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On the Moduli Space of a Spherical Polygonal Linkage

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Abstract. We give a "wall-crossing" formula for computing the topology of the moduli space of a closed n-gon linkage on $\mathbb{S}^2$. We do this by determining the Morse theory of the function ρ_n on the moduli space of n-gon linkages which is given by the length of the last side—the length of the last side is allowed to vary, the first $(n-1)$ side-lengths are fixed. We obtain a Morse function on the $(n-2)$-torus with level sets moduli spaces of n-gon linkages. The critical points of ρ_n are the linkages which are contained in a great circle. We give a formula for the signature of the Hessian of ρ_n at such a linkage in terms of the number of back-tracks and the winding number. We use our formula to determine the moduli spaces of all regular pentagonal spherical linkages.

1 Introduction

Our goal in this paper is to give a "wall-crossing" formula for determining the topology of the moduli space of a closed n-gon linkage on $\mathbb{S}^2$. We will give definitions in Section 2. The definitions of the configuration space and the moduli space $M(\Lambda, X)$ of a general linkage Λ in a constant curvature space X are given in [KM3].

Let $r = (r_1, r_2, \dots, r_n)$ be an n-tuple of real numbers satisfying $0 < r_i < \pi$. Let $N_{r'}$ be the moduli space of the free $(n-1)$-gon spherical linkage with side-lengths $r' := (r_1, \dots, r_{n-1})$, so $N_{r'}$ is the quotient by $\mathrm{SO}(3)$ of the subspace $\tilde{N}_{r'} \subset (\mathbb{S}^2)^n$ defined by

$$\tilde{N}_{r'} = \{u = (u_1, \dots, u_n) \in (\mathbb{S}^2)^n : d(u_i, u_{i+1}) = r_i, 1 \le i \le n-1\}.$$

Here d is the spherical distance. The points $u_1, u_2, \dots, u_n$ are called the vertices of the linkage $T \in \tilde{N}_{r'}$. Clearly $N_{r'} \cong (\mathbb{S}^1)^{n-2}$. We will study the Morse theory of the function $\rho_n \colon N_{r'} \to \mathbb{R}$ given by

$$\rho_n(u) = d(u_1, u_n).$$

We will restrict to u's such that $0 < \rho_n(u) < \pi$ so that ρ_n is differentiable. Notice that

$$M_r := \rho_n^{-1}(r_n) \subset N_{r'}$$

is the *moduli space* of *closed polygonal linkages* in $\mathbb{S}^2$ with the side-lengths $(r_1, \dots, r_n)$.

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Definition We define the closed n-gon linkage $P = P(T)$ associated to a free $(n-1)$-gon linkage T to be the linkage obtained by adding the length-minimizing geodesic segment[1] $(u_n, u_1) = e_n \subset \mathbb{S}^2$ joining u_n to u_1.

Thus r_n is the length of the new edge e_n. Hence, in terms of deformations of the closed n-gon P in $\mathbb{S}^2$, we can obtain $N_{r'}$ by fixing the lengths of the first $n-1$ sides and *letting the length of the last side vary*.

In order to state the Main Theorem we will need some definitions.

Definition A linkage in $\mathbb{S}^2$ is degenerate if it lies in a great circle γ of $\mathbb{S}^2$.

Suppose now that P is a degenerate closed n-gon linkage contained in a great circle γ. We orient γ and define $\epsilon_i \in \{\pm 1\}$ to be 1 if the orientation of the i-th edge of P agrees with that of γ and -1 otherwise. We say that the i-th edge of P is a *forward-track* if $\epsilon_i = 1$ and a *back-track* otherwise. We let $f = f(P)$ be the number of forward-tracks and $b = b(P)$ be the number of back-tracks so $f + b = n$. Define the winding number $w = w(P)$ by

$$\sum_{i=1}^{n} \epsilon_i r_i = 2\pi w.$$

The numbers b, f and w depend on the orientation of γ. We will deal with this below.

We will see that the critical points of ρ_n on $N_{r'}$ are the degenerate linkages. If T is a degenerate free $(n-1)$-gon linkage our goal is to give a formula for the signature of the Hessian $D^2\rho_n|_T$ in terms of $b(P)$, $f(P)$ and $w(P)$ where $P = P(T)$ is the associated closed n-gon linkage (see above). Clearly we must give a rule for orienting the great circle $\gamma \supset T$.

Definition **(orienting γ)** Suppose $u = (u_1, u_2, \ldots, u_n)$ is a closed degenerate linkage contained in a great circle γ. Orient γ so that the arc joining u_1 to u_n is positively directed. Thus an edge e_i is a back-track if it has the same direction as $e_n = (u_n, u_1)$.

We will prove the following theorem (with b, f and w defined using the above orientation of γ).

Main Theorem *Let $T \in N_{r'}$ be a degenerate free $(n-1)$-gon linkage and P be the associated degenerate closed n-gon linkage. Then the signature of $D^2\rho_n|_T$ is*

$$\big(b(P) + 2w(P) - 1,\ f(P) - 2w(P) - 1\big).$$

Remark The analogue of the Main Theorem for polygonal linkages in the Euclidean plane was proved in Lemma 11 of [KM1].

The Main Theorem reduces the description of the moduli spaces of spherical polygonal linkages to the combinatorics of the chambers of the polyhedron $D_n(\mathbb{S}^2)$ (see Section 2). These computations are manageable for $n = 4, 5, 6$ but become formidable for $n \geq 7$. In [G] the moduli spaces of all spherical n-gons for $n = 4, 5, 6$ are determined. In this paper we illustrate the wall-crossing formula by describing the moduli spaces of regular spherical pentagons.

[1] In what follows (a, b) will always denote the shortest geodesic segment connecting non-antipodal points a, b in $\mathbb{S}^2$.

This paper depends on the result of [KM2] *that ρ_n is a Morse function*. This result is what underlies the deformation arguments in Lemma 5.4 and Lemma 5.6. This paper completes the computation of the signature of $D^2\rho_n$ in Theorem 8.10 of that paper. In the appendix to this paper we patch up an error in [KM2] which allows us to apply the results of that paper that we need here.

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2 Preliminaries

Definition 2.1 A closed spherical n-gon $P = (e_1, \dots, e_n)$ is an n-tuple of oriented geodesic arcs e_j (in $\mathbb{S}^2$) of lengths between 0 and π (inclusive) such that the end-point of e_{i-1} is equal to the initial point of e_i, $0 \le i \le n$ (the indices are taken modulo n).

Definition 2.2 Let $\mathcal{P}_n(\mathbb{S}^2)$ be the space of closed n-gons on $\mathbb{S}^2$ with geodesic edges.

We let r_i be the length of e_i in the spherical metric. The arcs $e_1, \dots, e_n$ will be called the edges of P. We will use $u = (u_1, \dots, u_n)$ to denote the set of vertices of P, that is, the set of initial points of the edges e_i. We will soon restrict ourselves to n-gons P with the property that $0 < r_i < \pi$, $1 \le i \le n$. In this case P is determined by its vertices $u_1, \dots, u_n$ and we may write $P = u = (u_1, \dots, u_n)$.

Definition 2.3 Let $\rho\colon \mathcal{P}_n(\mathbb{S}^2) \to (\mathbb{R}_+)^n$ defined by $\rho(u) = r = (r_1, \dots, r_n)$ be the side length map. That is, the distances, $d(u_i, u_{i+1})$ in the spherical metric satisfy $d(u_i, u_{i+1}) = r_i$ for $1 \le i \le n$ where we consider $u_{n+1} = u_1$.

Definition 2.4 $D_n(\mathbb{S}^2) = \rho\big(\mathcal{P}_n(\mathbb{S}^2)\big)$ is the space of possible side lengths. We let $\tilde{M}_r := \rho^{-1}(r)$ be the configuration space of closed n-gon linkages in $\mathbb{S}^2$ with the side-lengths r.

It is immediate that $\tilde{M}_r$ is the set of real points of the affine variety over $\mathbb{R}$ (*i.e.*, $\tilde{M}_r$ is a real algebraic set) defined by

$$u_i \cdot u_{i+1} = \cos r_i, \quad 1 \le i \le n,$$

where $\vec{x} \cdot \vec{y}$ denotes the scalar product in $\mathbb{R}^3$. The group SO(3) acts on $\tilde{M}_r$ according to

$$g(u) = (gu_1, \dots, gu_n), \quad u \in \tilde{M}_r, \quad g \in \mathrm{SO}(3).$$

Definition 2.5 The moduli space M_r of n-gon linkages on $\mathbb{S}^2$ with side lengths $r = (r_1, \dots, r_n)$ is defined to be the quotient space of $\tilde{M}_r$ by SO(3).

We now prove that M_r has the structure of a real algebraic set—here we assume $0 < r_i < \pi$, $1 \le i \le n$. Let $\vec{\epsilon}_1, \vec{\epsilon}_2, \vec{\epsilon}_3$ denote the standard basis of $\mathbb{R}^3$.

Lemma 2.6 *Define $\Sigma_r \subset \tilde{M}_r$ by $\Sigma_r = \{u \in \tilde{M}_r : u_1 = \vec{\epsilon}_1,\, u_n = \cos r_n \vec{\epsilon}_1 + \sin r_n \vec{\epsilon}_2\}$. Then Σ_r is a cross-section to the orbits of* $\mathrm{SO}(3)$ *on $\tilde{M}_r$.*

Proof Obvious. ■

Since the quotient map $\tilde{M}_r \to M_r$ induces a homeomorphism from Σ_r to M_r and Σ_r is a real algebraic set, M_r is a real algebraic set by transport of structure. In what follows we identify M_r and Σ_r. Notice that

$$M_r = \rho_n^{-1}(r_n), \quad \rho_n \colon N_{r'} \to \mathbb{R}, \quad \rho_n(P) = r_n, \quad \text{where } r = (r_1, \dots, r_n).$$

We let $\mathcal{Q}_n(\mathbb{S}^2)$ be the quotient space of $\mathcal{P}_n(\mathbb{S}^2)$ by $\mathrm{SO}(3)$ and let $\pi \colon \mathcal{Q}_n(\mathbb{S}^2) \to (\mathbb{R}_+)^n$ be the map induced by ρ. Hence for $r \in (\mathbb{R}_+)^n$

$$M_r = \pi^{-1}(r).$$

Our strategy is to study how the fibers of π vary as r varies in $D_n(\mathbb{S}^2)$.

We have

Lemma 2.7

(i) The Zariski tangent space $T_u(\tilde{M}_r)$ is given by

$$T_u(\tilde{M}_r) = \ker d\rho|_u.$$

(ii) The Zariski tangent space $T_u(M_r)$ is given by

$$T_u(M_r) = \ker d\pi|_u.$$

Corollary 2.8 *The variety $\tilde{M}_r$ (resp. M_r) is smooth if and only if r is a regular value of ρ (resp. π).*

From [KM2], Theorem 1.1 we deduce

Theorem 2.9 *Let $P \in \mathcal{P}_n(\mathbb{S}^2)$ (resp. $\mathcal{Q}_n(\mathbb{S}^2)$). Then P is a critical point of ρ (resp. π) if and only if P is degenerate.*

3 The Results of A. Galitzer

In [G], A. Galitzer has described $D_n(\mathbb{S}^2)$. We will need some notation to describe her results. If $I \subset \{1, 2, \dots, n\}$ we let $\bar{I}$ denote the complement of I, $|I|$ be the cardinality of I and $r_I = \sum_{i \in I} r_i$. Define a polyhedron $K_n \subset \mathbb{R}^n$ by the system of inequalities

$$0 \le r_i \le \pi, \quad 1 \le i \le n, \quad \text{and}$$
$$r_I \le r_{\bar{I}} + (|I| - 1)\pi, \quad I \subset \{1, 2, \dots, n\}, \quad \text{with } |I| \text{ odd}.$$

Then Galitzer proves

Theorem 3.1 $K_n = D_n(\mathbb{S}^2)$.

In addition she proves that the codimension 1 faces of $D_n(\mathbb{S}^2)$ are given by the intersections of the hyperplanes corresponding to the above inequalities with K_n, *i.e.*, the above representation of K_n is irredundant.

The space $\mathcal{Q}_n$ is difficult to work with since the mapping π is not differentiable. To remedy this we let $\mathcal{P}_n^0$ denote the open subset of $\mathcal{P}_n$ corresponding to those n-gons such that successive vertices u_i, u_{i+1} ($i \in \mathbb{Z}/n$) do not coincide and are not antipodal. We let $\mathcal{Q}_n^0$ denote the quotient of $\mathcal{P}_n^0$ by $SO(3)$. Then $\mathcal{Q}_n^0$ is naturally a smooth manifold of dimension $2n-3$. Indeed, $\mathcal{Q}_n^0$ is naturally diffeomorphic to the submanifold $\Sigma \subset \mathcal{P}_n^0$ consisting of those n-gons with the vertex set $u = (u_1, \dots, u_n)$ satisfying

$$u_1 = \vec{\epsilon}_1, \quad u_n \cdot \vec{\epsilon}_3 = 0, \quad u_n \cdot \vec{\epsilon}_2 > 0 \quad \text{and} \quad 0 < d(u_i, u_{i+1}) < \pi, \quad 1 \leq i \leq n.$$

Recall $\vec{\epsilon}_1, \vec{\epsilon}_2, \vec{\epsilon}_3$ is the standard basis of $\mathbb{R}^3$.

Note that $\Sigma_r = M_r \cap S$ (see Lemma 2.6) and that $K_n^0 \subset \pi(\mathcal{Q}_n^0)$, where K_n^0 is the interior of K_n. We will henceforth replace π by its restriction to $\mathcal{Q}_n^0$.

We shall see shortly (Theorem 3.3) that the set of critical values of π inside K_n^0 is the union of certain hyperplane sections of K_n^0. We call these hyperplane sections *walls* of K_n. Connected components in K_n^0 of the complement of the union of the walls are called *chambers*. In [G], Galitzer determines the walls of K_n. We again summarize her results.

Let $I \subset \{1, \dots, n\}$ be any non-empty subset. For each nonnegative integer w let $H_{I,w}$ denote the hyperplane in $\mathbb{R}^n$ defined by the equation

$$r_I - r_{\bar{I}} = 2\pi w.$$

The intersection of such a hyperplane with K_n^0 is called a *wall*.

We then have the following lemma of Galitzer

Lemma 3.2 $H_{I,w} \cap K_n^0 \neq \varnothing \Leftrightarrow |I| \geq 2w + 2$.

Proof Suppose $r^* \in H_{I,w} \cap K_n^0$. Since $r^* \in H_{I,w}$ we have

$$r_I^* - r_{\bar{I}}^* = 2\pi w.$$

Assume first that $|I|$ is odd. Since $r^* \in K_n^0$ we also have

$$r_I^* - r_{\bar{I}}^* < (|I| - 1)\pi.$$

Hence $2\pi w < (|I| - 1)\pi$ and

$$|I| > 2w + 1.$$

Now assume that $|I|$ is even. We have the trivial inequality

$$r_I^* - r_{\bar{I}}^* < |I|\pi.$$

Since $r_I^* - r_{\bar{I}}^* = 2\pi w$ we obtain $2\pi w < |I|\pi$ and $|I| > 2w$. Hence $|I| \geq 2w + 1$, but $|I|$ is even, so we obtain $|I| \geq 2w + 2$.

To prove the converse we first note that there exists a cross-section $s_{I,w}\colon H_{I,w}\cap(0,\pi)^n \to \mathcal{Q}_n^0$ to the restriction of π to $\pi^{-1}(H_{I,w})$ defined inductively as follows. Let $r^* \in H_{I,w}\cap(0,\pi)^n$. The vertices u_1 and u_n are determined by the condition that the image of $s_{I,w}$ belongs to Σ_{r^*} (see Lemma 2.6). Place the vertex u_{n-1} on the equator so that e_{n-1} is a forward track (and $d(u_{n-1},u_n) = r^*_{n-1}$) if $n-1 \in I$ and on the other side of u_n if $n-1 \in \bar{I}$. Continue inductively. The resulting degenerate linkage closes up because $r^*_I - r^*_{\bar{I}} = 2\pi w$.

We claim that $H_{I,w}\cap(0,\pi)^n \neq \varnothing$ if and only if $|I| \geq 2w+1$. Necessity is easy, if r^* is in the intersection then

$$r^*_I - r^*_{\bar{I}} = 2\pi w \Rightarrow 2\pi w < r^*_I < \pi|I|.$$

We prove sufficiency by constructing r^* in the intersection so that $r^*_i = \rho, i \in I$ and $r^*_i = \delta, i \in \bar{I}$. Hence ρ and δ must satisfy $|I|\rho - |\bar{I}|\delta = 2\pi w$. Suppose first that $\delta = 0$. Then $\rho := 2\pi w/|I| < \pi$. Now choose $\epsilon > 0$ such that $\epsilon/|I| < \pi - \rho$ and $\epsilon/|\bar{I}| < \pi$. Change ρ to $\rho + \epsilon/|I|$ and δ to $\epsilon/|\bar{I}|$. Then r^* is in the intersection and the claim follows.

We now observe that the existence of the cross-section $s_{I,w}$ constructed above implies

$$H_{I,w}\cap(0,\pi)^n = H_{I,w}\cap K_n.$$

Put $\Delta := H_{I,w}\cap(0,\pi)^n$. Then Δ is the interior of a polyhedron of dimension $n-1$. Hence Δ cannot be contained in the $(n-2)$-skeleton of K_n. Thus Δ is either a face of dimension $n-1$ of K_n or else $H_{I,w}\cap K_n^0$ is nonempty. But if $H_{I,w}\cap K_n$ is a face of dimension $n-1$ it must be the face given by

$$r_I - r_{\bar{I}} = (|I|-1)\pi.$$

Consequently $2w = |I|-1$ and $|I| = 2w+1$. Thus $|I| \geq 2w+2$ implies that $H_{I,w}\cap K_n^0$ is nonempty. ■

The set of critical values of π is then determined by

Theorem 3.3 *Let $r \in K_n^0$. Then r is a critical value of π if and only if $r \in H_{I,w}$ for some $I, w \geq 0$ with $|I| \geq 2w+2$.*

Proof Clearly there exists a degenerate $u \in \pi^{-1}(r)$ if and only if r satisfies an equation of the form $r_I - r_{\bar{I}} = 2\pi w$. Now apply Theorem 2.9. ■

Remark 3.4 Since π is proper it is a fibration over each chamber and the topology of the fibers does not change within a chamber.

4 Recuttings and Flips of Spherical n-Gons

In this section we construct two groups acting on the space of spherical n-gons.

We first construct the group $\mathcal{R}$ of *recuttings*. Let $D_n'(\mathbb{S}^2) = \{r \in D_n(\mathbb{S}^2)$: all components of r are distinct$\}$. Let $\mathcal{P}_n'(\mathbb{S}^2) = \rho^{-1}\big(D_n'(\mathbb{S}^2)\big)\cap\mathcal{P}_n^0(\mathbb{S}^2)$. The permutation group S_n operates naturally on $D_n'(\mathbb{S}^2)$. We will construct a group $\mathcal{R}$ acting on $\mathcal{P}_n'(\mathbb{S}^2)$ and an epimorphism $\phi\colon \mathcal{R} \to S_n$ so that the projection ρ is ϕ-equivariant:

$$\rho(gP) = \phi(g)\rho(P) \quad P \in \mathcal{P}_n', \quad g \in \mathcal{R}.$$

We will call elements $g \in \mathcal{R}$ recuttings. Adler [A] defined recuttings for the Euclidean plane. Here we define the recuttings for the spherical case.

We define the *basic recuttings* $R_i \colon \mathcal{P}'_n(\mathbb{S}^2) \to \mathcal{P}'_n(\mathbb{S}^2)$, $1 \le i \le n$ as follows. Let $u \in \mathcal{P}'_n(\mathbb{S}^2)$ with $u = (u_1, u_2, \ldots, u_n)$. Take any geodesic arc connecting the points u_{i-1} and u_{i+1}, and look at its perpendicular bisector. The bisector is unique because $r_{i-1} \neq r_i$. Reflect the point u_i through this perpendicular line to exchange r_{i-1} and r_i. Leave all other vertices fixed. This is what we will call the *basic recutting* R_i at the i-th vertex.

The equation for the basic recutting at the i-th vertex is as follows. Set $R_i(u) = (w_1, w_2, \ldots, w_n)$. Then we have

$$w_i = u_i - 2\frac{u_i \cdot (u_{i+1} - u_{i-1})}{\|u_{i+1} - u_{i-1}\|^2}\,(u_{i+1} - u_{i-1})$$

and

$$w_j = u_j, \quad j \neq i.$$

Then the basic recuttings are well defined on the space $\mathcal{P}'_n(\mathbb{S}^2)$. We let $\mathcal{R}$ be the group generated by the basic recuttings. Since the generators act on $\mathcal{P}'_n(\mathbb{S}^2)$, so does $\mathcal{R}$. Notice that the action of $\mathcal{R}$ preserves the subset of degenerate polygons and their winding numbers and the orientation of their edges.

We next define the *basic flips* F_i, $1 \le i \le n$. We define $F_i \colon \mathcal{P}^0_n(\mathbb{S}^2) \to \mathcal{P}^0_n(\mathbb{S}^2)$, $1 \le i \le n$, by

$$F_i(u_1, \ldots, u_n) = (u_1, \ldots, -u_i, \ldots, u_n).$$

We note that F_i induces the map $\bar{F}_i \colon D_n(\mathbb{S}^2) \to D_n(\mathbb{S}^2)$ given by

$$\bar{F}_i(r_1, \ldots, r_n) = (r_1, \ldots, \pi - r_{i-1}, \pi - r_i, \ldots, r_n).$$

Note that flips F_i preserve the set of degenerate n-gons but change b and w by ± 1.

5 The Morse Theory of ρ_n

In this section we will prove the Main Theorem. We begin by discussing what we proved along these lines in [KM2]. Suppose $r^* \in K^0_n$ lies on the intersection of the walls

$$H_{I_1,w_1}, H_{H_2,w_2}, \ldots, H_{I_p,w_p}.$$

Choose a degenerate linkage u^* with $\pi(u^*) = r^*$. Let γ be the great circle containing u^*.

Definition 5.1 The vertical line segment L through r^* will be the line segment defined by

$$r_i = r^*_i, \quad 1 \le i \le n-1 \quad \text{and} \quad r^*_n - \delta \le r_n \le r^*_n + \delta.$$

We assume that δ is chosen so that L does not intersect any wall except at r^*. Let $X_L = \pi^{-1}(L)$.

Lemma 5.2 *X_L is a smooth submanifold of $\mathcal{Q}_n$ diffeomorphic to the $(n-2)$-torus. Moreover $X_L \cong N_{r'}$, where $r' := (r^*_1, \ldots, r^*_{n-1})$ (see Section 1).*

Proof We first observe that $\rho^{-1}(L)$ is diffeomorphic to $\mathbb{S}^2 \times (\mathbb{S}^1)^{n-1}$. Indeed a point in $\rho^{-1}(L)$ is a closed n-gon where the lengths of the first $(n-1)$-sides are prescribed to be $r_1^*, r_2^*, \ldots, r_{n-1}^*$ but the length of the n-th side is not determined. The operation of forgetting the n-th side gives an isomorphism to the moduli space of the free linkage with $(n-1)$-edges. The $\mathbb{S}^2$ factor comes from the position of the first vertex u_1, the circle factors come from the angles between successive edges. The quotient $\pi^{-1}(L) = \rho^{-1}(L)/\operatorname{SO}(3)$ can be obtained by fixing the position of the first edge. Clearly $X_L \cong N_{r'}$. ■

In [KM2], Theorem 8.10, we proved

Theorem 5.3 *$\rho_n|X_L$ is a Morse function with a finite collection of critical points $u^*_{(1)} \cup \cdots \cup u^*_{(p)}$, all located on the critical fiber M_{r^*}. Each critical point $u^*_{(i)}$ corresponds to a degenerate n-gon linkage in M_{r^*} with f_i forward-tracks, b_i back-tracks and the winding number w_i contained in a great circle γ_i. Then the signature of the Hessian of $\rho_n|X_L$ at $u^*_{(i)}$ is either $(f_i - 2w_i - 1, b_i + 2w_i - 1)$ or $(b_i + 2w_i - 1, f_i - 2w_i - 1)$ depending on the orientations of γ_i, $1 \leq i \leq p$.*

We now concentrate on a single critical point $u^* = T^*$ of ρ_n contained in a great circle γ with the associated closed polygon P^* which has f forward-tracks and winding number w. We orient γ as described in Section 2 (*i.e.*, in the direction of rotation from u_1 to u_n). Let L^* be a vertical segment through $\rho(u^*)$.

We begin the proof of the Main Theorem with

Lemma 5.4 *There exists a vertical line segment $L^\# \subset D_n(\mathbb{S}^2)$ and a degenerate free $(n-1)$-gon linkage $T^\#$ with $\pi(T^\#) = r^\# \in L^\#$ such that*

(i) The forward-tracks of the associated closed linkage $P(T^\#)$ are the first f edges of $T^\#$.
(ii) $w(T^\#) = w(T^*)$, $f\big(P(T^\#)\big) = f$.
(iii) signature $D^2(\rho_n|X_{L^\#})|_{T^\#}$ = signature $D^2(\rho_n|X_{L^*})|_{T^*}$.
(iv) $r^\#$ belongs to exactly one wall in $D_n(\mathbb{S}^2)$ and does not belong to any minor wall.

Proof The hyperplanes $r_i = r_j$ intersect the hyperplane $r_I - r_{\bar{I}} = 2\pi w$ transversally. Hence $H_{I,w} \cap D_n'(\mathbb{S}^2)$ is the complement of a union of hyperplane sections of $H_{I,w}$ and hence is dense. Thus there exists $\bar{r} \in H_{I,w}$ close to r^* such that components of $\bar{r}$ are distinct. We let $\bar{L}$ be the vertical segment passing through $\bar{r}$, $X_{\bar{L}} = \pi^{-1}(\bar{L})$ and $\bar{u} = s_{I,w}(\bar{r})$ (see Lemma 3.3). We claim

$$\text{signature } D^2(\rho_n|X_{\bar{L}})|_{\bar{u}} = \text{signature } D^2(\rho_n|X_{L^*})|_{u*}.$$

To see this let B be the line segment in $H_{I,w}$ joining $\bar{r}$ to r^*. For $b \in B$, let L_b be the vertical segment through b and $u_b = s_{I,w^{(b)}}$. We obtain the curve $D^2(\rho_n|X_{L_b})|_{u_b}$ which joins the two Hessians above. By Theorem 5.3 these quadratic forms are nondegenerate and the claim follows. The same argument proves that we can choose $\bar{r}$ which belongs to exactly one wall.

We now choose a permutation σ of the set $\{1, 2, \ldots, n\}$ which fixes n and sends $I := \{i_1, \ldots, i_f\}$ to $\{1, 2, \ldots, f\}$. Choose a recutting R in the subgroup of $\mathcal{R}$ generated by $\{R_2, \ldots, R_{n-2}\}$ such that $\phi(R) = \sigma$. Put $r^\# = \sigma(\bar{r})$ and $u^\# = R(\bar{u})$. The line segment $\bar{L}$ through $\bar{r}$ is carried by σ to the line segment $L^\#$ through $r^\#$. Hence the corresponding manifold $X_{\bar{L}}$ is carried to $X_{L^\#}$ by R. We claim

$$\text{signature } D^2(\rho_n|X_{L^\#})|_{u^\#} = \text{signature } D^2(\rho_n|X_{L'})|_{\bar{u}}.$$

Indeed since $\rho_n|X_{L^\#} = \rho_n \circ R|_{X_{\tilde{L}}}$ we find that

$$dR_{\tilde{u}}\colon T_{\tilde{u}}(X_{\tilde{L}}) \longrightarrow T_{u^\#}(X_{L^\#})$$

is an isometry of the quadratic form on the right-hand side to that on the left-hand side. ■

We can now reduce to the case $w = 0$.

Lemma 5.5 *There exists a flip F such that $\tilde{T} = F(T^\#)$ satisfies*

(i) $b(\tilde{T}) = b(T^\#) + 2w(T^\#)$
(ii) $w(\tilde{T}) = 0$
(iii) signature $D^2(\rho_n|X_{\tilde{L}})|_{\tilde{T}}$ = *signature* $D^2(\rho_n|X_{L^\#})|_{T^\#}$.

Here $\tilde{L} = \bar{F}(L^\#)$.

Proof We consider the case $w > 0$ (the case when $w < 0$ is treated similarly, just instead of flipping forward-tracks we flip back-tracks). We let F be the product of flips given by

$$F = F_2 \circ F_4 \circ \cdots \circ F_{2w}.$$

We note that since $f \geq 2w + 2 > 2w$ all the edges that are flipped are forward-tracks (and they become back-tracks after flipping). Thus (i) and (ii) are clear. The statement (iii) is proved in the same fashion as (iii) in the previous lemma. ■

We let K be the set of forward tracks of $\tilde{T}$ (or the associated closed n-gon linkage $\tilde{P}$). Hence $\tilde{r} = \pi(\tilde{P})$ is on the wall $H_{K,0}$.

We next deform $\tilde{r}$ along the wall $H_{K,0}$ to $\hat{r}$ such that $\hat{r}_1 + \hat{r}_2 + \cdots + \hat{r}_n < 2\pi$. The corresponding degenerate closed n-gon linkage $s_{K,0}(\hat{r}) = \hat{u}$ will have perimeter less than 2π. To accomplish this let $A \subset D_n(\mathbb{S}^2) \cap H_{K,0}$ be the line segment

$$A = \{\lambda\tilde{r} : \epsilon < \lambda < 1 + \epsilon\}.$$

Choose λ_0 such that $\sum_{i=1}^{n} \lambda_0\tilde{r}_i < 2\pi$. Let $\hat{r} = \lambda_0\tilde{r}$ and $\hat{L}$ be the vertical segment through $\hat{r}$. Put $\hat{u} = s_{K,0}(\hat{r})$.

Lemma 5.6 *The signature of* $D^2(\rho_n|X_{\hat{L}})|_{\hat{u}}$ *is equal to the signature of* $D^2(\rho_n|X_{\tilde{L}})|_{\tilde{u}}$.

Proof For $a \in A$ define L_a and u_a as in the proof of Lemma 5.4. We obtain the curve $D^2(\rho_n|X_{L_a})|_{u_a}$ and the proof goes as in Lemma 5.4. ■

Let $\hat{f}$ (resp. $\hat{b}$) be the number of forward-tracks (resp. back-tracks) of $\hat{u}$. By Lemma 5.5, $\hat{f} = f(P) - 2w(P)$ and $\hat{b} = b(P) + 2w(P)$.

We complete the proof of the Main Theorem by

Proposition 5.7 *The signature of* $D^2(\rho_n|X_{\hat{L}})|_{\hat{u}}$ *is* $(\hat{b} - 1, \hat{f} - 1)$.

The proposition will be a consequence of the next three lemmas. In what follows let $\hat{P} = \hat{u} = (\hat{u}_1, \hat{u}_2, \ldots, \hat{u}_n)$ be a degenerate closed n-gon linkage of perimeter less that 2π. We assume that $\pi(\hat{P})$ belongs to exactly one wall. Then any vertex u_i is connected to u_1 by a unique geodesic segment (u_1, u_i) which does not degenerate to a point.

Following [KK] we introduce local coordinates $\psi_2, \psi_3, \ldots, \psi_{n-1}$ on $X_{\hat{L}}$ by defining ψ_i to be the signed angle at u_i between the oriented segment (u_1, u_i) and the oriented edge e_i. For instance if $u_i = \vec{\epsilon}_2$, $u_{i+1} = -\vec{\epsilon}_1$ then $\psi_i = 0$. If $u_{i+1} = (\vec{\epsilon}_1 + \vec{\epsilon}_2)/\sqrt{2}$ then $\psi_i = \pi$. We then have

Lemma 5.8 *$\psi_2, \psi_3, \ldots, \psi_{n-1}$ are local coordinates near $\hat{u}$.*

Proof See [KK, Section 3]. ■

Remark 5.9 In [KK] the authors study free linkages in $\mathbb{S}^3$. Our coordinates are obtained from theirs by dropping their vector field Y. Thus we use an orthonormal frame (X, Z) where Z is the radial field.

We now have the clever observation of [KK], the reason for choosing the above coordinates.

Lemma 5.10

$$\frac{\partial^2 \rho_n}{\partial\psi_i \partial\psi_j}\Big|_{\hat{u}} = 0, \ i \neq j$$

Proof Assume $i < j$. Then by [KK, p. 84] we find that the restriction

$$\frac{\partial \rho_n}{\partial \psi_j}\Big|_{\psi_k = \hat{\psi}_k, \quad k \neq i}$$

of the partial derivative to the curve

$$\Gamma_k := \{\psi_k = \hat{\psi}_k, k \neq i\}$$

is identically zero as a function of ψ_i, this implies the lemma. Below we sketch a proof of vanishing of this derivative. We give the picture (Figure 1) in the Euclidean case with $\psi_j = 0$. We draw only the vertices u_1, u_i, u_j and u_n.

Pick a point u on the curve Γ_k. Then the points u_1, u_j, u_n belong to a common geodesic circle in $\mathbb{S}^2$. As ψ_j varies the line segment (u_j, u_n) rotates around u_j. Clearly the vertex u_n moves along a (small) circle tangent at $\psi_j = 0$ to the bigger circle which is the level set of ρ_n for the fixed values of ψ_i and $\psi_k = \hat{\psi}_k$, $k \neq i$. Hence $\frac{\partial \rho_n}{\partial \psi_j}|_{\Gamma_k}$ is identically zero as a function of ψ_i. ■

Lemma 5.11

(i) If $\hat{e}_i$ is a back-track then $\frac{\partial^2 \rho_n}{\partial \psi_i^2}|_{\hat{u}} > 0$.

(ii) If $\hat{e}_i$ is a forward-track then $\frac{\partial^2 \rho_n}{\partial \psi_i^2}|_{\hat{u}} < 0$.

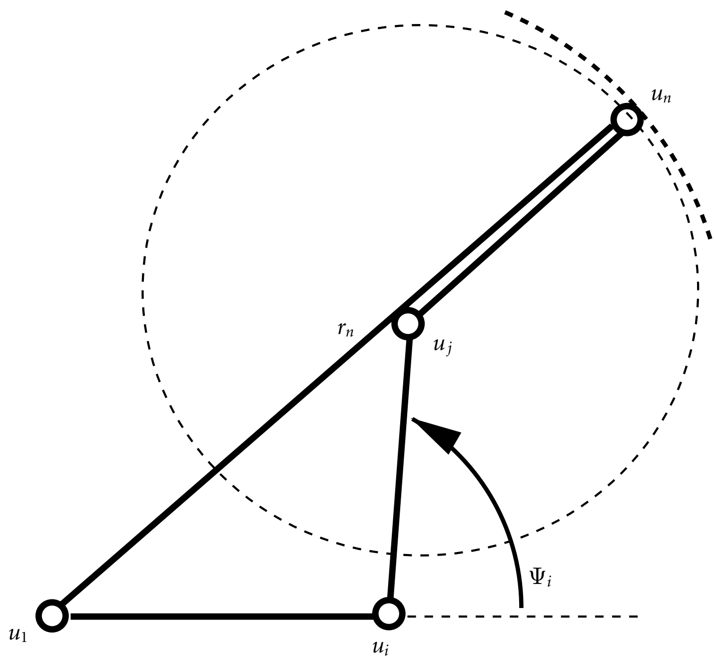

Figure 1: Vanishing of the derivative.

Proof We prove (i) and leave (ii) to the reader. We let ψ_i be a value close to $\hat{\psi}_i = \pi$ and consider the curve $\psi_j = \hat{\psi}_j$, $j \neq i$. We obtain the picture described on Figure 2 (again we have drawn the Euclidean case).
Here we have omitted all vertices except $u_1, u_i, u_{i+1}, u_{n-1}$ and u_n and assumed (in the Figure 2) that $\hat{\psi}_{i+1} = 0$ and $\hat{\psi}_{n-1} = \pi$.

We set $d(u_1, u_i) = a$, $d(u_{i+1}, u_n) = b$. From the spherical "law of cosines" (see [B, Proposition 18.6.8]) we have

$$\cos(r_n + b) = \cos a \cos r_i + \sin a \sin r_i \cos(\pi - \psi_i).$$

Differentiating implicitly we obtain

$$\frac{\partial^2 \rho_n}{\partial \psi_i^2}\Big|_{\hat{u}} = \frac{\sin a \sin \hat{r}_i}{\sin(\hat{r}_n + b)}.$$

Since the perimeter of $\hat{u}$ is less than 2π we have $a < \pi$, $\hat{r}_n + b < \pi$ and (i) follows. ■

With this, Proposition 5.7 and the Main Theorem are proved.

6 The Wall-Crossing Formula and Regular Spherical Pentagons

In this section we explain how the Main Theorem can be used to describe how the moduli spaces M_r change as we cross a wall. As an illustration of our technique we describe the moduli spaces of regular spherical pentagons.

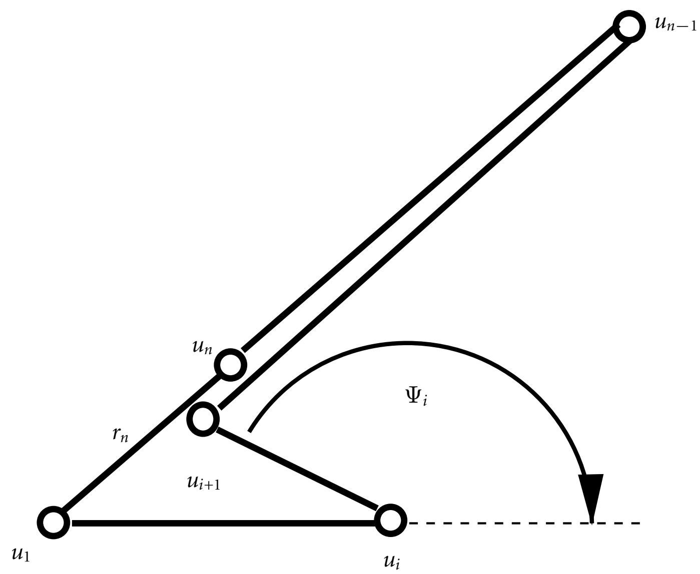

Figure 2: The sign of the second derivative.

We first claim that any wall-crossing can be effected by a vertical segment. Indeed as we have seen the walls are given by $r_I - r_{\bar{I}} = 2w\pi$ with $|I| \geq 2w + 2$. Let n_I be a normal vector to the above wall. Recall that the vector $\nu_n = (0, 0, \ldots, 0, 1)$ is parallel to a vertical segment through this wall. Since $\nu_n \cdot n_I \neq 0$ any vertical segment is transverse to a wall and the claim follows.

From the Main Theorem we obtain

Theorem 6.1 **(The wall-crossing formula)** *Suppose we cross the wall $H_{I,w}$ at $r_n = r_n^*$ along a vertical segment L with $r_n^* - \delta \leq r_n \leq r_n^* + \delta$. Then*

(i) $M_{r^*+\delta}$ is obtained from $M_{r^*-\delta}$ by attaching an $(f - 2w - 1)$-handle.
(ii) $M_{r^*-\delta}$ is obtained from $M_{r^*+\delta}$ by attaching some $(b + 2w - 1)$-handle.

We now apply our formula to describe the moduli spaces of regular spherical pentagons M_r with $r = (a, a, a, a, a)$. The description of the moduli space M_r for $\frac{2\pi}{5} < a < \frac{2\pi}{3}$ was first done in [G] by a different method. Assume first that $0 < a < \frac{2\pi}{5}$. Since the perimeter of P is less than 2π the moduli space $M_r = M_r(\mathbb{S}^2)$ is diffeomorphic to the corresponding Euclidean moduli space $M_r = M_r(\mathbb{R}^2)$ by [S]. Hence by [KM1, Theorem 2], M_r is the genus four surface, $0 < a < \frac{2\pi}{5}$.

Now as a goes from $\frac{2\pi}{5} - \delta$ to $\frac{2\pi}{5} + \delta$ we pass through the wall $r_1 + r_2 + r_3 + r_4 + r_5 = 2\pi$. We now describe what happens as we cross this wall using Theorem 6.1. Set $r_1 = r_2 = r_3 = r_4 = \frac{2\pi}{5}$ and let r_5 go from $\frac{2\pi}{5} - \delta$ to $\frac{2\pi}{5} + \delta$. The critical point $T \in N_r$ corresponding to the critical value $r_5 = \frac{2\pi}{5}$ is represented by the degenerate free 4-gon linkage with $P = P(T)$ obtained by dividing the equator γ into 5 equal parts proceeding anticlockwise around the equator and taking the first four segments. Our orientation rule requires us to orient the equator so that the positive direction is clockwise hence

$$b(P) = 5, \quad f(P) = 0, \quad w(P) = -1.$$

According to the main theorem the signature of $D^2\rho_5|_L$ is $(2, 1)$. Since ρ_5 increases as we cross the wall we obtain Theorem 6.1 of [G]:

$$M_r \text{ is the genus five surface,} \quad \text{if } \frac{2\pi}{5} < a < \frac{2\pi}{3}.$$

The point $r = (\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{2\pi}{3}, \frac{2\pi}{3}, \frac{2\pi}{3})$ lies on the intersection of five walls of the form

$$r_i + r_j + r_k + r_l - r_m = 2\pi.$$

There are two cases to consider, $m = 5$ and $m \neq 5$. We will analyse the first case and leave the second to the reader.

We will identify the equator of $\mathbb{S}^2$ with the unit circle on the complex plane. Let T be the degenerate free 4-gon linkage with vertices $(1, \omega, \omega^2, 1, \omega)$ where $\omega = \exp(2\pi i/3)$. By our orientation convention the unit circle has the usual (*i.e.*, counterclockwise) orientation and

$$b(P) = 1, \quad f(P) = 4, \quad w(P) = 1.$$

Hence $D^2\rho_5|_T$ has signature $(2, 1)$. The equation of the wall we are considering is $r_1 + r_2 + r_3 + r_4 - r_5 = 2\pi$. Let $\alpha(r_1, r_2, r_3, r_4, r_5) = r_1 + r_2 + r_3 + r_4 - r_5$. As a increases from $\frac{2\pi}{3} - \delta$ to $\frac{2\pi}{3} + \delta$ we pass from the half-space $\alpha < 2\pi$ to $\alpha > 2\pi$. Now to apply the Theorem we set $r_1 = r_2 = r_3 = r_4 = \frac{2\pi}{3}$. To cross from $\alpha < 2\pi$ to $\alpha > 2\pi$ we see that r_5 must *decrease* from $\frac{2\pi}{3} + \delta$ to $\frac{2\pi}{3} - \delta$. Thus we attach the "positive" or "ascending" disk of ρ_5 (*i.e.*, the unit disk in a maximal subspace of the tangent space at T on which the quadratic form $D^2\rho_5|_T$ is *positive-definite*) as we pass through the critical point $r_5 = \frac{2\pi}{3}$. Hence we attach a 2-handle. We attach 2-handles at the other 4 critical points of ρ_5 corresponding to the critical value $r_5 = \frac{2\pi}{3}$ and we obtain

$$M_r \approx \mathbb{S}^2, \quad \text{if } \frac{2\pi}{3} < a < \frac{4\pi}{5}.$$

We cross no more walls of $D_5(\mathbb{S}^2)$ until we reach the face given by $r_1+r_2+r_3+r_4+r_5 = 4\pi$ when $a = \frac{4\pi}{5}$. The critical value $r_5 = \frac{4\pi}{5}$ corresponds to the single critical point $u = (1, \zeta^2, \zeta^4, \zeta^6, \zeta^8)$ where $\zeta = \exp(2\pi i/5)$. We have $u_5 = \exp(-4\pi i/5)$. Hence γ is oriented n the clockwise direction. We obtain

$$b(P) = 5, \quad f(P) = 0, \quad w(P) = -2$$

and accordingly the signature of $D^2\rho_5|_T$ is $(0, 3)$. Hence P is locally rigid.

We can in fact determine the moduli space M_r as follows. Apply the flips F_1 and F_3 to change r to r^* with $r_1^* = r_2^* = r_3^* = r_4^* = \frac{\pi}{5}$, $r_5^* = \frac{4\pi}{5}$. This is a standard "Euclidean" rigid linkage and M_{r^*} = a point, as was to be expected since r is on a face.

Of course for $a > \frac{4\pi}{5}$, M_r is empty since we are outside $D_5(\mathbb{S}^2)$.

7 Appendix

The statement in Section 6 of [KM2] that $A^{\bullet}_{(2)}(M, adP)$ is a differential graded Lie algebra is false since the L^2-condition is not closed under bracket. Hence our proof that $B^{\bullet}(M, U; adP)$ is formal as a *differential graded Lie algebra* is not correct. However we can salvage all the results of [KM2] except the result that $B^{\bullet}(M, U; adP)$ is formal by the following "quick fix". First we apply the results of Section 5 of our paper [KM3] to deduce that the germ $(M_r, [P_0])$ is given by a single quadratic equation corresponding to the cup product: $q\colon H^1\big(B^{\bullet}(M, U; adP)\big) \to H^2\big(B^{\bullet}(M, U; adP)\big) = \mathbb{R}$.

Now we claim that the results of Section 7 of [KM2] do in fact compute q above. To see this we note first that the inclusion $B^{\bullet}(M, U, adP) \to A^{\bullet}_{(2)}(M, adP)$ is a quasi-isomorphism of *complexes*. The bracket of two elements of $A^1_{(2)}(M, adP)$ is integrable (but not necessarily square integrable) whence the integration pairing (using the trace on adP) is well-defined on $A^1_{(2)}(M, adP)$. By [Ga] it descends to cohomology and consequently agrees with q.

Remark 7.1 Formality of $B^{\bullet}(M, U; adP)$ follows from the recent result of P. Foth [F].

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