A combination theorem for Anosov subgroups

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Abstract

We prove an analogue of Klein combination theorem for Anosov subgroups by using a local-to-global principle for Morse quasigeodesics.

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1 Introduction

The combination theorems in geometric group theory provide tools to construct new groups with "nice" geometric properties out of old ones. The classical Klein combination theorem [Kle83] states that under certain assumptions the group $\langle \Gamma_1, \Gamma_2 \rangle$ generated by two Kleinian groups Γ_1 and Γ_2 is Kleinian, and is naturally isomorphic to the free product $\Gamma_1 * \Gamma_2$. In a series of articles [Mas65, Mas68, Mas71, Mas93], Maskit generalized the Klein combination theorem to amalgamated

free products and HNN extensions. These so called "Klein-Maskit combination theorems" have been generalized to the geometrically finite subgroups of the isometry groups of higher dimensional hyperbolic spaces by several mathematicians. For instance, in [BC08], Baker and Cooper proved the following theorem.

Theorem 1.1 (Virtual amalgam theorem, [BC08])**.** *If* Γ_1 *and* Γ_2 *are two geometrically finite subgroups of* $\text{Isom}(\mathbb{H}^n)$ *which have compatible parabolic subgroups, and if* $H = \Gamma_1 \cap \Gamma_2$ *is separable in* Γ_1 *and* Γ_2, *then there exists finite index subgroups* Γ'_1 *and* Γ'_2 *of* Γ_1 *and* Γ_2, *respectively, containing* H *such that the group* $\langle \Gamma'_1, \Gamma'_2 \rangle$ *generated by* Γ'_1 *and* Γ'_2 *is geometrically finite, and is naturally isomorphic to the amalgam* $\Gamma'_1 *_H \Gamma'_2$.

When Γ_1 and Γ_2 intersect trivially, the "compatibility condition" in the above theorem simply means that the limit sets of Γ_1 and Γ_2 in $\partial_\infty \mathbb{H}^n$ are disjoint. Since this case would be most relevant to our work, we state it separately.

Corollary 1.2. *If* Γ_1 *and* Γ_2 *are two geometrically finite subgroups of* $\text{Isom}(\mathbb{H}^n)$ *with disjoint limit sets in* $\partial_\infty \mathbb{H}^n$, *then there exists finite index subgroups* Γ'_1 *and* Γ'_2 *of* Γ_1 *and* Γ_2, *respectively, such that the group* $\langle \Gamma'_1, \Gamma'_2 \rangle$ *generated by* Γ'_1 *and* Γ'_2 *is geometrically finite and is naturally isomorphic* $\Gamma'_1 * \Gamma'_2$.

There are also certain generalizations of these combination theorems in the realm of subgroups of hyperbolic groups and, more generally, isometry groups of Gromov-hyperbolic spaces. In [Git99], Gitik proved that under certain conditions two quasiconvex subgroups of a δ-hyperbolic group "can be virtually amalgamated." In this regard, our main result (Theorem 1.3) are analogs of [Git99, Corollary 3]; see also Corollary 4 in the paper of Martínez-Pedroza–Sisto [MPS12] for a virtual amalgamation theorem in the relatively hyperbolic setting. Similarly, the generalization of our Theorem 1.3 given in Theorem 5.2 is an analog of [MP09, Theorem 1.1] and [MPS12, Theorem 2]. More specifically, these results of Martínez-Pedroza and Martínez-Pedroza–Sisto establish sufficient conditions for two subgroups A, B of a relatively hyperbolic group G to generate a relatively hyperbolic subgroup of G naturally isomorphic to $A *_{A \cap B} B$, where the conditions are in terms of a sufficiently high displacement for certain group elements. This setup is analogous to the one in Theorem 5.2 of the present paper where instead of passing to finite index subgroups we make certain high displacement assumptions.

In the present work, we prove a combination theorem for Anosov subgroups of semisimple Lie groups. Anosov representations of surface groups (and, more generally, fundamental groups of compact negatively curved manifolds) were introduced by Labourie [Lab06] to study the "Hitchin component" of the space of reducible representations in $PSL(n, \mathbb{R})$. Guichard and Weinhard [GW12] generalized this notion in the setting of representations of hyperbolic groups in real semisimple Lie groups. Anosov subgroups can be regarded as higher rank generalizations of convex-cocompact subgroups of isometry groups of negatively curved symmetric spaces. We refer the reader to [KL2] for *relativizations* of the class of Anosov subgroups: The relative notions developed there are analogous to the notion of geometrically finite isometry groups of rank 1 symmetric spaces which may contain parabolic elements. However, in the present paper we restrict our attention to Anosov subgroups and leave the discussion of combination theorems in the relative setting to a future investigation.

Our main result presents an analogue of Corollary 1.2 for Anosov subgroups. Let G be a semisimple Lie group, let P be a maximal parabolic subgroup conjugate to its opposite subgroups.

Our main result is the following.

Theorem 1.3 (Combination theorem)**.** *Let $\Gamma_1, \ldots, \Gamma_n$ be pairwise antipodal, residually finite[1] P-Anosov subgroups of G. Then there exist finite index subgroups Γ_i' of Γ_i, for $i = 1, \ldots, n$, such that the subgroup $\langle \Gamma_1', \ldots, \Gamma_n' \rangle$ generated by $\Gamma_1', \ldots, \Gamma_n'$ in G is P-Anosov, and is naturally isomorphic to the free product $\Gamma_1' * \cdots * \Gamma_n'$.*

The undefined term "antipodal" will be made precise later in the paper (Definition 4.22): This condition replaces the disjointness of the limit sets in Corollary 1.2. Moreover, the geometric finiteness in Corollary 1.2 is replaced by the Anosov condition.

In fact, our combination theorem is a special case of a more geometric result (Theorem 5.2), stated in terms of "sufficiently high displacements" and "sufficient antipodality" of the groups Γ_i at a point $x \in X = G/K$; with this condition, there is no need to pass to finite index subgroups.

Although we state main result (Theorem 1.3) in the language of Anosov representations, we never really use it in our proof. Instead, we use the language of *Morse subgroups*, and prove an equivalent statement in this context (Theorem 5.1).

In [KLPa], Kapovich, Leeb and Porti introduced a class of discrete subgroups of isometries of higher rank symmetric spaces. This class of subgroups generalizes the convex cocompact subgroups in the rank one Lie groups. In [KLPa] and in subsequent articles [KLPb, KLP17, KLP18], they introduced and proved several equivalent definitions of this class, and studied their geometric properties (e.g. structural stability, cocompactness etc.). Some of these equivalent definitions are given in terms of RCA subgroups, URU subgroups, Morse subgroups, asymptotically embedded subgroups, etc. In [KLPa], they also proved that the classes of Morse subgroups and Anosov subgroups are equal.

Theorem 1.4 (Morse $\Leftrightarrow$ Anosov, [KLPa])**.** *For a discrete subgroup Γ of G, the following are equivalent.*

 1. *Γ is $P_{\tau_{\mathrm{mod}}}$-Anosov.*

 2. *Γ is τ_{mod}-Morse.*

See also [KL1] and [KLP16] for detailed surveys on these results.

In [KLPa, Theorem 7.40] they used the local-to-global principle for Morse quasigeodesics to construct (free) Morse-Schottky subgroups of semisimple Lie groups (cf. also [Ben97]):

Theorem 1.5. *Suppose that $g_1, \ldots, g_n$ are hyperbolic isometries of a symmetric space $X = G/K$ of noncompact type, whose repelling/attracting points in the flag-manifold $G/P_{\tau_{mod}}$ are pairwise antipodal. Then for all sufficiently large N, the subgroup of G generated by $g_1^N, \ldots, g_n^N$ is τ_{mod}-Morse and free of rank k.*

While our main theorem contains this result as a special case when the subgroups $\Gamma_1, \ldots, \Gamma_n$ are cyclic, our proof uses some of the main ideas of the proof of [KLPa, Theorem 7.40].

Organization of this paper

In Section 2, we give a brief overview on symmetric spaces of noncompact type, Δ-valued distances and the triangle inequalities, τ_{mod}-regularities, parallel sets, ξ-angles, Θ-cones, and Θ-diamonds, mostly to set up our notations while leaving the details to the references. In Section 3, we prove

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[1]It suffices to assume that each Γ_i has trivial intersection with the center of G. See also the remark following Theorem 5.1.

several estimates on ξ-angles which will provide crucial ingredients for construction of Morse embeddings in the proof of our main result. In Section 4, or more specifically in 4.1 and 4.4, we discuss Morse properties. In Section 4.3, we introduce the replacement lemma (Theorem 4.11, and a generalized version Theorem 4.13) which is another important ingredient in the proof of our main result. In Section 4.5, we discuss the residual finiteness property of Morse subgroups. Finally, in Section 5, we state and prove our main result in terms of Morse subgroups (Theorem 5.1).

2   Geometric background

In this section, we first review some notions pertinent to geometry of *symmetric spaces of noncompact type*. A standard reference for this section is [Ebe96]. Then we briefly review various notions such as ideal boundaries, Tits metrics, τ_{mod}-regularity, Θ-cones, Θ-diamonds etc. enough to fix our notations and conventions. For a detailed exposition on these topics, we refer to [KLPa, KLPb].

2.1   Symmetric spaces of non-compact type

A *(global) symmetric space* X is a Riemannian manifold which has an *inversion symmetry* about each point, i.e. for each point $x \in X$, there exists an isometric involution $s_x : X \to X$ fixing x, called the *Cartan involution*, whose differential ds_x restricts to $-\mathrm{Id}$ on the tangent space $T_x X$. In this paper we consider only simply connected symmetric spaces.

Each symmetric space has a de Rham decomposition into irreducible symmetric spaces. A symmetric space X is said to be of *noncompact type* if it is nonpositively curved, simply connected and without a Euclidean factor. Under these assumptions, X is a *Hadamard manifold*, and is diffeomorphic to a Euclidean space.

A semisimple Lie algebra $\mathfrak{g}$ is called *compact* if its Killing form is negative definite. A semisimple Lie group G is compact if and only if its Lie algebra is compact. G is said to have *no compact factors* or of *noncompact type* if none of the factors of the direct sum decomposition of its Lie algebra $\mathfrak{g}$ into simple Lie algebras is compact, and the decomposition has no commutative factors.

Let G be a semisimple Lie group with no compact factors and with a finite center, let K be a maximal compact subgroup of G. Maximal compact subgroups of G are conjugate to each other. The coset space $X = G/K$ can be given a natural G-invariant Riemannian metric with respect to which it becomes a symmetric space of noncompact type. Moreover, under our assumptions, G is *commensurable* with the isometry group of X, $\mathrm{Isom}\,(X)$, in the sense that the homomorphism $G \to \mathrm{Isom}\,(X)$ has finite kernel and cokernel. The group G acts on $X = G/K$ transitively, so X is a homogeneous G-space.

In fact, any symmetric space of noncompact type arises as a quotient space as above. Let X be a symmetric space of noncompact type, and let $\mathrm{Isom}_0\,(X)$ be the identity component of $\mathrm{Isom}\,(X)$. Then $\mathrm{Isom}_0\,(X)$ is a semisimple Lie group with no compact factors and trivial center. We can identify X with the quotient $\mathrm{Isom}_0\,(X)/\mathrm{Isom}_0\,(X)_x$ where $\mathrm{Isom}_0\,(X)_x$ is the isotropy subgroup for some $x \in X$.

In the rest of this paper we reserve the letter X to denote a symmetric space of noncompact type. We identify X with G/K where G and K are as above. More assumptions on G will be

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made later on, see Section 2.3.

A *flat* in X is a totally geodesic submanifold of zero sectional curvature. A flat is called *maximal* if it is not properly contained in another flat. The group G acts transitively on the set of all maximal flats; the dimension of a maximal flat is called the *rank* of X.

A choice of a maximal flat will be called the *model flat*, and will be denoted by F_{mod}. F_{mod} is isometric to $\mathbb{E}^k$, where k is the rank of X. The image of the subgroup $G_{F_{\mathrm{mod}}} < G$ stabilizing the model flat in the group of isometric affine transformations $\mathrm{Isom}\,(F_{\mathrm{mod}})$ under restriction homomorphism is a semidirect product $\mathbb{R}^k \rtimes W$. Here W, called the *Weyl group*, is a (finite) group of isometries of F_{mod} generated by reflections fixing a chosen base point (origin) o_{mod}. A fundamental domain for the action $W \curvearrowright F_{\mathrm{mod}}$ is a certain convex cone with tip at o_{mod}, called the *model Weyl chamber*, and will be denoted by Δ.

2.2 Δ-valued distances

Given any two points $x, y \in X$, the unique oriented geodesic segment from x to y will be denoted by xy. All geodesics considered in this paper are unit speed parametrized. We denote the distance between x and y by $d(x, y)$.

Each oriented segment xy uniquely defines a vector v in Δ which can be realized as follows.

Any geodesic segment xy can be extended to a complete geodesic $f \subset X$ which is, in fact, a flat of dimension one. This geodesic f is contained in a maximal flat F. There exists an isometry $g \in G$ sending F to F_{mod}, x to o_{mod} and y to Δ. The vector $v \in \Delta$ is defined as the image $g(y)$; it is independent of the choice of g. This vector v is called the Δ*-valued distance* from x to y, and denoted by $d_\Delta(x, y)$.

It follows from our discussion that $d_\Delta(x, y)$ is a complete G-congruence invariant for an oriented segment xy or an ordered pair (x, y). Precisely, for two pairs of points (x, y) and (x', y'), there exists $g \in G$ satisfying $(gx, gy) = (x', y')$ if and only if $d_\Delta(x, y) = d_\Delta(x', y')$.

The Δ-valued distances satisfy a set of inequalities which are generalizations of the ordinary triangle inequality (see [KLM09]). For our purpose, the following form will be sufficient.

Triangle inequality for Δ-valued distances. *For any triple of points $x, y, z \in X$,*

$$\|d_\Delta(x, y) - d_\Delta(x, z)\| \le d(y, z),$$

where $d_\Delta(x, y) - d_\Delta(x, z)$ is realized as a vector in F_{mod} and $\|\cdot\|$ is the induced Euclidean norm.

2.3 Ideal boundaries and Tits buildings

Two geodesic rays in X are said to be *asymptotic* if they are within a finite Hausdorff distance from each other. The *ideal* or *visual boundary* $\partial_\infty X$ is the set of asymptotic classes of rays. Given $x \in X$ and an asymptotic class ζ, the unique ray emanating from x which is a member of the asymptotic class ζ will be denoted by $x\zeta$. For a fixed base point $x \in X$, the set $\partial_\infty X$ can be metrized by the *angle metric* $\angle_x$,

$$\angle_x(\zeta_1, \zeta_2) = \text{angle between the rays } x\zeta_1 \text{ and } x\zeta_2.$$

The *visual topology* on $\partial_\infty X$ induced by an angle metric $\angle_x$ is independent of the choice of a base point. In fact, $\partial_\infty X$ is homeomorphic to S^{n-1} where n is the dimension of X.

The natural *Tits metric* on the ideal boundary $\partial_\infty X$ can be defined as

$$\angle_{\mathrm{Tits}}(\zeta, \eta) = \sup_{x \in X} \angle_x(\zeta, \eta).$$

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This metric defines *Tits topology* on $\partial_\infty X$ which is finer than the visual topology, and $\partial_\infty X$ equipped with this topology is called the *Tits boundary* of X denoted by $\partial_{\text{Tits}} X$.

The Weyl group W acts as a reflection group on the Tits boundary $a_{\text{mod}} = \partial_{\text{Tits}} F_{\text{mod}} \cong S^{k-1}$, where k is the rank of X. The pair (a_{mod}, W) is called the *spherical Coxeter complex* associated with X. The quotient $\sigma_{\text{mod}} = a_{\text{mod}}/W$ is called the *spherical model Weyl chamber* which we identify as a fundamental chamber of (a_{mod}, W). Accordingly, we regard the model Weyl chamber Δ of F_{mod} as a cone in F_{mod} with tip at the origin o_{mod} and ideal boundary σ_{mod}.

The spherical Coxeter complex structure on a_{mod} induces a G-invariant spherical simplicial structure on $\partial_{\text{Tits}} X$. This simplicial complex, called the *spherical* or *Tits building* associated to X; we assume this building to be *thick*[2]. The facets of this simplicial complex are called *chambers* in $\partial_{\text{Tits}} X$ and the ideal boundaries of maximal flats are *apartments* in $\partial_{\text{Tits}} X$.

Each chamber is naturally identified with the model chamber σ_{mod} under the projection map (also called the *type map*)

$$\theta : \partial_{\text{Tits}} X \to \sigma_{\text{mod}}.$$

The type map is equivariant with respect to the isometric actions of $\text{Isom}(X)$ on $\partial_{\text{Tits}} X$ and σ_{mod}; hence, G acts on σ_{mod}.

From now on, we always assume that G acts on the model chamber σ_{mod} trivially. In particular, G preserves each de Rham factor of X and the type map θ amounts to the quotient map $\partial_{\text{Tits}} X \to \partial_{\text{Tits}} X/G$.

For an ideal point ζ, $\zeta_{\text{mod}} = \theta(\zeta) \in \sigma_{\text{mod}}$ is called the *type* of ζ. Accordingly, for a face τ of a chamber σ, the face $\tau_{\text{mod}} = \theta(\tau)$ of σ_{mod} is called the *type* of τ.

We denote the *opposition involution* on σ_{mod} by

$$\iota = -w_0,$$

where w_0 denotes the longest element in $W \curvearrowright a_{\text{mod}}$.

Two simplices τ_1, τ_2 of $\partial_{\text{Tits}} X$ are called *antipodal* or *opposite* if there exists a point $x \in X$ such that $s_x(\tau_1) = \tau_2$, where s_x is the Cartan involution with respect to x. Equivalently, two such simplices are contained in an apartment a such that the antipodal map $-\text{Id}$ (induced by a Cartan involution) sends τ_1 to τ_2. Their types are related by $\theta(\tau_1) = \iota\theta(\tau_2)$.

Throughout the paper, we will consider only ι-invariant faces τ_{mod} of σ_{mod}. For every such face we pick one and for all a fixed point $\xi = \xi_{\text{mod}}$ of ι in the interior of τ_{mod}. Then, for every simplex τ in $\partial_{\text{Tits}} X$ of type τ_{mod}, we define a point $\xi_\tau \in \tau$ by

$$\{\xi_\tau\} = \theta^{-1}(\xi_{\text{mod}}) \cap \tau.$$

For a type (face) τ_{mod} of σ_{mod} and a point $x \in X$, we define the ξ-*angle* between two simplices τ_1 and τ_2 of type τ_{mod} with respect to x by

$$\angle_x^\xi(\tau_1, \tau_2) = \angle_x(\xi_{\tau_1}, \xi_{\tau_2}).$$

Given τ_{mod}-regular segments xy_1, xy_2 in X, we define the ξ-angle

$$\angle_x^\xi(y_1, y_2) := \angle_x^\xi(\tau_1, \tau_2),$$

where $y_i \in V(x, \text{st}(\tau_i)), i = 1, 2$ (see the notion of τ_{mod}-regularity and $V(x, \text{st}(\tau))$ in the next subsection).

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[2]Here "thick" means every simplex of codimension 1 is a face of at least three maximal simplices.

The angular distance $\angle_x^\xi$ induces a *visual topology* on the space of simplices of type τ_{mod}. The group G acts transitively on this space. The stabilizers P_τ of simplices $\tau \subset \partial_{\mathrm{Tits}}X$ are called the *parabolic subgroups* of G. After identifying τ_{mod} with a simplex τ of type τ_{mod}, the space of simplices of type τ_{mod},

$$\mathrm{Flag}\,(\tau_{\mathrm{mod}}) = G/P_{\tau_{\mathrm{mod}}},$$

is called the *partial flag manifold*. The topology of $\mathrm{Flag}\,(\tau_{\mathrm{mod}})$ as a homogeneous G-space agrees with the visual topology.

2.4   Parallel sets, regularity, cones and diamonds

We often denote a pair of antipodal simplices by τ_+ and τ_-. Let $\tau_\pm$ be a pair of antipodal simplices of same type τ_{mod}. Every such pair $\tau_\pm$ is contained in a unique minimal[3] singular sphere $S \subset \partial_\infty X$. The *parallel set* of the pair $\tau_\pm$ is defined to be the union of all flats in X which are asymptotic to S, and denoted by $P(\tau_-, \tau_+)$. Equivalently, $P(\tau_-, \tau_+)$ is the union of all maximal flats F whose ideal boundaries $\partial_\infty F$ contain $\tau_\pm$. The parallel set $P(\tau_-, \tau_+)$ is a nonpositively curved symmetric space with Euclidean de Rham factor.

In the simplicial complex $\partial_{\mathrm{Tits}}X$, we define the *star* $\mathrm{st}\,(\tau)$, the *open star* $\mathrm{ost}\,(\tau)$ and the *boundary* $\partial\mathrm{st}\,(\tau)$ for a simplex $\tau \in \partial_{\mathrm{Tits}}X$ as

$$\mathrm{st}\,(\tau) = \text{minimal subcomplex of } \partial_{\mathrm{Tits}}X \text{ consisting of simplices } \sigma \supset \tau,$$

$$\mathrm{ost}\,(\tau) = \text{union of all open simplices whose closure intersects int}\,(\tau),$$

$$\partial\mathrm{st}\,(\tau) = \mathrm{st}\,(\tau) - \mathrm{ost}\,(\tau).$$

Accordingly, we denote the open star and boundary of the star of a model face τ_{mod} in the simplicial complex σ_{mod} by $\mathrm{ost}\,(\tau_{\mathrm{mod}})$ and $\partial\mathrm{st}\,(\tau_{\mathrm{mod}})$, respectively. Note that the simplicial map $\theta : \partial_{\mathrm{Tits}}X \to \sigma_{\mathrm{mod}}$ sends $\mathrm{ost}\,(\tau)$ and $\partial\mathrm{st}\,(\tau)$ to $\mathrm{ost}\,(\tau_{\mathrm{mod}}) \subset \sigma_{\mathrm{mod}}$ and $\partial\mathrm{st}\,(\tau_{\mathrm{mod}}) = \sigma_{\mathrm{mod}} - \mathrm{ost}\,(\tau_{\mathrm{mod}})$, respectively, where τ_{mod} is the type of τ.

For a subset $\Theta \subset \mathrm{ost}\,(\tau_{\mathrm{mod}})$, we define the τ_{mod}-*boundary* $\partial\Theta$ in the topological sense as a subset of $\mathrm{ost}\,(\tau_{\mathrm{mod}})$, where the topology is provided by the Tits metric. We define the interior $\mathrm{int}\,(\Theta)$ of Θ as $\Theta - \partial\Theta$. If Θ is compact, then $\epsilon_0(\Theta) := \angle_{\mathrm{Tits}}(\partial\mathrm{st}\,(\tau), \Theta) > 0$. Moreover, if Θ' and Θ are two compact subsets of $\mathrm{ost}\,(\tau_{\mathrm{mod}})$ such that $\Theta \subset \mathrm{int}\,(\Theta')$, a scenario we will often consider in our paper, then $\epsilon_0(\Theta, \Theta') := \angle_{\mathrm{Tits}}(\Theta, \partial\Theta') > 0$.

A subset Θ of σ_{mod} is called τ_{mod}-*Weyl-convex* if its symmetrization $W_{\tau_{\mathrm{mod}}}\Theta$ in a_{mod} is convex. Here $W_{\tau_{\mathrm{mod}}}$ denotes the stabilizer of the face τ_{mod} for the action $W \curvearrowright a_{\mathrm{mod}}$. For a ($\tau_{\mathrm{mod}}$-)Weyl-convex subset $\Theta \subset \mathrm{ost}\,(\tau_{\mathrm{mod}})$, we define the Θ-*star* of a simplex $\tau \in \partial_{\mathrm{Tits}}X$ as

$$\mathrm{st}_\Theta\,(\tau) = \theta^{-1}(\Theta) \cap \mathrm{st}\,(\tau).$$

The star $\mathrm{st}\,(\tau)$ and Θ-stars $\mathrm{st}_\Theta\,(\tau)$ of a simplex τ are convex subsets of $\partial_{\mathrm{Tits}}X$ with respect to the Tits metric (see [KLPa, KLP17]).

Define the τ_{mod}-*regular* part of the ideal boundary as $\partial_\infty^{\tau_{\mathrm{mod}}-\mathrm{reg}}X = \theta^{-1}\mathrm{ost}\,(\tau_{\mathrm{mod}})$. An ideal point ξ is called τ_{mod}-*regular* if $\xi \in \partial_\infty^{\tau_{\mathrm{mod}}-\mathrm{reg}}$. Given $x \in X$ and $\xi \in \partial_\infty X$, the geodesic ray $x\xi$ is called τ_{mod}-regular if $\xi \in \mathrm{ost}\,(\tau_{\mathrm{mod}})$. A geodesic segment xy is called τ_{mod}-*regular* if xy can be extended to a τ_{mod}-regular ray $x\xi$. For a Weyl-convex subset $\Theta \subset \mathrm{ost}\,(\tau_{\mathrm{mod}})$, in a similar fashion we define Θ-*regularities* for ideal points, rays and segments. Note that a segment xy is τ_{mod}-regular if and only if yx is $\iota(\tau_{\mathrm{mod}})$-regular.

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[3] "Minimal" means that the dimension of S matches with the dimension of the cells $\tau_\pm$.

Let τ_{mod} be an ι-invariant face of σ_{mod}, and Θ is an ι-invariant, Weyl-convex, compact subset of ost (τ_{mod}). Given a point $x \in X$ and a simplex τ of type τ_{mod}, the Θ-cone $V(x, \mathrm{st}_\Theta(\tau))$ with tip x is defined as the union of all Θ-regular rays $x\xi$ asymptotic to st (τ). For a Θ-regular segment xy, the Θ-diamond $\Diamond_\Theta(x, y)$ is defined as

$$\Diamond_\Theta(x, y) = V(x, \mathrm{st}_\Theta(\tau_+)) \cap V(y, \mathrm{st}_\Theta(\tau_-)) \subset P(\tau_-, \tau_+),$$

where $\tau_\pm$ are unique (unless $x = y$) pair of antipodal simplices in Flag (τ_{mod}) such that $y \in V(x, \mathrm{st}_\Theta(\tau_+))$ and $x \in V(y, \mathrm{st}_\Theta(\tau_+))$. The cones and diamonds are convex subsets of X, see [KLPa, KLP17].

The following lemma will be useful in the upcoming treatment.

Lemma 2.1. *Let* $\Theta, \Theta' \subset$ *ost* (τ_{mod}) *be compact subsets such that* Θ *is contained in the interior of* Θ', *and let* $xy \subset X$ *be a* Θ-*regular geodesic. Also, let* $x', y' \in X$ *points which satisfy* $d(x, x'), d(y, y') \leq D$ *for some* $D > 0$. *If* $d(x, y) \geq 2D/\sin\alpha$, *where* $\alpha = \angle_{\mathrm{Tits}}(\Theta, \partial\Theta') < \pi/2$, *then* $x'y'$ *is* Θ'-*regular.*

Proof. Let $xy \subset x\xi$ where ξ is a Θ-regular ideal point. Let y'' be a point on $x'\xi$ which satisfies $d(x, y) = d(x', y'')$. Then, $d(y, y'') \leq d(x, x') \leq D$, where the first inequality comes from the fact that $d(x(t), x'(t))$ is a non-increasing function, $x(t), x'(t)$ being unit speed parameterizations of $x\xi, x'\xi$, respectively. Hence, $d(y', y'') \leq d(y, y') + d(y, y'') \leq 2D$.

The triangle inequality for the Δ-lengths implies $\|d_\Delta(x', y') - d_\Delta(x', y'')\| \leq d(y', y'') \leq 2D$. Since $d_\Delta(x', y'')$ is Θ-regular, $d_\Delta(x', y')$ is Θ'-regular whenever $\angle(d_\Delta(x', y'), d_\Delta(x', y'')) \leq \alpha$ which happens whenever $d(x', y'') \geq 2D/\sin\alpha$. $\qquad\square$

3 Visual angle estimates

The key result in this section is Proposition 3.8 which will be used in the proof of Theorem 5.1 to construct *Morse quasigeodesics* (see Definition 4.3). In the first section, we first obtain some weaker results which would lead to the estimates in Proposition 3.8 in the later section.

In what follows, we always denote by τ_{mod} an ι-invariant face of the model chamber σ_{mod}. The sets denoted by Θ, Θ' etc. will always be ι-invariant, Weyl-convex, compact subset of ost (τ_{mod}). By ξ_{mod} we denote an ι-invariant point in the interior of τ_{mod}.

3.1 Small visual angles I

Define the *space of opposite simplices*

$$\mathcal{X} = (\mathrm{Flag}(\tau_{\mathrm{mod}}) \times \mathrm{Flag}(\tau_{\mathrm{mod}}))^{\mathrm{opp}} \underset{\mathrm{open}}{\subset} \mathrm{Flag}(\tau_{\mathrm{mod}}) \times \mathrm{Flag}(\tau_{\mathrm{mod}}),$$

which consists of all pairs of opposite simplices of Flag (τ_{mod}). This space has a transitive G-action which makes it a homogeneous G-space. The point stabilizer H of this action is the intersection of two opposite parabolic subgroups of G.

Throughout in this section x will be a fixed point of X. For a point $\omega = (\tau_+, \tau_-) \in \mathcal{X}$, let $P(\omega)$ denote the parallel set $P(\tau_+, \tau_-)$. We define a function $d_x^{\mathrm{opp}} : \mathcal{X} \to \mathbb{R}_{\geq 0}$ by

$$d_x^{\mathrm{opp}}(\omega) = d(x, P(\omega)).$$

Proposition 3.1. *The function* d_x^{opp} *is continuous.*

Proof. The proof is the same as of Lemma 2.21 of [KLP17]. Fix a point $\omega_0 \in \mathcal{X}$. From the fiber bundle theory, we have a fibration

$$H \longrightarrow G \xrightarrow{\mathrm{ev}_{\omega_0}} \mathcal{X},$$

where H denotes the point stabilizer of the transitive G action, and ev_{ω_0} denotes the evaluation map $\mathrm{ev}_{\omega_0}(g) = g \cdot \omega_0$. See [Ste99, Sections 7.4, 7.5]. For any $\omega \in \mathcal{X}$, there exists a neighborhood U such that ev_{ω_0} has a local section σ over U,

$$\sigma : U \to G, \quad \mathrm{ev}_{\omega_0} \circ \sigma = \mathrm{Id}_U.$$

It suffices to show that d_x^{opp} is continuous on such neighborhoods U.

Define a function $d' : X \times \mathcal{X} \to \mathbb{R}_{\geq 0}$ by $d'(x, \omega) = d_x^{\mathrm{opp}}(\omega)$. Note that the action of G on $X \times \mathcal{X}$ given by $g(x, \omega) = (gx, g\omega)$ leaves d' invariant. Therefore, on U,

$$
\begin{aligned}
d_x^{\mathrm{opp}}(\omega) = d'(x, \omega) = d'(x, \sigma(\omega)\omega_0) &= d'(\sigma(\omega)^{-1}x, \omega_0) \\
&= d_{\sigma(\omega)^{-1}x}^{\mathrm{opp}}(\omega_0) = d(\sigma(\omega)^{-1}x, P(\omega_0)),
\end{aligned}
$$

where the last function is continuous on U. Therefore, d_x^{opp} is continuous on U. $\qquad\square$

Definition 3.2 (Antipodal subsets). A pair of subsets Λ_1, Λ_2 of $\mathrm{Flag}\,(\tau_{\mathrm{mod}})$ is called *antipodal*, if any simplex $\tau_1 \in \Lambda_1$ is antipodal[4] to any simplex $\tau_2 \in \Lambda_2$ and vice versa.

Let Λ_1 and Λ_2 be a pair of compact, antipodal subsets of $\mathrm{Flag}\,(\tau_{\mathrm{mod}})$. Then, $\Lambda_1 \times \Lambda_2$ is a compact subset of $\mathcal{X}$.

Proposition 3.1 implies:

Corollary 3.3. *Let Λ_1 and Λ_2 be compact, antipodal subsets of $\mathrm{Flag}\,(\tau_{\mathrm{mod}})$. If Λ_1 and Λ_2 are antipodal, then, for any point $x \in X$, there is a number $D = D(\Lambda_1, \Lambda_2, x)$ such that*

$$d(x, P(\tau_1, \tau_2)) \leq D, \quad \forall \tau_1 \in \Lambda_1, \forall \tau_2 \in \Lambda_2.$$

Proposition 3.4. *Let $\Lambda_1, \Lambda_2 \subset \mathrm{Flag}\,(\tau_{\mathrm{mod}})$ be compact, antipodal subsets. There exists a function $f = f(\Lambda_1, \Lambda_2, x) : [0, \infty) \to [0, \pi]$ satisfying $f(R) \to 0$ as $R \to \infty$ such that for any $\tau_1 \in \Lambda_1$, $\tau_2 \in \Lambda_2$, and for any $z_1 \in x\xi_{\tau_1}$, $z_2 \in x\xi_{\tau_2}$ satisfying $d(z_1, x), d(z_2, x) \geq R$, we have*

$$\alpha_1 = \angle_{z_1}(x, z_2) \leq f(R), \quad \alpha_2 = \angle_{z_2}(x, z_1) \leq f(R).$$

Proof. Let $\bar{x} \in P(\tau_1, \tau_2)$ be the point closest to x. Recall that we denote the Cartan involution about a point $y \in X$ by s_y. Note that $s_{\bar{x}}$ preserves $P(\tau_1, \tau_2)$. Since τ_1 and τ_2 are antipodal, $s_{\bar{x}}(\tau_1) = \tau_2$. Hence $\angle_{\bar{x}}^{\xi}(\tau_1, \tau_2) = \pi$, i.e. $s_{\bar{x}}(\xi_{\tau_1}) = \xi_{\tau_2}$. Let $c : (-\infty, \infty) \to P(\tau_1, \tau_2)$, $c(0) = \bar{x}$, be the biinfinite geodesic passing through $\bar{x}$ and asymptotic to $c(+\infty) = \xi_{\tau_1}$ and $c(-\infty) = \xi_{\tau_2}$. For $i = 1, 2$, let $c_i : [0, \infty) \to X$ be the geodesic ray $x\xi_{\tau_i}$ (see Figure 1(a)). Since the functions $d(c(t), c_1(t))$ and $d(c(-t), c_2(t))$ are bounded convex functions, they are decreasing with maximum at $t = 0$. Therefore,

$$d(c(t), c_1(t)) \leq D, \ d(c(-t), c_2(t)) \leq D, \quad \forall t \in [0, \infty), \tag{3.1}$$

where $D > 0$ is a number as in Corollary 3.3.

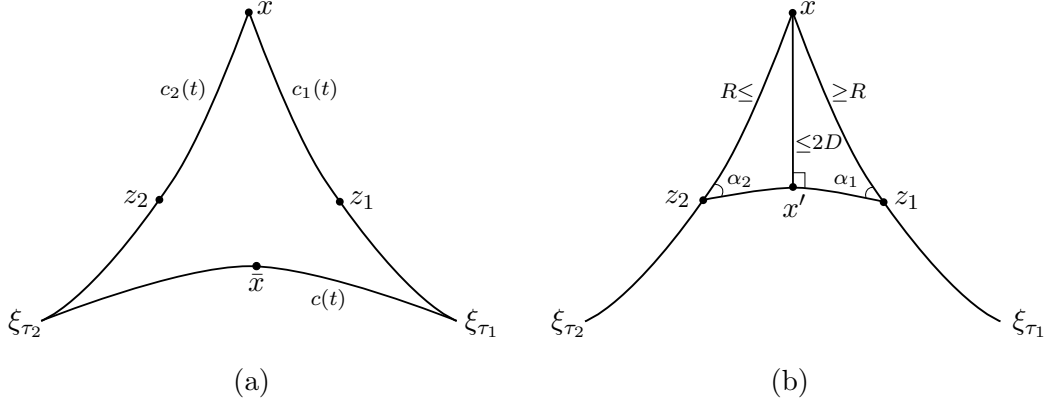[4]See Subsection 2.3 for the definition of antipodal simplices.

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Figure 1

For $R \geq 0$, let $c_1(t_1) = z_1 \in x\xi_{\tau_1}$ and $c_2(t_2) = z_2 \in x\xi_{\tau_2}$ be any points satisfying $t_1 = d(z_1, x) \geq R$ and $t_2 = d(z_2, x) \geq R$. By (3.1), the Hausdorff distance between the segments $z_1 z_2$ and $c(t_1)c(-t_2)$ is bounded above by D. Combining with $d(x, \bar{x}) \leq D$ we obtain

$$d(x, z_1 z_2) \leq 2D. \tag{3.2}$$

Let x' be the point on $z_1 z_2$ nearest to x. When $R \geq 2D + 1$, x' is in the interior of $z_1 z_2$. Consider geodesic triangles $\triangle_1 = \triangle(x, x', z_1)$ and $\triangle_2 = \triangle(x, x', z_2)$; the angle of $\triangle_1$ and $\triangle_2$ at the vertex x' is $\pi/2$. Let $\alpha_1 = \angle_{z_1}(x, x') = \angle_{z_1}(x, z_2)$ and $\alpha_2 = \angle_{z_2}(x, x') = \angle_{z_2}(x, z_1)$ (see Figure 1(b)). Let $\tilde{\triangle}_1$ and $\tilde{\triangle}_2$ be the Euclidean comparison triangles of $\triangle_1$ and $\triangle_2$, respectively; we denote the corresponding vertices of $\tilde{\triangle}_1$ and $\tilde{\triangle}_2$ by the same symbols. In the triangles $\tilde{\triangle}_1$, $\tilde{\triangle}_2$, since the angles at the vertex x' are at least $\pi/2$, we have

$$\tilde{\alpha}_i \leq \sin^{-1}\left(\frac{xx'}{xz_i}\right) \leq \sin^{-1}\left(\frac{2D}{R}\right), \qquad i = 1, 2,$$

where $\tilde{\alpha}_i$ denotes the angle corresponding to α_i. The second inequality in above comes from (3.2). Since the triangles $\triangle_1$ and $\triangle_2$ are thinner than the triangles $\tilde{\triangle}_1$ and $\tilde{\triangle}_2$, respectively, we have $\alpha_i \leq \tilde{\alpha}_i$. Therefore, when $R \geq 2D + 1$, $f(R)$ can be given by the following formula:

$$f(R) = \sin^{-1}\left(\frac{2D}{R}\right).$$

The domain of f can be extended to $R < 2D + 1$ continuously. However, the continuity of f is irrelevant; we can simply set $f(R) = \pi$ for $R < 2D + 1$. $\qquad\square$

We also give a ξ-angle version of the proposition above which will be useful in the next section.

Proposition 3.5. *Let $\Lambda_1, \Lambda_2 \subset \mathrm{Flag}\,(\tau_{\mathrm{mod}})$ be compact antipodal subsets. Given $\Theta \subset \mathrm{ost}\,(\tau_{\mathrm{mod}})$ containing ξ_{mod} in its interior, there exists $R_0 = R_0(x, \Lambda_1, \Lambda_2, \Theta, \xi)$ such that for any $\tau_1 \in \Lambda_1$, $\tau_2 \in \Lambda_2$, and for any $z_1 \in x\xi_{\tau_1}$, $z_2 \in x\xi_{\tau_2}$ satisfying $d(z_1, x), d(z_2, x) \geq R_0$, the segment $z_1 z_2$ is Θ-regular.*

Moreover, there exists a function $f_0 = f_0(x, \Lambda_1, \Lambda_2, \xi) : [0, \infty) \to [0, \pi]$ satisfying $f_0(R) \to 0$ as $R \to \infty$ such that for any $\tau_1 \in \Lambda_1$, $\tau_2 \in \Lambda_2$, and for any $z_1 \in x\xi_{\tau_1}$, $z_2 \in x\xi_{\tau_2}$ satisfying $d(z_1, x), d(z_2, x) \geq R \geq R_0$, we have

$$\angle^\xi_{z_1}(x, z_2), \angle^\xi_{z_2}(x, z_1) \leq f_0(R).$$

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Proof. Let $\alpha = \min \{\angle_{\mathrm{Tits}}(\xi, \zeta) \mid \zeta \in \partial\Theta\} > 0$. Using the triangle inequality for the Δ-lengths, we get

$$\|d_\Delta(x, z_1) - d_\Delta(x_1, z_1)\| \leq d(x, x_1),$$

for any point $x_1 \in X$. Specializing to $x_1 = x'$, the point on $z_1 z_2$ closest to x, we obtain

$$\|d_\Delta(x, z_1) - d_\Delta(x', z_1)\| \leq 2D.$$

Then $x' z_1$ is Θ-regular when $x z_1$ has length $\geq 2D/\sin\alpha$. Therefore, the constant R_0 can be given by

$$R_0 = \frac{2D}{\sin\alpha}. \tag{3.3}$$

This proves first part of the proposition.

For the second part, let $(\Theta_n)_{n\in\mathbb{N}}$ be a nested sequence of ι-invariant, Weyl-convex, compact subsets of $\mathrm{ost}(\tau_{\mathrm{mod}})$ such that ξ is an interior point of each Θ_n, and $\bigcap_{n=1}^\infty \Theta_n = \{\xi\}$. Let α_n be the Tits-distance from ξ to the boundary of Θ_n,

$$\alpha_n = \min \{\angle_{\mathrm{Tits}}(\xi, \zeta) \mid \zeta \in \partial\Theta_n\} > 0.$$

Clearly, $(\alpha_n)_{n\in\mathbb{N}}$ is a strictly decreasing sequence converging to zero. This implies that $R_0(\Theta_n)$ is strictly increasing which diverges to infinity, where R_0 is as in (3.3). If $R_0(\Theta_n) \leq d(x, z_1) < R_0(\Theta_{n+1})$, then the first part of the proposition implies that $z_2 z_1$ is Θ_n-regular, which then implies

$$\angle_{z_2}^\xi(x, z_1) \leq \angle_{z_2}(x, z_1) + \angle_{o_{\mathrm{mod}}}(\xi, d_\Delta(z_2, z_1))$$
$$\leq f(R) + \alpha_n,$$

where the function f is as in Proposition 3.4. Therefore, when $R_0(\Theta_n) \leq R < R_0(\Theta_{n+1})$, we may define

$$f_0(R) = f(R) + \alpha_n.$$

As in the case of f in Proposition 3.4, continuity of f_0 is irrelevant. $\qquad\square$

3.2 Small visual angles II

The Θ-cones (over a fixed simplex $\tau \in \mathrm{Flag}(\tau_{\mathrm{mod}})$) vary continuously with their tips. Here, the topology on the set of Θ-cones over a fixed simplex τ is given by their Hausdorff distances. Precisely, we have,

Theorem 3.6 (Uniform continuity of Θ-cones, [KLPa]). *The Hausdorff distance between two Θ-cones over a fixed $\tau \in \mathrm{Flag}(\tau_{\mathrm{mod}})$ is bounded by the distance between their tips,*

$$d_{\mathrm{Haus}}\left(V(x, \mathrm{st}_\Theta(\tau)), V(\bar{x}, \mathrm{st}_\Theta(\tau))\right) \leq d(x, \bar{x}).$$

Moreover, for diamonds, one also has the following form of uniform continuity. This will be useful in our paper, especially in the discussion of replacements (Section 4.3).

Theorem 3.7 (Uniform continuity of diamonds). *Given any Θ' with $\mathrm{int}(\Theta') \supset \Theta$, and any $\delta > 0$, there exists $c = c(\Theta, \Theta', \delta)$ such that for all Θ-regular segments xy and $x'y'$ with $d(x, x') \leq \delta$, $d(y, y') \leq \delta$, we have*

$$\Diamond_\Theta(x, y) \subset \mathcal{N}_c\left(\Diamond_{\Theta'}(x', y')\right).$$

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Proof. We will prove this theorem as a corollary of [KLPb, Theorem 5.16]: For every (Θ, B)-regular (L, A)-quasigeodesic $q : [a_-, a_+] \to X$ and points $x_\pm \in X$ within distance $\leq B$ from $q(a_\pm)$, the image of q is contained in the $D(L, A, \Theta, B)$-neighborhood of the diamond $\Diamond_{\tau_{\mathrm{mod}}}(x_-, x_+)$.

Remark. Using the hard theorem [KLPb, Theorem 5.16] in order to prove Theorem 3.7 is an overkill, but it is quicker than a direct argument. We refer the reader to Section 4 for the definition of (Θ, B)-regular quasigeodesics.

By appealing to the triangle inequalities for Δ-length, one gets a slightly more precise statement, namely, there exists $D(L, A, \Theta, \Theta', B)$ such that the image of q is contained in the $D(L, A, \Theta, \Theta', B)$-neighborhood of the diamond $\Diamond_{\Theta'}(x_-, x_+)$.

We observe that for every point $z \in \Diamond_\Theta(x, y)$ the broken geodesic segment

$$xz \star zy$$

is $(L, 0)$-quasigeodesic for some $L = L(\Theta)$, and is $(\Theta, 0)$-regular. Hence, according to the above sharpening of [KLPb, Theorem 5.16], the point z belongs to the $c = D(L, 0, \Theta, \Theta', \delta)$-neighborhood of the diamond $\Diamond_{\Theta'}(x', y')$, provided that

$$d(x, x') \leq \delta, d(y, y') \leq \delta.$$

Thus,

$$\Diamond_\Theta(x, y) \subset \mathcal{N}_c\left(\Diamond_{\Theta'}(x', y')\right). \qquad \square$$

Now we turn to the main estimate in this section.

Proposition 3.8 (Uniformly small visual angles). *Let $\Lambda_1, \Lambda_2 \subset \mathrm{Flag}(\tau_{\mathrm{mod}})$ be compact, antipodal sets, and let Θ' be a subset of $\mathrm{ost}(\tau_{\mathrm{mod}})$ containing Θ in its interior. Let $y_1 \in V(x, \mathrm{st}_\Theta(\tau_1))$ and $y_2 \in V(x, \mathrm{st}_\Theta(\tau_2))$ be any points, where $\tau_1 \in \Lambda_1$ and $\tau_2 \in \Lambda_2$ are any simplices. Then,*

1. *There exists a constant $R_1 = R_1(x, \Lambda_1, \Lambda_2, \Theta', \Theta)$ such that $y_1 y_2$ is Θ'-regular if $d(x, y_i) \geq R_1$.*

2. *There exists a function $f_1 = f_1(x, \Lambda_1, \Lambda_2, \Theta', \Theta, \xi) : [0, \infty) \to [0, \pi]$ satisfying $\lim_{R \to \infty} f_1(R) = 0$ such that if $d(x, y_i) \geq R \geq R_1$, then*

$$\angle_{y_1}^\xi(x, y_2), \angle_{y_2}^\xi(x, y_1) \leq f_1(R). \tag{3.4}$$

Proof. For part 1, we take an approach similar to the one given in the proof of Proposition 3.5. Let $\bar{x}$ be the nearest point projection of x into the parallel set $P(\tau_1, \tau_2)$, and for each $i = 1, 2$ let $\bar{y}_i$ denote the nearest point projection of y_i into $V(\bar{x}, \mathrm{st}_\Theta(\tau_i)) \subset P(\tau_1, \tau_2)$. Let $\alpha = \angle_{\mathrm{Tits}}(\Theta, \partial\Theta') > 0$, and $\alpha' = \angle_{\mathrm{Tits}}(\Theta, \partial\mathrm{st}(\tau_{\mathrm{mod}})) \geq \alpha$. Finally, let $D = D(\Lambda_1, \Lambda_2, x)$ be the constant given by Corollary 3.3.

Since $d(x, \bar{x}) \leq D$, we combine this with Theorem 3.6 to get

$$d(y_i, \bar{y}_i) \leq D, \quad i = 1, 2. \tag{3.5}$$

Then, using the triangle inequality for Δ-lengths we deduce

$$\|d_\Delta(y_1, y_2) - d_\Delta(\bar{y}_1, \bar{y}_2)\| \leq \|d_\Delta(y_1, y_2) - d_\Delta(\bar{y}_1, y_2)\| + \|d_\Delta(\bar{y}_1, y_2) - d_\Delta(\bar{y}_1, \bar{y}_2)\|$$
$$\leq d(y_1, \bar{y}_1) + d(y_2, \bar{y}_2) \leq 2D.$$

Since $\bar{y}_1 \bar{y}_2$ is Θ-regular, $y_1 y_2$ is Θ'-regular whenever $y_1 y_2$ has length $\geq 2D/\sin\alpha$. See Lemma 2.1. Moreover,

$$d(y_1, y_2) \geq d(\bar{y}_1, \bar{y}_2) - 2D \geq d(\bar{y}_i, \bar{x}) \sin\alpha' - 2D \geq (d(y_i, x) - 2D) \sin\alpha - 2D$$
$$= d(y_i, x) \sin\alpha - 2D(1 + \sin\alpha), \quad (3.6)$$

where the second inequality comes from triangle comparisons (note that $\angle_{\bar{x}}(\bar{y}_1, \bar{y}_2) \geq \alpha'$ because $\bar{y}_1$, $\bar{y}_2$ are in different Θ-cones with tip $\bar{x}$), and the third inequality follows from (3.5), $d(x, \bar{x}) \leq D$, $\pi/2 > \alpha' \geq \alpha$ and the polygon inequality. Using (3.6), we obtain: $d(y_1, y_2) \geq 2D/\sin\alpha$ whenever $d(x, y_1)$ or $d(x, y_2)$ is greater than $2D(1/\sin^2\alpha + 1/\sin\alpha + 1)$. We may set

$$R_1 = 2D\left(1 + \frac{1}{\sin\alpha} + \frac{1}{\sin^2\alpha}\right).$$

This proves part 1.

To prove part 2 we need the following lemmas.

Recall that $s_x : X \to X$ denotes the Cartan involution of X fixing x.

Lemma 3.9. *Let $\tau, \tau' \in \mathrm{Flag}\,(\tau_{\mathrm{mod}})$ be a pair of simplices, let $x \in X$ be any point, and let $y \in V(x, \mathrm{st}_\Theta(s_x \tau))$ be a point satisfying $d(x, y) \geq l$. For sufficiently small ϵ, $\epsilon \leq \epsilon_0(\xi_{\mathrm{mod}})$, we have: If $\angle_x^\xi(\tau, \tau') \leq \epsilon$, then*

$$\angle_y^\xi(\tau, \tau') \leq \epsilon'(\Theta, l)$$

with $\epsilon'(\Theta, l) \to 0$ as $l \to \infty$.

Proof. Let $\xi_+ \in \tau$, $\xi_- \in s_x \tau$, $\xi' \in \tau'$ be ξ_{mod}-regular points. Then

$$\angle_x^\xi(\tau, \tau') \leq \epsilon \implies \angle_x(\xi_+, \xi') \leq \epsilon \implies \angle_x(\xi_-, \xi') \geq \pi - \epsilon.$$

Using [KLPa, Lemma 2.44(ii)], there exists a function $\epsilon'(\Theta, l)$ satisfying $\lim_{l\to\infty} \epsilon'(\Theta, l) = 0$ such that

$$\angle_y(\xi_-, \xi') \geq \pi - \epsilon'(\Theta, l).$$

Then,

$$\angle_y^\xi(\tau, \tau') = \angle_y(\xi_+, \xi') = \pi - \angle_y(\xi_-, \xi') \leq \epsilon'(\Theta, l). \qquad \square$$

In the following, Θ'' will denote an auxiliary subset of $\mathrm{ost}\,(\tau_{\mathrm{mod}})$ such that $\mathrm{int}\,(\Theta'') \supset \Theta$. Let $\alpha'' = \angle_{\mathrm{Tits}}(\Theta, \partial\Theta'')$.

Lemma 3.10. *Let $\tau \in \mathrm{Flag}\,(\tau_{\mathrm{mod}})$ be any simplex, and $y \in V(x, \mathrm{st}_\Theta(\tau))$ be any point. If $z \in x\xi_\tau$ is any point such that $d(x, y) \sin\alpha'' \geq d(x, z)$, then*

$$y \in V(z, \mathrm{st}_{\Theta''}(\tau)).$$

Proof. Let F be a maximal flat asymptotic to τ containing x and y, and let y' be the nearest point projection of y into $x\xi_\tau$. Since $\xi \in \tau_{\mathrm{mod}}$, the Tits distance from ξ to any point in Θ is bounded above by $\pi/2 - \epsilon(\Theta)$ where $\epsilon(\Theta) > 0$. Then, the distance from x to y' is comparable to $d(x, y)$, i.e.

$$d(x, y) \cos(\theta) = d(x, y'), \quad \theta = \angle_x(y, y') \leq \pi/2 - \epsilon(\Theta).$$

Notice that $0 < \alpha'' \leq \theta + \alpha'' \leq \pi/2 - \epsilon(\Theta'')$. For any point $z' \in xy'$, $y \in V(z', \mathrm{st}_{\Theta''}(\tau))$ whenever $d(y, y') \leq d(z', y') \tan(\theta + \alpha'')$.

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Let $z' \in xy'$ be a point which satisfies $d(y, y') = d(z', y') \tan(\theta + \alpha'')$. In that case,

$$d(x, z') = d(x, y') - d(z', y') = d(x, y) \cos\theta - d(y, y') \cot(\theta + \alpha'')$$

$$= d(x, y) \cos\theta - d(x, y) \sin\theta \cot(\theta + \alpha'') = d(x, y) \frac{\sin\alpha''}{\sin(\theta + \alpha'')} \geq d(x, y) \sin\alpha''.$$

Moreover, if $z \in x\xi_\tau$ is the point satisfying $d(x, z) = d(x, y) \sin\alpha''$, then $z' \in V(z, \mathrm{st}_{\Theta''}(\tau))$, and from convexity of cones, $y \in V(z, \mathrm{st}_{\Theta''}(\tau))$. $\qquad\square$

Lemma 3.11. *There exists a function $f'_1(x, \Lambda_1, \Lambda_2, \Theta, \xi) : [0, \infty) \to [0, \pi]$ satisfying $f'_1(R) \to 0$ as $R \to \infty$ such that the following holds: For $\tau_1 \in \Lambda_1$, let $y_1 \in V(x, \mathrm{st}_\Theta(\tau_1))$ be any point. If $d(x, y_1) \geq R$, then*

$$\max_{\tau_2 \in \Lambda_2} \angle^\xi_{y_1}(x, \tau_2) \leq f'_1(R).$$

Proof. Let $\alpha'' = \angle_{\mathrm{Tits}}(\Theta, \partial\Theta'')$ as defined above, where Θ'' is some auxiliary subset of $\mathrm{ost}(\tau_{\mathrm{mod}})$. Using Lemma 3.10, if $z_1 \in x\xi_{\tau_1}$ satisfies $d(x, z_1) = d(x, y_1) \sin\alpha''$, then $y_1 \in V(z_1, \mathrm{st}_{\Theta''}(\tau_1))$. See Figure 2. Letting $d(x, z_2) \to \infty$ in Proposition 3.5, we get

$$\angle^\xi_{z_1}(s_x(\tau_1), \tau_2) = \angle^\xi_{z_1}(x, \tau_2) \leq f_0(R \sin\alpha''), \quad \forall \tau_2 \in \Lambda_2.$$

When R is sufficiently large, $R \geq R_2(x, \Lambda_1, \Lambda_2, \xi)$, then $f_0(R \sin\alpha) \leq \epsilon_0(\xi_{\mathrm{mod}})$, where $\epsilon_0(\xi_{\mathrm{mod}})$ is as in Lemma 3.9. Moreover, since $d(y_1, z_1) \geq R(1 - \sin\alpha'')$, Lemma 3.9 implies that

$$\angle^\xi_{y_1}(x, \tau_2) = \angle^\xi_{y_1}(s_x(\tau_1), \tau_2) \leq \epsilon'(\Theta'', R(1 - \sin\alpha'')), \quad \forall \tau_2 \in \Lambda_2.$$
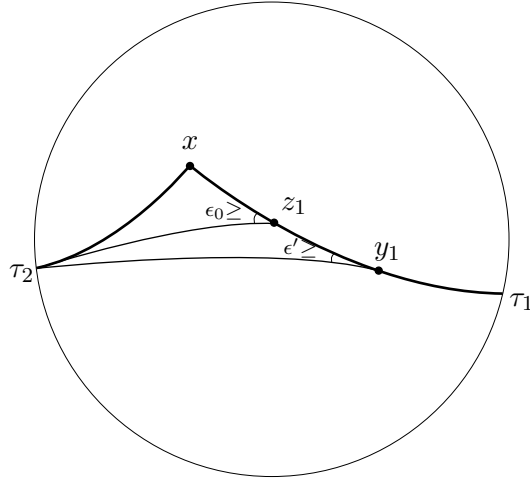

Figure 2: A schematic picture depicting small angles. The thick lines are Weyl cones $V(x, \mathrm{st}_\Theta(\tau_1))$ and $V(x, \mathrm{st}_\Theta(\tau_2))$, while the thin lines are the geodesic rays $z_1\xi_2$ and $y_1\xi_2$.

So, we may define

$$f'_1(R) = \begin{cases} \epsilon'(\Theta'', R(1 - \sin\alpha'')), & \text{if } R \geq R_2 \\ \pi, & \text{otherwise} \end{cases}.$$

$\qquad\square$

Now we are ready to prove the estimate (3.4). We first observe that the only property of the point $x \in X$ we have used to estimate the functions f in Proposition 3.4, R_0 and f_0 in Proposition 3.5 and subsequently f_1' in Lemma 3.11 is that

$$d(x, P(\tau_1, \tau_2)) \leq D, \quad \forall \tau_1 \in \Lambda_1, \forall \tau_2 \in \Lambda_2.$$

All these estimates for x work for any other point x_1 satisfying this inequality with the same number D. Moreover, all these estimates work if we replace Λ_1 or Λ_2 by their proper subsets. In particular, we may replace Λ_2 by any of its singleton subsets $\{\tau_2\}$, or replace x by a point $y_2 \in V(x, \mathrm{st}_\Theta(\tau_2))$. Here we use the fact that for a fixed τ_2 and a point y_2 in $V(x, \mathrm{st}_\Theta(\tau_2))$,

$$d(y_2, P(\tau_1, \tau_2)) \leq d(x, P(\tau_1, \tau_2)) \leq D, \quad \forall \tau_1 \in \Lambda_1.$$

Therefore, the same estimate $f_1'(x, \Lambda_1, \Lambda_2, \xi)$ works when x and Λ_2 (elsewhere) in Lemma 3.11 are replaced by y_2 and $\{\tau_2\}$, respectively. Precisely, whenever $y_1 y_2$ is a Θ'-regular,

$$\angle_{y_1}^\xi(y_2, \tau_2) \leq f_1'(d(y_1, y_2)), \tag{3.7}$$

where $f_1' = f_1'(x, \Lambda_1, \Lambda_2, \Theta', \xi)$. Θ'-regularity of $y_1 y_2$ is also guaranteed whenever, for $i = 1, 2$, $d(x, y_i) \geq R_1(x, \Lambda_1, \Lambda_2, \Theta', \Theta)$.

Therefore, if $R \geq R_1(x, \Lambda_1, \Lambda_2, \Theta', \Theta)$ and $d(x, y_1), d(x, y_2) \geq R$, we can use Lemma 3.11, (3.6) and (3.7) to get

$$\angle_{y_1}^\xi(x, y_2) \leq \angle_{y_1}^\xi(x, \tau_2) + \angle_{y_1}^\xi(y_2, \tau_2)$$
$$\leq f_1'(R) + f_1'(R \sin \alpha - 2D(1 + \sin \alpha)) \leq 2 f_1'(R \sin \alpha - 4D),$$

where $\alpha = \angle_{\mathrm{Tits}}(\Theta, \partial \Theta')$.

This completes the proof of part 2. $\qquad\square$

4 Morse condition

In this section, we discuss Morse quasigeodesics, Morse embeddings and Morse subgroups and their various properties. These notions were introduced in [KLPa], and it was proved that the notions of Morse subgroups and Anosov subgroups are equivalent (Theorem 1.4).

One of the important new result in this Section is the *replacement lemma* (see Theorems 4.11 and 4.13) which will be proved in Section 4.3. This will be an important ingredient in the proof of Theorem 5.1.

4.1 Stability of quasigeodesics

Recall that a *quasigeodesic* in a metric space (Y, d_Y) is a quasiisometric embedding of an interval $I \subset \mathbb{R}$ into Y. Quantitively speaking, an (L, A)-quasigeodesic in Y is a map, not necessarily continuous, $\gamma : I \to Y$ which satisfies

$$L^{-1}|a - b| - A \leq d_Y(\gamma(a), \gamma(b)) \leq L|a - b| + A,$$

where d_Y is the metric of Y. When Y is assumed to be a geodesic δ-hyperbolic space, the Morse lemma, proven for these spaces by Gromov [Gro87, Proposition 7.2.A], establishes stability of quasigeodesics. Precisely, an (L, A)-quasigeodesic in a δ-hyperbolic space stays within a uniform

neighborhood of a geodesic; the radius H of this neighborhood solely depends on the given parameters, namely L, A and δ,

$$H = L^2(A_1 A + A_2 \delta),$$

where A_1 and A_2 are universal constants, see [Shc13]. The stability of quasigeodesics can also be stated without referring to geodesics: An (L, A)-quasigeodesic path is *stable* if the image of any (L', A')-quasigeodesic with the same endpoints is uniformly close to the original path. Thus, any uniform quasigeodesic in a δ-hyperbolic space is stable. Morse lemma is a vital ingredient to prove the invariance of hyperbolicity under quasiisometries, see [DK18, Corollary 11.43].

One of the major differences between the coarse geometric nature of CAT(0) (or *non-positively curved*) and δ-hyperbolic spaces is that the quasigeodesics in CAT(0) spaces can be unstable, and thus the most naive generalization of the Morse lemma fails in the CAT(0) settings, already for the Euclidean plane. Some versions of the Morse lemma are known for CAT(0) spaces; in [Sul14] it has been shown that a quasigeodesic is stable if and only if it is *strongly contracting*. However, this class of quasigeodesics is too restrictive in the context of symmetric spaces.

Nevertheless, according to the main theorem of [KLPb], an analogue of the Morse lemma holds for τ_{mod}-*regular quasigeodesics*, with diamonds (or, cones, or parallel sets) replacing geodesic segments (rays, complete geodesics).

The letters B, D which appear bellow are non-negative numbers.

Definition 4.1 (Regular quasigeodesics)**.** A pair of points in X is called Θ-*regular* if the geodesic segment connecting them is Θ-regular. An (L, A)-quasigeodesic $\gamma : I \to X$ is called (Θ, B)-*regular* if for all $t_1, t_2 \in I$, $|t_1 - t_2| \geq B$ implies that $(\gamma(t_1), \gamma(t_2))$ is Θ-regular.

In [KLPb, Theorem 5.17], it is shown that (finite) regular quasigeodesics are special in the sense that they live very close to the diamonds. We state this result in the next theorem.

Theorem 4.2 (Morse Lemma for Symmetric Spaces of Higher Rank)**.** *Let* $\gamma : [a, b] \to X$ *be a* (Θ, B)-*regular* (L, A)-*quasigeodesic. There exists a constant* $D = D(L, A, \Theta, \Theta', B, X) > 0$ *such that the image of* γ *is contained in the* D-*neighborhood of a diamond* $\Diamond_{\Theta'}(x_1, x_2)$ *with tips satisfying* $d(\gamma(a), x_1) \leq D$, $d(\gamma(b), x_2) \leq D$.

Now we review the notion of Morse quasigeodesics.

Definition 4.3 (Morse quasigeodesics, [KLPa])**.** A (finite, semiinfinite, or biinfinite) (L, A)-quasigeodesic $\gamma : I \to X$ is called a (L, A, Θ, D)-*Morse quasigeodesic* if for all $t_1, t_2 \in I$, the image $\gamma([t_1, t_2])$ is D-close to a Θ-diamond $\Diamond_{\Theta}(x_1, x_2)$ with tips satisfying $d(x_i, \gamma(t_i)) \leq D$, for $i = 1, 2$.

Remark.

1. In light of this definition, Theorem 4.2 is equivalent to saying that *the uniformly regular uniform quasigeodesics are uniformly Morse.* Conversely, uniformly Morse quasigeodesics are also uniformly regular.

2. Note that it is not in general true that for an (L, A, Θ, D)-Morse quasigeodesic γ, the segment $\gamma(t_1)\gamma(t_2)$ is τ_{mod}-regular. However, when $t_2 - t_1$ is uniformly large, $\gamma(t_1)\gamma(t_2)$ becomes uniformly τ_{mod}-regular, and in this case one can say that $\gamma([t_1, t_2])$ lies in a uniform neighborhood of $\Diamond_{\Theta'}(\gamma(t_1), \gamma(t_2))$ for any subset $\Theta' \subset \mathrm{ost}(\tau_{\mathrm{mod}})$ containing Θ in its interior (cf. Theorem 3.7).

4.2 Straight sequences

We review some important tools for constructing Morse quasigeodesics.

Let Θ, Θ' be subsets of $\mathrm{Flag}\,(\tau_{\mathrm{mod}})$ such that

$$\mathrm{int}\,(\Theta') \supset \Theta.$$

Definition 4.4 (Straight-spaced sequences, [KLPa])**.** Let $\epsilon \geq 0$ be a number. A (finite, infinite, or biinfinite) sequence (x_n) is called (Θ, ϵ)-*straight* if, for each n, the segments $x_n x_{n+1}$ are Θ-regular and

$$\angle_{x_n}^{\xi}(x_{n-1}, x_{n+1}) \geq \pi - \epsilon.$$

Moreover, (x_n) is called *l-spaced* if $d(x_n, x_{n+1}) \geq l$ for all n.

Definition 4.5 (Morse sequence)**.** A sequence (x_n) is called (Θ, D)-*Morse* if the piecewise geodesic path formed by connecting consecutive points by geodesic segments is a (Θ, D)-Morse quasigeodesic.

Theorem 4.6 (Morse lemma for straight spaced sequences, [KLPa])**.** *For Θ, Θ', D, there exists l, ϵ such that any (Θ, ϵ)-straight l-spaced sequence (x_n) in X is D-close to a parallel set $P(\tau_+, \tau_-)$ of type τ_{mod}. Moreover, the nearest point projection $\bar{x}_n$ of x_n on $P(\tau_+, \tau_-)$ satisfies*

$$\bar{x}_{n\pm m} \in V(\bar{x}_n, \mathrm{st}_{\Theta'}(\tau_{\pm})), \quad \forall m \in \mathbb{N}.$$

Furthermore, the sequence (x_n) is a uniform Morse sequence with parameters depending only on the given data Θ, Θ', D.

4.3 Replacements

Here we define an alternative notion of stability of quasigeodesics, namely that *Morse property is stable under replacements*. See Theorem 4.11, and its generalized version Theorem 4.13.

We first develop an important tool which will be needed in the proof of these results.

Definition 4.7 (Longitudinal segments)**.** Let y_1, y_2 be any points in $P(\tau_-, \tau_+)$. The (oriented) segment $y_1 y_2$ is called Θ-*longitudinal* if $y_2 \in V(y_1, \mathrm{st}_{\Theta}(\tau_+))$. Moreover, $y_1 y_2$ is called (τ_{mod})-*longitudinal* if $y_2 \in V(y_1, \mathrm{ost}(\tau_+))$

Convexity of Θ-cones implies:

Lemma 4.8 (Concatenation of longitudinal segments)**.** *Let $x_1, x_2, x_3 \in P(\tau_-, \tau_+)$ be points such that $x_1 x_2$ and $x_2 x_3$ are Θ-longitudinal. Then $x_1 x_3$ is also Θ-longitudinal.*

Proposition 4.9 (Nearby diamonds)**.** *Let $\gamma : [a, b] \to X$ be an (L, A, Θ, D)-Morse qusaigeodesic, and let $\delta > 0$ be any number. Let $P(\tau_-, \tau_+)$ be a parallel set such that the image of γ is contained in $\mathcal{N}_{\delta}(P(\tau_-, \tau_+))$. Denote the nearest point projection of $\gamma(t)$ into $P(\tau_-, \tau_+)$ by $\bar{\gamma}(t)$. Suppose that $\bar{\gamma}(a)\bar{\gamma}(b)$ is longitudinal. Then, there exist $R' = R'(L, A, \Theta, \Theta', D, \delta)$, $D' = D'(L, A, \Theta, \Theta', D, \delta)$ such that the following holds: For any $t_1, t_2 \in [a, b]$, if $(t_2 - t_1) \geq R'$, then $\bar{\gamma}(t_1)\bar{\gamma}(t_2)$ is Θ'-longitudinal and $\gamma([t_1, t_2]) \subset \mathcal{N}_{D'}(\Diamond_{\Theta'}(\bar{\gamma}(t_1), \bar{\gamma}(t_2)))$.*

Proof. Let $\Theta'', \Theta''' \subset \tau_{\mathrm{mod}}$ be auxiliary subsets such that $\mathrm{int}\,(\Theta') \supset \Theta'''$, $\mathrm{int}\,(\Theta''') \supset \Theta''$, and $\mathrm{int}\,(\Theta'') \supset \Theta$. Note that when $(b - a)$ is sufficiently large, the triangle inequality for the Δ-lengths asserts that $\bar{\gamma}(a)\bar{\gamma}(b)$ is Θ''-regular, which in turn makes $\bar{\gamma}(a)\bar{\gamma}(b)$ Θ'''-longitudinal. Therefore, it follows that

$$\bar{\gamma}([a, b]) \subset \mathcal{N}_{c+\delta}(\Diamond_{\Theta'''}(\bar{\gamma}(a), \bar{\gamma}(b))) \subset \mathcal{N}_{c+\delta}(V(\bar{\gamma}(a), \mathrm{st}_{\Theta'''}(\tau_+))),$$

where $c = c(\Theta'', \Theta''', D + \delta)$ is the constant as in Theorem 3.7.

Let $t \in [a, b]$ be any point. From above, we get $d(\bar\gamma(t), V(\bar\gamma(a), \mathrm{st}_{\Theta'''}(\tau_+))) \le c + \delta$. Using the triangle inequality for the Δ-lengths again, we obtain that when $t - a \ge R \gg L(c + \delta)$, then

$$\bar\gamma(t) \in V(\bar\gamma(a), \mathrm{st}_{\Theta'}(\tau_+)),$$

i.e. $\bar\gamma(a)\bar\gamma(t)$ is Θ'-longitudinal. By reversing the direction of γ, we also get that when $b - t \ge R$, then $\bar\gamma(t)\bar\gamma(b)$ is Θ'-longitudinal.

For arbitrary $t_1, t_2 \in [a, b]$, we let $t = (t_2 - t_1)/2$. The same argument applied to the paths $\gamma([a, t])$, $\gamma([t, b])$ implies that when $t - t_1 \ge R$, and $t_2 - t \ge R$, then $\bar\gamma(t_1)\bar\gamma(t)$ and $\bar\gamma(t)\bar\gamma(t_2)$ are Θ'-longitudinal segments. Using Lemma 4.8, we get that $\bar\gamma(t_1)\bar\gamma(t_2)$ is Θ'-longitudinal.

Therefore, $\bar\gamma(t_1)\bar\gamma(t_2)$ is Θ'-longitudinal whenever $t_2 - t_1 \ge 2R$.

After enlarging Θ', the second part follows from Theorem 3.7. $\qquad\square$

We now turn to the discussion of replacements.

Definition 4.10 (Morse quasigeodesic replacements). Let $\gamma : I \to X$ be an (L, A, Θ, D)-Morse quasigeodesic, and let $[t_1, t_2]$ be a subinterval of I. Let $\gamma' : [t_1, t_2] \to X$ be another (L', A', Θ', D')-Morse quasigeodesic such that $\gamma|_{\{t_1, t_2\}} = \gamma'|_{\{t_1, t_2\}}$ (see Figure 3). An (L', A', Θ', D')-*Morse quasigeodesic replacement* of $\gamma|_{[t_1, t_2]}$ is the concatenation of $\gamma|_{I - (t_1, t_2)}$ with $\gamma'|_{[t_1, t_2]}$.
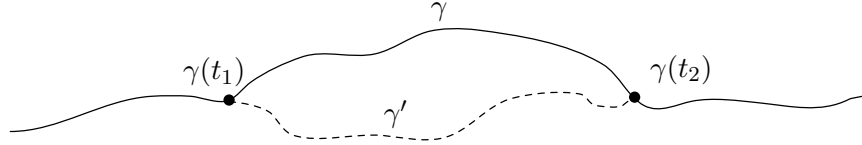

Figure 3: Replacement: The original path γ is depicted as a solid line, and the path γ' is depicted as a dashed line.

Theorem 4.11 (Replacement lemma). *Uniform Morse quasigeodesic replacements are uniformly Morse.*

Proof. Suppose that $I = [a, b]$ is some interval. Let $\gamma : I \to X$ be an (L, A, Θ, D)-Morse quasigeodesic, and let $\rho : I \to X$ be obtained by replacing $\gamma|_{[t_1, t_2]}$ by a (L', A', Θ', D')-Morse quasigeodesic $\gamma' : [t_1, t_2] \to X$. Let Θ'' be subset of $\mathrm{ost}(\tau_{\mathrm{mod}})$ which contains Θ and Θ'. Replacing the parameters (L, A, Θ, D) and (L', A', Θ', D') by $(L'', A'', \Theta'', D'')$, where $L'' = \max\{L, L'\}$, $A'' = \max\{A, A'\}$, $D'' = \max\{D, D'\}$, and some $\Theta'' \supset \Theta \cup \Theta'$, we may assume that $(L, A, \Theta, D) = (L', A', \Theta', D')$ to begin with.

By definition, there exists a diamond $\Diamond_\Theta(x_1, x_2)$ with $d(x_1, \gamma(a)) \le D$, $d(x_2, \gamma(b)) \le D$ such that $\gamma([a, b]) \subset \mathcal{N}_D(\Diamond_\Theta(x_1, x_2))$. Without loss of generality, we may assume that $x_1 \ne x_2$. The diamond $\Diamond_\Theta(x_1, x_2)$ spans a unique parallel set $P(\tau_-, \tau_+)$ such that $x_2 \in V(x_1, \mathrm{st}_\Theta(\tau_{\mathrm{mod}})) \tau_+$. We denote the nearest point projections of $\gamma(t)$ and $\gamma'(t)$ to $P(\tau_-, \tau_+)$ by $\bar\gamma(t)$ and $\bar\gamma'(t)$, respectively.

By the triangle inequality for the Δ-lengths we get that when $(b - a)$ is sufficiently large, $(b - a) \ge C(\Theta, \Theta', D)$, then $\bar\gamma(a)\bar\gamma(b)$ is Θ'-longitudinal[5]. Using Proposition 4.9, when $(t_2 - t_1) \ge R'$,

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[5]the nearest point projection might not send $\gamma(a)$ (resp. $\gamma(b)$) to x_1 (resp. x_2), but sends into a D-neighborhood of x_1 (resp. x_2).

then $\bar\gamma(t_1)\bar\gamma(t_2) = \bar\gamma'(t_1)\bar\gamma'(t_2)$ is also Θ'-longitudinal. From Theorem 3.7 we get a constant D' such that $\gamma'([t_1,t_2]) \subset \mathcal{N}_{D'}\left(P(\tau_-,\tau_+)\right)$.

We prove that any subpath $\rho|_{[r_1,r_2]}$ is uniformly close to a diamond. From above, if $(r_2-r_1) \geq R'$, for $r_1, r_2 \in I$, then $\bar\gamma(r_2) \in V(\bar\gamma(r_1), \mathrm{st}_{\Theta'}(\tau_+))$. This also holds for γ' and $r_1, r_2 \in [t_1, t_2]$ for a bigger R' because $\gamma'([t_1,t_1])$ is D'-close to $P(\tau_-, \tau_+)$, and $\bar\gamma'(t_1)\bar\gamma'(t_2)$ is longitudinal. Also, note that in this case, possibly after enlarging Θ', $\gamma|_{[r_1,r_2]}$ and $\gamma'|_{[r_1,r_2]}$ become uniformly close to $\diamondsuit_{\Theta'}(\bar\gamma(r_1), \bar\gamma(r_2))$ and $\diamondsuit_{\Theta'}(\bar\gamma'(r_1), \bar\gamma'(r_2))$, respectively (Theorem 3.7).

Clearly, when both r_1, r_2 belong to one of the sets $[a, t_1]$, $[t_1, t_2]$, $[t_2, b]$, then $\rho([r_1, r_2])$ is uniformly close to a diamond. Therefore, the following are the only nontrivial cases.

Case 1. $r_1 \in [a, t_1]$ and $r_2 \in [t_1, t_2]$.

In this case, if $(t_1 - r_1) \geq R'$ and $(r_2 - t_1) \geq R'$, then from the discussion above we get

$$\bar\gamma(t_1) \in V(\bar\gamma(r_1), \mathrm{st}_{\Theta'}(\tau_+)), \quad \bar\gamma'(r_2) \in V(\bar\gamma(t_1), \mathrm{st}_{\Theta'}(\tau_+)).$$

From convexity of cones, it follows that

$$\bar\gamma'(r_2) \in V(\bar\gamma(r_1), \mathrm{st}_{\Theta'}(\tau_+)).$$

Since $\diamondsuit_{\Theta'}(\bar\gamma(r_1), \bar\gamma(t_1))$ and $\diamondsuit_{\Theta'}(\bar\gamma'(t_1), \bar\gamma'(r_2))$ are subsets of $\diamondsuit_{\Theta'}(\bar\gamma(r_1), \bar\gamma'(r_2))$, $\rho|_{[r_1,r_2]}$ is uniformly close to $\diamondsuit_{\Theta'}(\bar\gamma(r_1), \bar\gamma'(r_2))$.

Now we prove the quasiisometric inequality for $\rho(r_1)$ and $\rho(r_2)$. Since the points $\bar\gamma(r_1)$ and $\bar\gamma'(r_2)$ belong to two opposite cones with tip $\bar\gamma(t_1) = \bar\gamma'(t_1)$,

$$\angle_{\bar\gamma(t_1)}\left(\bar\gamma(r_1), \bar\gamma'(r_2)\right) \geq \alpha',$$

where $\alpha' = \angle_{\mathrm{Tits}}(\Theta', \partial\mathrm{st}(\tau_{\mathrm{mod}}))$. Comparing the geodesic triangle $\triangle\left(\bar\gamma(r_1), \bar\gamma(t_1), \bar\gamma'(r_2)\right)$ with a Euclidean one, we get

$$d\left(\bar\gamma(r_1), \bar\gamma'(r_2)\right) \geq \frac{\sin\alpha'}{2}\left(d\left(\bar\gamma(r_1), \bar\gamma(t_1)\right) + d\left(\bar\gamma'(t_1), \bar\gamma'(r_2)\right)\right).$$

This, together with standard polygon inequality, implies

$$d(\rho(r_1), \rho(r_2)) = d\left(\gamma(r_1), \gamma'(r_2)\right) \geq \frac{\sin\alpha'}{2}\left(d\left(\gamma(r_1), \gamma(t_1)\right) + d\left(\gamma'(t_1), \gamma'(r_2)\right)\right)$$
$$- 2D'(1 + \sin\alpha') \geq \frac{\sin\alpha'}{2L}|r_1 - r_2| - \left(4D' + A\right).$$

In the last inequality, we are using the quasigeodesic data for the paths γ and γ'.

Case 2. $r_1 \in [t_1, t_2]$ and $r_2 \in [t_2, b]$.

This case is settled by reversing the direction of γ in the case 1.

Case 3. $r_1 \in [a, t_1]$ and $r_2 \in [t_2, b]$.

The quasiisometric inequality for $\rho(r_1)$ and $\rho(r_2)$ is clear, since

$$d\left(\rho(r_1), \rho(r_2)\right) = d\left(\gamma(r_1), \gamma(r_2)\right) \geq \frac{|r_1 - r_2|}{L} - A.$$

It remains only to show that the image $\rho([r_1, r_2])$ is uniformly close to a Θ'-diamond.

We know that $\gamma([r_1, r_2])$ is D-close to a diamond $\Diamond_\Theta (x_1, x_2)$ satisfying $d(x_i, \gamma(r_i)) \leq D$, and that $\gamma'([t_1, t_2])$ is D-close to a diamond $\Diamond_\Theta (y_1, y_2)$ satisfying $d(y_i, \gamma'(t_i)) \leq D$, for $i = 1, 2$. Since $\gamma(t_i) = \gamma'(t_i)$, it follows that the points y_1 and y_2 are $2D$-close to $\Diamond_\Theta (x_1, x_2)$. Let $P(\tau_-, \tau_+)$ be the unique parallel set spanned by $\Diamond_\Theta (x_1, x_2)$ satisfying $x_2 \in V(x_1, \mathrm{st}_\Theta(\tau_{\mathrm{mod}})) \tau_+$. Then,

$$y_1 y_2 \subset \mathcal{N}_{2D}(P(\tau_-, \tau_+)).$$

Let $\bar{y}_1, \bar{y}_2$ denote the projections of y_1, y_2, respectively, in $P(\tau_-, \tau_+)$. Note that the points y_1, y_2 are D-close to $\gamma([r_1, r_2])$. Using Proposition 4.9, it follows that when $d(y_1, y_2)$, or equivalently $(t_2 - t_1)$, is sufficiently large, then $\bar{y}_1 \bar{y}_2$ is Θ'-longitudinal. In addition, note that the points $\bar{y}_1, \bar{y}_2$ are $4D$-close to the cones $V(x_1, \mathrm{st}_\Theta(\tau_+)), V(x_2, \mathrm{st}_\Theta(\tau_-))$, respectively. Using the triangle inequality for the Δ-lengths it follows that when $d(x_1, \bar{y}_1)$ and $d(x_2, \bar{y}_2)$, or equivalently $(t_1 - r_1)$ and $(r_2 - t_2)$, are large enough, then $x_1 \bar{y}_1$ and $\bar{y}_2 x_2$ are Θ'-longitudinal. Therefore,

$$\bar{y}_1 \bar{y}_2 \subset \Diamond_{\Theta'}(x_1, x_2).$$

Using Theorem 3.7, there is a constant c which depends only on D, Θ', Θ'' such that

$$\Diamond_{\Theta''}(y_1, y_2) \subset \mathcal{N}_c(\Diamond_{\Theta'}(\bar{y}_1, \bar{y}_2)) \subset \mathcal{N}_c(\Diamond_{\Theta'}(x_1, x_2)).$$

Therefore, $\rho([r_1, r_2])$ is $(c + D)$-close to $\Diamond_{\Theta'}(x_1, x_2)$. $\qquad \square$

Remark. The replacement lemma is false if we relax the Morse condition. It is not generally true that a *uniform quasigeodesic replacement* of an (ordinary) quasigeodesic in a $\mathrm{CAT}(k)$ space, $k \geq 0$, is a uniform quasigeodesic. See the example below. However, if $k < 0$, then the ordinary quasigeodesics are Morse quasigeodesics, so the replacement lemma for ordinary quasigeodesics holds.

Example 4.12. Let $Y = \mathbb{R}^2$, γ be the x-axis. For $r \geq 0$, define a constant speed piecewise path $\gamma'_r : [-r, r] \to \mathbb{R}^2$ with endpoints at $(-r, 0)$ and $(r, 0)$ as in Figure 4. An easy calculation shows that γ' is a $(4, 0)$-quasigeodesic. Let ρ_r denote the replacements, for $r \geq 0$. Then $\rho_r(2r) = \rho_r(r - k_r)$, for some number $k_r > 0$ (observe the point $(2r, 0)$). However, if ρ_r is an (l, a)-quasigeodesic, then $d(\rho_r(2r), \rho_r(r - k_r)) \geq r/l - a$, which is false for large r's.
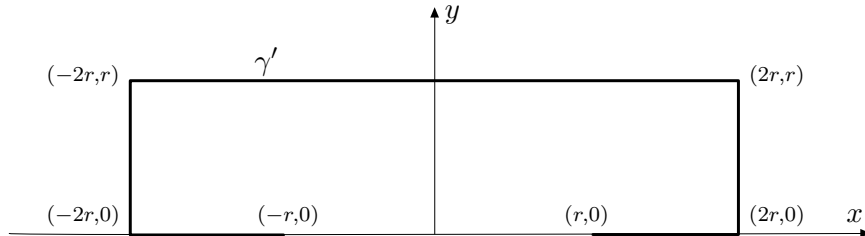

Figure 4

We can also replace a Morse quasigeodesic at multiple segments.

Theorem 4.13 (Generalized replacement lemma). *Let $\gamma : [a, b] \to X$ be an (L, A, Θ, D)-Morse quasigeodesic, and let $a = t_0 \le t_1 \le \cdots \le t_{r_0-1} \le t_{r_0} = b$. For $r = 1, \ldots, r_0$, let $\gamma_r : [t_{r-1}, t_r] \to X$ be an (L', A', Θ', D')-Morse quasigeodesic with $\gamma_r|_{\{t_{r-1}, t_r\}} = \gamma|_{\{t_{r-1}, t_r\}}$. Then the concatenation of γ_r's is an $(L'', A'', \Theta'', D'')$-Morse quasigeodesic where $(L'', A'', \Theta'', D'')$ depends only on (L, A, Θ, D) and (L', A', Θ', D').*

The proof of this theorem closely follows the proof of the previous one and, we are omitting the details.

4.4 Morse subgroups

We first review the notion of Morse subgroups of G. See [KLPa, Sections 7.4, 7.5].

For a finitely generated group H with a finite generating set A, we denote by $\mathrm{Cay}\,(H, A)$ the associated *Cayley graph* equipped with the word metric. We usually suppress "A" from the notation, and denote the Cayley graph by $\mathrm{Cay}\,(H)$. A finitely generated group H is called *hyperbolic* if $\mathrm{Cay}\,(H)$ is hyperbolic.

A metric space Y is called (l, a)-*quasigeodesic* if any pair of points can be connected by an (l, a)-quasigeodesic. Y is called *quasigeodesic*, if it is (l, a)-quasigeodesic for some constants l, a. For a finitely generated subgroup $H < G$ and a chosen base point $x \in X$, there is a natural map $o_x : \mathrm{Cay}\,(H) \to X$ induced by the orbit map $H \to Hx$. A subgroup $H < G$ is called *undistorted* (in G), if some (any) o_x is a quasiisometric embedding. General quasiisometric embeddings of a quasigeodesic space into a symmetric space tend to be "bad". However, one obtains a good control on these embeddings when they are *Morse*; we review this notion below.

Definition 4.14 (Morse embeddings, [KLPa]). *Let X be a symmetric space of noncompact type. A map $f : Y \to X$ from a quasigeodesic space Y is called Θ-Morse embedding if it sends uniform quasigeodesics in Y to uniform Θ-Morse quasigeodesics in X. Moreover, f is called τ_{mod}-Morse embedding if it is Θ-Morse embedding for some Θ.*

Now we state the notion of Morse subgroups.

Definition 4.15 (Morse subgroups, [KLPa]). *A finitely generated subgroup Γ of G is called τ_{mod}-Morse if, for an(y) $x \in X$, the map $o_x : \mathrm{Cay}\,(H) \to X$ of $\mathrm{Cay}\,(\Gamma)$ into X induced by $\Gamma \to Hx$ is τ_{mod}-Morse.*

Note that Morse subgroups are undistorted.

Every τ_{mod}-Morse subgroup Γ induces a canonical boundary embedding $\beta : \partial_\infty \Gamma \to \mathrm{Flag}\,(\tau_{\mathrm{mod}})$, see [KLPa, KLP17]. The image of β in $\mathrm{Flag}\,(\tau_{\mathrm{mod}})$ is called the *flag limit set* of Γ, and will be denoted by $\Lambda_{\tau_{\mathrm{mod}}}\,(\Gamma)$.

Moreover, τ_{mod}-Morse subgroups are *uniformly τ_{mod}-regular* (see [KLP17]) and, hence, the accumulation set in $\partial_\infty X$ of any orbit Γx contains only points whose types are elements of Θ, for some compact Weyl-convex subset $\Theta \subset \mathrm{ost}\,(\tau_{\mathrm{mod}})$. The smallest such Θ will be denoted by Θ_Γ.

Remark. A finitely generated uniformly τ_{mod}-regular and undistorted subgroup $\Gamma < G$ is called a τ_{mod}-*URU* subgroup. The equivalence of τ_{mod}-Morse and τ_{mod}-URU is the main result of [KLPb]; see also [KLP17].

Proposition 4.16. *Let Γ be a τ_{mod}-Morse subgroup, let Λ' be any compact set in $\mathrm{Flag}\,(\tau_{\mathrm{mod}})$ whose interior contains $\Lambda = \Lambda_{\tau_{\mathrm{mod}}}\,(\Gamma)$, and let Θ' be any compact set containing $\Theta = \Theta_\Gamma(x)$ in its interior. There exists a number $S > 0$ such that any $\gamma \in \Gamma$ satisfying $d(x, \gamma x) > S$ also satisfies $\gamma x \in V\,(x, \mathrm{st}_{\Theta'}(\tau))$, for some $\tau \in \Lambda'$.*

Proof. We first prove that there exists $S' > 0$ such that $d(x, \gamma x) > S'$ implies that $(x, \gamma x)$ is Θ'-regular. Suppose that S' does not exist; then, there is an unbounded sequence $(\gamma_i)_{i \in \mathbb{N}}$ in Γ such that $(x, \gamma_i x)$ is not Θ'-regular for all i. Then, $(\gamma_i x)_{i \in \mathbb{N}}$ subconverge to an ideal point whose type $\notin \text{int}(\Theta')$. This can not happen because the interior of Θ' contains Θ.

Next we prove that S exists. Assuming that it does not exist, we get an unbounded sequence $(\gamma_i')_{i \in \mathbb{N}}$ in Γ such that $\gamma_i' x \notin V(x, \text{st}_{\Theta'}(\tau))$, for all $i \in \mathbb{N}$ and $\tau \in \Lambda'$. After extraction we may assume that $(x, \gamma_i' x)$ is Θ'-regular, for all i. But then, $(\gamma_i' x)_{i \in \mathbb{N}}$ does not accumulate in any simplex in the interior of Λ' i.e. Γ has a limit simplex outside Λ, but this gives a contradiction. $\qquad \square$

4.5 Residual finiteness

An important feature shared by many finitely generated subgroups of G is the *residual finiteness* property which enables us to obtain finite index subgroups which avoid a given finite set of elements.

Definition 4.17 (Residual finiteness)**.** A group H is called *residually finite (RF)* if it satisfies one of the following equivalent conditions: (1) Given a finite subset $S \subset H \setminus \{1_H\}$, there exists a finite index subgroup $F < H$ such that $F \cap S = \emptyset$. (2) Given an element $h \in H \setminus \{1_H\}$, there exits a finite group Φ and a homomorphism $\phi : H \to \Phi$ such that $\phi(h) \neq 1_\Phi$. (3) The intersection of finite index subgroups of H is trivial.

Residual finiteness of Morse subgroups is a corollary to the following celebrated theorem.

Let R be a commutative ring with unity, and let $GL(N, R)$ denote the group (with multiplication) of non-singular $N \times N$ matrices with entries in R. Then,

Theorem 4.18 (A. I. Mal'cev, [Mal40])**.** *Finitely generated subgroups of $GL(N, R)$ are RF.*

As an application to this theorem, one obtains,

Corollary 4.19. *Each finitely generated subgroup $\Gamma < G$ which intersects the center of G trivially is RF.*

Proof. Under our assumptions, the adjoint representation $\Gamma \to GL(\mathfrak{g})$ is faithful. $\qquad \square$

For a subgroup $\Gamma < G$, we define the *norm* of Γ with respect to $x \in X$ as

$$\|\Gamma\|_x = \inf\{d(x, \gamma x) \mid 1_\Gamma \neq \gamma \in \Gamma\}.$$

Note that when $\|\Gamma\|_x > 0$, Γ is discrete. Residual finiteness implies the following useful lemma which we use to obtain subgroups whose nontrivial elements send x arbitrarily far:

Lemma 4.20. *Let Γ be a RF discrete subgroup of G. For any $R \in \mathbb{R}$, there exist a finite index subgroup $\Gamma' < \Gamma$ such that $\|\Gamma'\|_x \geq R$.*

Proof. Since Γ is discrete, the set $\Phi = \{\gamma \mid d(x, \gamma x) < R\}$ is finite. The assertion follows from the residual finiteness property. $\qquad \square$

Combining this lemma with Proposition 4.16, we get the following:

Corollary 4.21. *Let $\Gamma < G$ be a RF τ_{mod}-Morse subgroup, let Λ' be any compact set in $\text{Flag}(\tau_{\text{mod}})$ whose interior contains $\Lambda = \Lambda_{\tau_{\text{mod}}}(\Gamma)$, and let Θ' be any compact set containing $\Theta = \Theta_\Gamma$ in its interior. There exists $S_1 > 0$ such that for any $S \geq S_1$ there exists a finite index subgroup Γ' of Γ satisfying $\|\Gamma'\|_x > S$ which also satisfies the following: For any $\gamma' \in \Gamma'$ exists $\tau \in \Lambda'$ for which $\gamma' x \in V(x, \text{st}_{\Theta'}(\tau))$.*

Now we briefly turn to the discussion of pairwise antipodal subgroups before proving our main theorem in the next section.

Definition 4.22 (Antipodal Morse subgroups). A pair of τ_{mod}-Morse subgroups $\Gamma_1, \Gamma_2 < G$ is called *antipodal* if their flag limit sets in $\mathrm{Flag}\,(\tau_{\mathrm{mod}})$ are antipodal.

Proposition 4.23. *Let* $\Gamma_1, \ldots, \Gamma_n$ *be pairwise antipodal, RF* τ_{mod}-*Morse subgroups of G. Let* $\Theta \subset \mathrm{ost}\,(\tau_{\mathrm{mod}})$ *be a subset which contains the sets* Θ_{Γ_i}, *for* $i = 1, \ldots, n$, *in its interior. Then, there exists a collection* $\{\Lambda_1', \ldots, \Lambda_n'\}$ *of pairwise antipodal, compact subsets of* $\mathrm{Flag}\,(\tau_{\mathrm{mod}})$, *and a number* $S_2 > 0$ *such that for any* $S \geq S_2$ *there exists a collection of finite index subgroups* $\Gamma_1', \ldots, \Gamma_n'$ *of* $\Gamma_1, \ldots, \Gamma_n$, *respectively, satisfying* $\|\Gamma_1'\|_x \geq S, \ldots, \|\Gamma_n'\|_x \geq S$ *which also satisfies the following: For each* $i = 1, \ldots, n$, *and for each* $\gamma_i \in \Gamma_i'$, *there exists* $\tau_i \in \Lambda_i'$ *such that*

$$\gamma_i x \in V(x, \mathrm{st}_\Theta\,(\tau_i)).$$

Proof. Once we show that there exists a collection $\{\Lambda_1', \ldots, \Lambda_n'\}$ such that, for each i, the interior of Λ_i' contains the flag limit set Λ_i of Γ_i, the first part of the proposition follows from the Corollary 4.21. We may construct $\Lambda_1', \ldots, \Lambda_n'$ as follows:

Lemma 4.24. *Let* $\{\Lambda_1, \ldots, \Lambda_n\}$ *be a collection pairwise antipodal, compact subsets of* $\mathrm{Flag}\,(\tau_{\mathrm{mod}})$. *Then, there exists a collection* $\{\Lambda_1', \ldots, \Lambda_n'\}$ *of pairwise antipodal, compact subsets of* $\mathrm{Flag}\,(\tau_{\mathrm{mod}})$ *such that each* Λ_i *is contained in the interior of* Λ_i'.

Proof. The case $n = 2$ can be proved as follows. Let Λ_1, Λ_2 be a pair of antipodal, compact subsets of $\mathrm{Flag}\,(\tau_{\mathrm{mod}})$. Then,

$$\Lambda_1 \times \Lambda_2 \underset{\text{compact}}{\subset} (\mathrm{Flag}\,(\tau_{\mathrm{mod}}) \times \mathrm{Flag}\,(\tau_{\mathrm{mod}}))^{\mathrm{opp}} \underset{\text{open}}{\subset} \mathrm{Flag}\,(\tau_{\mathrm{mod}}) \times \mathrm{Flag}\,(\tau_{\mathrm{mod}}).$$

There is a open neighborhood of $\Lambda_1 \times \Lambda_2$ in $(\mathrm{Flag}\,(\tau_{\mathrm{mod}}) \times \mathrm{Flag}\,(\tau_{\mathrm{mod}}))^{\mathrm{opp}}$ of the form $U_1 \times U_2$. In particular, the subsets U_1 and U_2 are antipodal. Then any pair of compact subsets Λ_1' and Λ_2' of U_1 and U_2, respectively, containing Λ_1 and Λ_2 in their respective interiors, does the job.

We consider now the general case $n \geq 3$ and let $\{\Lambda_1, \ldots, \Lambda_n\}$ be a collection of subsets as in proposition. For Λ_1, using the lemma, we find a compact neighborhood Λ_1' of Λ_1 which is antipodal to the compact

$$\bigcup_{k=2}^{n} \Lambda_k.$$

Then, $\{\Lambda_1', \Lambda_2, \ldots, \Lambda_n\}$ is new collection pairwise antipodal, compact subsets of $\mathrm{Flag}\,(\tau_{\mathrm{mod}})$. The same argument yields a compact neighborhood Λ_2' of Λ_2 antipodal to $\Lambda_1', \Lambda_3, \ldots, \Lambda_k$. We continue inductively. $\square$

This completes the proof of the proposition. $\square$

5 A combination theorem

In this section, we prove our main result.

Theorem 5.1 (Combination theorem). *Let* $\Gamma_1, \ldots, \Gamma_n$ *be pairwise antipodal, RF* τ_{mod}-*Morse subgroups of G. Then, there exist finite index subgroups* $\Gamma_i' < \Gamma_i$, *for* $i = 1, \ldots, n$, *such that* $\langle \Gamma_1', \ldots, \Gamma_n' \rangle$ *is* τ_{mod}-*Morse, and is naturally isomorphic to* $\Gamma_1' * \cdots * \Gamma_n'$

Proof. We first fix our notations. We denote the τ_{mod}-flag limit sets of $\Gamma_1, \ldots, \Gamma_n$ by $\Lambda_1, \ldots, \Lambda_n$, respectively. Let $\Theta \subset \mathrm{ost}\,(\tau_{\mathrm{mod}})$, let $\{\Lambda_1', \ldots, \Lambda_n'\}$ be a collection of compact, pairwise antipodal subsets in $\mathrm{Flag}\,(\tau_{\mathrm{mod}})$, and let $S_2 > 0$ be as in Proposition 4.23. As always, the point x will be treated as a fixed base point in X. Finally, $\Theta \subset \Theta' \subset \Theta''$ are ι-invariant, Weyl-convex, compact subsets of $\mathrm{ost}\,(\tau_{\mathrm{mod}})$ such that

$$\mathrm{int}\,(\Theta'') \supset \Theta', \quad \mathrm{int}\,(\Theta') \supset \Theta.$$

By Proposition 4.23, for each $S > S_2$ there exist finite index subgroups $\Gamma_1', \ldots, \Gamma_n'$ of $\Gamma_1, \ldots, \Gamma_n$, respectively, of norms $\|\Gamma_i'\|_x \geq S$, such that for each $i = 1, \ldots, n$, and each $\gamma_i \in \Gamma_i'$,

$$\gamma_i x \in V(x, \mathrm{st}_\Theta\,(\tau_i)), \tag{5.1}$$

for some $\tau_i \in \Lambda_i'$. Let A_i be a finite generating set of Γ_i', for each $i = 1, \ldots, n$. This choice endows each Γ_i' with a word metric, and thus yields a Θ-Morse embedding $o_x^i : \mathrm{Cay}\,(\Gamma_i', A_i) \to X$ induced by the orbit map $\Gamma_i' \to \Gamma_i' x$. We take the standard generating set $A = A_1 \cup \cdots \cup A_n$ of the abstract free product $\Gamma' = \Gamma_1' * \cdots * \Gamma_n'$. We obtain a natural homomorphism $\varphi : \Gamma' \to G$. When S sufficiently large we prove that $o_x : \mathrm{Cay}\,(\Gamma', A) \to X$ is a Θ'-Morse embedding, i.e. we prove that the geodesics of $\mathrm{Cay}\,(\Gamma', A)$ are mapped to uniform Morse quasigeodesics in X. This not only will prove that φ is injective, but also will show that the subgroup $\langle \Gamma_1', \ldots, \Gamma_n' \rangle$ of G generated by $\Gamma_1', \ldots, \Gamma_n'$ is τ_{mod}-Morse.

Claim. *There exists $S_0 > 0$ such that if $S \geq S_0$, then the map $o_x : \mathrm{Cay}\,(\Gamma', A) \to X$ sends (finite) geodesics to uniform Morse quasigeodesics.*

Proof. Given any $\gamma \in \Gamma'$, there is a natural embedding of $\mathrm{Cay}\,(\Gamma_i')$ into $\mathrm{Cay}\,(\Gamma')$ given by the right multiplication map $\gamma_i \mapsto \gamma_i \gamma$. Any geodesic in $\mathrm{Cay}\,(\Gamma')$ is a concatenation of paths which are images of the geodesics under the maps above. By equivariance, it suffices to study the geodesics in Γ' starting at $1_{\Gamma'}$. Any geodesic ψ with starting point $1_{\Gamma'}$ in $\mathrm{Cay}\,(\Gamma')$ can be written as

$$\psi \;:\; 1_{\Gamma'}, \; \gamma_{k_1}, \; \gamma_{k_2}\gamma_{k_1}, \ldots, \tag{5.2}$$

where the path joining $\gamma_{k_r}\gamma_{k_{r-1}} \ldots \gamma_{k_1}$ and $\gamma_{k_{r-1}} \ldots \gamma_{k_1}$ in $\mathrm{Cay}\,(\Gamma')$ is the image of a geodesic segment in $\mathrm{Cay}\,(\Gamma_i')$ connecting the identity to γ_{k_r} under the map $(\cdot) \mapsto (\cdot)(\gamma_{k_{r-1}} \ldots \gamma_{k_1})$, assuming that $\gamma_{k_r} \in \Gamma_i'$. We group together γ_{k_r}'s in above to avoid two consecutive ones coming from same Γ_i's.

The (finite) sequence (5.2) is mapped to $x, \; \gamma_{k_1}x, \; \gamma_{k_2}\gamma_{k_1}x, \ldots$ under the map o_x; to avoid cumbersome notations, we denote $\gamma_{k_r}\gamma_{k_{r-1}} \ldots \gamma_{k_1}$ by g_r, denote $\gamma_{k_r}\gamma_{k_{r-1}} \ldots \gamma_{k_1}x$ by p_r and assume that the index r of this sequence varies between 0 and r_0. Using these notations, we have

$$g_r = \gamma_{k_r}g_{r-1}, \quad g_r x = p_r. \tag{5.3}$$

Let $m_1 = p_0$, $m_{r_0} = p_{r_0}$, and, for $2 \leq r \leq r_0 - 1$, let m_r denote the midpoint of p_{r-1} and p_r (see Figure 5).

It follows from (5.1) that all the segments $p_{r-1}p_r$ in X are Θ-regular and have length at least S. Moreover, it follows from (5.3) that, for any $1 \leq r \leq r_0 - 1$, precomposing the right multiplication action $g_r^{-1} \curvearrowright \Gamma$ with o_x maps the hinge $p_{r-1}p_rp_{r+1}$ to $(\gamma_{k_r}^{-1}x)(x)(\gamma_{k_{r+1}}x)$ which is of the form $(\gamma_i x)(x)(\gamma_j x)$, for some $\gamma_i \in \Gamma_i'$, $\gamma_j \in \Gamma_j'$, $i \neq j$. From (5.1), we get that $\gamma_i x \in V(x, \mathrm{st}_\Theta\,(\tau_i))$ and $\gamma_j x \in V(x, \mathrm{st}_\Theta\,(\tau_j))$, for some $\tau_i \in \Lambda_i'$ and $\tau_j \in \Lambda_j'$. To simplify our notation, the corresponding images of m_r and m_{r+1} are denoted by m_i' and m_j', respectively.
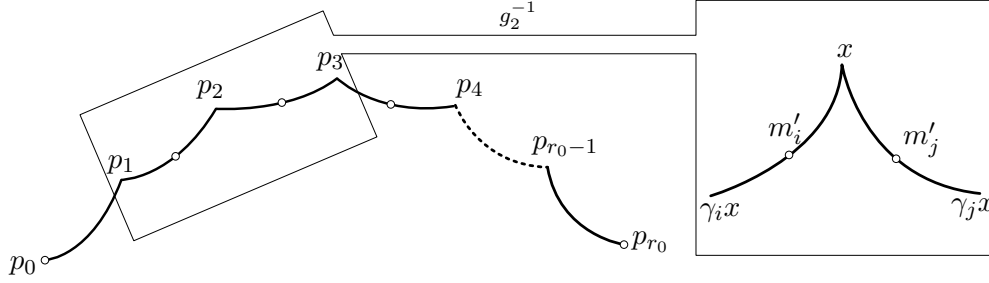
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Figure 5: Morse embedding of quasigeodesics. The hollow points represent the midpoint sequence (m_i).

Let $D = \max\{D(\Lambda'_i, \Lambda'_j, x) \mid 1 \leq i < j \leq n\}$, where $D(\Lambda'_i, \Lambda'_j, x)$ is the constant given by Corollary 3.3. Moreover, let $R_1(x, \Lambda'_i, \Lambda'_j, \Theta', \Theta)$ and $f_1(x, \Lambda'_i, \Lambda'_j, \Theta', \Theta, \xi)$ be the quantities as in Proposition 3.8. Define

$$R_1 = \max_{i,j,\ i \neq j} \left\{ R_1(x, \Lambda'_i, \Lambda'_j, \Theta', \Theta) \right\},$$

and

$$f_1 = \max_{i,j,\ i \neq j} \left\{ f_1(x, \Lambda'_i, \Lambda'_j, \Theta', \Theta, \xi) \right\}.$$

Note that $d(x, m'_i), d(x, m'_j) \geq S/2$. Using part 1 of Proposition 3.8, when $S/2 \geq R_1$, then $m'_i m'_j$ is Θ'-regular. Using part 2 of the same proposition we get

$$\angle^{\xi}_{m'_i}(x, m'_j), \angle^{\xi}_{m_j}(x, m'_i) \leq f_1(S/2).$$

Moreover, using and (3.6), we obtain

$$d(m'_i, m'_j) \geq (S \sin \alpha)/2 - 4D,$$

where $\alpha = \angle_{\text{Tits}}(\Theta, \partial\Theta')$. Therefore, when $S \geq 2R_1$, the sequence (m_r) is $(\Theta', 2f_1(S/2))$-straight and $((S \sin \alpha)/2 - 4D)$-spaced.

For any $\delta' > 0$, Theorem 4.6 applied to Θ', Θ'' and δ' concludes that there exists $S_0 \gg R_1$ such that when $S \geq S_0$, then the sequence (m_r) is δ'-close to a parallel set $P(\tau_-, \tau_+)$ such that the nearest point projection map sends $m_r m_{r+1}$ to a Θ''-longitudinal segment. This proves that the piecewise geodesic path $p_0 p_1 \ldots p_{r_0}$ is a uniform Morse quasigeodesic for sufficiently small δ'.

Finally, we prove that $o_x \circ \psi$ is uniformly Morse. By invoking the Morse property of Γ'_i's, we get that each segment of $o_x \circ \psi$ connecting a consecutive pair p_r and p_{r+1} is a uniform Morse quasigeodesic. Therefore, $o_x \circ \psi$ is obtained by replacing the geodesic segments $p_r p_{r+1}$ of the path $p_0 p_1 \ldots p_{r_0}$ by uniform Morse quasigeodesics. From the generalized replacement lemma (Theorem 4.13), it follows that $o_x \circ \psi$ is also a uniform Morse quasigeodesic. □

This completes the proof of the theorem. □

Remark. The RF condition in the above theorem can be relaxed by integrating the content of Corollary 4.19 into the hypothesis. Precisely, instead of requiring Γ_i's to be RF one may require that Γ_i's intersect the center of G trivially. When $G \cong \text{Isom}_0(X)$, this happens automatically, because $\text{Isom}_0(X)$ is centerless.

Below is a more general form of Theorem 5.1 which does not involve passing to finite index subgroups, but instead requires "sufficient antipodality and sparseness" of the subgroups Γ_i. Let

$$(\underbrace{\text{Flag}\,(\tau_{\text{mod}}) \times ... \times \text{Flag}\,(\tau_{\text{mod}})}_{n\ \text{times}})^{\text{opp}}$$

denote the subset of $(\text{Flag}\,(\tau_{\text{mod}}))^n$ consisting of n tuples of pairwise antipodal flags. For a subset

$$A \subset (\underbrace{\text{Flag}\,(\tau_{\text{mod}}) \times ... \times \text{Flag}\,(\tau_{\text{mod}})}_{n\ \text{times}})^{\text{opp}}$$

and for $x \in X$ define the subset $O_{A,x} \subset X^n$ consisting of n-tuples $(x_1, ..., x_n)$ such that for some $(\tau_1, ..., \tau_n) \in A$, we have $x_i \in V(x, \text{st}\,(\tau_i))$, $i = 1, ..., n$.

Theorem 5.2. *For each compact*

$$A \subset (\underbrace{\text{Flag}\,(\tau_{\text{mod}}) \times ... \times \text{Flag}\,(\tau_{\text{mod}})}_{n\ times})^{\text{opp}},$$

and $\Theta \subset \text{ost}\,(\tau_{\text{mod}})$, there exists a constant $S = S(A, \Theta, x)$ such that the following holds. Let $\Gamma_1, ..., \Gamma_n$ be P-Anosov subgroups of G such that:
  a. $\|\Gamma_i\|_x \geq S$, $i = 1, ..., n$.
  b. For each $\gamma_i \in \Gamma_i, i = 1, ..., n$, the segment $x\gamma_i(x)$ is Θ-regular.
  c. For each n-tuple of nontrivial elements $\gamma_i \in \Gamma_i - \{1\}$, $i = 1, ..., n$, we have

$$(\gamma_1(x), ..., \gamma_n(x)) \in O_{A,x}.$$

Then the subgroup of G generated by $\Gamma_1, ..., \Gamma_n$ is P-Anosov, and is naturally isomorphic to the free product $\Gamma_1 * ... * \Gamma_n$.

Proof. The proof is very similar to the one of Theorem 5.1. The conclusion of Proposition 4.23 now becomes a hypothesis on the subgroups Γ_i, so no passage to finite index subgroups is required. Hence, the rest of the proof of Theorem 5.1 goes through. $\qquad\square$

Remark. We should note that this theorem is in the spirit of the "quantitative ping-pong lemma" of Breuillard and Gelander, see [BG08, Lemma 2.3].

As a last remark, we note that the traditional Klein-Maskit combination theorems are stated not in terms of group actions on symmetric spaces but in terms of their actions on the sphere at infinity; they also do not involve passing to finite index subgroups. The following is a reasonable combination conjecture in the setting of Anosov subgroups:

Conjecture 5.3. *Let $A_1, ..., A_n \subset \text{Flag}\,(\tau_{\text{mod}})$ be nonempty compact subsets such that any two distinct elements of*

$$A := \bigcup_{i=1}^{n} A_i$$

are antipodal. Suppose that $\Gamma_1, ..., \Gamma_n$ are $P_{\tau_{\text{mod}}}$-Anosov subgroups of G such that for all $i = 1, ..., n$ and all $\gamma \in \Gamma_i - \{1\}$ we have

$$\gamma(A - A_i) \subset \text{int}\,(A_i).$$

Then the subgroup Γ of G generated by $\Gamma_1, ..., \Gamma_n$ is $P_{\tau_{\text{mod}}}$-Anosov.

Note that under the above assumptions, Γ is naturally isomorphic to the free product $\Gamma_1 * ... * \Gamma_n$, see e.g. [Tit72].

List of notations

- $\angle_x^{\xi}(x_1, x_2) = \xi$-angle between τ_{mod}-regular segments xx_1 and xx_2 (see Section 2.3)

- $\diamondsuit_{\Theta}(x_1, x_2) = \Theta$-diamond with tips at x_1 and x_2 (see Section 2.4)

- $\iota = $ the opposition involution (see Section 2.1)

- $\mathcal{N}_D(\cdot) = $ open D-neighborhood

- $\mathrm{ost}(\tau) = $ open star of τ in the visual boundary (see Section 2.4)

- $\mathrm{st}(\tau) = $ star of τ in the visual boundary (see Section 2.4)

- $V(x, \mathrm{st}_{\Theta}(\tau)) = \Theta$-cone asymptotic to τ with tip at x (see Section 2.4)

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