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On monodromy of complex projective structures

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Summary. We prove that for any nonelementary representation $\rho: \pi_1(S) \to \mathrm{SL}(2, \mathbb{C})$ of the fundamental group of a closed orientable hyperbolic surface S there exists a complex projective structure on S with the monodromy ρ.

1. Introduction

Let S be a smooth closed surface. A *complex projective structure* σ on S is a maximal atlas such that all transition maps belong to the group $\mathrm{PSL}(2, \mathbb{C})$. Each complex projective structure σ on S defines a homomorphism $\rho: F = \pi_1(S) \to \mathrm{SL}(2, \mathbb{C})$ which is called the *monodromy representation* of σ. The projection of this representation into $\mathrm{PSL}(2, \mathbb{C})$ is unique up to conjugation. An important class of complex projective structures is given by *uniformization*. Suppose that $\Gamma \subset \mathrm{SL}(2, \mathbb{C})$ is a torsion-free Kleinian group acting discontinuously on a nonempty domain $D \subset \bar{\mathbb{C}}$. Then the canonical complex projective structure on D projects to a complex projective structure on $S = D/\Gamma$. In this case the monodromy representation is an epimomorphism $\rho: \pi_1(S) \to \Gamma$ with the kernel $\pi_1(D)$. However complex projective structure does not have to appear this way, in particular the monodromy representation can be nondiscrete.

Recall that a representation $\rho: G \to \mathrm{SL}(2, \mathbb{C})$ is *nonelementary* if there is no invariant point or geodesic in $\mathbb{H}^3 \cup \bar{\mathbb{C}}$ for the action of the group $\rho(G)$. It is well-known that if $\rho: F \to \mathrm{SL}(2, \mathbb{C})$ is a monodromy representation of a complex projective structure on a closed surface S of negative Euler characteristic then the representation ρ must be nonelementary, see [1, p. 297–305], [13], [12], [18]. In this paper we prove that this is the only restriction on monodromy representations.

Theorem 1. *Suppose that S is a closed orientable surface of the genus $g > 1$. Let $\rho: \pi_1(S) \to \mathrm{SL}(2, \mathbb{C})$ be a nonelementary representation. Then there exists a complex projective structure on S with the monodromy homomorphism ρ.*

This theorem was conjectured by R. Gunning in [13] and a proof was announced by D. Gallo in [7]. Some particular cases of Theorem 1 were established earlier. Under the assumption that the monodromy group $\rho(F)$ is contained in $\mathrm{SL}(2, \mathbb{R})$ Theorem 1 was proven in the preprint [8]. Suppose that ρ factors through a homomorphism onto a free group of rank g so that the images in $\mathrm{SL}(2, \mathbb{C})$ of free generators are loxodromic. Under these assumptions Theorem 1 was proven by D. Hejhal in [15]. D. Hejhal also conjectured that "generic" representations into $\mathrm{SL}(2, \mathbb{C})$ are monodromy representations. Our proof of Theorem 1 is based on ideas of [4], [7], [8] and [9].

The idea of the proof of Theorem 1 is to combine the "continuity method" of [4] with combinatorial arguments of [7], [8] using properties [9] of the representation variety $\mathrm{Hom}(F, \mathrm{SL}(2, \mathbb{C}))/\mathrm{SL}(2, \mathbb{C})$. Namely, we connect the representation ρ with a Fuchsian representation r_0 by a special family of homomorphisms $r_t: F \to \mathrm{SL}(2, \mathbb{C}), 0 \leqq t \leqq 1, r_1 = \rho$ (Theorem 2). This is done by generalizing arguments of [8] and [9]. The map from the space of all complex projective structures on S into the representation variety $\mathrm{Hom}(F, \mathrm{SL}(2, \mathbb{C}))/\mathrm{SL}(2, \mathbb{C})$ is open [15]. Thus the hyperbolic structure on S with the monodromy r_0 belongs to a family of complex projective structures c_t with the monodromy r_t ($0 \leqq t < t_1$, where $t_1 \leqq 1$). If the family of structures c_t *degenerates* as $t \to t_1$, then using *grafting* of c_t we "regenerate" c_t and pass through the point of degeneration t_1, retaining the family r_t of the monodromy representations (Theorem 3). Then we repeat the process. The families c_t and r_t are chosen so that there are only finitely many points of degeneration. Therefore, eventually we get a complex projective structure with the monodromy ρ. In Section 7 we consider the possibility of extending these arguments to an *arbitrary* family of representations $F \to \mathrm{SL}(2, \mathbb{C})$. In the same section we also discuss relation between degenerations of complex projective structures and properties of unstable bundles.

It is unclear at this moment whether one can avoid combinatorial arguments in the proof of Theorem 1 using instead harmonic maps or pleated surfaces. In Section 4.3 we prove the following

Corollary. *Suppose that S is a closed orientable hyperbolic surface, $r: \pi_1(S) \to P\mathrm{SL}(2, \mathbb{C})$ is a nonelementary representation. Then there exists an r-equivariant pleated map $f: \mathbb{H}^2 = \tilde{S} \to \mathbb{H}^3$.*

Note that the existence of an equivariant harmonic map was established by S. Donaldson and K. Corlette.

2. Definitions and notations

2.1. We shall consider the extended complex plane $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ as the sphere at infinity of the 3-dimensional hyperbolic space $\mathbb{H}^3$. Thus the group

$\mathrm{PSL}(2, \mathbb{C})$ is identified with the group of orientation-preserving isometries of $\mathbb{H}^3$. Denote by $\tau: \mathrm{SL}(2, \mathbb{C}) \to \mathrm{PSL}(2, \mathbb{C})$ the projectivization. In this paper we shall assume that the hyperbolic plane $\mathbb{H}^2$ is embedded in $\bar{\mathbb{C}}$ as the upper-half plane

$$\{z \in \bar{\mathbb{C}}: \mathrm{Im}(z) > 0\}$$

with the metric $ds = |dz|/\mathrm{Im}(z)$. Then the hyperbolic metric in $\mathbb{H}^2$ is invariant under the group $\mathrm{PSL}(2, \mathbb{R})$ of conformal automorphisms of $\mathbb{H}^2$. If $X \subset \mathbb{H}^2$ then we shall denote by $N_\varepsilon(X)$ the ε-neighborhood of X in the hyperbolic plane $\mathbb{H}^2$.

Let G be a subgroup of $\mathrm{PSL}(2, \mathbb{C})$. The group G acts discontinuously at $z \in \mathbb{C}$ provided that there exists a neighborhood U of z such that $gU \cap U = \emptyset$ for all but finitely many $g \in G$. Denote by $\Omega(G)$ the *region of discontinuity* of G, i.e. the set of points $z \in \bar{\mathbb{C}}$ such that G acts discontinuously at z. If $\Omega(G) \neq \emptyset$ then the group G is called *Kleinian*. Let $g \in \mathrm{SL}(2, \mathbb{C})$ be an element so that $\tau(g)$ is different from the identity. If $\tau(g)$ has only one fixed point in $\bar{\mathbb{C}}$ then g is called *parabolic*. An element $g \neq \pm 1$ is parabolic if and only if $\mathrm{Tr}^2(g) = 4$. For any parabolic element g the Moebius transformation $\tau(g)$ is conjugate in $\mathrm{PSL}(2, \mathbb{C})$ to the translation $z \mapsto z + 1$.

An element g of $\mathrm{SL}(2, \mathbb{C})$ is called *loxodromic* if $\mathrm{Tr}(g) \notin [-2, 2]$. For any loxodromic element g the Moebius transformation $\tau(g)$ is conjugate in $\mathrm{PSL}(2, \mathbb{C})$ to a dilation $z \mapsto \lambda \cdot z$, where $\lambda \in \mathbb{C}$, $|\lambda| \neq 1$. Therefore g has two fixed points in $\bar{\mathbb{C}}$, one of them is attractive, another is repulsive.

An element g is called *elliptic* if $\mathrm{Tr}(g) \in (-2, 2)$ or $g = \pm 1$. An element g is elliptic if and only if $\tau(g)$ is conjugate to a rotation $z \mapsto e^{i\theta} z$. If $g \in \mathrm{PSL}(2, \mathbb{C}) - \{1\}$ has a lift $\tilde{g}$ into $\mathrm{SL}(2, \mathbb{C})$ which is loxodromic (resp. parabolic, elliptic) then the element g itself will be called loxodromic (resp. parabolic, elliptic).

Given an element $g \in \mathrm{SL}(2, \mathbb{C})$ we shall denote by $\hat{g}$ its projection to $\mathrm{PSL}(2, \mathbb{C})$; if $\rho: \Gamma \to \mathrm{SL}(2, \mathbb{C})$ is a representation then $\hat{\rho}$ will denote the composition $\tau \circ \rho$.

Consider the projective model for the hyperbolic space $\mathbb{H}^3 \subset RP^3$, then

$$\mathrm{PSL}(2, \mathbb{C}) \subset \mathrm{PSL}(3, \mathbb{R})$$

If $\hat{g} \in \mathrm{PSL}(2, \mathbb{C})$ is an elliptic element then the axis of $\hat{g}$ is the set of points $z \in RP^3$ such that $\hat{g}z = z$. Suppose that $g \in \mathrm{SL}(2, \mathbb{C})$ is a loxodromic element. Then the *axis* of g to be denoted by $Axis(g)$ is a geodesic in $\mathbb{H}^3$ which connects the fixed points of $\hat{g}$. The *translational length* $l(g)$ of the element g is the hyperbolic distance between z and $\hat{g}(z)$ for any $z \in Axis(g)$.

A subgroup $\Gamma \subset \mathrm{SL}(2, \mathbb{C})$ is called *elementary* if it has an invariant point or geodesic in $\bar{\mathbb{C}} \cup \mathbb{H}^3$. Any elementary group is either relatively compact or is not Zariski dense (over $\mathbb{C}$) in $\mathrm{SL}(2, \mathbb{C})$.

If G is a finitely-generated group then $\mathrm{Hom}(G, \mathrm{SL}(2, \mathbb{C}))^0$ will denote the space of all nonelementary representations of G into $\mathrm{SL}(2, \mathbb{C})$. The quotient $R(G)^0 = \mathrm{Hom}(G, \mathrm{SL}(2, \mathbb{C}))^0/\mathrm{SL}(2, \mathbb{C})$ is an algebraic variety.

The commutator of elements $a, b \in G$ will be denoted by $[a, b] = aba^{-1}b^{-1}$. For every surface S we denote by $\tilde{S}$ the universal cover of S, then the fundamental group $F = \pi_1(S)$ acts on $\tilde{S}$ as the group of covering transformations. All surfaces in this paper are assumed to be orientable.

Suppose that S is an oriented surface, α is a simple loop on S, then D_α denotes the *Dehn twist* on S along α (see [3]). Let a, b be a pair of closed loops on S. By $i(a, b)$ we denote the *geometric intersection number between a and b*, i.e. the minimal number of points of intersection for all loops a', b' homotopic to a, b resplectively. Therefore, α is homotopic to a simple loop iff $i(\alpha, \alpha) = 0$. Suppose that $i(a, b) = 1$. Then $i(a^n \cdot b, a^n \cdot b) = 0$ for all $n \in \mathbb{Z}$. Let a, b be a pair of smooth simple loops so that $a \cap b$ is a single point q where a is tangent to b. Suppose that a, b are oriented so that at the point of tangency q they have opposite directions. Then $i(ab, ab) = 0$. A compact subsurface $S' \subset S$ is called *incompressible* if each component of $\partial S'$ is homotopically nontrivial in S.

A compact surface Σ is called "*pants*" (or "*pair of pants*") if it is homeomorphic to

$$\{z \in \mathbb{C}: |z| \leqq 4, |z - 2| \geqq 1, |z + 2| \geqq 1\} \tag{1}$$

Suppose that T is a 2-dimensional torus, $D \subset T$ is an embedded closed disc. Then the surface $T - \mathrm{int}(D)$ is called a "handle". Let $C \subset X$ be a smooth simple curve on a surface X. Then a *coorientation* v on C is a nonvanishing smooth vector-field along C such that at each point $q \in C$ the vector v_q and the tangent space $T_q(C)$ span the whole tangent plane $T_q(X)$.

Suppose that we are given a collection $D_1, D_1', \ldots, D_r, D_r'$ of disjoint closed topological discs in $\bar{\mathbb{C}}$. Let $g_j \in \mathrm{PSL}(2, \mathbb{C}), j = 1, \ldots, r$, be a family of Moebius transformations such that $g_j(D_j) = \bar{\mathbb{C}} - int(D_j')$. The group G generated by $g_1, \ldots, g_r$ is called a *Schottky group*. This group is always Kleinian, it is isomorphic to the free group on r generators $\mathbb{F}_r$. Each Schottky group G can be isomorphically lifted to $\tilde{G} \subset \mathrm{SL}(2, \mathbb{C})$, the group $\tilde{G}$ will be also called a Schottky group. Suppose that a Schottky group G has an invariant closed disc $U \subset \bar{\mathbb{C}}$ and the rank of the group G is equal 2. Then the quotient $(U \cap \Omega(G))/G$ is either a pair of pants or a handle. Conversely, if a torsion-free Kleinian group G has an invariant closed disc $U \subset \bar{\mathbb{C}}$ and $(U \cap \Omega(G))/G$ is homeomorphic to a pair of pants (or a handle) then G is a Schottky group of rank 2.

A Kleinian group G is called *Fuchsian* if it has an invariant round disc Δ in $\bar{\mathbb{C}}$ (we do not require $\Lambda(G)$ to be the whole circle $\partial\Delta$). A Kleinian group G will be called *quasifuchsian* if its limit set is a topological circle and G preserves the orientation on $\Lambda(G)$.

2.2. Suppose that σ is a complex projective structure of a surface S. Then σ defines a local diffeomorphism *dev* from the universal covering $\tilde{S}$ to the extended complex plane $\bar{\mathbb{C}}$. Locally the map *dev* is a complex projective diffeomorphism with respect to the complex projective structures on $\tilde{S}$ and $\bar{\mathbb{C}}$.

The map *dev* is called the *developing map* of σ. Assume that the fundamental group $F = \pi_1(S)$ acts on $\tilde{S}$ as the group of covering transformations. Then the developing map *dev* induces a homomorphism $\rho\colon F \to \mathrm{SL}(2,\mathbb{C})$ which satisfies the property:

$$\hat{\rho}(g) \circ dev = dev \circ g \text{ for any } g \in F \tag{2}$$

The representation ρ is called the *monodromy representation* of the structure σ. The representation $\hat{\rho}$ is unique up to conjugation in $\mathrm{PSL}(2,\mathbb{C})$. The group $\rho(F)$ is called the *monodromy group*. Conversely, suppose that we are given a local homeomorphism $dev\colon \tilde{S} \to \bar{\mathbb{C}}$ and a representation ρ which satisfy (2). Consider the pull back $dev^*(can)$ of the canonical complex projective structure from $\bar{\mathbb{C}}$ to $\tilde{S}$. The group F acts as a group of automorphisms of $dev^*(can)$, thus the projection of $dev^*(can)$ to $\tilde{S}/F$ is a complex projective structure σ. The map *dev* is a developing map of this structure.

3. Outline of the proof of Theorem 1

The proof of Theorem 1 consists of 3 main steps. Suppose that S is a closed oriented surface of the genus $g > 1$ and we are given a nonelementary representation $\rho\colon \pi_1(S) = F \to \mathrm{SL}(2,\mathbb{C})$. We shall identify $\tilde{S}$ with the hyperbolic plane and F with a Fuchsian group so that $S = \mathbb{H}^2/F$.

3.1. Step I. **Theorem 2.** There exists a decomposition of the surface S into the union of pairs of pants P_j and a continuous family of representations $r_t\colon F \to \mathrm{SL}(2,\mathbb{C})$ so that:

(a) $r_0 = id$, $r_1 = \rho$;

(b) for every $t \in [0,1]$ the restriction of r_t to each subgroup $\pi_1(P_j)$ is an isomorphism between Schottky groups $\pi_1(P_j) = F_j$ and $r_t(F_j)$.

3.2. Step II. Consider the annulus $A = \{z \in \mathbb{C}\colon 1 \leqq |z| \leqq R\}$ with the boundary curves $\alpha^- = \{z\colon 1 = |z|\}$, $\alpha^+ = \{z\colon R = |z|\}$. Suppose that

$$g_t^{\pm}\colon \alpha^{\pm} \to T^2, \quad t \in [0,1] \tag{3}$$

is a smooth "generic" family of C^2-smooth embeddings into the two-dimensional torus T^2. Here "generic" means that for all but finitely many $t \in [0,1]$ the oriented curves $\gamma_t^+ = g_t^+(\alpha^+)$ and $\gamma_t^- = g_t^-(\alpha^-)$ are transversal. Assume that the conformal structure c_t on the torus T depends continuously on t. Suppose that for $t = 0$ the map $g_0^+ \cup g_0^-$ can be extended to a smooth embedding ϕ of A into T^2.

Choose two continuous families $v_t^{\pm}$ of coorientations on $\gamma_t^{\pm}$ so that they "agree" with the map ϕ at $t = 0$. This means that the preimages of $v_{t=0}^{\pm}$ under the derivative of ϕ are directed "inward" the domain $A \subset \mathbb{C}$.

Theorem 3. *There exists a continuous family of local diffeomorphisms* $g_t\colon A \to T^2$, $t \in [0, 1]$, *such that* $g_t|_{\alpha^\pm} = g_t^\pm$ *and the coorientations* $v_t^\pm$ *agree with the maps* g_t.

Step III. Now we can explain how Theorems 2 and 3 imply Theorem 1.

3.3. Let p be the universal cover $p\colon \tilde{S} = \mathbb{H}^2 \to S$. Denote by $\{P_j\colon 1 \leqq j \leqq 2g-2\}$ the collection of pants in the decomposition of S given by Theorem 2.

For each P_j choose a connected component $\tilde{Q}_j$ of $p^{-1}P_j$. Let F_j be the stabilizer of $\tilde{Q}_j$ in F. Denote by r_{jt} the restriction of r_t to F_j. Then for each j there exists a continuous family of quasiconformal homeomorphisms

$$f_{jt}\colon \bar{\mathbb{C}} \to \bar{\mathbb{C}}$$

realizing the isomorphisms r_{jt} so that $f_{j0} = id$ and f_{jt} are C^2-smooth diffeomorphisms in $\Omega(F)$ (see [5]).

Let B be the union of boundary curves of all pants P_j and $\tilde{B} = p^{-1}B$. Choose a sufficiently small positive number ε so that any closed hyperbolic disc in $\mathbb{H}^2$ of the radius 2ε can intersect not more than one component of $\tilde{B}$.

3.4. Remove from all the components $int\tilde{P}_j$ of $\mathbb{H}^2 - \tilde{B}$ the ε-neighborhoods of $\partial\tilde{P}_j$; put $\tilde{P}_j^* = \tilde{P}_j - N_\varepsilon(\tilde{B})$. Choose representatives $B_{jk} \subset \tilde{B}$ for the cosets in $B = \tilde{B}/F$; the index jk means that B_{jk} is the common boundary arc of two adjacent connected components $\tilde{P}_j$, $\tilde{P}_k$ of $\mathbb{H}^2 - \tilde{B}$.

For each domain $\tilde{P}_j^* = \gamma\tilde{Q}_i^*$ (where $\gamma \in F$) we define the developing map

$$f_t = r_t(\gamma) \circ f_{jt} \circ \gamma^{-1} \tag{4}$$

This definition does not depend on the choice of γ since f_{jt} are r_t-equivariant. Thus, we have a r_t-equivariant continuous family of local diffeomorphisms

$$f_t\colon \mathbb{H}^2 - N_\varepsilon(\tilde{B}) \to \bar{\mathbb{C}} \tag{5}$$

Denote by C_{jk} the ε-neighborhood $N_\varepsilon(B_{jk})$ of B_{jk} in $\mathbb{H}^2$. Let $\tilde{\alpha}_{jk} \subset \partial\tilde{P}_j$, $\tilde{\alpha}_{kj} \subset \partial\tilde{P}_k$ be the arcs bounding C_{jk}; let $\langle b_{jk}\rangle$ be the stabilizer of C_{jk} in F.

Our aim is to define a continuous family of local homeomorphisms $h_{jk,t}\colon C_{jk} \to \bar{\mathbb{C}}$ which satisfy the following properties:

(a) $h_{jk,t}$ are equivariant with respect to the representations

$$r_t|_{\langle b_{jk}\rangle} \tag{6}$$

(b) $h_{jk,t}$ coincide with the restriction of f_t to the boundary of C_{jk} so that for each t the map d_t defined as the union of f_t and $h_{jk,t}$ is locally injective near all components of $\partial N_\varepsilon(\tilde{B})$.

3.5. The construction of the local homeomorphisms $h_{jk,t}\colon C_{jk} \to \bar{\mathbb{C}}$ is an application of Theorem 3. Consider the projections of f_t:

$$g_t^+ : \alpha_{jk} = \tilde{\alpha}_{jk}/\langle b_{jk}\rangle \to \Omega(\langle \hat{r}_t(b_{jk})\rangle)/\langle \hat{r}_t(b_{jk})\rangle = T^2 \tag{7}$$

$$g_t^- : \alpha_{kj} = \tilde{\alpha}_{kj}/\langle b_{jk}\rangle \to \Omega(\langle \hat{r}_t(b_{jk})\rangle)/\langle \hat{r}_t(b_{jk})\rangle = T^2$$

Denote by $A \subset S$ the annulus $p(C_{jk})$ bounded by $\alpha^+ = \alpha_{jk}$ and $\alpha^- = \alpha_{kj}$. Then without loss of generality we can assume that g_t^+ and g_t^- are "generic" (in the sense of Theorem 3). The map ϕ is the projection of the identity map to the annulus A. The coorientation on $\tilde{\alpha}_{jk}$ is given by a $\langle b_{jk}\rangle$-invariant vector-field directed inward the crescent C_{jk}; the coorientation on $f_t(\tilde{\alpha}_{jk})$ is given by the image under f_t of the coorientation on $\tilde{\alpha}_{jk}$. Then we apply Theorem 3 and lift the family of maps g_t (given by Theorem 3) to a family of local homeomorphisms $h_{jk,t}\colon C_{jk} \to \bar{\mathbb{C}}$.

Denote by $d_t\colon \mathbb{H}^2 \to \bar{\mathbb{C}}$ the extension of f_t via $h_{jk,t}$ as above. Then, for each $\gamma \in F$ we define d_t on the domain γC_{jk} as

$$d_t = r_t(\gamma) \circ h_{jk,t} \circ \gamma^{-1} \tag{8}$$

The local injectivity of the map

$$d_t\colon \mathbb{H}^2 \to \bar{\mathbb{C}} \tag{9}$$

near ∂C_{kj} follows from the fact that g_t agrees with the coorientation of the loops $g_t^{\pm}(\alpha^{\pm})$. The map d_t is a local homeomorphism on $\mathbb{H}^2 - N_\varepsilon(\tilde{B})$ and on $N_\varepsilon(\tilde{B})$.

Therefore the map $d_1\colon \mathbb{H}^2 \to \bar{\mathbb{C}}$ is a local homeomorphism which is equivariant with respect to the representation $\rho = r_1$. Thus the map d_1 is a developing map of a complex projective structure σ on S with the monodromy $\rho = r_1$.

This finishes the proof of Theorem 1. □

4. Proof of Theorem 2

4.1. The representation space

$$R(F) = \mathrm{Hom}(F, \mathrm{SL}(2,\mathbb{C}))/\mathrm{SL}(2,\mathbb{C})$$

is connected according to [9]. The subset of elementary representations has real codimension $\geqq 2$ in $R(F)$. Therefore there exists a continuous family ρ_t of nonelementary representations of F into $\mathrm{SL}(2,\mathbb{C})$ such that $\rho_1 = \rho$, ρ_0 is a Fuchsian representation corresponding to the uniformization of S, i.e. $S = \mathbb{H}^2/F$. Choose a set $\mathscr{G} = \{a_1, a_2, \ldots, a_g, b_1, \ldots, b_g\}$ of canonical generators of F, so that $[a_1, b_1] \cdot \ldots \cdot [a_g, b_g] = 1$. Then each loop

$$a_j a_i, a_j b_i, b_j b_i \ (j \neq i) \tag{10}$$

is simple and nonseparating. Our first goal is to prove

Proposition 1. *The group F has a canonical system of generators $\mathscr{G}$ such that the elements $a_1, b_1 \in \mathscr{G}$ have loxodromic images under the representation ρ and the group $\langle \rho(a_1), \rho(b_1)\rangle$ is not elementary.*

This Proposition is analogous to [8] *and it is a generalization of the well known fact that each nonelementary group in* $\mathrm{PSL}(2,\mathbb{C})$ *contains a nonabelian Schottky subgroup. The proof of this statement occupies Sections* 4.2, 4.3.

4.2. Suppose that images under ρ of all canonical generators and their products (10) are elliptic. Consider the projective model of $\mathbb{H}^3$ in RP^3 and denote by $L(\rho x) \subset A^3 \subset RP^3$ the *axis* of the elliptic element $\hat{\rho}x$; $x \in \mathcal{G}$. This fixed-point set is either a line in RP^3 or it is the whole space RP^3 (if $\hat{\rho}(x) = 1$).

The group generated by $\rho x, \rho y$ such that $\mathrm{Tr}(\rho x), \mathrm{Tr}(\rho y), \mathrm{Tr}(\rho(xy)) \in \mathbb{R}$ is either conjugate to a subgroup of $\mathrm{SL}(2, \mathbb{R})$ or $\mathrm{SU}(2)$ or to a group of upper-triangular matrices. Thus, for each pair of different generators $x, y \in \mathcal{G}$ we have:

$$RP^3 \ni L(\rho x) \cap L(\rho y) \cap L(\rho(xy)) \neq \emptyset$$

This implies that all one-dimensional axes $L(\rho x)$, $x \in \mathcal{G}$:

(i) either have a common point q,

(ii) or they are contained in one plane P.

In the case (i) we have 2 possibilities:

(ia) $q \in cl(\mathbb{H}^3)$, in this case the group $\rho(F)$ has a fixed point q in $cl(\mathbb{H}^3)$ and hence $\rho(F)$ is elementary.

(ib) $q \notin cl(\mathbb{H}^3)$. Denote by q^* the dual plane to q (with respect to $\partial\mathbb{H}^3$). Therefore $q^* \cap \mathbb{H}^3$ is a hyperbolic plane which is invariant under $\rho(F)$.

Consider the case (ib).

Let $x \in \{a_2, \ldots, b_g\}$ be an element such that $\rho\langle a_1, x\rangle$ is not Abelian. Then we can change the basis $\mathcal{G}$:

$$b_1 \mapsto b_1 x \tag{11}$$

The loop $b_1 x$ is simple, nonseparating and $i(a_1, b_1 x) = 1$. Hence we can consider $a_1, b_1 := b_1 x$ as elements of another canonical basis. After this change of the basis we may assume that $\rho(a_1)$ and $\rho(b_1)$ do not commute. Therefore we can apply the following

Lemma 1. [8]. *If $x, y \in \mathrm{PSL}(2, \mathbb{R})$ are elliptic elements which do not commute, then there exists a number $m \in \mathbb{Z}$ such that:*
either $x^m y$, or xy^m is loxodromic.

Proof. This lemma was proven algebraically in [8]. In Section 6 we give a geometric proof of Lemma 1. □

Lemma 1 implies that some nonseparating simple loop $a_1 b_1^m \in F$ (or $a_1^m b_1$) has loxodromic image under ρ. Take this loop as an element a_1 of a new basis of the fundamental group F.

Consider the case (ii). Suppose that ρx, ρy do not commute and $\rho(xy)$ is elliptic. There exists $z \in \mathcal{G} - \{x, y\}$ such that the intersection of axes of $\rho(x)$, $\rho(y)$, $\rho(z)$ is empty. The axis $L(\rho(xy))$ doesn't belong to P and so the product $\rho(w = xyz)$ is loxodromic. Again the loop w is simple and does not separate S.

We conclude that in the case when all elements $\rho(a_j)$, $\rho(b_i)$ are elliptic one can change the basis $\mathcal{G}$ so that image of one element of the new basis is loxodromic.

4.3. To consider the parabolic case we shall need the following

Lemma 2. *Suppose that $a \in \mathrm{SL}(2, \mathbb{C})$ is parabolic and $b \in \mathrm{SL}(2, \mathbb{C})$ is an element such that a, b generate a nonelementary group. Then there exists a positive integer n_0 such that the element $a^n b$ is loxodromic for each $n > n_0$.*

Proof. See Section 6. □

Suppose now that the image of the canonical generator a_1 is parabolic. Then we use the change of the basis (11) to find a new basis $\mathscr{G}$ so that $\rho(a_1)$ is parabolic and the elements $\rho(a_1), \rho(b_1)$ generate a nonelementary group. Therefore, according to Lemma 2, for sufficiently large m the element $\rho(a_1^m b_1)$ is loxodromic. Again, the product $a_1^m b_1$ is represented by a simple nonseparating loop.

Thus, in any case, there is a simple nonseparating loop $a_1 \subset S$ such that $\rho(a_1)$ is loxodromic. Applying the transformation (11) we can change the basis $\mathscr{G}$ so that the group $\rho\langle a_1, b_1 \rangle$ is not Abelian and $\rho(b_1)$ is loxodromic. However this group can be elementary. Then $\hat{\rho}(a_1), \hat{\rho}(b_1)$ have a common fixed point w in $\bar{\mathbb{C}}$. We may assume that this point is attractive for the element $\hat{\rho}(a_1)$. Let $q \neq w$ be the second fixed point of $\hat{\rho}(a_1)$. Since w is not fixed by the group $\hat{\rho}(F)$, there exists an element $x \in \{a_2, b_2, \ldots, a_g, b_g\}$ such that $\hat{\rho}(x)(w) \neq w$. It follows that for large values of n the element $\rho(x a_1^n b_1)$ is loxodromic and its fixed-point set does not intersect $\{w, q\}$. The loop $b_1' = x a_1^n b_1$ is homotopic to a simple closed curve so that $i(a_1, b_1') = 1$. This means that we can use the change of the basis $b_1 \mapsto b_1'$ to find $\mathscr{G}$ so that the group $\langle \rho(a_1), \rho(b_1) \rangle$ is not elementary and both $\rho(a_1), \rho(b_1)$ are loxodromic. This concludes the proof of Proposition 1.

Remark 1. In these arguments we actually did not use the fact that the representation $\hat{\rho}$ has a lift into $\mathrm{SL}(2, \mathbb{C})$. Therefore as a corollary of the above discussion we get the following

Corollary. *Suppose tnat S is a closed orientable hyperbolic surface, $\hat{\rho}: \pi_1(S) \to$ $\mathrm{PSL}(2, \mathbb{C})$ is a nonelementary representation. Then there exists a $\hat{\rho}$-equivariant pleated map $f: \mathbb{H}^2 = \tilde{S} \to \mathbb{H}^3$.*

Proof. There exists a simple closed loop $a_1 \subset S$ such that $\hat{\rho}(a_1)$ is loxodromic transformation. Thus we can apply the arguments in [21, Section 6] to construct an equivariant pleated map f. □

Remark 2. Note that the conditions of Corollary are necessary and sufficient for existence of an equivariant nondegenerate harmonic map $h: \tilde{S} \to \mathbb{H}^3$.

4.4. Splitting the surface

In this section we shall need the following lemmas:

Lemma 3. (Cf. [8]) *Suppose that $a, b \in \mathrm{SL}(2, \mathbb{C})$ are such that b is loxodromic and $aba^{-1} \neq b^{-1}$. Then there exists a positive integer n_0 such that the element $b^n a$ is loxodromic either for all $n \geqq n_0$ or for all $n \leqq -n_0$.*

Proof. See Section 6. $\square$

Lemma 4. (Cf. [8]) *Suppose that $a, b, c \in \mathrm{SL}(2, \mathbb{C})$ are such that b, c are loxodromic and the group $\langle a, c \rangle$ is not elementary. Then there exists a positive integer m_0 so that the element $c^m b c^{-m} a$* is loxodromic for all $m \geqq m_0$.

Proof. See Section 6. $\square$

Proposition 2. *There exists a decomposition of S into the union of pairs of pants P_j such that:*

The restriction of ρ to each $\pi_1(P_j)$ is nonelementary and images of the peripheral elements of $\pi_1(P_j)$ are loxodromic.

Proof. Suppose that $S' \subset S$ is a compact connected incompressible subsurface (with or without boundary) which is different from a handle. Let α, β be simple nonseparating loops on S' so that $i(\alpha, \beta) = 1$, images of α, β are loxodromic and the group $\langle \rho(\alpha), \rho(\beta) \rangle$ is not elementary. Assume also that for any boundary loop $b \subset \partial S'$ the image of b under ρ is loxodromic.

Then there exists a simple homotopically nontrivial loop ξ on S' so that:

(i) either ξ is the product of two boundary loops a, b (**Splitting I**);

(ii) or ξ is a nonseparating loop which has zero geometric intersection number with α, β (**Splitting II**), see Fig. 1.

We choose a base-point x_0 on the loop α and (in the case (i)) connect b to x_0 by a "tail" t so that $b' = t \cdot b \cdot t^{-1}$ is homotopic to a simple loop and $i(b', \beta) = 0$. Denote by δ a simple loop homotopic $\alpha b'$. This loop is nonseparating. In the case (ii) instead of a boundary curve take any loop b such that $i(b, \xi) = 1$, $i(b, \alpha) = i(\beta) = 0$, connect b with a by a "tail" t and let $b' = t \cdot b \cdot t^{-1}$, $\delta = \alpha b'$. We can always assume that $\rho(\delta)$ is loxodromic and $\rho\langle \delta, b \rangle$ is not elementary (using the base-change $\alpha \to \beta^n a, \beta \to \beta$ if necessary). Then:

$$D_\delta^m(\xi) = \begin{cases} \delta^m \xi, & \text{in the case (ii)} \\ a\delta^m b \delta^{-m}, & \text{in the case (i)} \end{cases}$$

Notice that

$$D_\delta^m(\beta) = \delta^m \beta \tag{12}$$

In any case, for sufficiently large m the elements $\rho D_\delta^m(\xi)$, $\rho D_\delta^m(\beta)$ are loxodromic and the group

$$\langle \rho(a), \rho(D_\delta^m(\beta)) \rangle \tag{13}$$

is not elementary (see Lemma 3, Lemma 4). Then split the surface S' along the simple loop $D_\delta^m(\xi)$.

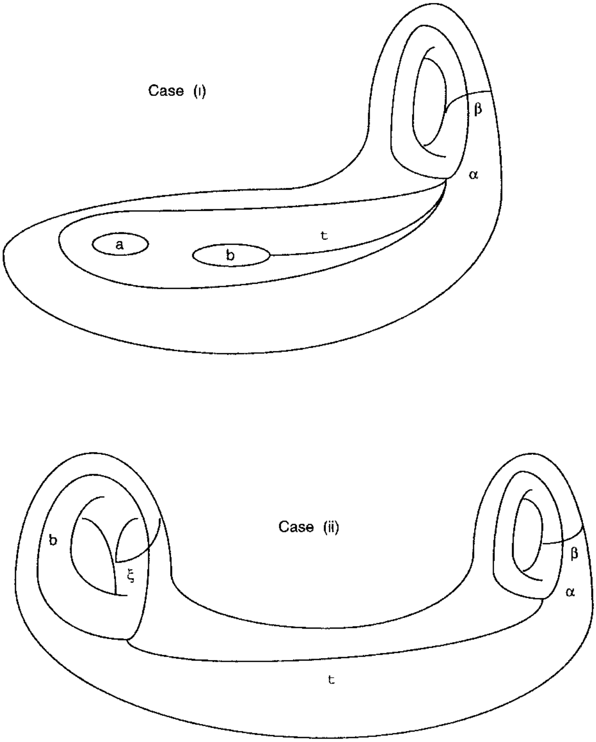

Fig. 1.

In the case (i) the result S'' will consist of two surfaces: a pair of pants P and a surface S'' with at least one nonseparating loop $D_\delta^m \beta$ which has loxodromic image. Then the group

$$\rho(\pi_1(P)) = \langle \rho(a), \rho(D_\delta^m(\beta)) \rangle \tag{14}$$

is not elementary. The group $\rho(\pi_1(S''))$ is not elementary as well since

$$\rho\langle \delta, D_\delta^m(\beta) = \delta^m \beta \rangle \tag{15}$$

is not elementary.

In the case (ii) the surface S'' is connected and has the same properties as in the case (i).

Now, using the **Splittings I** and **II**, we start decomposition of the surface S. To begin with we consider $S' := S$ and the pair of loops $\{\alpha, \beta\} = \{a_1, b_1\}$ as constructed in Section 4.3. Then we apply the **Splitting II** to split all but one handles of S (Fig. 2).

Then, applying the **Splitting I**, we "chop off" the splitted handles (Fig. 3). The result is a collection of $g-1$ pairs of pants K_j ($j = 1, \dots, g-1$) and a surface S' of the genus 1 with $g-1$ boundary components.

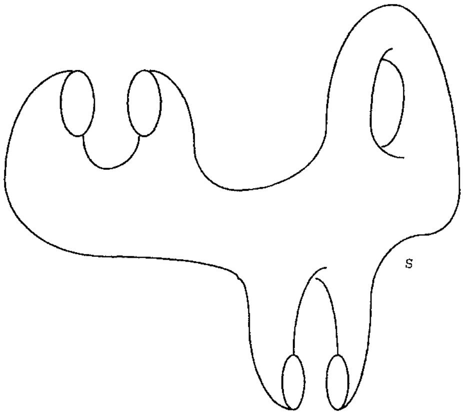

Fig. 2.

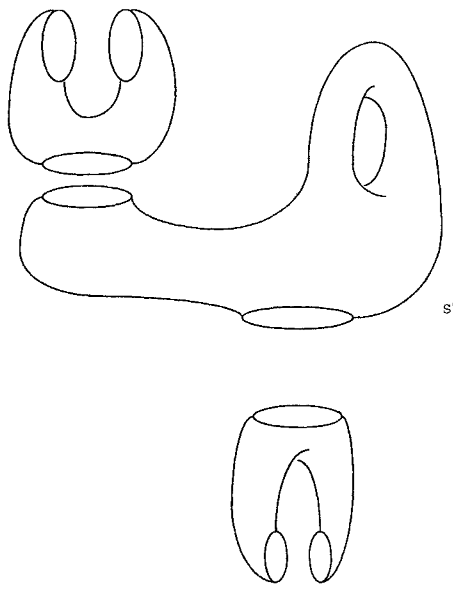

Fig. 3.

Using the **Splitting I** we decompose the rest of the surface into the union of pairs of pants Q_j and a handle T' with a boundary loop δ. The group $\rho(\pi_1(T'))$ is not elementary and $\rho(\delta)$ is loxodromic. Split T' along a simple nonseparating loop α which has loxodromic image under ρ and such that the fixed-point sets of $\hat{\rho}(\alpha)$, $\hat{\rho}(\delta)$ are disjoint (Fig. 4). The fundamental group of the surface $K_g = T' - \alpha$ has nonelementary image under ρ.

This finishes the first decomposition of the surface S. As the result we obtain a decomposition of S into the union of pairs of pants K_i, Q_j such that:

(a) The union of Q_j is a sphere with holes S_0.

(b) One boundary component of each K_i is adjacent to ∂S_0 along a boundary curve of Q_i, $i = 1, \dots, g$.

(c) By identifying 2 other boundary loops of each K_j we obtain a collection of handles

$$H_1, H_2, \dots, H_g$$

in the surface S.

(d) For each pair of pants P in this decomposition the group $\rho(\pi_1(P))$ is not elementary and images of all boundary loops are loxodromic.

This finishes the proof of Proposition 2. □

4.5. *Deformations of representations*

Recall that in Section 4.4 we have constructed a special decomposition of the surface S into collection of pairs of pants.

Proposition 3. *There exists a continuous family of representations* $r_t\colon F \to \mathrm{SL}(2, \mathbb{C})$ *such that*:

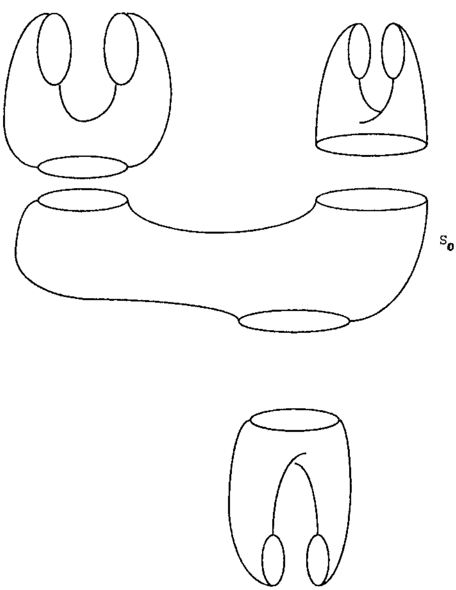

Fig. 4.

(a) $r_0 = \rho_0$, $r_1 = \rho_1$;

(b) *for each pair of pants P in the decomposition of S, for each boundary loop γ of P and for each $t \in [0, 1]$ the element $r_t(\gamma)$ is loxodromic and the restriction of r_t on $\pi_1(P)$ is nonelementary.*

Proof. The construction of the family r_t is based on a generalization of methods and results of [9]. Let P be a pair of pants, $\mathbb{F}_2 = \pi_1(P)$. Denote by a, b, ab the boundary loops of P. The variety of nonelementary representations

$$R(\mathbb{F}_2)^0 = \mathrm{Hom}(\mathbb{F}_2, \mathrm{SL}(2, \mathbb{C}))^0/\mathrm{SL}(2, \mathbb{C}) \tag{16}$$

is parameterized by

$$\mathbb{C}^3 = \{(\mathrm{Tr}(h(a)), \mathrm{Tr}(h(b)), \mathrm{Tr}(h(ab))) \colon h \in \mathrm{Hom}(\mathbb{F}_2, \mathrm{SL}(2, \mathbb{C}))^0\} \tag{17}$$

see [9].

Lemma 5. *Suppose that λ_t is a family of representations of $\mathbb{F}_2$ into $\mathrm{SL}(2, \mathbb{C})$ such that λ_1, λ_0 are nonelementary and the restrictions of λ_1, λ_0 to a, b, ab are loxodromic. Then the curve λ_t in $(\mathbb{F}_2)$ is homotopic (relative to $\{0, 1\}$) to a curve μ_t of nonelementary representations such that:*
$\mu_t(a), \mu_t(b), \mu_t(ab)$ are loxodromic for all t.

Proof. The existence of the path μ_t follows from the fact that the interval $[-2, 2]$ of traces of non-loxodromic elements doesn't separate $\mathbb{C}$. □

Recall the properties of the decomposition of S that was constructed in Section 4.4. The surface S is the union of "handles" H_j and the "sphere with holes" S_0. The graph dual to the decomposition of S_0 into the union of pairs of pants Q_j is a tree.

Therefore Lemma 5 implies that we can change the family of representations

$$\rho_t|_{\pi_1(S_0)}$$

so that the new path of representations r_t^0 satisfies the properties:

(1) r_t^0 coincides with the restriction of ρ_t to $\pi_1(S_0)$ for $t = 0, 1$;

(2) for each Q_j and every $t \in [0, 1]$ the group $r_t^0(\pi_1(Q_j))$ is nonelementary and every $\gamma \subset \partial Q_j$ has loxodromic image $r_t^0(\gamma)$.

4.6. Now we have to extend r_t^0 to representations of the fundamental groups of the "handles" H_i. First, using [9], we can extend r_t^0 to **some** smooth family of representations r_t' of $\pi_1(S)$ so that $r_t' = \rho_t$ for $t = 0, 1$.

Suppose that H is a handle, r_t' is a smooth family of nonelementary representations of $\pi_1(H)$ into $\mathrm{SL}(2, \mathbb{C})$, $\alpha \subset H$ is a simple nonseparating loop, δ is the boundary curve of H. Let K denote the surface $H - \alpha$. Assume that

(a) $r_t'(\delta)$ are loxodromic for all t and $r_0'(\alpha)$ and $r_1'(\alpha)$ are also loxodromic;

(b) the restrictions $r_t|_{\pi_1(K)}$ are nonelementary for $t = 0, 1$.

Lemma 6. *Under the conditions above there exists a smooth family of representations,* $r_t\colon \mathbb{F}_2 = \pi_1(H) \to \mathrm{SL}(2, \mathbb{C})$ *such that:*

(a) $r_t(\alpha)$ *are loxodromic for all* t;

(b) $r_t(\delta) = r_t'(\delta)$ *and the restrictions* $r_t|_{\pi_1(K)}$ *are nonelementary for* $t \in [0, 1]$.

Proof. Denote by β an oriented simple closed curve on H such that $i(\alpha, \beta) = 1$; let $\gamma = \alpha\beta$, then $\delta = [\alpha, \beta]$. Consider the restriction map

$$\mathrm{Res}\colon R(\mathbb{F}_2)^0 \to R(\langle\delta\rangle)$$

The variety $R(\langle\delta\rangle)^\lambda$ of loxodromic representations in $R(\langle\delta\rangle)$ contains the curve $\langle r_t'(\delta)\rangle$. Take any point $r \in R(\langle\delta\rangle)^\lambda$, thus $\mathrm{Tr}(r(\delta)) = u \neq \pm 2$. The preimage $\mathrm{Res}^{-1}(r)$ is biholomorphic to the quadric

$$\mathcal{Q} = \{(x, y, z) \in \mathbb{C}^3\colon x^2 + y^2 + z^2 - xyz = u + 2\}$$

where $x = \mathrm{Tr}(h(\beta))$, $y = \mathrm{Tr}(h(\gamma))$, $z = \mathrm{Tr}(h(\alpha))$ for $h \in \mathrm{Res}^{-1}(r)$, see [9].

Remark 3. The quadric $\mathcal{Q}$ is smooth since $u^2 \neq 4$.

Claim 1. For fixed $u \neq \pm 2$ there exists a nonseparating compact real curve $J = J(u) \subset \mathbb{C}$ such that for every $x_0^2 \notin J \cup \{4\}$ the real curve

$$C_{x_0}[-2, 2] = \{(x, y, z) \in Q\colon x = x_0;\ z \in [-2, 2]\}$$

does not separate the complex curve $C_{x_0} = \{(x, y, z) \in \mathcal{Q}\colon x = x_0\}$.

Proof. For any $x_0 \neq \pm 2$ the curve C_{x_0} is nonsingular. The projection $z\colon C_{x_0} \to \mathbb{C}$ is a 2-fold ramified covering. Therefore $z^{-1}([-2, 2])$ does not separate C_{x_0} if the set critical values $C(z)$ of the projection z does not intersect the segment $[-2, 2]$. The set of values of x_0^2 such that $C(z) \cap [-2, 2] \neq \emptyset$ is a simply-connected curve

$$J = \{4(z^2 - u - 2)/(z^2 + 4) \text{ such that } z \in [-2, 2]\} \qquad \square \tag{18}$$

To construct the curve r_t we first perturb r_t' to a curve of representations r_t'' which is transversal to

$$R_p(\mathbb{F}_2)^0 = \{r \in \mathrm{Hom}(\mathbb{F}_2, \mathrm{SL}(2, \mathbb{C}))^0 \mid \mathrm{Tr}(r(\delta)) \in [-2, 2]\}/\mathrm{SL}(2, \mathbb{C}) \tag{19}$$

Therefore $\{t \in [0, 1] \mid r_t'' \in R_p(\mathbb{F}_2)\} = \{t_1 < t_2 \ldots < t_s\}$ where $0 < t_1 \leqq t_s < 1$. Denote by u_t the number $\mathrm{Tr}(r_t''(\delta))$. We can assume that for each t_j

$$\mathrm{Tr}(r_{t_j}''(\beta)) \notin J(u_{t_j}) \tag{20}$$

Then we use Claim 1 to change the curve r_t'' to a curve r_t near all points t_j so that $\mathrm{Tr}(r_t(\alpha)) \notin [-2, 2]$. All representations in the curve r_t belong to $R(\mathbb{F}_2)^0$ and they are loxodromic on the elements δ, α. Therefore, the restriction of each r_t to $\pi_1(K)$ is nonelementary. $\square$

We apply Lemma 6 to all handles H_j in S. As the result we obtain a path r_t which satisfies the conditions (a) and (b) of Proposition 3. This finishes the proof of Proposition 3. □

4.7. Final decomposition of the surface S

Let K_i, Q_i be pairs of pants in the decomposition of S which have one common boundary curve α. Then $E = int(K_i) \cup int(Q_i) \cup \alpha$ is a sphere with 4 holes.

Proposition 4. *There is a decomposition of E into the union of 2 pairs of pants P, R such that the restrictions of each r_t to $\pi_1(P)$, $\pi_1(R)$ are isomorphisms of Schottky groups.*

Proof. The proof of this Proposition is similar to [8]. Let $c, b \in \pi_1(Q_i)$, $g, h \in \pi_1(K_i)$ be primitive peripheral elements of $\pi_1(E)$ (see Fig. 5). Denote by δ a simple loop on E which is freely homotopic to $b \cdot g$. The Dehn twists D_α^n act on $\pi_1(E)$ as follows:

$$D_\alpha^n(c) = c, \quad D_\alpha^n(b) = b, \quad D_\alpha^n(g) = g' = \alpha^n g \alpha^{-n}, \quad D_\alpha^n(h) = h' = \alpha^n h \alpha^{-n} \tag{21}$$

Our goal is to prove that there exists a number n such that for all $t \in [0, 1]$ the groups $\langle r_t(g'), r_t(b) \rangle$ and $\langle r_t(h'), r_t(c) \rangle$ are Schottky groups of the rank 2. Recall that for a loxodromic element $f \in \mathrm{PSL}(2, \mathbb{C})$ the translational length $l(f)$ is $min\{d(x, fx) : x \in \mathbb{H}^3\}$. We shall need the following:

Lemma 7. *Let* $\sinh(L(\varepsilon)/2) \cdot \sinh(\varepsilon) = 1$. *Suppose that* $g_1, g_2 \in \mathrm{PSL}(2, \mathbb{C})$ *are loxodromic elements such that*

$$\min\{l(g_1), l(g_2)\} \geqq \varepsilon > 0, \quad \mathrm{dist}(\mathrm{Axis}(g_1), \mathrm{Axis}(g_2)) \geqq L(\varepsilon)$$

Then the group $\langle g_1, g_2 \rangle$ is a Schottky group or rank 2.

Proof. Let $[X_1, X_2]$ the shortest segment between $A_1 = \mathrm{Axis}(g_1)$ and $A_2 = \mathrm{Axis}(g_2)$ so that $X_j \in A_j$. Denote by γ the geodesic in $\mathbb{H}^3$ which contains $[X_1, X_2]$. Set

$$V_j = B_j \cup B_j' = \{z \in \mathbb{H}^3 : d(X_j, z) \geqq d(z, g_j X_j)\} \cup \{z \in \mathbb{H}^3 : d(X_j, z) \geqq d(z, g_j^{-1} X_j)\} \tag{22}$$

The distance between X_j and $g_j X_j$ is at least ε. Therefore the diameter of the orthogonal projection $\pi(V_j)$ of V_j onto γ is at most q where

$$\sinh(q) = 1/\sinh(\varepsilon) \tag{23}$$

Thus, since $\mathrm{dist}(\mathrm{Axis}(g_1), \mathrm{Axis}(g_2)) \geqq L(\varepsilon) \geqq L = 2q$ then $\pi(V_1) \cap \pi(V_2) = \emptyset$. This implies that the intersection between V_1 and V_2 is empty. The real boundary in $\bar{\mathbb{C}}$ of V_j is the disjoint union of two discs D_j, D_j' so that

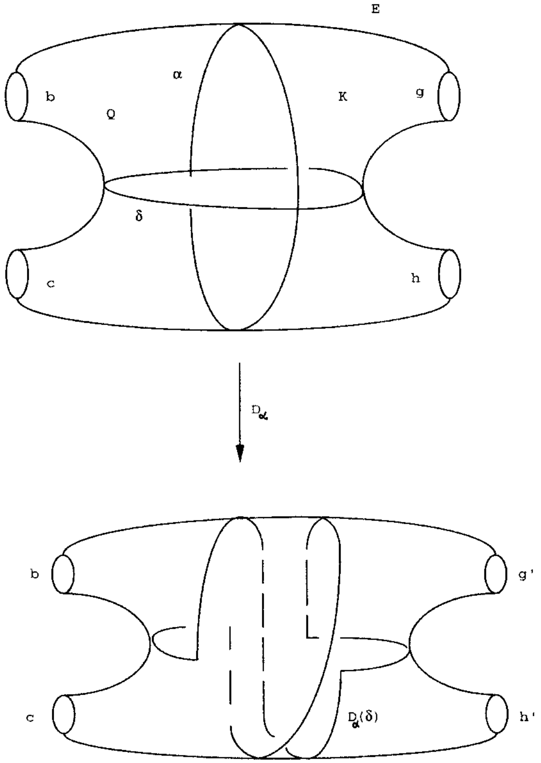

Fig. 5.

$g_j(int\,D_j) = ext(D_j')$. Moreover, all the discs D_1, D_1', D_2, D_2' are disjoint. Thus the group generated by g_1, g_2 is a Schottky group of rank 2. □

The assertion (b) of Proposition 3 implies that for each t the spherical distance from any of the fixed point of $\hat{r}_t(b), \hat{r}_t(c), \hat{r}_t(g), \hat{r}_t(h)$ to any of the fixed point of $\hat{r}_t(\alpha)$ is greater than some positive number v which is independent of t. Moreover, $\min\{\ell(r_t(\alpha)), t \in [0,1]\} \geqq \mu > 0$ for some μ.

Denote by ε the number

$$\min\{\ell(r_t(b)), \ell(r_t(c)), \ell(r_t(h)), \ell(r_t(g)) \colon t \in [0,1]\} \tag{24}$$

Therefore there exists a number n which does not depend on t such that the distance from $\hat{r}_t(\alpha^n)\mathrm{Axis}(r_t(g))$ to $\mathrm{Axis}(r_t(b))$ and from $\hat{r}_t(\alpha^n)\mathrm{Axis}(r_t(h))$ to $\mathrm{Axis}(r_t(c))$ is at least $L(\varepsilon)$ for every t. However

$$\hat{r}_t(\alpha^n)\mathrm{Axis}(r_t(g)) = \mathrm{Axis}(r_t(g')), \quad \hat{r}_t(\alpha^n)\mathrm{Axis}(r_t(h)) = \mathrm{Axis}(r_t(h')) \tag{25}$$

Hence we can find a number n independent on t such that for every t the distance between the axes of $\hat{r}_t(g'), \hat{r}_t(b)$ and $\hat{r}_t(h'), \hat{r}_t(c)$ is not less than $L(\varepsilon)$.

This means that the conditions of Lemma 7 are satisfied and the groups $\langle r_t(g'), r_t(b)\rangle$ and $\langle r_t(h'), r_t(c)\rangle$ are Schottky groups of rank 2.

The simple loop $\delta(n) = D^n_\alpha(\delta)$ separates g', b from h', c. We split E along $\delta(n)$ to obtain a new decomposition of E into the union of pair of pants P, R (Fig. 5). This finishes the proof of Proposition 4. $\square$

Recall that according to Proposition 2 the surface S is the union of pairs of pants K_i, Q_i, $i = 1, \ldots, g$, where each K_i and Q_i share a common boundary loop. We apply Proposition 4 to each pair K_i, Q_i to get a new decomposition of S. This decomposition and the family of representations r_t satisfy the properties (a) and (b) in Theorem 2.

This concludes the proof of Theorem 2. $\square$

5. Proof of Theorem 3

Let $0 < t_1 < t_2 < \cdots < t_k \leqq 1$ be the set of points where the curves γ_t^+ and γ_t^- are not transversal.

5.1. Step 1. For $0 \leqq t < t_1$ we define a continuous family of smooth extensions $g_{1,t}$ of $g_t^\pm$ as follows.

Let h_t be *any* continuous family of smooth embeddings $A \to T^2$ defined for $0 \leqq t < t_1$ which satisfies the properties:

(a) the restriction of each h_t to the boundary of A coincides with $g_t^\pm$, and

(b) h_t agrees with the coorientation of the curves $\gamma_t^\pm$.

We recall that the conformal structure c_t on the torus T^2 depends continuously on t. Let $int(A)$ be conformally-equivalent to $\mathbb{H}^2/\langle q\rangle$ and $int(h_t A)$ be conformally equivalent to $\mathbb{H}^2/\langle q_t\rangle$ where $q_t \in \mathrm{PSL}(2, \mathbb{R})$ depends continuously on the parameter t. Then $g_t^\pm$ lifts to a diffeomorphism

$$\tilde{g}_t^\pm : \partial_\infty \mathbb{H}^2 - \Lambda(\langle q\rangle) \to \partial_\infty \mathbb{H}^2 - \Lambda(\langle q_t\rangle) \tag{26}$$

which continuously depends on t and is equivariant with respect to the isomorphism $\langle q\rangle \to \langle q_t\rangle$. Thus, $\tilde{g}_t^\pm$ admits a canonical equivariant extension to a diffeomorphism

$$\tilde{g}_t : \mathbb{H}^2 \to \mathbb{H}^2 \tag{27}$$

which depends continuously on the parameter t (see [6]). Then the projection of $\tilde{g}_t$ to A defines a smooth extension $g_{1,t}: A \to T^2$ of the map $g_t^\pm$.

5.2. Grafting. A general description of the *grafting* can be found in [10], [17], here we consider only a particular case. Denote by $p_{1,t}: (T^2, c_{2,t}) \to (T^2, c_t)$ a holomorphic 4-fold covering whose defining subgroup in $\pi_1(T^2)$ contains the homotopy class of γ_t^+. The family $c_{2,t}$ of conformal structures on the torus T^2 depends continuously on t. We choose lifts $g_{2,t}^\pm: \alpha^\pm \to (T^2, c_{2,t})$ of the

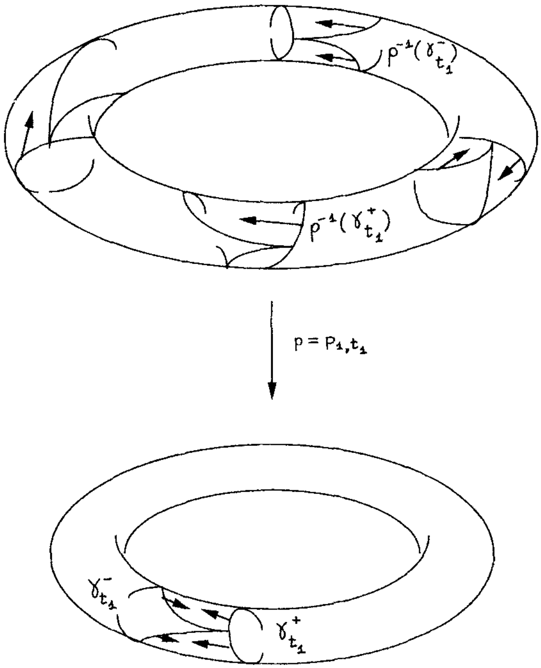

Fig. 6.

maps $g_t^\pm$ under these coverings so that $\gamma_{2,t}^- = g_{2,t}^-(\alpha^-)$ and $\gamma_{2,t}^+ = g_{2,t}^+(\alpha^+)$ do not intersect each other for all $0 \leqq t < t_2$. See Fig. 6. The coorientations on curves $\gamma_{2,t}^\pm$ are obtained by pull-back of the coorientations on $\gamma_t^\pm$.

5.3. *Step 2.* Now we can apply Step 1 to the family $\gamma_{2,t}^\pm$.

We continue this process until we pass through all singular values $t_1, t_2, \ldots, t_k$.

As the result we obtain a continuous family of homeomorphic embeddings

$$g_{k,t}\colon A \to (T^2, c_{k,t})$$

The restriction of $g_{k,t}$ to the boundary of A coincides with the lift of $g_t^\pm$ via the covering

$$(T^2, c_{k,t}) \xrightarrow{p_{k-1,t}} (T^2, c_{k-1,t}) \longrightarrow \cdots \xrightarrow{p_{1,t}} (T^2, c_t) \tag{28}$$

and these maps "agree" with the coorientation.

We define g_t to be

$$g_t = p_{1,t} \circ \cdots \circ p_{k-1,t} \circ g_{k,t} \tag{29}$$

This family of local homeomorphisms has all required properties. □

6. Products of matrices

6.1. Proof of Lemma 1. Consider the group H generated by $x, y, xy = z$. Let X, Y, Z be the fixed points for action of x, y, z in $\mathbb{H}^2$. All these points are distinct. Take the geodesic l_1 through X, Y, the geodesic l_2 through Y, Z, and the geodesic l_3 through X, Z. These geodesics bound a triangle Δ in $\mathbb{H}^2$. Denote by R_j the reflection in $\mathbb{H}^2$ with the fixed-point set l_j. Then $x = R_3 R_1$, $z = R_2 R_3$, $y = R_1 R_2$ (cf. [11]). In particular, the angles α_x, α_y at the vertices X, Y of Δ are equal to one half of the rotational angles of x, y. Suppose now that $\alpha_x \leqq \alpha_y$. Then there is a number m such that:

$$\pi - \alpha_y \leqq m\alpha_x < \pi \tag{30}$$

Let l_{mx} be the geodesic through X so that the angle between l_{mx} and l_1 is $m\alpha_x$. Then $R_{mx} R_1 = x^m$ where R_{mx} is the reflection in l_{mx}. On the other hand, the condition (30) implies that l_{mx} and l_2 do not intersect even on the boundary of $\mathbb{H}^2$ (since the sum of angles in any hyperbolic triangle is less than π). Thus the element $x^m y = R_{mx} R_2$ is hyperbolic. □

6.2. Proof of Lemma 2. Applying conjugation we can assume that

$$a = \begin{pmatrix} \pm 1 & 1 \\ 0 & \pm 1 \end{pmatrix}, \quad b = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \tag{31}$$

$$\operatorname{Tr}(a^n b) = \pm(\alpha + \delta) + n\gamma \tag{32}$$

The number γ is different from zero since a, b generate a nonelementary group. Therefore, for $n > 2|\alpha + \delta + 2|/|\gamma|$ the trace of the matrix $a^n b$ does not belong to the interval $[-2, 2]$. □

6.3. Proof of Lemma 3. Applying conjugation we can assume that $\hat{b}\colon z \mapsto \lambda z$, $|\lambda| > 1$. Note that $\hat{a}\colon z \mapsto c/z$ would imply that a, b anticommute. Therefore $|\operatorname{Tr}(b^n \cdot a)| \to \infty$ as $n \to \infty$ or $n \to -\infty$. □

6.4. Proof of Lemma 4. If the element a is not elliptic, then the conclusion of Lemma follows from the Klein Combination Theorem (see [16]). Thus suppose that a is elliptic and $\hat{a}\colon z \mapsto e^{i\theta} z$ and $e^{i\theta} \neq \pm 1$. Let I, I' be a pair of disjoint discs in $\bar{\mathbb{C}}$ such that $\hat{b}\colon \operatorname{int}(I) \to \operatorname{ext}(I')$; put $I_m = c^m(I)$, $I'_m = c^m(I')$. The attractive fixed point α of the element $\hat{c}$ is neither zero nor infinity. As $m \to \infty$ the discs I_m and I'_m accumulate to α. Therefore for sufficiently large m the union $I_m \cup I'_m$ lies between two rays emanating from zero: R_1 and $R_2 = \hat{a}(R_1)$. The disc $J_m = (\hat{a})^{-1} I_m$ satisfies the property:

$$\tau(c^m b c^{-m} a)\colon \operatorname{int}(J_m) \to \operatorname{ext}(I'_m) \tag{33}$$

We conclude that the element $c^m b c^{-m} a$ is loxodromic since $J_m \cap I'_m = \emptyset$. □

7. Degeneration and regeneration of complex projective structures

7.1. In this section we discuss the behavior of a degenerating family of complex projective structures after *grafting*. The operation of *grafting* was originally introduced by B. Maskit for structures with Fuchsian monodromy [17]. A general definition was given later by W. Goldman [10]. Fix a complex projective structure c with the developing map d and monodromy ρ. Let L be a union of disjoint simple closed homotopically nontrivial curves λ_j on S. Suppose that for each curve λ_j and for each component $\tilde{\lambda}_j$ of its lift to $\tilde{S}$ the restriction of d to $\tilde{\lambda}_j$ is injective and $\rho(\lambda_j)$ is loxodromic. Then split S along L and for each λ_j split $\bar{\mathbb{C}}$ along $d(\tilde{\lambda}_j)$. Glue the quotients $(\bar{\mathbb{C}} - cl(d(\tilde{\lambda}_j))/\langle \rho\lambda_j \rangle$ to the surface $S - L$ along λ_j. The surface obtained by gluing has a natural complex projective structure which is denoted by $gr(c, L)$ and is said to be obtained by *grafting* of c along L. The monodromy representation of the structure $gr(c, L)$ is equal to ρ.

The space $C(S)$ of "marked" complex projective structures on S is a fiber bundle over the Teichmuller space $p\colon C(S) \to T(S)$. Each fiber $p^{-1}(\zeta)$ is the space of holomorphic quadratic differentials $Q(\zeta)$ on the marked Riemann surface (S, ζ). Denote by $hol\colon C(S) \to \mathrm{Hom}(F, \mathrm{SL}(2, \mathbb{C}))^0/\mathrm{SL}(2, \mathbb{C})$ the monodromy map. This map is a local homeomorphism which is not a covering [15]. The space $C(S)$ has a "natural compactification" $\overline{C(S)}$ which is the projective compatification along the fibers $Q(\zeta)$ and the compactification along $T(S)$ by measured foliations.

We are left with the following challenging problems.

Problem 1. Describe points $z \in \overline{C(S)} - C(S)$ such that there is a continuous path $c\colon [0, 1] \to \overline{C(S)}$ with the properties: $c([0, 1)) \subset C(S)$ and there exists a limit

$$\lim_{t \to 1} \mathrm{hol}(c(t)) \in \mathrm{Hom}(F, \mathrm{SL}(2, \mathbb{C}))^0/\mathrm{SL}(2, \mathbb{C}) \tag{34}$$

This would measure the defiation of *hol* from a covering map. In the very interesting particular case when $c(t)$ is contained in the space $QF(S)$ of quasifuchsian complex projective structures, the answer is given by the "double limit" theorem of W. Thurston [20]. However Thurston uses different compactification of $QF(S)$ which can not be generalized to $C(S)$.

Problem 2. For given $r \in \mathrm{Hom}(F, \mathrm{SL}(2, \mathbb{C}))^0/\mathrm{SL}(2, \mathbb{C})$ describe $\mathrm{hol}^{-1}(r)$.

Actually, two problems are closely related since the difficulty in solving Problem 2 lies in the failure of *hol* to be a covering.

7.2. Problem 2 was solved by W. Goldman [10] in the case of faithful quasifuchsian representations r. Every structure in $\mathrm{hol}^{-1}(r)$ can be obtained from a "quasifuchsian structure" c by "grafting".

Not so much is known about Problem 1. The space

$$R(F)^0 = \mathrm{Hom}(F, \mathrm{SL}(2,\mathbb{C}))^0/\mathrm{SL}(2,\mathbb{C})$$

has a natural structure of a smooth algebraic variety. Consider the holomorphic family $\pi\colon V \to R(F)^0$ of holomorphic bundles, where $\pi^{-1}([r])$ is the flat holomorphic $\mathbb{C}^2$-bundle over S with the monodromy r. Then $\mathrm{hol}(Q(\zeta))$ is the set of points $[r]$ in $R(F)^0$ for which $\pi^{-1}([r])$ is maximally unstable (see [12, Proposition A4]).

Thus, the upper-semicontinuity theorem for cohomology (see [2]) implies that $\mathrm{hol}(Q(\zeta))$ is an analytic subvariety in $R(F)^0$ and therefore it is properly embedded (cf. [19]). On the other hand, $R(F)^0$ has a holomorphic foliation where each leaf consists of holomorphically equivalent flat bundles. It follows from [12, 14] that the subvariety $\mathrm{hol}(Q(\zeta))$ is a leaf of this foliation and hence it is smooth. The restriction of hol to $Q(\zeta)$ is injective, therefore it is a proper map.

Thus the degeneration of a family of marked complex projective structures $c(t)$ in Question 1 implies that the underlying marked complex structures also degenerate. The last can happen either because of the action of the modular group on $T(S)$ or because of "pinching" of S along a finite family of simple disjoint loops α_j, $j = 1, \dots, q$. Suppose that there exists a system of simple loops $\{l_1, \dots, l_q\} = L \subset S$ such that:

(a) the grafting along L is possible for all $0 \leqq t < 1$,

(b) $i(\alpha_j, L) \neq 0$ for each j and

(c) the elements $\rho_1(l_j)$ are loxodromic.

Remark 4. There are examples when such system of curves does not exist, see Section 7.3.

Then the curves α_j are not pinched as $t \to 1$ in the family of complex structures $p(gr(c_t, L))$. Indeed, the limit of the complex structures $(S, p(c_t))$ (as $t \to 1$) is a stable singular curve S_1 where the loops α_j are pinched to singular points. The application of grafting to S_1 along L results in a nonsingular complex curve S_1'.

7.3. Example. Suppose that a family of representation $r_t\colon F \to \mathrm{SL}(2,\mathbb{C})$ consists of quasifuchsian representations for $0 \leqq t < 1$ so that $r_0(F)$ is a Fuchsian group that we shall identify with F.

Assume also that the image of r_1 is a "regular b-group", so that an element $a \in F$ is the only accidental parabolic element for r_1 (up to conjugation in F), see [16] for definitions. The discontinuity domain of F consists of two components D, D^*; suppose that D is the component such that the representation r_1 cannot be induced by a homeomorphism $f\colon D \subsetneq \Omega(r_1(F))$. However, for each $0 \leqq t < 1$ there are homeomorphisms $f_t\colon D \subsetneq \Omega(r_t(F))$ so that $r_t(\gamma) \circ f_t = f_t \circ \gamma$ for all $\gamma \in F$ and f_t depends continuously on t. Thus, f_t are developing maps for a family of complex projective structures c_t on S with the monodromy r_t. Let $\sigma_t = gr(c_t, A)$ where A is a simple loop on S representing a. Then, the families of structures c_t, σ_t degenerate as $t \to 1$ since the underlying complex structures are "pinched" along A. Denote by d_t the family of developing maps for σ_t. Let β be any simple homotopically nontrivial loop on S. For each component $\tilde{\beta}$ of $p^{-1}(\beta) \subset D$ the image $d_t(\tilde{\beta})$ is not a simple arc in $\bar{\mathbb{C}}$. Thus for each t the grafting of σ_t along β is impossible. Therefore, it is impossible to "regenerate" σ_t (as $t \to 1$) using grafting. There are two orientation classes of complex projective structures with the monodromy r_1. One can prove that any two structures with the monodromy r_1 and the same orientation can be related by a sequence of grafting and its inverse.

We shall discuss the problem of regeneration of complex projective structures in details in another paper.

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References

1. P. Appell, E. Goursat, P. Fatou: Théorie des Functions Algébraiques. Vol. 2, Paris, Gauthier-Villars 1930.
2. C. Banica, O. Stanasila: Algebraic methods in the global theory of complex spaces. London, New York: Wiley, 1976
3. J. Birman: Braids, links, and mapping class groups, Princeton University Press, 1975
4. S. Choi, W. Goldman: Convex real projective structures on closed surfaces are closed. Proceedings of Amercian Math. Society **118** (1993) N 4, 1227–1236
5. V. Chuckrow: Schottky groups with applications to Kleinian groups. Ann. Math., **88** (1968) 47–61
6. A. Douady, C. Earle: Conformally natural extension of homeomorphisms of the circle. Acta Math. **157** (1986) 23–48
7. D. Gallo: Complex projective structures with prescribed monodromy. Bull. AMS, **20** (1989) 31–34
8. D. Gallo, W. Goldman, M. Porter: Projective structures with monodromy in PSL(2, $\mathbb{R}$) (Preprint 1987)
9. W. Goldman: Topological components of spaces of representations. Invent. Math. **93** (1988) 557–607
10. W. Goldman: Projective structures with Fuchsian holonomy. J. Diff. Geom. **25** (1987) 297–326
11. L. Greenberg: Homomorphisms of triangle groups in PSL(2, $\mathbb{C}$). In: Riemann surfaces and related topics. Stony Brook, Ann. Math. Stud., vol. **97**, 1980
12. R. Gunning: Special coordinate coverings of Riemann Surface. Math. Ann. **170**, 67–86 (1967)
13. R. Gunning: On affine and complex projective structures. In: Riemann Surfaces and Related Topics. Stony Brook, Ann. Math. Stud., vol. 97, 1980
14. R. Gunning: Lectures on Vector bundles over Riemann Surfaces. Princeton University Press, 1967
15. D. Hejhal: Monodromy groups and linearly polymorphic functions. Acta Math. **135** (1975) 1–55
16. B. Maskit: Kleinian groups, Springer Verlag, 1987
17. B. Maskit: On a class of Kleinian groups. Ann. Ac. Sci. Fenn., Ser. A, **442** (1969) 1–8
18. S. Matsumoto: Foundations of flat conformal structures. Adv. Stud. Pure Appl. Math. **20** (1992), Aspects of Low Dimensional Manifolds, p. 167–251
19. M. Narasimhan, C. Seshadri: Stable and unitary vector bundles on a compact Riemann surface. Ann. Math. **82** (1965) 540–567
20. W. Thurston: Hyperbolic structures on 3-manifolds, II: Surface groups and 3-manifolds which fiber over the circle (Preprint)
21. W. Thurston: Hyperbolic structures on 3-manifolds, I, Ann. Math. **124** (1986), 203–246