. When l is fixed, by abusing the notation, we will denote by $B_r(Z)$ (and $S_r(Z)$) the intersection of the corresponding ball (or the sphere) with Y_l. We will only use these concepts when Z is a vertex or a chamber Δ of a spherical building. The balls $B_r(\Delta) \subset Y_l$ will serve as *Schubert cycles* in the Grassmannian Y_l, while the spheres $S_r(\Delta)$ will play the role of (open) Schubert cells.

5. Weighted configurations and geodesic polygons

Weighted configurations

Let Y be a spherical building modeled on (S, W). We recall that $\angle$ denotes the angular metric on Y. Given a collection $\mu_1, \ldots, \mu_m$ of nonnegative real numbers ("weights") we define a *weighted configuration* on Y as a map

$$\psi : \{1, \ldots, m\} \to Y, \quad \psi(i) = \xi_i \in Y.$$

We thus get m points $\xi_i, i = 1, \ldots, m$ on Y assigned the weights $\mu_i, i = 1, \ldots, m$. We will use the notation

$$\psi = (\mu_1\xi_1, \ldots, \mu_m\xi_m).$$

Let $\theta : Y \to \Delta_{\mathrm{sph}}$ denote the *type-projection* to the spherical Weyl chamber. Given a weighted configuration $\psi = (\mu_1\xi_1, \ldots, \mu_m\xi_m)$ on Y, we define $\theta(\psi)$, the *type of* ψ, to be the m-tuple of vectors

$$(\lambda_1, \ldots, \lambda_m) \in \Delta^m,$$

where $\lambda_i = \mu_i\theta(\xi_i)$.

Following [KLM1], for a finite weighted configuration $\psi = (\mu_1\xi_1, \ldots, \mu_m\xi_m)$ on Y, we define the function

$$\mathrm{slope}_\psi : Y \to \mathbb{R}, \qquad \mathrm{slope}_\psi(\eta) = -\sum_{i=1}^{m} \mu_i \cos(\angle(\eta, \xi_i)).$$

Definition 3. A weighted configuration ψ is called *semistable* if the associated slope function is ≥ 0 on Y.

It is shown in [KLM1] that slope_ψ coincides with the Mumford's numerical stability function for weighted configurations on generalized flag-varieties. What is important is the fact that the above notion of stability, unlike the stability conditions in algebraic and symplectic geometry, does not require a group action on a smooth manifold. (Actually, it does not need any group at all.)

Vector-valued distance functions

Let X be a Euclidean building modeled on (A, W_{af}). We next define the Δ-valued distance function d_Δ on X, following [KLM1], [KLM2]. (The reader should not confuse d_Δ with the W_{af}-valued distance on X that could be used in order to axiomatize buildings; see [W].) Here $\Delta = \Delta_{\mathrm{af}}$ is an (affine) Weyl chamber of (A, W_{af}). We first define the function d_Δ on A. Let $o \in A$ denote the point fixed by W. We regard o as the origin in the affine space A, thus giving A the structure of a vector space V. Given two points $x, y \in A$, we consider the vector $v = \overrightarrow{xy} = y - x$ and project it to a vector $\bar{v} \in \Delta$ via the map

$$V \to V/W = \Delta.$$

Then $d_\Delta(x, y) := \bar{v}$. It is clear from the construction that W_{af} preserves d_Δ. Suppose now that $x, y \in X$. By the Connectedness Axiom, there exists an apartment $\phi : A \to X$ whose image contains x and y. We then set

$$d_\Delta(x, y) := d_\Delta(\phi^{-1}(x), \phi^{-1}(y)) \in \Delta.$$

Since the transition maps between the charts are in W_{af}, it follows that the distance function d_Δ on X is well-defined. Note that d_Δ is, in general, nonsymmetric:

$$d_\Delta(x, y) = \lambda \iff d_\Delta(y, x) = \lambda^*, \quad \lambda^* = -w_\circ(\lambda), \tag{2}$$

where $w_\circ \in W$ is the longest element. Hence, unless $w_\circ = -1$, $d_\Delta(x, y) \neq d_\Delta(y, x)$.

A (*closed*) *geodesic m-gon* on X is an m-tuple of points $x_1, \ldots, x_m$, the *vertices* of the polygon. Since (by the CAT(0) property of X) for every two points $x, y \in X$ there exists a unique geodesic segment $\overline{xy}$ connecting x to y, the choice of vertices uniquely determines a closed 1-cycle in X, called a *geodesic polygon.* We will use the notation $x_1 \cdots x_m$ for this polygon. The Δ-*side-lengths* of this polygon are the vectors $\lambda_i = d_\Delta(x_i, x_{i+1})$, where i is taken modulo m.

The following is proven in [KLM2]:

Theorem 4. *Let Y be a thick spherical building modeled on (S, W) and X be a thick Euclidean building modeled on $(A, W_{\mathrm{af}} = \Lambda \rtimes W)$, for an arbitrary Λ. Then:*

There exists a weighted semistable configuration ψ of type $(\lambda_1, \ldots, \lambda_m)$ on Y if and only if there exists a closed geodesic m-gon $x_1 \cdots x_m$ in X with the Δ-side-lengths $(\lambda_1, \ldots, \lambda_m)$.

In particular, the existence of a semistable configuration (or a geodesic polygon) depends only on W and nothing else. The way it will be used in our paper is to construct special spherical buildings modeled on $(S^1, I_2(n))$ (buildings satisfying Axiom A), to which certain "transversality arguments" from [KLM1] apply.

Definition 4. Given a thick spherical building X with the Weyl group W, we let $\mathcal{K}_m(X)$ denote the set of vectors $\overrightarrow{\lambda} = (\lambda_1, \ldots, \lambda_m)$ in Δ^m, so that X contains a semistable weighted configuration of the type $\overrightarrow{\lambda}$. We will refer to $\mathcal{K}_m(X)$ as the *Stability Cone* of X. (These cones are also known as *Eigenvalue Cones* in the context of Lie groups and Lie algebras.) When W is fixed, we will frequently abbreviate $\mathcal{K}_m(X)$ to $\mathcal{K}_m$ since this cone depends only on the dihedral group W.

Note that the conicality of X is clear since a positive multiple of a semistable weighted configuration is again semistable. What is not obvious is that $\mathcal{K}_m(X)$ is a convex polyhedral cone. We will see in Section 12 (as a combination of the results of this paper and [KLM1]) that this is indeed always the case.

6. *m*-pods

Fix an integer $n \geq 2$. Let $r_1, \ldots, r_m$ be positive integers such that

$$r_i + r_j \geq n, \quad \forall i \neq j.$$

Given this data, we define an m-pod T as follows.

Let B denote the bipartite graph which is the disjoint union of the edges Δ_1, …, Δ_m. These edges will be the *bases* of the m-pod T. Add to B the vertex z of type $l \in \{1,2\}$, the *center* of T. Now, connect z to the appropriate vertices $x_i \in \Delta_i$ by the paths p_i of the combinatorial lengths r_i, so that

$$r_i \equiv \text{type}(x_i) + \text{type}(z) \pmod 2, \quad i = 1, \ldots, m.$$

(The above equation uniquely determines each x_i.) The resulting graph is the m-pod T. The paths p_i are the *legs* of T. It is easy to define the type of the vertices of T (extending those of $x_1, \ldots, x_m, z$), so that T is a bipartite graph.

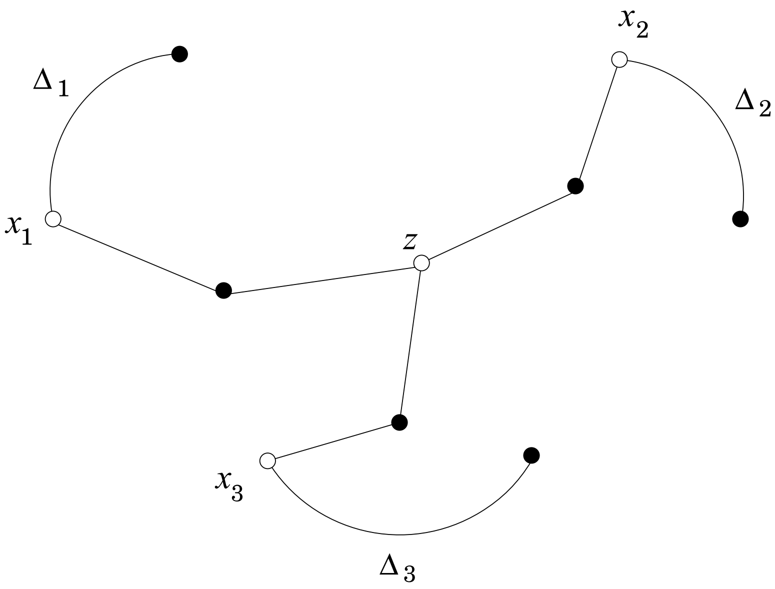

FIGURE 1. A 3-pod, $n = 3$ or $n = 4$

Suppose now that Y is a bipartite graph of girth $\geq 2n$, $\Delta_1, \ldots, \Delta_m \subset Y$ are mutually antipodal edges and $r_1, \ldots, r_m$ are positive integers so that

$$r_i + r_j \geq n, \quad \forall i \neq j.$$

We define a new bipartite graph Y' by attaching the m-pod T with legs of the lengths r_i, $i = 1, \ldots, m$, and with the bases $\Delta_1, \ldots, \Delta_m$:

$$Y' := Y \cup_B T, \qquad B = \Delta_1 \cup \cdots \cup \Delta_m,$$

where the attaching map identifies the bases of T with the edges $\Delta_i \subset Y$ preserving the type.

Lemma 5. *The graph Y' still has girth $\geq 2n$.*

Proof. Since $r_i + r_j \geq n$ for $i \neq j$ and the edges $\Delta_i \subset Y$ are mutually antipodal, the only thing we need to avoid is having $i \neq j$ so that $r_i + r_j = n$ and $d_Y(x_i, x_j) = n-1$. Suppose such i, j exist. Then

$$\mathrm{type}(x_i) + \mathrm{type}(x_j) \equiv r_i + r_j = n \pmod 2$$

and

$$\mathrm{type}(x_i) + \mathrm{type}(x_j) \equiv d_Y(x_i, x_j) = n - 1 \pmod 2.$$

Contradiction. $\square$

7. Buildings and free constructions

We define a class of rank 2 spherical buildings X with the Weyl group $W = I_2(n)$ satisfying:

Axiom A.

1. Each vertex of X has infinite valence; in particular, X is thick.
2. For each $m \geq 3$ the following holds. Let $\Delta_i, i = 1, \ldots, m$ be pairwise antipodal chambers in X and let $0 < r_i \leq n - 1$, $i = 1, \ldots, m$, be integers so that

$$r_i + r_j \geq n, \quad \forall i \neq j.$$

Then there exist infinitely many vertices $\eta \in X$ of both types, so that

$$d(\eta, \Delta_i) \leq r_i.$$

In other words, the intersection of metric spheres

$$I = \bigcap_i S_{r_i}(\Delta_i)$$

contains infinitely many vertices of both types.

Remark 2.

1. For the purposes of the proof of Theorem 19, it suffices to have property (2) for a fixed infinite collection $\Delta_1, \Delta_2, \ldots$ of pairwise antipodal chambers. Moreover, it suffices to assume that the intersection I contains at least 2 vertices of each type (rather than an infinite number). However, for the purposes of developing "Schubert precalculus", it is important to have Axiom A as stated above.

2. Clearly, Axiom A fails for finite buildings. However, it also fails for some infinite buildings. For instance, it fails for the Tits buildings associated with the complex algebraic groups $\mathrm{Sp}(4, \mathbb{C})$ and $\mathrm{G}_2(\mathbb{C})$.

Buildings satisfying Axiom A constitute the class of "sufficiently rich" buildings mentioned in the Introduction: For these buildings we will develop "Schubert precalculus" later in the paper.

Lemma 6. *Let X be a thick rank 2 spherical building satisfying Axiom A. Let $\Delta_1, \ldots, \Delta_m$ be pairwise antipodal chambers in X. Then there exists a chamber Δ_{m+1} antipodal to all chambers $\Delta_1, \ldots, \Delta_m$.*

Proof. Let $r_i := n - 1$. Then, by Axiom A, there exists a vertex $x \in X$ so that

$$d(x, \Delta_i) = r_i, \quad i = 1, \ldots, m.$$

For each i we let $x_i \in \Delta_i$ be the vertex realizing $d(x, \Delta_i)$. Since X has infinite valence at x, there exists a vertex $y \in X$ incident to x, which does not belong to any of the geodesics $\overline{xx_i}, i = 1, \ldots, m$. It is then clear that $d(y, \Delta_i) = n$, $i = 1, \ldots, m$. Therefore, the chamber $\Delta_{m+1} := \overline{xy}$ is antipodal to all chambers $\Delta_1, \ldots, \Delta_m$. $\square$

Remark 3. It is not hard to prove that if X is a thick building with Weyl group $I_2(n)$, then the conclusion of the above lemma holds for $m = 2$ without any extra assumptions.

We now prove the existence of thick buildings satisfying Axiom A.

Theorem 7. *For each n there exists a thick spherical building X with Weyl group $W \cong I_2(n)$, with countably many vertices and satisfying Axiom A. Moreover, every (countable) graph of girth $\geq 2n$ embeds in a (countable) building satisfying Axiom A.*

Proof. We first recall the *free construction* of rank 2 spherical buildings (see [Ti], [Ro], [FS]):

Let Z be a connected bipartite graph of girth $\geq 2n$. Given every pair of vertices $z, z' \in Z$ of distance $n + 1$ from each other, we add to Z an edge-path p of the combinatorial length $n - 1$ connecting z and z'; similarly, for every pair of vertices in Z of distance n from each other, we add an edge-path q of the combinatorial length n connecting z and z'. Let $\overline{Z}$ denote the graph obtained by attaching paths p and q to Z in this manner. The notion of *type* extends to the vertices of the paths p and q so that the new graph $\overline{Z}$ is again bipartite. One easily sees that the bipartite graph $\overline{Z}$ again has girth $\geq 2n$ and that each vertex has valence ≥ 2. The *free construction* based on a connected graph Z_0 of girth $\geq 2n$ consists in the inductive application of the bar-operation: $Z_{i+1} := \overline{Z_i}$. Then the direct limit of the resulting graphs is a thick building. We modify the above procedure by supplementing it with the operation $Z \hookrightarrow Z'$ described below.

Let Z be a bipartite graph of girth $\geq 2n$. We define a new graph Z' as follows. For every vertex-type $l = 1, 2$, every $m \geq 3$, every m-tuple of mutually antipodal edges Δ_i in Z and integers $0 < r_i < n - 1$, $i = 1, \dots, m$, satisfying

$$r_i + r_j \geq n, \quad \forall i \neq j,$$

we attach to Z an m-pod T with the bases $\Delta_1, \dots, \Delta_m$, center of the type l and the legs of the lengths $r_1, \dots, r_m$, respectively. Denote the graph obtained from Z by attaching all these m-pods by Z'. Then Z' is a bipartite graph. Applying Lemma 5 repeatedly, we see that Z' still has girth $\geq 2n$.

We now proceed with the inductive construction of the building Y. We start with X_0, which is an arbitrary connected bipartite graph of girth $\geq 2n$.

Then set $X_1 := X_0'$ (by attaching m-pods for all m to all m-tuples of pairwise antipodal chambers). Take $X_2 := \overline{X_1}$ (i.e., it is obtained from X_1 as in the free construction) and continue this 2-step process inductively: for every even $N = 2k$ set $X_{N+1} := X_N'$ and $X_{N+2} := \overline{X_{N+1}}$.

Let Y denote the increasing union of the resulting graphs. Then, clearly, Y is a connected infinite bipartite graph.

Lemma 8. *Y is a thick building modeled on W, satisfying Axiom A.*

Proof. 1. Clearly, Y has girth $\geq 2n$. Note that for each N, the natural inclusion $X_N \to X_{N+1}$ is 1-Lipschitz (distance-decreasing). Moreover, by the construction, the maps $X_N \to X_{N+1}$ are $n-1$-isometric in the sense of Section 3. By the construction, if $x, y \in X_N$ (N is odd) are vertices within distance $d \geq n+1$, then

$$d_{X_N}(x, y) > d_{X_{N+1}}(x, y)$$

(as there will be a pair of vertices u, v within distance $n+1$ on the geodesic $\overline{xy} \subset X_n$, the distance $d(u, v)$ in X_{N+1} becomes $n-1$). Thus,

$$d_{X_{N+2s}}(x, y) \leq n, \qquad \text{where} \quad s = d - (n+1).$$

Therefore, Y has the diameter n. For every vertex $y \in Y$ there exists a vertex $y' \in Y$ which has the (combinatorial) distance n from y. Therefore, attaching the q-paths in the bar-operation assures that there are infinitely many half-apartments in Y connecting y to y'. In particular, there are infinitely many apartments containing y and y'. This implies that Y is a thick (with each vertex having infinite valence) spherical building with the Weyl group $W = I_2(n)$.

2. In order to check Axiom A, let $\Delta_1, \dots, \Delta_m \subset Y$ be antipodal chambers and $r_1, \dots, r_m$ be positive integers so that

$$r_i + r_j \geq n, \quad \forall i \neq j.$$

Then there exists k_0 so that $\Delta_1, \dots, \Delta_m \subset X_{k_0}$. Since the maps $X_k \to Y$ are distance-decreasing, it follows that there exists $k_1 \geq k_0$ so that $\Delta_1, \dots, \Delta_m \subset X_{k_1}$

are antipodal. Therefore, by the construction, for every odd step of the induction there will be two (new) m-pods with the legs of the lengths $r_1, \ldots, r_m$ and centers of the type $l = 1, 2$ attached to the bases $\Delta_1, \ldots, \Delta_m$. Therefore, the intersections

$$\bigcap_{i=1}^{m} S_{r_i}(\Delta_i) \subset X_k, \qquad k \geq k_1,$$

will contain at least $(k - k_1)/2$ vertices of both types. Since the maps $\iota : X_N \to X_{N+k}, k \geq 0$ are $(n-1)$-isometric, $\iota(S_r(\Delta)) \subset S_r(\Delta) \subset X_{N+k}$. Therefore, Y satisfies Axiom A. $\square$

This concludes the proof of Theorem 7. $\square$

One can modify the above construction by allowing the transfinite induction, but we will not need this. More interestingly, one can modify the construction of Y to obtain a rank 2 spherical building X which satisfies the following *universality property* (with n fixed):

Axiom U. Let G be an arbitrary finite connected bipartite graph of girth $\geq 2n$, let $H \subset G$ be a (possibly disconnected) subgraph and $\phi : H \to X$ be a morphism (a distance-decreasing embedding preserving the type of vertices). Then ϕ extends to a morphism $G \to X$.

The Axiom U is somewhat reminiscent of the Kirszbraum's property; see, e.g., [LPS].

Thus, Axiom A is a special case of the Axiom U, defined with respect to a particular class of graphs G (i.e., m-pods), their subgraphs (the sets of vertices x_i of valence 1) and maps ϕ (sending x_i's to vertices of antipodal chambers). With this in mind, the construction of (countably infinite) buildings satisfying Axiom U is identical to the proof of Theorem 7.

8. Highly homogeneous buildings satisfying Axiom A

The goal of this section is to show that the "highly homogeneous" buildings constructed by K. Tent in [Te] satisfy Axiom A. This will give an alternative proof of Theorem 7.

We need several definitions. From now on, fix an integer $n \geq 2$.

Definition 5. Let G be a finite graph with the set of vertices $V(G)$ and the set of edges $E(G)$. Define the *weighted Euler characteristic* of G as

$$y(G) = (n-1)|V(G)| - (n-2)|E(G)|.$$

Define the class of finite graphs $\mathcal{K}$ as the class of bipartite graphs G satisfying the following:

(1) $\mathrm{girth}(G) \geq 2n$.
(2) If G contains a subgraph H which in turn contains an embedded $2k$-cycle, $k > 2n$, then

$$y(H) \geq 2n + 2.$$

We convert $\mathcal{K}$ to a category, also denoted $\mathcal{K}$, by declaring morphisms between graphs in $\mathcal{K}$ to be label-preserving embeddings of bipartite graphs which are $(n-1)$-isometric maps with respect to the combinatorial metrics on graphs.

A bipartite graph U is called a *$\mathcal{K}$-homogeneous universal model* if it satisfies the following:

(1) U is terminal for the category $\mathcal{K}$, i.e., every finite subgraph in U belongs to $\mathcal{K}$ and for every graph $G \in \mathcal{K}$ there exists an $(n-1)$-isometric embedding $G \to U$.

(2) If $G \in \mathcal{K}$ and $\phi, \psi : G \to U$ are $(n-1)$-isometric embeddings, then there exists an automorphism
$$\alpha : U \to U$$
so that $\alpha \circ \phi = \psi$.

The main result of [Te] is

Theorem 9. *The category $\mathcal{K}$ admits a $\mathcal{K}$-homogeneous universal model X.*

Most of the proof of the above theorem deals with establishing that the category $\mathcal{K}$ satisfies the following *amalgamation (or pull-back) property*:

Every diagram

$$\begin{tikzcd} & X_1 \\ X_0 \arrow[ur] \arrow[dr] & \\ & X_2 \end{tikzcd}$$

extends to a commutative diagram

$$\begin{tikzcd} & X_1 \arrow[dr] & \\ X_0 \arrow[ur] \arrow[rr] \arrow[dr] & & X_3 \\ & X_2 \arrow[ur] & \end{tikzcd} .$$

The universal graph X as in Theorem 9 is then shown in [Te] to be a rank 2 thick spherical building with the Weyl group $W = I_2(n)$, such that the automorphism group $Aut(X)$ of X acts transitively on the set of apartments in X, so that the stabilizer of every apartment is infinite and contains W. Moreover, $Aut(X)$ also acts transitively on the set of simple $2(n+1)$-cycles in X.

Proposition 10. *The universal graph X as above satisfies Axiom A.*

Proof. Let $r_1, \ldots, r_m$ be positive integers so that

$$r_i + r_j \geq n, \forall i \neq j.$$

Let T be an m-pod with the bases $\Delta_1, \ldots, \Delta_m$ and the legs of the length $r_1, \ldots, r_m$. Since T contains no embedded cycles, it is an object in $\mathcal{K}$. In particular, the graph

B which is the disjoint union of the bases $\Delta_1, \dots, \Delta_m$, is also an object in $\mathcal{K}$. Since the chambers Δ_i are super-antipodal in T (i.e., Δ_i, Δ_j are at the distance $\geq n-1$ for all $i \neq j$), it follows that the embedding $B \to T$ is a morphism in $\mathcal{K}$. Then, by repeatedly using the amalgamation property, we can amalgamate N copies of T along B to obtain a graph G_N which is again an object in $\mathcal{K}$.

Let $\Delta_1, \dots, \Delta_m$ be a collection of antipodal chambers in X. Then the identity embedding $\psi : \Delta_1 \cup \cdots \cup \Delta_m \to X$ is also a morphism in $\mathcal{K}$.

We claim that

$$\bigcap_{i=1}^{m} B_{r_i}(\Delta_i)$$

contains infinitely many vertices of each type $l = 1, 2$.

Indeed, the disjoint union of the chambers Δ_i determines a bipartite graph $B \subset X$. Form an m-pod T with the union of bases B, legs of the lengths $r_1, \dots, r_m$ and the center z of the type l. Let G_N be the graph obtained by amalgamating N copies of T as above along the bases. Clearly,

$$\bigcap_{i=1}^{m} B_{r_i}(\Delta_i) \subset G_N$$

contains N vertices of the type l, the centers of the m-pods T. Since $G_N \in \mathcal{K}$, and X is terminal with respect to $\mathcal{K}$, it follows that there exists an $(n-1)$-isometric embedding $\phi : G_N \to X$. Because ϕ is distance-decreasing, the intersection

$$\bigcap_{i=1}^{m} B_{r_i}(\phi(\Delta_i)) \subset X$$

also contains N vertices of the type l.

We thus obtain two morphisms $\phi, \psi : B \to X$, where ψ is the identity embedding. By the property 2 of a $\mathcal{K}$-*homogeneous universal model*, there exists an automorphism $\alpha : X \to X$ so that $\alpha \circ \phi = \psi$. Therefore,

$$\bigcap_{i=1}^{m} B_{r_i}(\Delta_i) \subset X$$

contains at least N vertices of the type l, namely, the images of the centers of the m-pods $T \subset G_N$ under $\alpha \circ \phi$. Since N was chosen arbitrarily, the proposition follows. $\square$

9. Intersections of balls in buildings satisfying Axiom A

In this section we prove several basic facts about cardinalities of intersections of balls in buildings satisfying Axiom A.

Lemma 11. *Suppose that X is a thick spherical building with the Weyl group $W = I_2(n)$. Let $r_1 + r_2 = n - 1$ and Δ_1, Δ_2 be nonantipodal chambers (i.e., they are within distance $\leq n - 2$). Then $B_{r_1}(\Delta_1) \cap B_{r_2}(\Delta_2)$ contains vertices of both types.*

Proof. Let $A \subset X$ denote an apartment containing Δ_1, Δ_2. It suffices to consider the case when the distance between the chambers is exactly $n-2$ (as the chambers get closer the intersection only increases). We will assume that $r_1 > 0, r_2 > 0$ and will leave the remaining cases to the reader. Then A will contain unique vertices x, y (of distinct type) so that

$$d(x,\Delta_1) = r_1, \quad d(x,\Delta_2) = r_2 - 1, \quad d(y,\Delta_1) = r_1 - 1, \quad d(y,\Delta_2) = r_2.$$

Thus, $x, y \in B_{r_1}(\Delta_1) \cap B_{r_2}(\Delta_2)$. (Note that if $d(\Delta_1, \Delta_2) = n-2$ then $\{x, y\} = B_{r_1}(\Delta_1) \cap B_{r_2}(\Delta_2)$.) $\square$

Lemma 12. *For every thick spherical building X with the Weyl group $W = I_2(n)$, and every pair of antipodal chambers $\Delta_1, \Delta_2 \subset X$, and nonnegative integers r_1, r_2 satisfying $r_1 + r_2 = n-1$, the intersection*

$$B_{r_1}(\Delta_1) \cap B_{r_2}(\Delta_2)$$

consists of exactly two vertices, one of each type.

Proof. Let $A \subset X$ be an apartment containing Δ_1, Δ_2. It is clear that the intersection

$$B_{r_1}(\Delta_1) \cap B_{r_2}(\Delta_2) \cap A$$

consists of exactly two vertices u, v, one of each type. Let $\alpha \subset A$ denote the subarc of length $n-1$ connecting vertices x_i of the chambers $\Delta_i, i = 1, 2$, so that $u \in \alpha$. Suppose there is a vertex $z \in X \setminus A$, $\mathrm{type}(z) = \mathrm{type}(u)$, so that

$$d(z, \Delta_i) = r_i, i = 1, 2.$$

Then it is clear that $d(x_i, z) = r_i$, $i = 1, 2$ and we thus obtain a path (of length $n-1$)

$$\beta = \overline{x_1 z} \cup \overline{z x_2}$$

connecting x_1 to x_2. Since $d(x_1, x_2) = n-1$, it follows that β is a geodesic path in X. Thus, we have two distinct geodesics $\alpha, \beta \subset X$ of the length $n-1$ connecting x_1, x_2. The union $\alpha \circ \beta$ is a (possibly nonembedded) homologically nontrivial cycle of length $2(n-1)$ in X. This contradicts the fact that X has girth $2n$. $\square$

Corollary 13. *Under the assumptions of Lemma 12, let $\Delta_1, \ldots, \Delta_m$ be antipodal chambers in X. Then*

(1) *If $r_1, \ldots, r_m$ are nonnegative integers so that $r_1 + r_2 = n-1$ and $r_3 = \cdots = r_m = n-1$, then the intersection of balls*

$$I := \bigcap_i B_{r_i}(\Delta_i)$$

consists of exactly two vertices (one of each type).

(2) *If $r_1, \ldots r_m$ are integers so that $\sum_i r_i < (n-1)(m-1), 0 \le r_i \le n-1, i = 1, \ldots, m$ then the above intersection of balls I is empty.*

Proof. The first assertion follows from Lemma 12, since $B_{r_i}(\Delta_i) = X, i \geq 3$. To prove the second assertion we note that there are $i \neq j \in \{1, \ldots, m\}$ so that $r_i + r_j < n - 1$. Therefore, $B_{r_i}(\Delta_i) \cap B_{r_j}(\Delta_j) = \varnothing$ since $d(\Delta_i, \Delta_j) = n - 1$. □

Lemma 14. *Let X be a building with the Weyl group $W = I_2(n)$, satisfying Axiom A. Suppose that $r_i, i = 1, \ldots, m$ are positive integers so that*

$$r_k \leq n - 1, k = 1, \ldots, m, \tag{3}$$

and

$$\sum_i r_i \geq (n-1)(m-1). \tag{4}$$

Then for every m-tuple of antipodal chambers $\Delta_1, \ldots \Delta_m$ in X, one of the following mutually exclusive cases occurs:

(a) *Either the intersection*

$$\bigcap_i B_{r_i}(\Delta_i)$$

contains infinitely many vertices of both types $l = 1, 2$.

(b) *Or (4) is the equality, for two indices, $i \neq j$, $r_i + r_j = n - 1$ and for all $k \notin \{i, j\}$ the inequality (3) is the equality.*

Proof. If $r_i + r_j \geq n$ for all $i \neq j$, the assertion follows from Axiom A (namely, the alternative (a) holds). Suppose that, say, $r_1 + r_2 \leq n - 1$. Then

$$\sum_{i=1}^{m} r_i \leq (n-1) + \sum_{i=3}^{m} r_i \leq (n-1)(m-1).$$

Since $\sum_i r_i \geq (n-1)(m-1)$, we see that $r_1 + r_2 = n - 1$, $r_3 = \cdots = r_m = n - 1$ and $\sum_{i=1}^m r_i = (n-1)(m-1)$. The fact that (a) and (b) cannot occur simultaneously follows from Lemma 12. □

Recall that X_l denotes the set of vertices of type l in X. By combining Lemma 14 and Corollary 13, we obtain

Corollary 15. *Suppose that X satisfies Axiom A. Let $\Delta_1, \ldots, \Delta_m$ be antipodal chambers in X and let $r_1, \ldots, r_m$ be nonnegative integers so that $r_i \leq n - 1$, $i = 1, \ldots, m$. Then the following are equivalent:*

(1) *$\bigcap_i B_{r_i}(\Delta_i) \cap X_l$ is a single point for $l = 1, 2$.*

(2) *After renumbering the indices, $r_1 + r_2 = n - 1$ and $r_3 = \cdots = r_m = n - 1$.*

Moreover, if $\sum_i r_i \geq (n-1)(m-1)$ then $\bigcap_i B_{r_i}(\Delta_i)$ contains vertices of both types.

10. Pre-rings

A *pre-ring* is an algebraic system R with the usual properties of a ring, except that the operations are only partially defined. (By analogy with groupoids, the pre-rings should be called *ringoids*; however, this name is already taken for something else.)

The standard examples of pre-rings which are used in calculus are $\widehat{\mathbb{R}} = \mathbb{R} \cup \pm\infty$ and $\widehat{\mathbb{C}} = \mathbb{C} \cup \infty$. Below is a similar example which we will use in this paper. For a ring R define the pre-ring $\widehat{R} := R \cup \infty$. The algebraic operations in $\widehat{R}$ are extended from the ring R as follows:

1. Addition and multiplication are commutative and associative; 0 and 1 are neutral elements with respect to the addition and multiplication.

2. Moreover, we have

addition	$x \neq \infty$	∞
$y \neq \infty$	$x + y$	∞
∞	∞	undefined

multiplication	0	$x \in R \setminus \{0\}$	∞
0	0	0	0
$y \in R \setminus \{0\}$	0	xy	∞
∞	0	∞	∞

.

Remark 4. It is customary to assume that $0 \cdot \infty$ is undefined, but in the situation we are interested in (where pre-rings will appear as degenerations of rings), we can assume that $0 \cdot \infty = 0$.

11. Schubert precalculus

From now on, we fix a thick spherical building X satisfying Axiom A. (Much of our discussion however, uses only the fact that X is a spherical building with the Weyl group $I_2(n)$.)

Our next goal is to introduce a *Schubert precalculus* in X. According to a theorem of Kramer and Tent [KT], for $n \notin \{2, 3, 4, 6\}$, there are no thick spherical buildings with the Weyl group $I_2(n)$ that admit structure of an algebraic variety defined over an algebraically closed field. Since we are interested in general $n \geq 2$, this forces the algebro-geometric features of the buildings described below to be quite limited.

Let $l \in \{1, 2\}$ be a type of vertices of X. We will think of the set X_l of vertices of type l as the lth "Grassmannian". Let $\Delta \subset X$ be a chamber and $0 \leq r \leq n-1$ be an integer. We define the "Schubert cell" $C_r(\Delta) \subset X_l$ to be the r-sphere $S_r(\Delta)$ in X_l centered at Δ and having radius r:

$$C_r(\Delta) = \{x \in X_l \mid d(x, \Delta) = r\}.$$

(We suppress the dependence on l in the notation for the Schubert cell.) The number r is the "dimension" of the cell. We define the "Schubert cycle" $\overline{C_r(\Delta)}$, the "closure" of the Schubert cell $C_r(\Delta)$, as the closed r-ball centered at Δ:

$$\overline{C_r(\Delta)} := B_r(\Delta) \cap X_l.$$

The number r is the "dimension" of this cycle. Thus, each r-dimensional Schubert cycle is the union of $r+1$ Schubert cells which are the "concentric spheres". By taking $r = n-1$, we see that X_l is a Schubert cycle of dimension $n-1$.

There is much more to be said here, but we defer this discussion to another paper.

Homology

The coefficient system for our homology pre-ring is the pre-ring $\widehat{R}$ defined in Section 10. The simplest case will be when $R = \mathbb{Z}/2$; then $\widehat{R}$ consists of three elements: $0, 1, \infty$. This example will actually suffice for our purposes, but our discussion here is more general. We will suppress the coefficients in the notation for $H_*(X_l, \widehat{R})$ in what follows.

Let $W = I_2(n)$. We declare $d = n - 1$ to be the formal dimension of X_l. Set $r^* := d - r$ for $0 \le r \le d$. Fix a (positive) chamber $\Delta_+ \subset X$.

Using the Schubert precalculus we define the *homology pre-ring* $H_*(X_l)$ ($l = 1, 2$) with coefficients in $\widehat{R}$, by declaring its (additive) generators in each dimension $0 \le r \le n-1$ to be the Schubert classes $[\overline{C_r(\Delta)}]$, where Δ are chambers in X. We declare

$$C_r := [\overline{C_r(\Delta)}] = [\overline{C_r(\Delta_+)}]$$

for every Δ and set

$$H_r(X_l) = 0, \quad \text{for} \ \ r < 0, \quad \text{and} \ \ r > d.$$

The "fundamental class" in $H_d(X_l)$ is represented by $X_l = B_d(\Delta_+)$. We declare a collection of cycles $\overline{C_{r_i}(\Delta_i)}, i \in I$, to be *transversal* if the chambers $\Delta_i, i \in I$ are pairwise antipodal. Using this notion of transversality we define the *intersection product* on $H_*(X_l)$ as follows.

Consider two antipodal chambers Δ_1, Δ_2. For $0 \le r_1, r_2 \le n-1$,

$$\overline{C_{r_1}(\Delta_1)} \cap \overline{C_{r_2}(\Delta_2)} = B_{r_1}(\Delta_1) \cap B_{r_2}(\Delta_2),$$

is the "support set" of the product class

$$[\overline{C_{r_1}(\Delta_1)}] \cdot [\overline{C_{r_2}(\Delta_2)}] \in H_{r_3}(X_l),$$

where $r_3^* = r_1^* + r_2^*$, i.e., $r_3 = r_1 + r_2 - (n-1)$. The product class itself is a multiple $a \cdot [\overline{C_{r_3}(\Delta_+)}]$ of the standard generator. To compute $a \in \widehat{R}$, we declare that the classes $c = C_{r_3}$ and $c^* = C_{r_3^*}$ are "Poincaré dual" to each other:

$$c = PD(c^*),$$

as their dimensions add up to the dimension d of the fundamental class. Therefore, take a chamber Δ_3 antipodal to both Δ_1, Δ_2: it exists by Lemma 6. Then $a \in \widehat{R}$ is the cardinality of the intersection:

$$\overline{C_{r_1}(\Delta_1)} \cap \overline{C_{r_2}(\Delta_2)} \cap \overline{C_{r_3^*}(\Delta_3)}. \tag{5}$$

Remark 5. Here and in what follows we are abusing the terminology and declare the cardinality of an infinite set to be ∞.This is justified, for instance, by the fact that Theorem 7 yields buildings that have countably many vertices and our convention amounts to $\aleph_0 = \infty \in \widehat{R}$.

As we will see below, the cardinality of the intersection is $0, 1$ or ∞; these cardinalities are naturally identified with the elements of $\widehat{R}$.

One can easily check (see below) that a does not depend on the choice of cycles representing the given homology classes. In particular, the fundamental class is the unit in the pre-ring $H_*(X_l)$.

We then compute a using the results of Section 9:

1. If $r_1 + r_2 < n-1$ then $a = 0$ (Corollary 13(2)).

2. If $r_1 + r_2 = n-1$ then $a = 1$: the Schubert cycles $\overline{C_{r_i}(\Delta_i)}, i = 1, 2$, are Poincaré dual to each other (Lemma 12).

3. Suppose now that $r_1 + r_2 > n-1$. We will apply Lemma 14 to the triple intersection (5); observe that $r_1+r_2+r_3^* = 2(n-1)$, i.e., inequality (3) in Lemma 14 is the equality in this case. Then, by Lemma 14:

3a. If $r_1, r_2 < n-1$ then $a = \infty$.

3b. If $r_i = n-1$ for some $i = 1, 2$, then $r_{3-i} + r_3^* = n-1$ and $a = 1$.

Thus, the triple intersection (5) is finite $\iff$ it consists of a single vertex in X_l $\iff$ two of the three classes among $C_{r_1}, C_{r_2}, C_{r_3^*}$ are Poincaré dual to each other and the remaining class is the fundamental class.

Lemma 16. *Let $C_{r_i} \in H_{r_i}(X_l), i = 1, \dots, m$ be the generators (Schubert classes) so that*

$$r_1 + \cdots + r_m = d(m-1) \iff \sum_{i=1}^{m} r_i^* = d,$$

i.e., the product of these classes (in some order) equals $a[pt]$, where $pt = \overline{C_0(\Delta_+)}$. Then $a \in \widehat{R}$ is the cardinality of the intersection

$$\bigcap_{i=1}^{m} B_{r_i}(\Delta_i),$$

where $\Delta_1, \dots, \Delta_m$ are pairwise antipodal chambers (which exist by Lemma 6).

Proof. First of all, without loss of generality we may assume that none of the classes C_{r_i} is the unit $[X_l]$ in $H_*(X_l)$. Note that, since $r_1 + \cdots + r_m = d(m-1)$, in the computation of the product of $C_{r_1}, \dots, C_{r_m}$ we will never encounter the multiplication by zero. Then (after permuting the indices), the product of the classes $C_{r_1}, \dots, C_{r_m}$ will be of the form

$$\cdots (C_{r_1} \cdot C_{r_2}) \cdots .$$

By the definition, $C_{r_1} \cdot C_{r_2} = a_{12} C_r$, where

$$r^* = r_1^* + r_2^*.$$

The element $a_{12} \in \widehat{R}$ is the cardinality of the intersection

$$B_{r_1}(\Delta_1) \cap B_{r_2}(\Delta_2) \cap B_{r^*}(\Delta),$$

where $\Delta_1, \Delta_2, \Delta$ are pairwise antipodal. In view of the above product calculations 1–3, and the fact that $r_1 \neq d, r_2 \neq d$, we see that $a_{12} = \infty$ (since $a_{12} = 0$ is excluded), unless $r = d$, $r_1 = r_2^*$ and, therefore, $c_2 = PD(c_1)$. In the latter case, $B_{r^*}(\Delta) = X_l$ and, hence, a_{12} is the cardinality (equal to 1) of the intersection

$$B_{r_1}(\Delta_1) \cap B_{r_2}(\Delta_2).$$

Since

$$\sum r_i = d(m-1), \quad 0 \le r_i \le d, \quad i = 1, \ldots, m,$$

we conclude that $r_3 = \cdots = r_m = d$. Thus, $m = 2$ and $a = a_{12} = 1$ in this case.

If $a_{12} = \infty$ then it follows from the definition of $\widehat{R}$ that $a = \infty$, since, in the computation of the product of $C_{r_1}, \ldots, C_{r_m}$ we will never multiply by zero. On the other hand, in this case the classes C_{r_1}, C_{r_2} are not Poincaré dual to each other and Lemma 14 implies that the intersection

$$\bigcap_{i=1}^{m} B_{r_i}(\Delta_i) \subset X_l$$

is also infinite. Lemma follows. $\square$

Corollary 17. *$H_*(X_l, \widehat{R})$ is a pre-ring.*

Proof. The only thing which is unclear from the definition is that the product is associative. To verify associativity, we have to show that

$$((C_{r_1} C_{r_2}) C_{r_3}) \cdot C_{r_4} = (C_{r_1}(C_{r_2} C_{r_3})) \cdot C_{r_4} \tag{6}$$

where $C_{r_i} \in H_{r_i}(X_l)$ are the generators and

$$r_1 + r_2 + r_3 + r_4 = (4-1)(n-1).$$

However, the equality (6) immediately follows from the above lemma. $\square$

Similarly to the definition of the Schubert precalculus on the Grassmannians X_l, we define the Schubert precalculus on the "flag-manifold" $\mathrm{Fl}(X)$ associated with X, i.e., the set of edges $E(X)$ of the graph X underlying the building X. The set $E(X)$ will be identified with the set of mid-points of the edges. We have two projections

$$p_l : E(X) \to X_l, \quad l = 1, 2,$$

sending each edge to its end-points. We will think of these projections as "$\mathbb{P}^1$-bundles." Accordingly, we define Schubert cycles in $\mathrm{Fl}(X)$ by pull-back of Schubert cycles in X_l via p_l:

$$\overline{C_{r,l}(\Delta)} := p_l^{-1}\left(\overline{C_{r-1}(\Delta)}\right), \quad r = 1, \ldots, n,$$

while 0-dimensional cycles in $\mathrm{Fl}(X)$ are, of course, just the edges of X. In terms of the metric geometry of X, the cycles $\overline{C_{r+1,l}}(\Delta)$ are described as follows. Fix a chamber Δ. Define the Schubert cell $C_{r,l}(\Delta)$ to be the set of chambers $\Delta' \subset X$ so that the distance between the midpoints $\mathrm{mid}(\Delta), \mathrm{mid}(\Delta')$ of Δ, Δ' equals r and the minimal distance $r-1$ between Δ, Δ' is realized by a vertex of type l in Δ'. Here the convention is that $C_{r,l}(\Delta) = C_{r,l+1}(\Delta)$ for $r=0, r=n=\mathrm{girth}(X)/2$, since for these values of r the minimal distance is realized by vertices of both types. The corresponding Schubert cycles $\overline{C_{r,l}(\Delta)}$ are defined by adding to $C_{r,l}(\Delta)$ all the chambers Δ'' contained in the geodesics connecting $\mathrm{mid}(\Delta), \mathrm{mid}(\Delta')$, for $\Delta' \in C_{r,l}(\Delta)$. The notion of transversality as in the case of X_l, is given by taking antipodal chambers. The Poincaré Duality is defined by

$$PD([\overline{C_{r,l}(\Delta)}]) = [\overline{C_{n-r,3-l}(\Delta)}], \quad l=1,2.$$

The reader will verify that this is consistent with the property that the intersection

$$\overline{C_{r,l}(\Delta_1)} \cap \overline{C_{n-r,3-l}(\Delta_2)}$$

is a single point. We declare that the homology classes $[\overline{C_{r,l}(\Delta)}]$ are independent of Δ and set up the notation

$$C_{r,l} := C_w := [\overline{C_{r,l}(\Delta)}],$$

where $w \in W$ is the unique element such that $w(\Delta) \in C_{r,l}(\Delta)$. Then the Poincaré Duality takes the form

$$PD(C_w) = C_{w_\circ w},$$

where $w_\circ \in W$ is the longest element.

We declare that $C_{r,l}$, $r=0,\dots,n, l=1,2$, form a basis of $H_*(\mathrm{Fl}(X))$, where $r = \dim(C_{r,l})$. We also require the pull-back maps p_l to be pre-ring homomorphisms. It remains to define the intersection products of the form

$$C_{r_1,1} \cdot C_{r_2,2}, \quad 0 \le r_1, r_2 \le n.$$

Analogously to the product in $H_*(X_l)$, we take two antipodal chambers Δ_1, Δ_2 and set

$$C_{r_1,1} \cdot C_{r_2,2} = a_1 C_{r_3,1} + a_2 C_{r_3,2}, \quad a_l \in \widehat{R}, l=1,2,$$

where $r_3 := r_1 + r_2 - n$ (i.e., $(n-r_1)+(n-r_2) = n - r_3$). In order to compute a_l's we take the third chamber Δ_3 antipodal to Δ_1, Δ_2, and let a_l denote the cardinality of the intersection

$$\overline{C_{r_1,1}(\Delta_1)} \cap \overline{C_{r_2,2}(\Delta_2)} \cap \overline{C_{r_3,3-l}(\Delta_3)}.$$

With these definitions,we obtain a homology pre-ring $H_*(\mathrm{Fl}(X), \widehat{R})$ abbreviated to $H_*(X, \widehat{R})$ or even $H_*(X)$. The proof of the following proposition is similar to the case of $H_*(X_l)$ and is left to the reader:

Proposition 18. *Let X be a thick building with the Weyl group $I_2(n)$, satisfying Axiom A. Then $H_*(X)$ is an associative and commutative pre-ring, generated by the elements $C_{r,l}, l=1,2, r=0,\dots,n$, subject to the relations:*

(1) $C_{0,1} = C_{0,2}$ (*the class of a point* [pt]),
$C_{n,1} = C_{n,2} = 1$ *is the unit in* $H_*(X)$ (*the "fundamental class"*);
(2) $C_{r_1,l} \cdot C_{r_2,l} = 0$ *if* $r_1 + r_2 \le n$, $l=1,2$;
(3) $C_{r_1,l} \cdot C_{r_2,l} = \infty$ *if* $n < r_1 + r_2$, $l=1,2$;
(4) $C_{r_1,1} \cdot C_{r_2,2} = 1$ *if* $r_1 + r_2 = n$;
(5) $C_{r_1,1} \cdot C_{r_2,2} = 0$ *if* $r_1 + r_2 < n$;
(6) $C_{r_1,1} \cdot C_{r_2,2} = \infty C_{r_3,1} + \infty C_{r_3,2}$ *if* $r_1 \ne n, r_2 \ne n, r_1 + r_2 > n$, *where* $r_3 = (r_1 + r_2) - n$.

12. The stability inequalities

Suppose that X is a rank 2 thick spherical building with the Weyl group $W \cong I_2(n)$, satisfying Axiom A. We continue with the notation from Section 11. Recall that $\angle$ is a path metric on X so that the length of each chamber is π/n.

We start with few simple observations. Let $C_r(\Delta)$ be a Schubert cell in X_l and $\eta \in C_r(\Delta)$, i.e., $d(\eta, \Delta) = r$. Then the point ζ in Δ nearest to η has the type $l + r$ (mod 2). In particular, ζ depends only on the cell $C_r(\Delta)$ (and not on the choice of η in the cell). Let $\xi \in \Delta$ be a point within $\angle$-distance τ from ζ. Then

$$\angle(\eta, \xi) = r\frac{\pi}{n} + \tau.$$

In particular, this angle is completely determined by the angle τ, by the type of η and the fact that we are dealing with the Schubert cell $C_r(\Delta)$. In particular, it follows that for each $\eta \in \overline{C_{r-1}(\Delta)} = \overline{C_r(\Delta)} \setminus C_r(\Delta)$, we have

$$\angle(\eta, \xi) < r\frac{\pi}{n} + \tau,$$

where ξ is defined as above. We now introduce the following system of inequalities WTI (weak triangle inequalities) on m-tuples of vectors $\overrightarrow{\lambda} = (\lambda_1, \ldots, \lambda_m) = (\mu_1\xi_1, \ldots, \mu_m\xi_m)$, with $\xi_i^0 \in \Delta_+$ and $\mu_i \in \mathbb{R}_+$.

Each Grassmannian X_l (or, equivalently, the choice of a vertex ζ of the standard spherical chamber Δ_+) will contribute a subsystem WTI_l of the triangle inequalities. Consider all possible m-tuples $(w_1, \ldots, w_m)$ of elements of W, so that all but two w_i's are equal to $w_\circ$ (the longest element of W) and the remaining elements w_i, w_j are "Poincaré dual" to each other ($w_i = PD_l(w_j)$), i.e., their relative lengths $r_i = \ell_l(w_i), r_j = \ell_l(w_j)$ in W/W_l satisfy

$$r_i + r_j = n - 1.$$

In other words, the corresponding Schubert cycles

$$C_{r_i} = [\overline{C_{r_i}(\Delta_+)}], \quad C_{r_j} = [\overline{C_{r_j}(\Delta_+)}]$$

in X_l have complementary dimensions and thus are Poincaré dual to each other:

$$C_{r_i} = PD(C_{r_j}).$$

See Section 11. Equivalently, we are considering m-tuples of integers $0 \le r_k \le m - 1$, which, after permutation of indices, have the form

$$(r_1, \ldots, r_m) = (r_1, r_2 = n - 1 - r_1, n - 1, \ldots, n - 1).$$

Note that $\ell_l(w_\circ) = n - 1$, thus $\ell_l(w_i) = r_i, i = 1, \ldots, m$.

Lastly, for every such tuple

$$\overrightarrow{w} = (w_1, \ldots, w_m) = (w_\circ, \ldots, w_\circ, w_i, \ldots, PD(w_i), \ldots, w_\circ)$$

we impose on the vector $\overrightarrow{\lambda}$ the inequality

$$\sum_j \langle \lambda_j, w_j(\zeta) \rangle = \sum_j \mu_j \cdot \cos\angle(\xi_j, w_j(\zeta)) \le 0 \tag{7}$$

denoted $\mathrm{WTI}_{l,\overrightarrow{w}}$. The collection of all these inequalities constitutes the system of inequalities WTI.

Theorem 19. *For any rank 2 thick spherical building X satisfying Axiom A with the Weyl group $I_2(n)$, one has:*

(i) *The Stability Cone $\mathcal{K}_m(X)$ (see Definition 4) is cut out by the inequalities* WTI.

(ii) *Moreover, if $\overrightarrow{\lambda} \in \mathcal{K}_m(X)$, then there exists a semistable weighted configuration $\psi = (\mu_1\xi'_1, \dots, \mu_m\xi'_m)$ on X of the type $\overrightarrow{\lambda}$ so that the points $\xi'_i, i = 1, \dots, m$, belong to mutually antipodal chambers in X.*

Proof. Our proof essentially repeats the one in [KLM1, Theorem 3.33]. We present it here for the sake of completeness.

1 (Existence of a semistable configuration). We begin by taking a collection of chambers $\Delta_1, \dots, \Delta_m \subset X$ in "general position," i.e., they are mutually antipodal. (In [KLM1] one instead takes a generic configuration of Schubert cycles in the generalized Grassmannian, representing the given homology classes.) Then for each $i = 1, \dots, m$ we place the weight μ_i at the point $\xi'_i \in \Delta_i$ that has the same type as ξ_i. We claim that the resulting weighted configuration ψ in X is semistable. Suppose not. Then, according to the "Harder–Narasimhan Lemma" [KLM1, Theorem 3.22], there exists $l \in \{1, 2\}$ so that in the Grassmannian X_l there exists a unique point η with the minimal (negative) slope with respect to ψ:

$$\mathrm{slope}_\psi(\eta) = -\sum_i \mu_i \cos(\angle(\eta, \xi'_i)) < 0,$$

i.e.,

$$\sum_i \mu_i \cos(\angle(\eta, \xi'_i)) > 0.$$

Consider the Schubert cells

$$C_{r_i}(\Delta_i), \quad i = 1, \dots, m,$$

where $r_i = d(\Delta_i, \eta)$ is the (combinatorial) distance between the chamber Δ_i and the vertex $\eta \in X_l$. Thus,

$$\eta \in J = \bigcap_{i=1}^m C_{r_i}(\Delta_i) \subset \bar{J} = \bigcap_{i=1}^m B_{r_i}(\Delta_i) \subset X_l.$$

By the observations in the beginning of this section, the function slope_ψ is constant on J. Since slope_ψ attains a unique minimum on X_l, it follows that $J = \{\eta\}$. Moreover, if

$$\eta' \in \bar{J} \setminus J,$$

then

$$\mathrm{slope}_\psi(\eta') = -\sum_i \mu_i \cos(\angle(\eta', \xi'_i)) < \mathrm{slope}_\psi(\eta),$$

which contradicts the minimality of η. Therefore, the intersection $\bar{J}$ is the single point η. Thus, the product in $H_*(X_l)$ of the Schubert classes $[\overline{C_{r_i}(\Delta_i)}]$, $i =$

$1, \ldots, m$, is $[pt]$ and the latter occurs exactly when (after permuting the indices) the n-tuple $(r_1, \ldots, r_m)$ has the form

$$(r_1, \ldots, r_m) = (r_1, r_2 = r_1^*, n-1, \ldots, n-1),$$

see Corollary 15. Let $\overrightarrow{w} = (w_1, w_2, w_3, \ldots, w_m) = (w_1, w_\circ w_1, w_\circ, \ldots, w_\circ)$ be the corresponding tuple of elements of the Weyl group W. Note that

$$\angle(\xi_k', \eta) = \angle(\xi_k, w_k(\zeta))$$

since $\eta \in C_{r_k}(\Delta_k)$ and $w_k(\zeta) \in C_{r_k}(\Delta_+)$, $k = 1, \ldots, m$. Therefore,

$$0 > \mathrm{slope}_\psi(\eta) = -\sum_i \mu_i \cos(\angle(\eta, \xi_i')) = -\sum_i \mu_i \cos(\angle(\xi_k, w_k(\zeta))).$$

The inequality $\mathrm{WTI}_{l,\overrightarrow{w}}$ however requires that

$$\sum_i \mu_i \cos(\angle(\xi_i, w_i(\zeta))) \leq 0.$$

Contradiction. Therefore, ψ is a semistable configuration.

2. Suppose that $\psi = (\mu_1 \xi_1', \ldots, \mu_m \xi_m')$ is a weighted semistable configuration on X of the type

$$\overrightarrow{\lambda} = (\mu_1 \xi_1, \ldots, \mu_m \xi_m).$$

Consider an m-tuple $\overrightarrow{w} = (w_1, w_2, w_3, \ldots, w_m) = (w_1, w_\circ w_1, w_\circ, \ldots, w_\circ)$ of elements of W as in the definition of the inequalities WTI (after permuting the indices we can assume that the tuple has this form). We will show that $\overrightarrow{\lambda}$ satisfies the inequality $\mathrm{WTI}_{l,\overrightarrow{w}}$ for $l = 1, 2$. Fix l and let $\zeta \in \Delta_+$ denote the vertex of type l. Let $r_1, \ldots, r_m$ be the relative lengths of $w_1, \ldots, w_m$ in W/W_l. Let $\Delta_i \subset X$ denote a chamber containing ξ_i'. Note that $\overline{C_{r_i}(\Delta_i)} = X_l$ for each $i \geq 3$ since $r_i = n-1$. According to Lemmas 11, 12, the intersection

$$\bigcap_{k=1}^m \overline{C_{r_k}(\Delta_k)} = \overline{C_{r_1}(\Delta_1)} \cap \overline{C_{r_2}(\Delta_2)} \subset X_l$$

contains a vertex $\eta \in X_l$ (possibly nonunique since Δ_1, Δ_2, a priori, need not be antipodal). Therefore,

$$d(\eta, \Delta_i) \leq r_i = d(w_i(\zeta), \Delta_+), \quad i = 1, \ldots, m.$$

Accordingly,

$$\angle(\eta, \xi_i') \leq \angle(w_i(\zeta), \xi_i), \quad i = 1, \ldots, m$$

since $\xi_i = \theta(\xi_i') \in \Delta_+$. Therefore,

$$0 \leq \mathrm{slope}_\psi(\eta) = -\sum_i \mu_i \cos(\angle(\eta, \xi_i')) \leq -\sum_i \mu_i \cos(\angle(w_i(\zeta), \xi_i)),$$

and

$$\sum_i \mu_i \cos(\angle(w_i(\zeta), \xi_i)) \leq 0$$

and, thus, $\overrightarrow{\lambda}$ satisfies $\mathrm{WTI}_{l,\overrightarrow{w}}$. $\square$

Corollary 20. *Theorem* 19(i) *holds for* **all** *1-dimensional thick spherical buildings* (*not necessarily satisfying Axiom A*) *with the Weyl group $I_2(n)$.*

Proof. We consider two thick spherical buildings X, X', where X satisfies Axiom A. According to Theorem 4, $\mathcal{K}_m(X) = \mathcal{K}_m(X')$. The corollary follows from Theorem 19 and the existence of buildings satisfying Axiom A. □

We now convert the system of weak triangle inequalities WTI to the form which appears in Theorem 1. For

$$\overrightarrow{w} = (w_1, \ldots, w_m) = (w_1, w_\circ w_1, w_\circ, \ldots, w_\circ),$$

and $\lambda_i = m_i\xi_i, i = 1, \ldots, n$, we set $w := w_1^{-1}$. Then, for $i \geq 3$,

$$\langle \lambda_i, w_i(\zeta)\rangle = \langle w_i^{-1}\lambda_i, \zeta\rangle = \langle w_\circ\lambda_i, \zeta\rangle = -\langle \lambda_i^*, \zeta\rangle,$$

while

$$\langle \lambda_2, w_2(\zeta)\rangle = \langle w_2^{-1}(\lambda_2), \zeta\rangle = -\langle w(\lambda_2^*), \zeta\rangle,$$
$$\langle \lambda_1, w_1(\zeta)\rangle = \langle w(\lambda_1), \zeta\rangle.$$

Therefore, the inequality

$$\sum_j \langle \lambda_j, w_j(\zeta)\rangle \leq 0$$

is equivalent to

$$\langle w(\lambda_1), \zeta\rangle - \langle w(\lambda_2^*), \zeta\rangle \leq \langle \sum_{j=3}^m \lambda_j^*, \zeta\rangle.$$

Since these inequalities hold for both vertices ζ of Δ_+, we obtain

$$w(\lambda_1 - \lambda_2^*) \leq_{\Delta^*} \sum_{j=3}^m \lambda_j^*, \quad w \in W.$$

This proves Theorem 1. □

Corollary 21. *Let X be a thick spherical building. Then the Stability Cone $\mathcal{K}_m(X)$ is a convex polyhedral cone.*

Proof. It suffices to consider the case when X does not have a spherical factor, i.e., its Coxeter complex (S, W) is essential: W has no global fixed points in S. The assertion of the corollary was proven in [KLM1], [KLM2] for all thick spherical buildings X with the *crystallographic* Weyl group W, i.e., W appearing as Weyl groups of complex semisimple Lie groups. If $W = W_1 \times \cdots \times W_k$ is a finite Coxeter group (with W_i Coxeter groups with connected Dynkin diagrams) which is a Weyl group of a thick spherical building X, then each W_i is either crystallographic or is a finite dihedral group $I_2(n)$, see [Ti]. It is immediate from the definition of semistability that

$$\mathcal{K}_m(X) = \mathcal{K}_m(X_1) \times \cdots \times \mathcal{K}_m(X_k),$$

where $X_1, \ldots, X_k$ are *irreducible factors of* X with respect to its joint decomposition into irreducible spherical subbuildings: the X_i's are thick irreducible spherical buildings with essential Coxeter complexes and Weyl groups W_i, $i = 1, \ldots, k$. It therefore follows from the above result of [KLM1, KLM2] and Theorem 19 that each $\mathcal{K}_m(X_i)$, and, hence, $\mathcal{K}_m(X)$, is a convex polyhedral cone. □

13. The universal dihedral cohomology algebra A_t

In this section we construct a family of algebras A_t, $t \in \mathbb{C}^\times$ as a universal deformation of the cohomology ring of the flag variety for each rank 2 complex Kac–Moody group G (including Lie groups $G = \mathrm{SL}_3, Sp_4, G_2$). It turns out that the complexification $\mathbb{C}\otimes A_t$ is isomorphic to the coinvariant algebra of the dihedral group $W = W_t$ acting on $\mathbb{C}^2$ with the parameter t, i.e., $t^2 + t^{-2}$ is the trace of the generator of the maximal normal cyclic subgroup of W.

For each integer $k \geq 0$ define the *t-integer* $[k]_t$ by

$$[k]_t := \frac{t^k - t^{-k}}{t - t^{-1}} = t^{1-k} + t^{3-k} + \cdots + t^{k-3} + t^{k-1}.$$

It is well-known (and easy to see) that for $k, \ell \geq 0$ one has

$$[k]_t[\ell]_t = [|k-\ell|+1]_t + [|k-\ell|+3]_t + \cdots + [|k+\ell|-1]_t. \tag{8}$$

Now define the t-factorials $[m]_t! := [1]_t[2]_t\cdots[\ell]_t$ and the t-binomial coefficients by:

$$\begin{bmatrix} m \\ k \end{bmatrix}_t = \frac{[m]_t!}{[k]_t![m-k]_t!}.$$

Note that, like the usual binomials, t-binomials $\left[\begin{smallmatrix} m \\ k \end{smallmatrix}\right]_t$ extend naturally to $k \in \mathbb{N}$ and $m \in \mathbb{R}_+$, although we will use them only for $k \in \mathbb{N}, m \in \mathbb{Z}$. The t-binomial coefficients satisfy the symmetry

$$\begin{bmatrix} n \\ k \end{bmatrix}_t = \begin{bmatrix} n \\ n-k \end{bmatrix}_t$$

and the Pascal recursion:

$$\begin{bmatrix} m \\ k \end{bmatrix}_t = t^k \begin{bmatrix} m-1 \\ k \end{bmatrix}_t + t^{k-m} \begin{bmatrix} m-1 \\ k-1 \end{bmatrix}_t.$$

Proposition 22. *Each t-binomial coefficient* $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_t$ *belongs to* $\mathbb{Z}[t+t^{-1}]$.

Proof. We need the following result.

Lemma 23. *For all* $k, \ell \geq 0$ *we have*

$$\begin{bmatrix} \ell+k \\ k \end{bmatrix}_t = \sum_{0\leq m\leq k\ell} c_m \cdot [m+1]_t \tag{9}$$

where each $c_m \in \mathbb{Z}_{\geq 0}$.

Proof. Let $V_1 = \mathbb{C}^2$ be the natural $\mathrm{SL}_2(\mathbb{C})$-module. Denote $V_\ell = S^\ell V_1$ so that $\dim V_\ell = \ell+1$. Clearly, each V_ℓ is a simple module. For each $k \geq 0$ let $V_{\ell,k} = S^k V_\ell$; then $\dim V_{\ell,k} = \binom{\ell+k}{k}$. Recall that for each finite-dimensional $\mathrm{SL}_2(\mathbb{C})$-module V the character $\mathrm{ch}(V)$ is a function of $t \in \mathbb{C}^\times$ defined by

$$\mathrm{ch}(V) = \mathrm{Tr}\left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}\Big|_V\right).$$

It is easy to see that $\mathrm{ch}(V_\ell)=[\ell+1]_t$ and $\mathrm{ch}(V_{\ell,k})=\left[{\ell+k \atop k}\right]_t$. Using the decomposition of $V_{\ell,k}$ into simple SL_2-modules

$$V_{\ell,k}=\sum_{0\le m\le k\ell} c_m\cdot V_m$$

where each $c_m\in\mathbb{Z}_{\ge 0}$ and applying $\mathrm{ch}(\cdot)$ to it, we obtain (9). Lemma follows. □

Observe, furthermore, that the obvious recursion $[m+1]_t=[2]_t[m]_t-[m-1]_t$, which is a particular case of (8), proves (by induction) that each t-number $[m+1]_t$ belongs to $\mathbb{Z}[t+t^{-1}]=\mathbb{Z}[[2]_t]$.

Combining this observation with (9), we finish the proof of the proposition. □

Let A' be the algebra over $\mathbb{C}(t)$ generated by σ_1,σ_2 subject to the relations

$$\sigma_1\sigma_2=\sigma_2\sigma_1,\quad (\sigma_1-t\sigma_2)(\sigma_1-t^{-1}\sigma_2)=0\ .$$

It is convenient to rewrite the second relation as:

$$[2]_t\sigma_1\sigma_2=\sigma_1^2+\sigma_2^2. \tag{10}$$

Lemma 24. *The following relations hold in A':*

$$[k+\ell]_t\sigma_1^k\sigma_2^\ell=[k]_t\sigma_1^{k+\ell}+[\ell]_t\sigma_2^{k+\ell} \tag{11}$$

for all $k,\ell\ge 0$. In particular, the monomials σ_i^k, $i\in\{1,2\}$, $k\ge 0$ form a $\mathbb{C}(t)$-linear basis of A'.

Proof. We proceed by induction in $\min(k,\ell)$. Indeed, if $k=0$ or $\ell=0$, we have nothing to prove. Otherwise, using (10) and the inductive hypothesis, we obtain:

$$\begin{aligned}
[k+\ell]_t\sigma_1^k\sigma_2^\ell &= [k+\ell]_t(\sigma_1\sigma_2)\sigma_1^{k-1}\sigma_2^{\ell-1}=\frac{[k+\ell]_t}{[2]_t}(\sigma_1^2+\sigma_2^2)\sigma_1^{k-1}\sigma_2^{\ell-1}\\
&=\frac{[k+\ell]_t}{[2]_t}(\sigma_1^{k+1}\sigma_2^{\ell-1}+\sigma_1^{k-1}\sigma_2^{\ell+1})\\
&=\frac{1}{[2]_t}([k+1]_t\sigma_1^{k+\ell}+[\ell-1]_t\sigma_2^{k+\ell}+[k-1]_t\sigma_1^{k+\ell}+[\ell+1]_t\sigma_2^{k+\ell})\\
&=\frac{1}{[2]_t}([k+1]_t+[k-1]_t)\sigma_1^{k+\ell}+\frac{1}{[2]_t}([\ell-1]_t+[\ell+1]_t)\sigma_2^{k+\ell}\\
&=[k]_t\sigma_1^{k+\ell}+[\ell]_t\sigma_2^{k+\ell}
\end{aligned}$$

by (8).

Furthermore, the relations (11) guarantee that the monomials σ_i^k, $i\in\{1,2\}$, $k\ge 0$ span A'. To verify their linear independence, let us compute the Hilbert series $h(A',z)$ of A'. Clearly, the Hilbert series of the polynomial algebra $\mathbb{C}(t)[\sigma_1,\sigma_2]$ is $1/(1-z)^2$ and the Hilbert series of any principal ideal I in $\mathbb{C}(t)[\sigma_1,\sigma_2]$ generated by a quadratic polynomial is $z^2/(1-z)^2$. Therefore, the Hilbert series of the quotient algebra $\mathbb{C}(t)[\sigma_1,\sigma_2]/I$ is

$$\frac{1}{(1-z)^2}-\frac{z^2}{(1-z)^2}=\frac{1+z}{1-z}=1+\sum_{k\geq 1}2z^k.$$

Applying this to our algebra $A'=\mathbb{C}(t)[\sigma_1,\sigma_2]/\langle(\sigma_1-t\sigma_2)(\sigma_1-t^{-1}\sigma_2)\rangle$ we see that each graded component of A' is 2-dimensional, which verifies the linear independence of the monomials. The lemma is proved. $\square$

Denote by $\sigma_i^{[k]}:=\frac{1}{[k]_t!}\sigma_i^k$, $i=1,2$, $k\geq 0$ the divided powers of σ_i, $i=1,2$. Denote by A the subalgebra of A' generated over $\mathbb{Z}[t+t^{-1}]$ by all $\sigma_i^{[k]}$, $i\in\{1,2\}$, $k\geq 0$.

Proposition 25. *The following relations hold in A:*

$$\sigma_1^{[k]}\sigma_1^{[\ell]}=\begin{bmatrix}k+\ell\\ k\end{bmatrix}_t\sigma_1^{[k+\ell]},\quad \sigma_2^{[k]}\sigma_2^{[\ell]}=\begin{bmatrix}k+\ell\\ k\end{bmatrix}_t\sigma_2^{[k+\ell]},\tag{12}$$

for all $k,\ell\geq 0$;

$$\sigma_1^{[k]}\sigma_2^{[\ell]}=\begin{bmatrix}k+\ell-1\\ k-1\end{bmatrix}_t\sigma_1^{[k+\ell]}+\begin{bmatrix}k+\ell-1\\ \ell-1\end{bmatrix}_t\sigma_2^{[k+\ell]}\tag{13}$$

for all $k,\ell\geq 0$.

In particular, the monomials $\sigma_i^{[k]}$, $i=1,2$, $k\geq 0$, form a $\mathbb{Z}[t+t^{-1}]$-linear basis in A, and the relations (12) *and* (13) *are defining for A.*

Proof. We have

$$\sigma_i^{[k]}\sigma_i^{[\ell]}=\frac{1}{[k]_t![\ell]_t!}\sigma_i^{k+\ell}=\frac{[k+\ell]_t}{[k]_t![\ell]_t!}\sigma_i^{[k]}\sigma_i^{[\ell]}$$

for $i\in\{1,2\}$, $k\geq 0$, which verifies (12). Furthermore, (11) implies that

$$\begin{aligned}\sigma_1^{[k]}\sigma_2^{[\ell]}&=\frac{1}{[k]_t![\ell]_t!}\sigma_1^k\sigma_2^\ell=\frac{1}{[k]_t![\ell]_t![k+\ell]_t}([k]_t\sigma_1^{k+\ell}+[\ell]_t\sigma_2^{k+\ell})\\&=\frac{[k+\ell-1]_t!}{[k-]_t![\ell]_t!}\sigma_1^{[k+\ell]}+\frac{[k+\ell-1]_t!}{[k]_t![\ell-1]_t!}\sigma_2^{[k+\ell]},\end{aligned}$$

which verifies (13).

Since all structure constants of A are t-binomial coefficients, Proposition 22 guarantees that A is defined over $\mathbb{Z}[t+t^{-1}]$.

Since, as a $\mathbb{Z}[t+t^{-1}]$-module, A is spanned by all products of various $\sigma_i^{[k]}$ and each such monomial is a $\mathbb{Z}[t+t^{-1}]$-linear combination of divided powers $\sigma_i^{[\ell]}$, $i\in\{1,2\}$, $\ell\geq 0$ by (12) and (13), we see that the divided powers span A as a $\mathbb{Z}[t+t^{-1}]$-module. It is also clear that the divided powers $\sigma_i^{[\ell]}$, $i\in\{1,2\}$, $\ell\geq 0$ are $\mathbb{Z}[t+t^{-1}]$-linearly independent because that was the case in A' by Lemma 24. Therefore, relations (12) and (13) are defining. The proposition is proved. $\square$

Now we will use the standard algebraic trick of specializing a formal parameter t into a nonzero complex number t_0. Clearly, this is impossible to do for A' because it is defined over $\mathbb{C}(t)$, but it is a perfectly reasonable to do so for the algebra A which is defined over $\mathbb{Z}[t+t^{-1}]$. Indeed, for each $t_0 \in \mathbb{C}^\times$ we define $\tilde{A}_{t_0} = R_0 \otimes_R A$, where $R = \mathbb{Z}[t+t^{-1}]$, $R_0 = \mathbb{Z}[t_0 + t_0^{-1}] \subset \mathbb{C}$, where R_0 is regarded as an R-module via the evaluation homomorphism $R \to R_0$ which takes t to t_0. By the construction, $\tilde{A}_{t_0}$ is a free $\mathbb{Z}[t_0 + t_0^{-1}]$-module, e.g., it has a basis $\sigma_i^{[k]}$, $i \in \{1,2\}$, $k \geq 0$.

With a slight abuse of notation, from now on we will denote by t a nonzero complex number so that $\tilde{A}_t$, $t \in \mathbb{C}^\times$ is the family of unital $\mathbb{Z}[t+t^{-1}]$-algebras with the presentation (12) and (13) (and $\sigma_1^{[0]} = \sigma_2^{[0]} = 1$).

For each $t \in \mathbb{C}^\times \setminus \{-1,1\}$ define $n_t \in \mathbb{Z} \sqcup \{\infty\}$ to be the order of t^2 in the multiplicative group $\mathbb{C}^\times$. If $t = \pm 1$, we set $n_{\pm 1} := \infty$. Thus, $n_t = \infty$ unless t^2 is a primitive nth root of unity and $n > 1$, in which case, $n_t = n$.

Note that if $n_t = n < \infty$, then $[n]_t = 0$ and $[n-k]_t = -t^n [k]_t$ for $0 \leq k \leq n$. In turn, this implies $\left[{m \atop k}\right]_t = 0$ for all $m \geq n_t$, $1 \leq k \leq m-1$ and

$$\begin{bmatrix} n-1 \\ k \end{bmatrix}_t = -t^n \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_t$$

hence

$$\begin{bmatrix} n-1 \\ k \end{bmatrix}_t = (-t^n)^k = 1,$$

which are most of the structure constants in (12) and (13). In particular, the following relations hold in $\tilde{A}_t$

$$\sigma_1^{[k]} \sigma_1^{[n-k]} = \sigma_2^{[k]} \sigma_2^{[n-k]} = 0, \qquad \sigma_1^{[k]} \sigma_2^{[n-k]} = \sigma_{12}^{[n]},$$

for all $1 \leq k < n = n_t$, where

$$\sigma_{12}^{[n]} := (-t^n)^{k-1} \sigma_1^{[n]} + (-t^n)^k \sigma_2^{[n]}.$$

Now define the algebra A_t, $t \in \mathbb{C}^\times$ over $\mathbb{Z}[t+t^{-1}] \subset \mathbb{C}$ as follows:

If $n_t = \infty$, then $A_t := \tilde{A}_t$.

If $n_t = n < \infty$ (i.e., $t^2 \neq 1$ is the nth primitive root of unity), then A_t is a subalgebra of $\tilde{A}_t$ generated by all $\sigma_1^{[k]}, \sigma_2^{[k]}$, $k = 0, 1, \ldots, n-1$ and by $\sigma_{12}^{[n]}$.

It is easy to see that in both cases the algebra A_t is $\mathbb{Z}$-graded via $\deg \sigma_i^{[k]} = k$. Moreover, in the second case, $\deg \sigma_{12}^{[n]} = n$ is the top degree in A_t, as $[n]_t = 0$.

For $t \in \mathbb{C}^\times$ let $W_t := \langle s_1, s_2 : s_1^2 = s_2^2 = 1, (s_1 s_2)^{n_t} = 1 \rangle$ be the dihedral group. Here it is understood that for $t = \pm 1$ we have the relation $s_1 s_2 = 1$ and for t which is not a root of unity, we have the tautological relator $(s_1 s_2)^0 = 1$. Define the W_t-action on the *weight lattice*

$$\Lambda_t = \mathbb{Z}[t+t^{-1}] \cdot \sigma_1 + \mathbb{Z}[t+t^{-1}] \cdot \sigma_2$$

by:

$$s_i(\sigma_j) = \sigma_j - \delta_{ij}(2\sigma_j - (t+t^{-1})\sigma_{3-i}) \tag{14}$$

for all $i, j \in \{1, 2\}$.

Recall that if W is a group acting on a vector space V, then the *coinvariant algebra* $S(V)_W$ is the quotient $S(V)/\langle S(V)_+^W\rangle$, where $S(V)_+^W$ stands for all W-invariants in the algebra of the constant-term-free polynomials

$$S(V)_+ = \sum_{k>0} S^k(V).$$

(The computations of $S(V)_W$ below, in the case of W_t with t a root of unity, present a very special case of the computation of coinvariant algebras for arbitrary finite groups; see, e.g., [H].)

The following proposition explains the origin of the algebra A_t:

Proposition 26. *For each $t \in \mathbb{C}^\times$ the algebra $\mathbb{C} \otimes A_t$ is naturally isomorphic to the coinvariant algebra of W_t acting on the vector space $V = \mathbb{C}\otimes\Lambda_t$. In particular, W_t naturally acts on A_t via:*

$$s_i(\sigma_j^{[k]}) = \sigma_j^{[k]} - \delta_{ij}(2\sigma_j^{[k]} - (t^k + t^{-k})\sigma_{3-i}^{[k]}) \tag{15}$$

for all $i, j \in \{1, 2\}$, $0 \leq k < n_t$ and (whenever $1 < n_t = n < \infty$)

$$s_i(\sigma_{12}^{[n]}) = -\sigma_{12}^{[n]}$$

for $i = 1, 2$.

Proof. Denote $z_1 = \sigma_1 - t\sigma_2$, $z_2 = t^{-1}\sigma_2 - \sigma_1$ and let

$$e_2 = -z_1 z_2 = (\sigma_1 - t\sigma_2)(\sigma_1 - t^{-1}\sigma_2) = \sigma_1^2 + \sigma_2^2 - (t + t^{-1})\sigma_1\sigma_2$$

(see (10)). It is easy to see that under the action (14), one has

$$s_1(z_1) = z_2, \qquad s_1(z_2) = z_1, \qquad s_2(z_1) = t^2 z_2, \qquad s_2(z_2) = t^{-2} z_2. \tag{16}$$

Hence, e_2 is invariant under the W_t-action.

Now assume that $[k]_t! \neq 0$ for all k, i.e., $n_t = \infty$. Then the algebra $\mathbb{C} \otimes A_t$ is just the quotient of $\mathbb{C}[\sigma_1, \sigma_2]$ by the quadratic ideal generated by e_2.

On the other hand, it is easy to see, using (16), that the W_t-invariant algebra $\mathbb{C}[\sigma_1, \sigma_2]^{W_t}$ is generated by e_2. Therefore, the coinvariant algebra $\mathbb{C}[\sigma_1, \sigma_2]_{W_t}$ is also the quotient $\mathbb{C}[\sigma_1, \sigma_2]/\langle e_2\rangle$. This proves the proposition in the case when $n_t = \infty$.

Assume that now $n_t = n < \infty$ or, equivalently, $[k]_t! \neq 0$ for $k < n$ and $[k]_t! = 0$ for $k \geq n$. Therefore, Proposition 25 guarantees that $\mathbb{C} \otimes A_t$ is a commutative algebra generated by σ_1, σ_2 subject to the relations

$$e_2 = 0, \qquad \sigma_1^n = \sigma_2^n = 0. \tag{17}$$

(In fact, $\sigma_{12}^n = \sigma_1\sigma_2^{n-1}$ by (13) because $[n-1]_t = -t^n$.)

Again, it is easy to see, using (16), that the W_t-invariant algebra $\mathbb{C}[\sigma_1, \sigma_2]^{W_t}$ is generated by e_2 and $e_n = z_1^n + z_2^n$. Therefore, the coinvariant algebra $\mathbb{C}[\sigma_1, \sigma_2]_{W_t}$

is the quotient $\mathbb{C}[\sigma_1,\sigma_2]/\langle e_2,e_n\rangle$. To finish the proof it suffices to show that the ideals $\langle\sigma_1^n,\sigma_2^n\rangle$ and $\langle e_n\rangle$ are equal in $\mathbb{C}[\sigma_1,\sigma_2]/\langle e_2\rangle$. Indeed, taking into account that $z_1z_2=0$ in $\mathbb{C}[\sigma_1,\sigma_2]/\langle e_2\rangle$ and that

$$\sigma_2=\frac{z_1+z_2}{t^{-1}-t},\qquad \sigma_1=\frac{t^{-1}z_1+tz_2}{t^{-1}-t},$$

we obtain:

$$\begin{aligned}\sigma_1^n&=\frac{(z_1+z_2)^n}{(t^{-1}-t)^n}=\frac{z_1^n+z_2^n}{(t^{-1}-t)^n},\\ \sigma_2^n&=\frac{(t^{-1}z_1+tz_2)^n}{(t^{-1}-t)^n}=\frac{t^{-n}z_1^n+t^nz_2^n}{(t^{-1}-t)^n}=t^n\frac{z_1^n+z_2^n}{(t^{-1}-t)^n}\end{aligned}$$

because $t^{2n}=1$. This proves the equality of ideals, hence the equality of quotients $\mathbb{C}\otimes A_t=\mathbb{C}[\sigma_1,\sigma_2]_{W_t}$.

In particular, this verifies that W_t naturally acts on $\mathbb{C}\otimes A_t$. To obtain (15), note that W_t preserves the component $\mathbb{C}\cdot\sigma_1^k+\mathbb{C}\cdot\sigma_2^k\subset\mathbb{C}\otimes A_t$ which is the kth symmetric power of $\mathbb{C}\cdot\sigma_1+\mathbb{C}\cdot\sigma_2$ (if $n<n_t$), and therefore, W_t acts on the former space in the same way as in the latter space, i.e., by (14) where t is replaced with t^k. The proposition is proved. □

It is convenient to label the above basis of A_t by the elements of the dihedral group W_t:

$$\sigma_w=\begin{cases}\sigma_i^{[k]} & \text{if } \ell(w)=k<n_t \text{ and } \ell(ws_i)<\ell(w),\\ \sigma_{12}^{[n_t]} & \text{if } \ell(w)=n_t<\infty\end{cases}\tag{18}$$

for $w\in W_t$, where $\ell:W\to\mathbb{Z}_{\ge 0}$ is the length function.

The following result is an equivalent reformulation of Proposition 25.

Proposition 27. *For each $t\in\mathbb{C}^\times$ the elements σ_w, $w\in W_t$ form a $\mathbb{Z}[t+t^{-1}]$-linear basis of A_t and the following relations are defining*:

- *If $\ell(u)+\ell(v)>n_t$, then $\sigma_u\sigma_v=0$.*
- *If $u=\underbrace{\cdots s_js_i}_{k}$, $v=\underbrace{\cdots s_js_i}_{\ell}$ and $k+\ell\le n_t$ and $\{i,j\}=\{1,2\}$, then*

$$\sigma_u\sigma_v=\begin{bmatrix}k+\ell\\ k\end{bmatrix}_t\sigma_w$$

where $w=\underbrace{\cdots s_js_i}_{k+\ell}$ (e.g., the right-hand side is 0 if $k+\ell=n_t$ and $k,\ell>0$).

- *If $u=\underbrace{\cdots s_2s_1}_{k}$, $v=\underbrace{\cdots s_1s_2}_{\ell}$ and $k+\ell<n_t$, then*

$$\sigma_u\sigma_v=\begin{bmatrix}k+\ell-1\\ k-1\end{bmatrix}_t\sigma_{w_1}+\begin{bmatrix}k+\ell-1\\ \ell-1\end{bmatrix}_t\sigma_{w_2}$$

where

$$w_1=\underbrace{\cdots s_2s_1}_{k+\ell},\qquad w_2=\underbrace{\cdots s_1s_2}_{k+\ell}.$$

- *If* $u = \underbrace{\cdots s_2 s_1}_{k}$, $v = \underbrace{\cdots s_1 s_2}_{\ell}$ *and* $k + \ell = n_t$, $k \le \ell$, *then*
$$\sigma_u \sigma_v = \begin{bmatrix} n_t - 1 \\ k - 1 \end{bmatrix}_t \sigma_{w_\circ} = \sigma_{w_\circ}$$
where $w_\circ = \underbrace{\cdots s_2 s_1}_{n_t}$ *is the longest element of the (finite) group* W_t.

Note that when $\theta = t + t^{-1} \in \mathbb{R}$, all structure constants of A_t are real numbers. We can refine this as follows.

Corollary 28. *The structure constants of* A_t *are nonnegative if and only if either* $t = e^{\pi\sqrt{-1}/n}$ *or* $t > 0$.

Proof. Indeed, the structure constants are t-binomials, which are nonnegative for $t = e^{\pi\sqrt{-1}/n}$ or $t > 0$, since $[m]_t \ge 0$ for $1 \le m \le n$. On the other hand, if, say, $n = n_t < \infty$ but t is not of the form $e^{\pi\sqrt{-1}/n}$, then there exists $1 \le m \le n$ so that $[m]_t < 0$. □

Remark 6. The above corollary is just one of many hints pointing to the existence of (possibly noncommutative, in view of nonintegrality of the structure constants) complex-algebraic varieties serving as flag-manifolds for noncrystallographic finite dihedral groups.

Let G be a complex Kac–Moody group with the Cartan matrix

$$\begin{pmatrix} 2 & -a_{12} \\ -a_{21} & 2 \end{pmatrix},$$

where a_{12} and a_{21} are arbitrary positive integers (if $a_{12}a_{21} \le 3$, then G is a finite-dimensional simple Lie group of rank 2). Let $t \in \mathbb{C}^\times$ be such that $t + t^{-1} = \sqrt{a_{12}a_{21}}$. In particular, the Weyl group of G is naturally isomorphic to W_t. Let $B \subset G$ be a Borel subgroup. It is well-known (see, e.g., [KK]) that the cohomology algebra $H^*(G/B)$ has a basis of Schubert classes $[X_w]$, $w \in W_t$.

The following is the main result of the section.

Theorem 29. *Let* G *and* B *be as above and* $c_1, c_2 \in \mathbb{C}^\times$ *be any numbers such that* $c_1/c_2 = \sqrt{a_{12}/a_{21}}$ *and* $\mathbb{Z}[c_1, c_2] \supset \mathbb{Z}[t + t^{-1}]$. *Then the association*

$$[X_w] \mapsto c_i^{\lceil k/2 \rceil} c_{3-i}^{\lfloor k/2 \rfloor} \cdot \sigma_w \tag{19}$$

for all $w \in W_t$, *where* $i \in \{1, 2\}$ *is such that* $\ell(ws_i) < \ell(w) = k$, *defines a* W_t-*equivariant isomorphism*

$$H^*(G/B, \mathbb{Z}[c_1, c_2]) \xrightarrow{\sim} \mathbb{Z}[c_1, c_2] \otimes A_t. \tag{20}$$

Proof. It suffices to prove that (19) defines a W_t equivariant isomorphism

$$H^*(G/B, \mathbb{C}) \xrightarrow{\sim} \mathbb{C} \otimes A_t. \tag{21}$$

Recall that the action of the Weyl group W of G on the root space $Q_{\mathbb{C}} = \mathbb{C}\cdot\alpha_1 + \mathbb{C}\cdot\alpha_2$ is given by:

$$s_i(\alpha_j) = \begin{cases} -\alpha_i & \text{if } i = j, \\ \alpha_i + a_{ij}\cdot\alpha_j & \text{if } i \neq j, \end{cases}$$

for $i, j \in \{1, 2\}$.

It follows from [KK, Prop. 3.10] that the algebra $H^*(G/B, \mathbb{Z})$ satisfies the following Chevalley formula:

$$[X_w][X_{s_i}] = \sum_{w_1, w_2, j} \omega_i^\vee(w_2^{-1}(\alpha_j)) \cdot [X_{w_1 s_j w_2}], \tag{22}$$

where the summation is over all $w_1, w_2 \in W$, and $j \in \{1, 2\}$ such that $w = w_1 w_2$, $\ell(w) = \ell(w_1) + \ell(w_2)$, and $\ell(w_1 s_j w_2) = \ell(w) + 1$. Here $\omega_i^\vee$, $i \in \{1, 2\}$, denotes the dual basis in Q^* of the basis α_1, α_2.

In particular, if $\ell(ws_i) < \ell(w)$, then the only nonzero summand in the right-hand side of (22) corresponds to $w_1 = 1$ and $w_2 = w$, and j such that $\ell(s_j w) = \ell(w) + 1$. Furthermore, if $\ell(ws_i) > \ell(w)$, then the right-hand side has two summands, first of which comes with $w_2 = 1$, $w_1 = w$, and the second with $w_1 = 1$, $w_2 = w$. Therefore,

$$[X_{\underbrace{\cdots s_j s_i}_{k}}][X_{s_i}] = \omega_i^\vee(\underbrace{s_i s_j \cdots s_{i'}}_{k}(\alpha_{3-i'}))[X_{\underbrace{\cdots s_j s_i}_{k+1}}], \tag{23}$$

$$[X_{\underbrace{\cdots s_i s_j}_{k}}][X_{s_i}] = [X_{\underbrace{\cdots s_j s_i}_{k+1}}] + \omega_i^\vee(\underbrace{s_i s_j \cdots s_{i'}}_{k}(\alpha_{3-i'}))[X_{\underbrace{\cdots s_i s_j}_{k+1}}] \tag{24}$$

for all $k < n_t$ and i, j such that $\{i, j\} = \{1, 2\}$, where i' stands for the appropriate index i or j (depending on $k \mod 2$). In particular, if $k = 1$, we obtain:

$$\begin{aligned} [X_{s_1}]^2 &= \omega_1^\vee(s_1(\alpha_2))[X_{s_2 s_1}], \\ [X_{s_2}]^2 &= \omega_2^\vee(s_2(\alpha_1))[X_{s_2 s_1}], \\ [X_{s_1}][X_{s_2}] &= [X_{s_1 s_2}] + [X_{s_2 s_1}] \end{aligned}$$

which implies the following quadratic relation in $H^*(G/B, \mathbb{Z})$:

$$a_{21}[X_{s_1}]^2 + a_{12}[X_{s_2}]^2 = a_{12}a_{21}[X_{s_1}][X_{s_2}]. \tag{25}$$

To utilize the identities (23) and (24), we need the following obvious result.

Lemma 30. *Let* $w = \underbrace{\cdots s_i s_j}_{k} \in W_t$, *where* $\{i, j\} = \{1, 2\}$. *Then*

$$w(\alpha_i) = [k + 1 - \varepsilon_k]_t \alpha_i + \sqrt{\frac{a_{ji}}{a_{ij}}} \cdot [k + \varepsilon_k]_t \alpha_j,$$

where $t+t^{-1}=\sqrt{a_{12}a_{21}}$ *and* $\varepsilon_k=\begin{cases}1 & \text{if } k \text{ is odd,}\\ 0 & \text{if } k \text{ is even.}\end{cases}$

By combining this lemma with (23), we obtain:

$$[X_{\underbrace{\cdots s_j s_i}_{k}}][X_{s_i}]=\left(\sqrt{\frac{a_{ij}}{a_{ji}}}\right)^{\varepsilon_k}[k+1]_t\cdot[X_{\underbrace{\cdots s_j s_i}_{k+1}}], \tag{26}$$

$$[X_{\underbrace{\cdots s_i s_j}_{k}}][X_{s_i}]=[X_{\underbrace{\cdots s_j s_i}_{k+1}}]+\left(\sqrt{\frac{a_{ij}}{a_{ji}}}\right)^{\varepsilon_k}[k+1]_t\cdot[X_{\underbrace{\cdots s_i s_j}_{k+1}}]. \tag{27}$$

Furthermore, (26) implies that

$$[X_{s_i}]^k=\left(\sqrt{\frac{a_{ij}}{a_{ji}}}\right)^{\lfloor k/2\rfloor}[k]_t!\cdot[X_{\underbrace{\cdots s_j s_i}_{k}}]. \tag{28}$$

In turn, this implies that $H^*(G/B,\mathbb{C})$ is generated by $[X_{s_1}]$, $[X_{s_2}]$, satisfying (25) and the relations

$$[X_{s_1}]^{n_t}=[X_{s_2}]^{n_t}=0 \tag{29}$$

if $n_t<\infty$. Pick $r_1,r_2\in\mathbb{C}^\times$ such that $r_1/r_2=\sqrt{a_{21}/a_{12}}$ and define

$$\varphi:\sigma_1\mapsto r_1[X_{s_1}],\qquad \varphi:\sigma_2\mapsto r_2[X_{s_2}].$$

In view of the relation (25), we obtain

$$\varphi(\sigma_1)^2+\varphi(\sigma_2)^2=\sqrt{a_{12}a_{21}}\,\varphi(\sigma_1\sigma_2).$$

Since $t+t^{-1}=\sqrt{a_{12}a_{21}}$, we conclude that φ preserves the defining quadratic equation (10) of A_t. The equation (29) implies that φ preserves the last two relators in (17) provided that $n=n_t<\infty$. Thus, φ extends a surjective homomorphism of algebras $\varphi:\mathbb{C}\otimes A_t\to H^*(G/B,\mathbb{C})$.

Clearly, this homomorphism is an isomorphism because it preserves the natural $\mathbb{Z}$-grading and because the respective graded components of both algebras are of the same dimension. Furthermore, let us show that for each $w\in W_t$ one has:

$$\varphi(\sigma_w)=r_i^{\lceil \ell(w)/2\rceil}r_j^{\lfloor \ell(w)/2\rfloor}[X_w]^{\ell(w)} \tag{30}$$

where $i\in\{1,2\}$ is such that $\ell(ws_i)<\ell(w)$ and $\{i,j\}=\{1,2\}$. Indeed, if $\ell(w)<n_t$, then

$$\begin{aligned}\varphi(\sigma_w)&=r_i^{\ell(w)}\frac{1}{[k]_t!}[X_{s_i}]^{\ell(w)}=r_i^{\ell(w)}\left(\frac{r_j}{r_i}\right)^{\lfloor \ell(w)/2\rfloor}[X_w]^{\ell(w)}\\&=r_i^{\lceil \ell(w)/2\rceil}r_j^{\lfloor \ell(w)/2\rfloor}[X_w]^{\ell(w)}.\end{aligned}$$

If $\ell(w) = n_t < \infty$ (i.e., w is the longest element of W), then $[n_t]_t = 0$ and, using (13) with $k = 1$ and (27) respectively, we obtain:

$$\sigma_{ws_1}\sigma_1 = \sigma_w, \qquad [X_{ws_1}][X_{s_1}] = [X_w].$$

Thus applying φ to the first of these relations, we obtain (taking into account that $r_1 = r_2$ when n_t is odd and using the already proved case of (30) with $w' = ws_1$, $i = 2$):

$$\begin{aligned}\varphi(\sigma_w) &= \varphi(\sigma_{ws_1}\sigma_1) = \varphi(\sigma_{ws_1})\varphi(\sigma_1)\\ &= r_2^{\lceil (n_t-1)/2\rceil} r_1^{\lfloor (n_t-1)/2\rfloor}[X_{ws_1}]r_1[X_{s_1}] = (r_1r_2)^{n_t/2}[X_w].\end{aligned}$$

Finally, taking $r_i = 1/c_i$, $i = 1, 2$, we see that the isomorphism φ^{-1} is given by (19) and its restriction to $H^*(G/B, \mathbb{Z}[c_1, c_2])$ becomes (20). The W-equivariancy of both φ and φ^{-1} follows. $\square$

Remark 7. A computation of the rings $H^*(G/B, \mathbb{Z})$ for rank 2 complex Kac–Moody groups G appeared in [Kit, Sect. 10]. We are grateful to Shrawan Kumar for this reference.

Remark 8. We can take

$$c_i = \sqrt{\frac{a_{i,3-i}}{\gcd(a_{12}, a_{21})}}, \qquad i = 1, 2,$$

in Theorem 29. Then $\mathbb{Z}[c_1, c_2] \supset \mathbb{Z}[t+t^{-1}]$ because $t+t^{-1} = c_1c_2\cdot\gcd(a_{12}, a_{21})$. In particular, if the Cartan matrix is symmetric, i.e., $a_{12} = a_{21}$, then the isomorphism (20) is over $\mathbb{Z}$ because $c_1 = c_2 = 1$ and $\mathbb{Z}[c_1, c_2] = \mathbb{Z}[t + t^{-1}] = \mathbb{Z}$.

In view of Theorem 29, we will refer to the algebra A_t as the *universal dihedral cohomology* and to the basis $\{\sigma_w\}$ as the *universal Schubert classes.* Under under various specializations of t it computes either cohomology rings of complex flag manifolds associated with complex Kac–Moody groups, or cohomology rings of “yet to be defined” flag-manifolds for noncrystallographic finite dihedral groups or nondiscrete infinite dihedral groups.

We call a complex number t *admissible* if either

(1) (finite case) $t = e^{\pm\pi\sqrt{-1}/n}$ for some $n \in \mathbb{Z}_{>0}$, or
(2) (hyperbolic case) t is a positive real number.

Then for every admissible t, $[k]_t > 0$ for all $0 \le k < n_t$. For an admissible t let $W_t^{(i)} = \{w \in W_t \mid \ell(ws_{3-i}) = \ell(w) + 1\}$, $i = 1, 2$.

Notation 31. *Denote by $B_t^{(i)}, i = 1, 2,$ the subalgebras of A_t generated by $X^{(i)} = \{\sigma_w \mid w \in W_t^{(i)}\}$.*

The subalgebras $B_t^{(i)}$ play the role of the cohomology rings of the “Grassmannians” $Y_i, i = 1, 2$ of spherical buildings Y modeled on (S^1, W_t), where t is a root of unity. It follows from Proposition 25 that $X^{(i)}$ is a basis of $B_t^{(i)}$, e.g.,

$\dim B_t^{(i)} = |W_t^{(i)}| = n_t - 1$, and that, moreover, the ring $B_t^{(i)}$ is naturally isomorphic to the cohomology ring

$$H^*(\mathbb{CP}^n), \qquad n = n_t.$$

Similarly, we will think of the algebra A_t, $t = e^{\pm\pi\sqrt{-1}/n}$, as the cohomology ring of the "flag manifold" $\mathrm{Fl}(Y)$.

14. Belkale–Kumar type filtration of A_t

In this section, we construct a filtration on A_t (and its subalgebras $B_t^{(i)}, i = 1, 2$) in the sense of Proposition 32, using a Belkale–Kumar type function $\varphi : W_t \to \mathbb{R}$. In the case when t is the nth primitive root of unity, the associated graded algebra $A_{t,0} = grA_t$ will play the role of the Belkale–Kumar cohomology of spherical buildings Y with finite Weyl group $I_2(n)$, which is "Poincaré dual" to the homology pre-ring $H_*(\mathrm{Fl}(Y))$ defined by the Schubert precalculus on Y.

Definition 6. Let $\mathbf{k}$ be a field and A be an associative $\mathbf{k}$-algebra with a basis $\{b_x \mid x \in X\}$ so that

$$b_x b_y = \sum_{z\in X} c_{x,y}^z b_z \tag{31}$$

for all $x, y \in X$, where $c_{x,y}^z \in \mathbf{k}$ are structure constants. Furthermore, given an ordered abelian semi-group Γ (e.g., $\Gamma = \mathbb{R}$), we say that a function $\varphi : X \to \Gamma$ is *concave* if

$$\varphi(x) + \varphi(y) \geq \varphi(z)$$

for all $x, y, z \in X$ such that $c_{x,y}^z \neq 0$.

Proposition 32. *In the notation* (31), *for each concave function* $X \to \Gamma$ *we have*:

(a) *A is filtered by Γ via* $A_{\leq\gamma} := \sum_{x\in X\,:\,\varphi(x)\leq\gamma} \mathbf{k} \cdot b_x$.

(b) *The multiplication in the associated graded algebra $A_0 = grA$ is given by*:

$$b_x \circ b_y = \sum_{z\in X\,:\,\varphi(z)=\varphi(x)+\varphi(y)} c_{xy}^z b_z. \tag{32}$$

for all $x, y \in X$, where $c_{xy}^z \in \mathbf{k}$ are the structure constants of A.

Proof. Part (a). Assume that $\varphi(x) \leq \gamma_1$, $\varphi(y) \leq \gamma_2$, i.e., $b_x \in A_{\leq\gamma_1}$, $b_y \in A_{\leq\gamma_2}$. Then each z such that $c_{xy}^z \neq 0$ satisfies $\varphi(z) \leq \varphi(x) + \varphi(y) \leq \gamma_1 + \gamma_2$, i.e., $b_z \in A_{\gamma_1+\gamma_2}$. Therefore, $b_x b_y \in A_{\gamma_1+\gamma_2}$. This proves (a). Part (b) immediately follows. $\square$

Remark 9. The algebra A_0 is the *Belkale–Kumar degeneration* of A. It was introduced by Belkale and Kumar in [BKu] in the special case of cohomology rings of flag-manifolds G/B, where G is a complex semisimple Lie group and B is its Borel subgroup. In order to relate our definition to that of [BKu], note that, given a concave function φ, Belkale and Kumar define the deformation A_τ of $A = H^*(G/B, \mathbb{C})$ by

$$b_x \odot_\tau b_y := \sum_{z \in X} \tau^{\varphi(x)+\varphi(y)-\varphi(z)} c_{xy}^z b_z.$$

Setting $\tau = 1$ one recovers the original algebra A, while sending τ to zero one obtains the degeneration $A_0 = \mathrm{gr}(A)$ of A.

Our goal is to generalize the function φ defined in [BKu] to the case of algebras A_t (for *admissible* values of t), so that our function φ will specialize to the Belkale–Kumar function in the case $n = 3, 4, 6$. Note that concavity of φ was proven in [BKu] as a consequence of the complex-algebraic nature of the variety G/B. In our case, such variety does not exist and we prove concavity by a direct calculation.

For $t \in \mathbb{C}^\times$ define the action of the dihedral group W_t on the 2-dimensional *root lattice*

$$Q = Q_t = \mathbb{Z}[t + t^{-1}] \cdot \alpha_1 + \mathbb{Z}[t + t^{-1}] \cdot \alpha_2$$

by:

$$s_i(\alpha_j) = \begin{cases} -\alpha_i & \text{if } i = j, \\ \alpha_i + [2]_t \cdot \alpha_j & \text{if } i \neq j \end{cases}$$

for $i, j \in \{1, 2\}$.

The above action extends to the *weight lattice*

$$\Lambda = \Lambda_t = \mathbb{Z}[t + t^{-1}] \cdot \omega_1 + \mathbb{Z}[t + t^{-1}] \cdot \omega_2$$

by:

$$s_i(\omega_j) = \omega_j - \delta_{ij} \iota(\alpha_i)$$

for all $i, j \in \{1, 2\}$, which is consistent with (14). Here $\iota : Q \to \Lambda$ is a $\mathbb{Z}[t + t^{-1}]$-linear map given by:

$$\iota(\alpha_1) = 2\omega_1 - (t + t^{-1})\omega_2, \qquad \iota(\alpha_2) = 2\omega_2 - (t + t^{-1})\omega_1.$$

For each $i \in \{1, 2\}$ define the map $[\,\cdot\,]_i : W_t \to Q$ recursively by $[1]_i = 0$ and

$$[s_j w]_i = \delta_{ij} \alpha_i + s_j([w]_i).$$

Note that the map $[\,\cdot\,]_i$ satisfies:

$$\iota([w]_i) = \omega_i - w(\omega_i).$$

Define the functions $\Phi_i : W_t \to \mathbb{Z}[t + t^{-1}]$, $i = 1, 2$, by

$$\Phi_i(w) = |[w]_i| \tag{33}$$

where $|g_1\alpha_1 + g_2\alpha_2| = g_1 + g_2$.

Proposition 33. *For any $w \in W_t$, $i = 1, 2$, we have:*

$$\Phi_i(w) = \begin{cases} \left[{\ell(w)+1 \atop 2}\right]_q & \text{if } \ell(ws_i) < \ell(w), \\ \left[{\ell(w) \atop 2}\right]_q & \text{if } \ell(ws_i) > \ell(w), \end{cases} \tag{34}$$

where $q = t^{1/2}$ and $\ell : W \to \mathbb{Z}$ is the word-length function on W with respect to the generating set s_1, s_2. In particular, the function $\Phi := \Phi_1 + \Phi_2$ is given by the formula:

$$\Phi(w) = ([\ell(w)]_q)^2. \tag{35}$$

Proof. We need the following obvious result:

Lemma 34. *For each $k \in \mathbb{Z}$ denote*

$$\alpha_k := \begin{cases} \alpha_1 & \text{if } k \text{ is odd,} \\ \alpha_2 & \text{if } k \text{ is even.} \end{cases}$$

Let $w = \underbrace{\cdots s_j s_i}_{k} \in W_t$, where $\{i, j\} = \{1, 2\}$. Then

$$w(\alpha_j) = [k]_t \alpha_{i+k} + [k+1]_t \alpha_{j+k}, \qquad [w]_i = \alpha_i + [2]_t \alpha_{i+1} + \cdots + [k]_t \alpha_{i+k-1}.$$

Proof. The assertion directly follows from Lemma 30 with $a_{12} = a_{21} = t + t^{-1}$. □

Furthermore, using the second identity of Lemma 34 we obtain for any $w \in W_t$ with $\ell(ws_i) < \ell(w)$:

$$|[w]_i| = [1]_t + [2]_t + \cdots + [\ell(w)]_t \ .$$

Using the fact that $[m]_t = [2m]_q/[2]_q$ for $q = t^{1/2}$ and any m, we obtain

$$\begin{aligned} |[w]_i| = [1]_t + [2]_t + \cdots + [\ell(w)]_t &= \frac{1}{[2]_q}([2]_q + [4]_q + \cdots + [2\ell(w)]_q) \\ &= \frac{1}{[2]_q}[\ell(w)]_q[\ell(w)+1]_q, \end{aligned}$$

which proves (34), since $\Phi_i(w) = \Phi_i(ws_{3-i}) = |[w]_i|$.

We now prove (35). Indeed, for any $w \in W_t$ let i be such that $\ell(ws_i) < \ell(w)$. Applying part (34), we obtain:

$$\Phi(w) = |[w]_i| + |[w]_{3-i}| = \begin{bmatrix} \ell(w)+1 \\ 2 \end{bmatrix}_q + \begin{bmatrix} \ell(w) \\ 2 \end{bmatrix}_q = ([\ell(w)]_q)^2 \ .$$

The proposition is proved. □

The following theorem is the main result of the section.

Theorem 35. *The functions*

$$\varphi_i : X^{(i)} \to \mathbb{R}, \qquad \varphi_i(\sigma_w) = -\Phi_i(w),$$

$i = 1, 2$, *and*

$$\varphi : X \to \mathbb{R}, \qquad \varphi(\sigma_w) = -\Phi(w)$$

(*see Proposition* 33) *are both concave in the sense of Definition* 6; *in particular, they define filtrations on* $B_t^{(i)}$ *and* A_t, *respectively, in the sense of Proposition* 32. *Moreover, the equalities*

$$\varphi(\sigma_u) + \varphi(\sigma_v) = \varphi(\sigma_w), \qquad \varphi_i(\sigma_u) + \varphi_i(\sigma_v) = \varphi_i(b_w)$$

are achieved if and only if either:

1. *For the function* φ, $u = 1$ *or* $v = 1$, *or* $\ell(u) + \ell(v) = \ell(w) = n$, *provided that* $n < \infty$.
2. *For the function* φ_i, $u = 1$ *or* $v = 1$, *or* $\ell(u) + \ell(v) = \ell(w) = n - 1$, *provided that* $n < \infty$.

Proof. Recall that a function $f : I \to \mathbb{R}$ defined on an interval $I \subset \mathbb{R}$ is called *superadditive* (resp. subadditive) if

$$f(x + y) \geq f(x) + f(y), \quad \text{resp.} \quad f(x + y) \leq f(x) + f(y) \tag{36}$$

for all $x, y, x + y \in I$. If f is convex, continuous, and $f(0) = 0$ then f is superadditive on $I = \mathbb{R}_+$; see [HP, Theorem 7.2.5]. Moreover, it follows from the proof of [HP, Theorem 7.2.5] that if f is strictly convex then (36) is a strict inequality unless $xy = 0$.

Let $t \in \mathbb{C}$ be an admissible number, $n := n_t$; let $q := t^{1/2}$ so that $q \in \mathbb{R}_+$ if $t > 0$ and $q = e^{\sqrt{-1}Q}$, $Q = \pi/2n$ if t is a root of unity. Define the functions

$$F(x) = \begin{bmatrix} x+1 \\ 2 \end{bmatrix}_q, \quad 0 \leq x \leq n - 1,$$
$$G(x) := ([x]_q)^2, \quad 0 \leq x \leq n,$$

where x are nonnegative real numbers.

Proposition 36. *The functions* F *and* G *are superadditive. Moreover, inequality* (36) *is equality if and only if* $xy(n - 1 - x - y) = 0$ (*for the function* F) *and* $xy(n - x - y) = 0$ (*for the function* G).

Proof. We have

$$F(x) = \frac{[x]_q[x+1]_q}{[2]_q} = \frac{f(x)}{(q - q^{-1})^2(q + q^{-1})}, \quad f(x) = (q^x - q^{-x})(q^{x+1} - q^{-x-1}),$$
$$G(x) = \frac{g(x)}{(q - q^{-1})^2}, \quad g(x) = (q^x - q^{-x})^2.$$

In particular, $F(0) = G(0) = 0$.

We first consider the hyperbolic case (i.e., $q > 0$). Then the denominators of both F and G are positive and numerators are equal to

$$f(x) = q^{2x+1} + q^{-2x-1} - q - q^{-1}, \qquad g(x) = q^{2x} + q^{-2x} - 2.$$

It is elementary that both functions are strictly convex on $[0, \infty)$ because $f''(x) > 0$ and $g''(x) > 0$. Hence, F and G are superadditive with equality in (36) if and only if $xy = 0$.

We therefore assume now that q is a root of unity. One can check that in this case F and G are neither convex nor concave on their domains, so we have to use a direct calculation in order to show superadditivity. The denominators of the functions F and G are both negative since they equal to $-8\sin^2(Q)\cos(Q)$ and $-4\sin^2(Q)$, respectively.

Consider the functions $f(x)$ and $g(x)$. It is easy to see that

$$\begin{aligned} f(z) - f(x) - f(y) &= (q^x - q^{-x})(q^y - q^{-y})(q^{x+y+1} + q^{-x-y-1}), \\ g(z) - g(x) - g(y) &= (q^x - q^{-x})(q^y - q^{-y})(q^z + q^{-z}). \end{aligned}$$

Therefore, if $x, y, z \in [0, n-1]$ with $z = x + y$ then:

$$f(z) - f(x) - f(y) \le 0$$

with equality if and only if $xy(n - 1 - z) = 0$ and

$$g(z) - g(x) - g(y) \le 0$$

with equality if and only if $xy(n - z) = 0$ because

$$(q^x - q^{-x})(q^y - q^{-y}) = -4\sin(Qx)\sin(Qy) \le 0, \quad q^u + q^{-u} = 2\cos(Qu) \ge 0$$

for any $x, y, u \in [0, n]$.

Thus both functions f and g are subadditive. Since the denominators in F and G are constant and negative, these functions are superadditive with equality in (36) if and only if $xy = 0$ or $x + y = n - 1$ (for F) and $x + y = n$ (for G). $\square$

We can now finish the proof of Theorem 35. We have

$$\Phi_i(w) := |[w]_i| = F(\ell(w)), \qquad w \in W^{(i)},\ 0 \le \ell(w) \le n - 1$$

and

$$\Phi(w) = |[w]_1| + |[w]_2| = G(\ell(w)), \quad w \in W,\ 0 \le \ell(w) \le n.$$

Observe that, since A_t is graded by the length function of W_t,

$$c_{uv}^w \ne 0 \ \Rightarrow\ \ell(w) = \ell(u) + \ell(v),$$

where c_{uv}^w are the structure constants:

$$\sigma_u \cdot \sigma_v = \sum_w c_{uv}^w \sigma_w.$$

Therefore, superadditivity of the functions F and G is equivalent to concavity of the functions $\varphi = -\Phi$ and $\varphi_i = -\Phi_i$. The equality cases in Theorem 35 immediately follow as well. $\square$

Corollary 37. *The rings A_t and $B_t^{(i)}$, $i=1,2$, admit Belkale–Kumar degenerations* $\mathrm{gr}(A_t)$ *and* $\mathrm{gr}(B_t^{(i)})$ *given by the functions φ and φ_i, respectively.*

Remark 10. We do not know a natural topological interpretation for the rings $\mathrm{gr}(A_t)$ and $\mathrm{gr}(B_t^{(i)})$.

15. Interpolating between homology pre-ring and the ring $\mathrm{gr}(A_t)$

Let $\mathbf{k}$ be a field. In this section we construct an interpolation between the homology pre-rings $H_*(X,\widehat{\mathbf{k}}), H_*(X_l,\widehat{\mathbf{k}})$ and the $\mathbf{k}$-algebras $\mathrm{gr}(A_t), \mathrm{gr}(B_t^{(l)})$, $t=e^{\pi\sqrt{-1}/n}$, which are the Belkale–Kumar degenerations of $A_t, B_t^{(l)}$ introduced in Section 14. Thereby, we link the geometrically defined homology pre-rings and the algebraically defined cohomology rings of X, X_l, $l=1,2$.

Below we again abuse the terminology and use the notation ∞ for the infinity in the one-point compactification of $\mathbb{R}$ and for the element of $\widehat{\mathbf{k}}$. Accordingly, we equip $\mathbf{k}$ with the discrete topology and set

$$\lim_{\tau\to\infty} f(\tau)a = \infty,$$

whenever $a\in\mathbf{k}^\times$ and $\lim_{\tau\to\infty} f(\tau)=\infty$.

1. Interpolation for A_t. Using the Belkale–Kumar function $\varphi=-\Phi$ as in the previous section, we define the (trivial) family of algebras $A_{t,\tau}$ as in Remark 9, with multiplication given (for $\tau>0$) by

$$\sigma_u \odot_\tau \sigma_v := \sum_{w:\ell(w)=\ell(u)+\ell(v)} \tau^{\varphi(u)+\varphi(v)-\varphi(w)} c_{uv}^w \sigma_w,$$

where c_{uv}^w are the structure constants in A_t. Then, as $\tau\to 0$, the algebra $A_{t,\tau}$ degenerates to $\mathrm{gr}(A_t)$. Now, let $\tau\to\infty$. Recall that $\varphi(u)+\varphi(v)-\varphi(w)>0$ unless it equals to zero (Proposition 36); the latter corresponds to the *degenerate cases*, i.e., products of Poincaré dual classes σ_u,σ_v or classes where $\sigma_u=1$ or $\sigma_v=1$. Therefore, the limit pre-ring $A_{t,\infty}$ has structure constants $\hat{c}_{uv}^w$ equal to $0,1,\infty$.

Here $\hat{c}_{uv}^w=0$ occurs unless $\ell(w)=\ell(u)+\ell(v)$, and $u,v,w\in W^{(i)}, i=1,2$; in the latter case $\hat{c}_{uv}^w=\infty$ except for the degenerate cases, in which the structure constants are equal to 1. Hence, in view of Proposition 18, we obtain a degree-preserving isomorphism of pre-rings $A_{t,\infty}\cong H_*(X,\widehat{\mathbf{k}})$ given by

$$\sigma_w \mapsto C_{w_\circ w}, \quad w\in W.$$

2. Interpolation for $B_t^{(l)}$, $l=1,2$. The argument here is identical to the case of A_t, except the isomorphism is given by

$$\sigma_w \mapsto C_{n-1-r}\in H_*(X_l,\widehat{\mathbf{k}}), \quad r=\ell_l(w). \tag{37}$$

We conclude that the relation between $H^*_{BK}(X,\mathbf{k}) := \mathrm{gr}(A_t)$ and $H_*(X,\widehat{\mathbf{k}})$ is that of "mirror partners": they are different degenerations of a common ring A_t.

16. Strong triangle inequalities

In this section we introduce a redundant system of inequalities equivalent to WTI. This equivalence will be used in the following section.

Let $W = I_2(n)$ with the affine Weyl chamber $\Delta \subset \mathbb{R}^2$, $\mathbf{k}$ a field and $\widehat{\mathbf{k}}$ the corresponding pre-ring. Define the subset

$$\Sigma_{A,m} \subset W^m$$

consisting of m-tuples $(u_1, \dots, u_m)$ of elements of W so that

$$\prod_i C_{u_i} = a \cdot C_{\mathbf{1}}, \qquad a \in \widehat{\mathbf{k}}^{\times}, \tag{38}$$

in the pre-ring $H_*(X, \widehat{\mathbf{k}})$, where X is a thick spherical building with the Weyl group W satisfying Axiom A. We then define cones $K(\Sigma_{A,m}) \subset \Delta^m$ by imposing the inequalities

$$\sum_i u_i^{-1}(\lambda_i) \leq_{\Delta^*} 0$$

for the m-tuples $(u_1, \dots, u_m) \in \Sigma_{A,m}$. We will refer to the defining inequalities of $K(\Sigma_{A,m})$ as *Strong Triangle Inequalities*, or STI.

Recall that $\mathcal{K}_m = \mathcal{K}_m(X) \subset \Delta^m$ is the Stability Cone of X, cut out by the inequalities WTI; see Section 12. Then, clearly,

$$K(\Sigma_{A,m}) \subset \mathcal{K}_m$$

since the system STI contains the WTI. The following is the main result of this section:

Theorem 38.

$$K(\Sigma_{A,m}) = \mathcal{K}_m.$$

Proof. Observe that $u_i \neq \mathbf{1}$ for $i = 1, \dots, m$, for otherwise the product in the left-hand side of (38) is zero. We first establish some inequalities concerning the relative lengths of elements of W:

Proposition 39. *Let $w_i \in W \setminus \{\mathbf{1}\}$, $i = 1, \dots, m$, be such that*

$$\prod_{i=1}^m C_{w_i} = C_{\mathbf{1}}$$

in the pre-ring $H_*(X, \widehat{\mathbf{k}})$. Then for $k = 1, 2$, we have:

$$\sum_{i=1}^m \ell_k(w_i) \geq (m-1)(n-1).$$

In other words, for $r_i := \ell_k(w_i)$,

$$\prod_{i=1}^m C_{r_i} \neq 0$$

in the pre-ring $H_*(X_k, \widehat{\mathbf{k}})$.

Proof. Let $u_i, u \in W^{(j)}$ be such that

$$\prod_{i=1}^{s} C_{u_i} = aC_u, \quad a \neq 0,$$

in the pre-ring $H_*(X, \widehat{\mathbf{k}})$. Then

$$\sum_{i=1}^{s} (n - \ell(u_i)) = n - \ell(u).$$

Since $\ell(u_i) = \ell_k(u_i) + \delta_{jk}, \ell(u) = \ell_k(u) + \delta_{jk}$, it follows that

$$\sum_{i=1}^{s} \ell_k(u_i) = \ell_k(u) + (s-1)(n - \delta_{jk}). \tag{39}$$

We next observe that if $w_i \in W^{(j)}$, then

$$\prod_{i=1}^{m} C_{w_i}$$

is never a nonzero multiple of $C_{\mathbf{1}}$. Hence, after permuting the indices, for the elements w_i as in the proposition, we obtain:

$$w_1, \dots, w_{m'} \in W^{(1)}, \qquad w_{m'+1}, \dots, w_m \in W^{(2)}$$

and for $m = m' + m''$, we have $1 \leq m', m'' \leq m - 1$. Therefore,

$$\prod_{i=1}^{m'} C_{w_i} = a' C_{w'}, \qquad \prod_{i=m'+1}^{m} C_{w_i} = a'' C_{w''}, \tag{40}$$

where $a', a'' \neq 0$ in $\widehat{\mathbf{k}}$, and $w' \in W^{(1)}, w'' \in W^{(2)}$. Moreover,

$$C_{w'} C_{w''} = C_{\mathbf{1}}$$

in $H_*(X, \widehat{\mathbf{k}})$. Therefore, by applying equations (39) to the product decompositions (40), we obtain

$$\sum_{k=1}^{m} \ell_k(w_i) = \ell_k(w') + \ell_k(w'') + mn - 2n + 1 - M$$

where $M = m'\delta_{1k} + m''\delta_{2k} \leq m - 1$. Since $\ell(w') + \ell(w'') = n$, it follows that

$$\ell_k(w') + \ell_k(w'') = n - \delta_{1k} - \delta_{2k} = n - 1.$$

Hence, we obtain

$$\sum_{k=1}^{m} \ell_k(w_i) = (m-1)n - M \geq (m-1)(n-1). \quad \square$$

We are now ready to prove the theorem. We have to show that every $\overrightarrow{\lambda} = (\lambda_1, \ldots, \lambda_m) \in \mathcal{K}_m$ satisfies the inequality

$$\sum_{i=1}^{m} w_i^{-1}(\lambda_i) \leq_{\Delta^*} 0$$

for every $(w_1, \ldots, w_m) \in \Sigma_{A,m}$. The latter is equivalent to two inequalities

$$\sum_{i=1}^{m} \langle \lambda_i, w_i(\zeta_k) \rangle = \sum_{i=1}^{m} \langle w_i^{-1}(\lambda_i), \zeta_k \rangle \leq 0, \qquad k = 1, 2,$$

where $\zeta_k, k = 1, 2$ are the vertices of the fundamental domain $\Delta_{\mathrm{sph}} \subset S^1$ of W.

Suppose that this inequality fails for some k and an m-tuple $(u_1, \ldots, u_m) \in \Sigma_{A,m}$. Since $\overrightarrow{\lambda} \in \mathcal{K}_m$, according to Theorem 19, there exists a semistable weighted configuration $\psi = (\mu_i \xi_i)$ in X of the type $\overrightarrow{\lambda}$, so that the points ξ_i belong to mutually antipodal spherical chambers $\Delta_1, \ldots, \Delta_m$ in X. Since

$$\prod_{i=1}^{m} C_{w_i} = C_{\mathbf{1}},$$

for $r_i := \ell_k(w_i)$, by combining Corollary 15 and Proposition 39, we see that the intersection

$$\bigcap_{i=1}^{m} C_{r_i}(\Delta_i)$$

contains a vertex η of type ζ_k. Thus, as in the proof of Theorem 19,

$$\mathrm{slope}_{\psi}(\eta) = -\sum_{j} \langle \lambda_j, w_j(\zeta_k) \rangle < 0.$$

This contradicts the semistability of ψ. □

17. Triangle inequalities for associative commutative algebras

We now put the concept of stability inequalities into a more general context by associating a system of monoids $K_m(A)$ (generalizing the Stability Cones) to certain commutative and associative rings (which generalize the rings A_t). One advantage of this formalism is to eliminate dependence on the existence of the longest element $w_\circ \in W$ and to get more natural sets of inequalities. We also establish linear isomorphisms of cones $K_m(A_t)$ (defined "cohomologically") applying the above formalism to the algebras A_t and the Stability Cones $\mathcal{K}_{m+1}(Y)$ (defined "homologically"). We conclude this section by showing that the system WTI is irredundant.

Let Λ be a free abelian group (or a free module over an integral domain).

Definition 7. We say that a family of sub-monoids $K_m \subset \Lambda^{m+1}$, $m \geq 1$ is *coherent* if:

(1) The natural S_m-action on the first m-factors of Λ^{m+1} preserves K_m.
(2) For any $(\lambda_1, \dots, \lambda_m; \mu) \in \Lambda^{m+1}$ and $0 < \ell < m$ the following are equivalent:
  - $(\lambda_1, \dots, \lambda_m; \mu) \in K_m$.
  - There exists $\mu' \in \Lambda$ such that $(\lambda_1, \dots, \lambda_m; \mu') \in K_m$ and

$$(\mu', \lambda_{m+1}, \dots, \lambda_\ell; \mu) \in K_{\ell+1-m}.$$

This definition is a natural generalization of the Stability Cone $\mathcal{K}_{m+1}(Y)$ (which describes $m+1$-tuples of Δ-valued side-lengths of polygons in Euclidean buildings $\mathfrak{Y}$ (see Section 5). The first property generalizes the fact that the existence of a polygon with the Δ-side-lengths $(\lambda_1, \dots, \lambda_{m+1})$ is equivalent to the existence of a polygon with the Δ-side-lengths $(\lambda_{\sigma(1)}, \dots, \lambda_{\sigma(m)}, \lambda_{m+1})$, where $\sigma \in S_m$ is a permutation. The second property generalizes the fact that gluing polygons in $\mathfrak{Y}$ along a common side produces a new polygon.

Below we will interpret a coherent family of sub-monoids as above, as a commutative and associative (multivalued) operad.

For any subsets $S \subset \Lambda^{m+1} = \Lambda^m \times \Lambda$, $S' \subset \Lambda^{\ell+1} = \Lambda \times \Lambda^\ell$ define the set $S' \circ S' \subset \Lambda^{m+k} = \Lambda^m \times \Lambda^\ell$ to be the set of all $(\lambda, \lambda') \in \Lambda^m \times \Lambda^\ell$ such that there exists $\mu \in \Lambda$ such that $(\lambda, \mu) \in S$ and $(\mu, \lambda') \in S'$. In other words, if we regard S, S' as correspondences $\Lambda^m \to \Lambda$ and $\Lambda \to \Lambda^\ell$, then $S \circ S'$ is their composition. The following is immediate:

Lemma 40. *The second coherence condition is equivalent to:*

$$K_m \circ K_\ell = K_{m+\ell-1}$$

for all $m, \ell \geq 1$.

The following result is obvious.

Lemma 41. *If K_m, $m \geq 0$ is a coherent family, then each K_m, $m \geq 3$, is the set of all $(\lambda_1, \dots, \lambda_m; \mu) \in \Lambda^{m+1}$ such that there exists a sequence $\mu_1, \dots, \mu_m = \mu$ of elements in Λ such that: $(\lambda_1, \lambda_2; \mu_1) \in K_2$ and $(\mu_k, \lambda_{k+2}; \mu_{k+1}) \in K_2$ for $k = 1, \dots, m-1$.*

We explain the naturality of the coherence condition below. To any submonoid $K_m \subset \Lambda^{m+1}$, $m \geq 1$ we associate an m-ary operation on subsets of Λ as follows. For any subsets $S_1, S_2, \dots, S_m \subset \Lambda$ define $S_1 \bullet S_2 \bullet \dots \bullet S_m \subset \Lambda_+$ to be the image of the intersection $S_1 \times \dots \times S_m \times \Lambda \cap K_m$ under the projection to the $(m+1)$st factor. In particular, if each $S_i = \{\lambda_i\}$ is a single element set, then

$$\lambda_1 \bullet \dots \bullet \lambda_m = \{\mu \in \Lambda_+ \mid (\lambda_1, \dots, \lambda_m; \mu) \in K_m\}.$$

In general,

$$S_1 \bullet \dots \bullet S_m = \bigcup_{(\lambda_1, \dots, \lambda_m) \in S_1 \times \dots \times S_m} \lambda_1 \bullet \dots \bullet \lambda_m.$$

Lemma 42. *If a family of submonoids $K_m \subset \Lambda^{m+1}$, $m \geq 1$ is coherent, then the above operations are:*

(a) *commutative, i.e.,* $S_{\sigma(1)} \bullet \cdots \bullet S_{\sigma(m)} = S_1 \bullet \cdots \bullet S_m$ *for any permutation* σ *of* $\{1, \ldots, m\}$*;*
(b) *associative, i.e.,* $S_1 \bullet \cdots \bullet S_k \bullet (S_{k+1} \bullet \cdots \bullet S_\ell) \bullet S_{\ell+1} \bullet \cdots \bullet S_m = S_1 \bullet \cdots \bullet S_m$ *for all* $1 \le k \le \ell \le m$ *(i.e., informally speaking, these operations comprise a symmetric associative operad; see e.g.,* [MSS] *).*

Proof. Part (a) is an obvious consequence of the first coherence condition.

We will now prove (b). Because of the already established commutativity, it suffices to verify the assertion for $k = 0$. Also it suffices to proceed in the case when each $S_i = \{\lambda_i\}$ is a one-element set. That is, it suffices to prove that

$$(\lambda_1 \bullet \cdots \bullet \lambda_\ell) \bullet \lambda_{\ell+1} \bullet \cdots \bullet \lambda_m = \lambda_1 \bullet \cdots \bullet \lambda_m.$$

The left-hand side is the set of all $\mu \in \Lambda$ such that $(\mu', \lambda_{\ell+1}, \ldots, \lambda_m; \mu) \in K_{m+1-\ell}$ for some $\mu' \in \Lambda$ satisfying $(\lambda_1, \ldots, \lambda_\ell; \mu) \in K_\ell$. By the second coherence condition, this is the set of all $\mu \in \Lambda$ such that $(\lambda_1, \ldots, \lambda_m; \mu) \in K_m$. But this set is exactly the right-hand side of the above equation. This proves (b).

The lemma is proved. □

We now construct families of monoids associated with some associative commutative algebras. Let $\preceq$ be a partial order on Λ compatible with the addition. This is equivalent to choosing a submonoid $\mathcal{M}$ (the "positive root cone") such that $-\mathcal{M} \cap \mathcal{M} = \{0\}$, so that $\lambda \preceq \mu$ if and only if $\mu - \lambda \in \mathcal{M}$ (therefore, $\mathcal{M} = \{\lambda \in \Lambda : 0 \preceq \lambda\}$).

Let A be a commutative associative $\mathbf{k}$-algebra as in Section 14 with the basis labeled by a set $X \subset \mathrm{End}(\Lambda)$ (i.e., the basis acts linearly on Λ). We define the structure coefficients $c^y_{x_1,\ldots,x_m} \in \mathbf{k}$ via

$$b_{x_1} \cdots b_{x_m} = \sum_{y \in X} c^y_{x_1,\ldots,x_m} b_y$$

for all $x_1, \ldots, x_n \in X$.

Given this data, we define:

- The dominant cone Λ_+ to be the set of all $\lambda \in \Lambda$ such that $x(\lambda) \preceq \lambda$ for all $x \in X$.
- For each $m \ge 0$ a subset $K_m(A) \subset \Lambda_+^{m+1} = \Lambda_+^m \times \Lambda_+$ to be the set of all $(\lambda_1, \ldots, \lambda_m; \mu) \in \Lambda_+^{m+1}$ such that

$$y(\mu) \preceq x_1(\lambda_1) + \cdots + x_m(\lambda_m) \tag{41}$$

  for all $x_1, \ldots, x_m, y \in X$ such that $c^y_{x_1,\ldots,x_m} \neq 0$ (with the convention that $K_0(A) = \Lambda_+$).

The following is immediate:

Lemma 43. *The set* $K_m(A)$ *is a submonoid of* Λ^{m+1} *invariant under the* S_m*-action on the first* m *factors.*

Lemma 44. *In the notation of Lemma* 40 *we have:*

$$K_m(A) \circ K_l(A) \subseteq K_{m+l-1}(A) \tag{42}$$

for all $m, l \geq 1$.

Proof. Indeed, let $(\lambda_1, \ldots, \lambda_{m+l-1}; \mu) \in K_m(A) \circ K_l(A)$. This means that there exists $\mu_1 \in \Lambda_+$ such that $(\lambda_1, \ldots, \lambda_m; \mu_1) \in K_m(A)$ and $(\mu_1, \lambda_{m+1}, \ldots, \lambda_{m+l-1}; \mu) \in K_l(A)$. Or, equivalently,

$$\begin{aligned} y_1(\mu_1) &\preceq x_1(\lambda_1) + \cdots + x_m(\lambda_m), \\ y(\mu) &\preceq y_1'(\mu_1) + x_{m+1}(\lambda_{m+1}) + \cdots + x_{m+l-1}(\lambda_{m+l-1}) \end{aligned} \tag{43}$$

for all $x_1, \ldots, x_{m+l-1}, y_1, y \in X$ such that $c^{y_1}_{x_1,\ldots,x_m} \neq 0$ and $c^{y}_{y_1',x_{m+1},\ldots,x_{m+l-1}} \neq 0$. Now fix arbitrary $x_1, \ldots, x_{m+l-1}, y \in X$ such that $c^{y}_{x_1,\ldots,x_{m+l-1}} \neq 0$. Because of the associativity of multiplication in A this implies the existence of y_1 such that $c^{y_1}_{x_1,\ldots,m} \neq 0$ and $c^{y}_{y_1,x_{m+1},\ldots,x_{m+l-1}} \neq 0$. Therefore, we can take $y_1' = y_1$ in (43) and add the inequalities (43). Hence, after canceling the term $y_1(\mu_1)$, we obtain

$$y(\mu) \preceq x_1(\lambda_1) + \cdots + x_{m+l-1}(\lambda_{m+l-1}).$$

The latter inequality holds for all $x_1, \ldots, x_{m+l-1}, y \in X$ such that $c^{y}_{x_1,\ldots,x_{m+l-1}} \neq 0$, hence $(\lambda_1, \ldots, \lambda_{m+l-1}; \mu) \in K_{m+l-1}(A)$. The lemma is proved. □

Thus, in view of Lemmas 40, 43, and 44 the coherence of $K_m(A)$, $m \geq 1$ depends entirely on whether or not the inclusion (42) is an equality.

Problem 45. *Classify all commutative and associative algebras* A *with basis labeled by* $X \subset \mathrm{End}(\Lambda)$ *such that*

$$K_m(A) \circ K_l(A) \supseteq K_{m+l-1}(A). \tag{44}$$

We now specialize to the case associated with finite dihedral Weyl groups. Let $W = W_t$, where $t = e^{\pi\sqrt{-1}/n}$, acting on the 2-dimensional real vector space V. We assume that $\mathbb{R} \otimes \Lambda = V^*$; let $\mathcal{M} = \Delta^* \subset V^*$ be the dual cone to the positive (affine) Weyl chamber $\Delta \subset V$ of W (with respect to the simple roots α_1, α_2), i.e., $\Delta^* = \{\mu \mid \langle \lambda, \mu \rangle \geq 0, \forall \lambda \in \Delta\}$). We take the based ring $A := A_t$, with the basis $\{\sigma_w | w \in W_t\}$; accordingly, we take the based rings $B^{(i)} := B_t^{(i)}$, $i = 1, 2$. Thus, for $\zeta \in V$, $\lambda \in \mathbb{R} \otimes \Lambda$, we have

$$\langle \sigma_w(\lambda), \zeta \rangle = \langle w^{-1}(\lambda), \zeta \rangle = \langle \lambda, w(\zeta) \rangle.$$

Let $A_0, B_0^{(i)}$ be the associated graded algebras of $A, B^{(i)}$ with respect to the filtrations defined by the concave function φ, φ_i given by (33) as in Theorem 35. Define

$$K_m(B) = K_m(B^{(1)}) \cap K_m(B^{(2)}), \qquad K_m(B_0) = K_m(B_0^{(1)}) \cap K_m(B_0^{(2)}).$$

Clearly,

$$K_m(A) \subset K_m(B) \subset K_m(B_0), \qquad K_m(A) \subset K_m(A_0) \subset K_m(B_0).$$

The following is the main result of the section. This is an analogue of the main result of [BKu] in the context of arbitrary finite dihedral groups. Recall that $\lambda^* = -w_\circ \lambda$; see (2).

Theorem 46. *Assume that* $t = e^{\pi\sqrt{-1}/n}$. *Then for each* $m \geq 2$ *we have:*

$$K_m(B_0) = K_m(B) = K_m(A_0) = K_m(A).$$

Moreover, the above cones are isomorphic to the Stability Cone $\mathcal{K}_{m+1}(Y)$ *for any thick spherical building* Y *with the Weyl group* W *via the linear map*

$$\Theta : (\lambda_1, \ldots, \lambda_m; \mu) \mapsto (\mu_1 = \lambda_1^*, \ldots, \mu_m = \mu_m^*, \mu_{m+1} = \mu).$$

Proof. Our goal is to relate the defining inequalities for the cone $K_m(A)$ to Strong Triangle Inequalities; it will then follow that

$$K_m(B_0) = K_m(A).$$

Set $PD(w) = w_\circ w$ in W. Observe that for $u_1, \ldots, u_m, v \in W$ and $\lambda_1, \ldots, \lambda_m, \mu \in V^*$,

$$\sum_{i=1}^{m} \sigma_{u_i}(\lambda_i) \geq_{\Delta^*} \sigma_v(\mu) \iff$$
$$\sum_{i=1}^{m} u_i^{-1}(\lambda_i) \geq_{\Delta^*} -v^{-1} w_\circ \mu^* = -PD(v)^{-1}\mu^* \iff$$
$$\sum_{i=1}^{m} u_i^{-1}(\lambda_i) + PD(v)^{-1}\mu^* \geq_{\Delta^*} 0 \iff$$
$$\sum_{i=1}^{m+1} u_i^{-1}(\lambda_i) \geq_{\Delta^*} 0$$

where $u_{m+1} := PD(v)$ and $\lambda_{m+1} := \mu^*$. Setting $w_i := PD(u_i)$, $\mu_i := \lambda_i^*$, we see that

$$\sum_{i=1}^{m+1} u_i^{-1}(\lambda_i) \geq_{\Delta^*} 0 \iff \sum_{i=1}^{m+1} w_i^{-1}(\mu_i) \leq_{\Delta^*} 0.$$

Moreover,

$$c^{v}_{u_1,\ldots,u_m} \neq 0 \text{ in } A \iff$$
$$c^{w_\circ}_{u_1,\ldots,u_{m+1}} \neq 0 \text{ in } A \iff$$
$$\prod_{i=1}^{m+1} C_{w_i} = aC_{\mathbf{1}}, \quad a \neq 0 \text{ in } H_*(Y, \widehat{\mathbf{k}}). \tag{45}$$

Recall that the system of inequalities

$$\sum_{i=1}^{m+1} w_i^{-1}(\mu_i) \leq_{\Delta^*} 0, \ \forall (w_1, \ldots, w_{m+1}), \quad \text{so that (45) holds}$$

is the system of Strong Triangle Inequalities. Therefore, the maps $u_i \mapsto PD(u_i)$, $i = 1, \ldots, m+1$, and

$$(\lambda_1, \ldots, \lambda_m; \mu) \mapsto (\mu_1 = \lambda_1^*, \ldots, \mu_m = \mu_m^*, \mu_{m+1} = \mu)$$

determine a natural bijection between the set of defining inequalities for the cone $K_m(A)$ and the set of Strong Triangle Inequalities. Similarly, we obtain a bijection between the defining inequalities of $K_m(B_0)$ and the set of Weak Triangle Inequalities. However, Strong Triangle Inequalities and Weak Triangle Inequalities determine the same cone, the Stability Cone $\mathcal{K}_{m+1}$; see Theorem 38. Therefore, the map

$$\Theta : (\lambda_1, \ldots, \lambda_m; \mu) \mapsto (\mu_1 = \lambda_1^*, \ldots, \mu_m = \mu_m^*, \mu_{m+1} = \mu)$$

determines the linear isomorphisms of the cones

$$K_m(A) \to \mathcal{K}_{m+1}, \qquad K_m(B_0) \to \mathcal{K}_{m+1}.$$

In particular, $K_m(A) = K_m(B) = K_m(B_0)$. The theorem follows. $\square$

Corollary 47. *$K_m(A)$ is invariant under $* : \lambda \to \lambda^*, \lambda \in \Delta$.*

Proof. Let Y be a thick spherical building as above. Then $*$ extends to an isometry $* : Y \to Y$. Since isometries preserve the (semi)stability condition, it follows that $\mathcal{K}_{m+1} = \mathcal{K}_{m+1}(Y)$ is invariant under $*$. Since Θ is $*$-equivariant, it follows that $K_m(A)$ is invariant under $*$ as well. $\square$

Corollary 48. *For the algebra A as above we have*

$$K_m(A) \circ K_l(A) = K_{m+l-1}(A).$$

Proof. Let $\mathfrak{Y}$ denote a thick Euclidean building modeled on $(\mathbb{R}^2, W)$. In view of the above theorem, we can interpret $K_k(A)$ as the set of $m+1$-tuples $(\lambda_1, \ldots, \lambda_m; \mu)$ which are Δ-valued side-lengths of "disoriented" geodesic $k+1$-gons $P = y_0 \ldots y_k$ in $\mathfrak{Y}$, so that

$$d_\Delta(y_{i-1}, y_i) = \lambda_i, \ 1 \le i \le k, \qquad d_\Delta(y_0, y_k) = \mu.$$

(Note that the last side of P has the orientation opposite to the rest.) For $k = m+l-1$, subdivide such a polygon by the diagonal $\overline{y_0 y_l}$ in two disoriented polygons

$$P' := y_0 y_1 \ldots y_l, \qquad P'' = y_0 y_l \ldots y_k.$$

Then the Δ-side lengths of these polygons are given by the tuples

$$(\lambda_1, \ldots, \lambda_l; \mu') \in K_l(A), \qquad (\mu', \lambda_{l+1}, \ldots, \lambda_k; \mu'') \in K_m(A),$$

where

$$\mu' = d_\Delta(y_0, y_l), \qquad \mu'' = \mu.$$

Hence, $K_{m+l-1}(A) \subset K_m(A) \circ K_l(A)$. $\square$

Theorem 49. *The system of inequalities* (1) *is irredundant.*

Proof. The system of inequalities (1) is nothing but the linear system

$$\langle w(\lambda_i - \lambda_j^*), \zeta_l \rangle \leq \Big\langle \sum_{k\neq i, k\neq j} \lambda_k^*, \zeta_l \Big\rangle, \qquad l = 1, 2, \;\; w \in W. \tag{46}$$

Fix regular vectors $\lambda_l \in \Delta, l = 1, \dots, m, l \neq i, l \neq j$ (i.e., vectors from the interior of the Weyl chamber Δ); then pick a vector $\lambda_j \in \Delta$ so that the distance from λ_j to the boundary of Δ is at least $|\lambda_K^*|$, where

$$\lambda_K^* := \sum_{k\neq i,j} \lambda_k^*.$$

Note that the vector λ_K^* is again regular. Set

$$P = P_{\lambda_1,\dots,\lambda_{i-1},\lambda_{i+1},\dots\lambda_m} = \lambda_j^* + \mathrm{Hull}(W \cdot \lambda_K^*) \subset \Delta.$$

Here *Hull* denotes the convex hull in $\mathbb{R}^2$. Then, for fixed $\lambda_l, l \neq i$ as above, the solution set to the Weak Triangle Inequalities (1) is exactly the polygon P. Since λ_K^* is regular, P is a $2m$-gon. Moreover, for each side of P exactly one of the defining inequalities (46) is an equality. $\square$

Belkale–Kumar inequalities

In the context of complex algebraic reductive groups G, Belkale and Kumar [BKu] gave a certain description of the Stability Cone $\mathcal{K}_{m+1}$ using the rings $H^*_{BK}(G/P)$, where P runs through the set of standard maximal parabolic subgroups of G, corresponding to the fundamental weights. In the context of rank 2 spherical buildings X, using our language, the system of Belkale–Kumar inequalities, imposed on vectors

$$(\lambda_1, \dots, \lambda_m; \mu) \in \Delta^{m+1},$$

reads as follows: For every $(x_1, \dots, x_m; y) \in (W^{(k)})^{m+1}, k = 1, 2,$ so that $c^y_{x_1,\dots,x_m} \neq 0$ in $\mathrm{gr}(B^{(k)})$, we impose the inequality:

$$\sum_{i=1}^m \langle \zeta_l, x_i(\lambda_i)\rangle \geq \langle \zeta_l, y(\mu)\rangle.$$

We now observe that under the map $H_*(X_k, \widehat{\mathbf{k}}) \to H^*_{BK}(X_l, \mathbf{k}) := \mathrm{gr}(B^{(k)})$, determined by the inverse to the map (37), the "infinities" in $H_*(X_k, \widehat{\mathbf{k}})$ correspond to zeroes in $H^*_{BK}(X, \mathbf{k})$. Accordingly, the structure constants equal to 1 match structure constants equal to 1. Since the system WTI is irredundant, we conclude that the system of Belkale–Kumar inequalities for $W = I_2(n)$ is also irredundant. Hence, Theorem 49 is an analogue (for $W = I_2(n)$) of a much deeper theorem by N. Ressayre [Re], who proved the irredundancy of Belkale–Kumar inequalities for arbitrary reductive groups.

References

[B] W. Ballmann, *Lectures on Spaces of Nonpositive Curvature*, DMV Seminar, Vol. 25, Birkhauser Verlag, Basel, 1995.

[BK] A. Berenstein, M. Kapovich, *Affine buildings for dihedral groups*, Geometria Dedicata, to appear.

[BKu] P. Belkale, S. Kumar, *Eigenvalue problem and a new product in cohomology of flag varieties*, Invent. Math. **166** (2006), no. 1, 185–228.

[BS] A. Berenstein, R. Sjamaar, *Projections of coadjoint orbits, moment polytopes, and the Hilbert–Mumford criterion*, J. Amer. Math. Soc. **13** (2000), 433–466.

[BH] M. Bridson, A. Haefliger, *Metric Spaces of Nonpositive Curvature*, Grundlehren der Mathematischen Wissenschaften, Vol. 319, Springer-Verlag, Berlin, 1999.

[C] C. Ramos-Cuevas, *Generalized triangle inequalities in thick Euclidean buildings of rank* 2, preprint, `arXiv:1009.1316v1`, 2010.

[FS] M. Funk, K. Strambach, *Free constructions*, in: *Handbook of Incidence Geometry*, North-Holland, Amsterdam, 1995, pp. 739–780.

[HP] E. Hille, R. Phillips, *Functional Analysis and Semi-Groups*, 3d printing of the revised edition of 1957, American Mathematical Society Colloquium Publications, Vol. XXXI, American Mathematical Society, Providence, RI, 1974. Russian transl.: Э. Хилле, Р. Филлипс, *Функциональный анализ и полугруппы*, ИЛ, M., 1962.

[H] H. Hiller, *Geometry of Coxeter groups*, Research Notes in Mathematics, Vol. 54, Pitman, Boston, 1982.

[KLM1] M. Kapovich, B. Leeb, J. Millson, *Convex functions on symmetric spaces, side lengths of polygons and the stability inequalities for weighted configurations at infinity*, J. Diff. Geom. **81** (2009), 297–354.

[KLM2] M. Kapovich, B. Leeb, J. Millson, *Polygons in buildings and their refined side lengths*, Geom. Funct. Anal. **19** (2009), no. 4, 1081–1100.

[KLM3] M. Kapovich, B. Leeb, J. Millson, *The Generalized Triangle Inequalities in Symmetric Spaces and Buildings with Applications to Algebra*, Memoirs of AMS, Vol. 192, 2008.

[Kit] N. Kitchloo, *On the topology of Kac–Moody groups*, preprint, `arXiv:0810.0851`, 2008.

[KL] B. Kleiner, B. Leeb, *Rigidity of quasi-isometries for symmetric spaces and Euclidean buildings*, Publ. Math. Inst. Hautes Études Sci. **86** (1997), 115–197.

[K] A. Klyachko, *Stable bundles, representation theory and Hermitian operators*, Selecta Math. **4** (1998), 419–445.

[KK] B. Kostant, S. Kumar, *The nil Hecke ring and the cohomology of G/P for a Kac–Moody group G*, Adv. Math. **62** (1986), 187–237.

[KT] L. Kramer, K. Tent, *Algebraic polygons*, J. Algebra **182** (1996), no. 2, 435–447.

[LPS] U. Lang, B. Pavlović, V. Schroeder, *Extensions of Lipschitz maps into Hadamard spaces*, Geom. Funct. Anal. **10** (2000), no. 6, 1527–1553.

[MSS] M. Markl, S. Snider, J. Stasheff, *Operads in Algebra, Topology and Physics*, Math. Surveys Monogr., Vol. 96, Amer. Math. Soc., Providence, RI, 2002.

[P] A. Parreau, *Immeubles affines: construction par les normes et étude des isométries*, in: *Crystallographic Groups and their Generalizations'* (Kortrijk, 1999), Contemp. Math., Vol. 262, Amer. Math. Soc., Providence, RI, 2000, pp. 263–302.

[Re] N. Ressayre, *Geometric invariant theory and the generalized eigenvalue problem*, Invent. Math. **180** (2010), no. 2, p. 389–441.

[Ro] M. Ronan, *Lectures on Buildings*, Perspectives in Mathematics, Vol. 7, Academic Press, 1989.

[Te] K. Tent, *Very homogeneous generalized n-gons of finite Morley rank*, J. London Math. Soc. (2) **62** (2000), no. 1, 1–15.

[Ti] J. Tits, *Endliche Spiegelungsgruppen, die als Weylgruppen auftreten*, Invent. Math. **43** (1977), no. 3, 283–295.

[W] R. Weiss, *Structure of Affine Buildings*, Annals of Math. Studies, Vol. 168, Princeton University Press, Princeton, NJ, 2008.