# On the Ahlfors Finiteness Theorem

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The goal of this note is to give a proof of the Ahlfors Finiteness theorem which requires just the bare minimum of the complex analysis: (a) the existence theorem for the Beltrami equation and (b) the Rado-Cartan uniqueness theorem for holomorphic functions. However our proof does require some (by now standard) 3-dimensional topology and Greenberg's algebraic trick to deal with the triply-punctured spheres. The key ideas of the proof are due to Ahlfors [Ahl64] and Carleson & Gamelin [CG93, pp. 72–72].

For a Kleinian group $G$ we let $\Omega(G)$ denote the discontinuity domain of $G$ and $\Lambda(G)$ the limit set of $G$. In this note all Kleinian groups are assumed to be nonelementary, i.e. their limit sets are infinite. Under such assumption the limit set is known to be perfect, i.e. every point is an accumulation point. Recall that a Kleinian group $G$ is called *analytically finite* if $\Omega(G)/G$ is an orbifold of finite conformal type, i.e. has only finite number of cone points after removing which we get a surface conformally equivalent to a hyperbolic surface of finite area.

**Theorem 1.** *(L. Ahlfors.) Every finitely generated Kleinian group $G \subset PSL(2, \mathbb{C})$ is analytically finite.*

*Proof.* In what follows we will need the following boundary version of the uniqueness theorem for holomorphic functions.

**Theorem 2.** *(T.Rado, H. Cartan, see e.g. [Nar, Ch. 11, §8, Theorem 2].) Suppose that $f$ is a holomorphic function in a connected open subset $D \subset \mathbb{C}$ and the boundary $\partial D$ contains a nonisolated point $z_0$ so that the following holds. There exists a neighborhood $U$ of $z_0$ in $\mathbb{C}$ such that for each $z \in U \cap \partial D$,*

$$\lim_{w \to z} f(w) = 0.$$

*Then $f$ is identically zero in $D$.*

We now begin the proof the Ahlfors' theorem.

**Step 1.** Recall that according to Selberg Lemma, the group $G$ contains a finite index torsion-free subgroup $G'$. It is clear that $G'$ is finitely generated, hence analytical finiteness of $G'$ would imply analytical finiteness of $G$. Thus we assume that $G$ is torsion-free. We next note that it suffices to prove analytical finiteness for finitely generated Kleinian groups $G$ such that each component of $\Omega(G)$ is contractible. To prove this implication we apply the Loop Theorem to the pair

$$(M(G), S(G))$$

where $S(G) = \Omega(G)/G$ and $M(G) = (\mathbb{H}^3 \cup \Omega(G))/G$. The Loop Theorem implies that $S(G)$ is the conformal connected sum of a finite number of surfaces $S(G_j)$, $j = 1, ..., k$, where each component of $\Omega(G_j)$ is simply-connected for every $j$. If we know that each $S(G_j)$ has finite conformal type, this would imply that $S(G)$ has finite conformal type as well and we are done.

**Step 2.**

**Claim 3.** *Suppose that $G \subset Isom(\mathbb{H}^2)$ is a finitely generated Kleinian group such that $\Lambda(G) = \mathbb{S}^1$. Then the Riemann surface $\mathbb{H}^2/G$ is has finite conformal type (equivalently, this is a surface of finite hyperbolic area).*

*Proof.* It is elementary that (any) Dirichlet fundamental polygon $P$ of $G$ in $\mathbb{H}^2$ has finitely many sides (see for instance [CB88]). Since $\Lambda(G) = \mathbb{S}^1$ it follows that $P$ is a finitely-sided polygon of finite area, its accumulation set in $\mathbb{S}^1$ consists of a finite number of vertices. Now the claim trivially follows. ☐

**Step 3.** This is the most interesting part of the proof.

**Proposition 4.** *For each component $\Omega_0 \subset \Omega(G)$ the stabilizer $G_0$ of $\Omega_0$ in $G$ has the property: $\Lambda(G_0) = \partial\Omega_0$.*

*Proof.* Suppose the assertion is false. Then there exists a point $z_0 \in \partial\Omega_0 - \Lambda(G_0)$, moreover, a whole neighborhood $U$ of $z_0$ in $\mathbb{C}$ is disjoint from $\Lambda(G_0)$. It is clear that $z_0$ cannot be an isolated point of $\partial\Omega_0$ (since $\Lambda(G)$ is perfect).

We pick a base-point point $x \in \Omega_0$ which is not fixed by any element of $G_0$, then choose a sufficiently small disk $D_\epsilon \subset \Omega_0$ centered at $x$. (The disk is chosen so that its images under the elements of $G_0$ are disjoint.) Consider an infinite-dimensional space $V$ of quasiconformal homeomorphisms $f : D_\epsilon \to D_\epsilon$ which fix three distinct points $z_1, z_2, z_3$ in $\partial D_\epsilon$ and so that the restriction mapping

$$V \to Homeo(\partial D_\epsilon), \quad f \mapsto f|\partial D_\epsilon$$

is injective. (For instance, start with the infinite-dimensional space $W$ of piecewise-linear homeomorphisms $\eta : \partial D_\epsilon \to \partial D_\epsilon$ which fix the points $z_1, z_2, z_3$ and take $V$ to be the space of the radial extensions of $\eta$'s.) Let $\mu_f$ denote the Beltrami differential of $f \in V$. For each $f$ extend $\mu_f$ $G$-invariantly from $D_\epsilon$ to the $G$-orbit of this disk and by zero to the rest of the 2-sphere. We will use the notation $\nu_f$ for this extension. Let $h_f$ denote the normalized (at three limit points of $G$) solution of the Beltrami equation

$$\bar{\partial} h = \nu_f \partial h, \quad h_f = h.$$

**Claim 5.** *The mapping $A : f \mapsto h_f|\Lambda(G)$ is injective.*

*Proof.* Suppose that $f_1, f_2 \in V$ are such that $A(f_i)$ coincide, $i = 1, 2$. Let $h_i := h_{f_i}, i = 1, 2$. Recall that $\nu_i = \nu_{f_i}$ are zero on $\Sigma := \Omega_0 - G_0(D_\epsilon)$, hence each $h_i$ is conformal in that part of $\Omega_0$. On the other hand, the disks in $G_0(D_\epsilon)$ do not accumulate to the points of the set $U \cap \partial\Omega_0$ (since this set is disjoint from the limit set of $G_0$). Hence $U \cap \partial\Omega_0 = U \cap \partial\Sigma$ (provided that $D_\epsilon$ is sufficiently small).

Therefore the holomorphic function $(h_1 - h_2)|\Sigma$ tends to zero as $w \to z \in U \cap \partial\Sigma$. Applying Theorem 2 we conclude that the functions $h_1$ and $h_2$ are equal on $\Sigma$, in particular they are equal on $\partial D_\epsilon$. On the other hand, $h_i|D_\epsilon$ satisfy the same Beltrami equation as $f_i$. It follows that $h_i|D_\epsilon = \varphi_i \circ f_i$ for conformal mappings $\varphi_i$ of $D_\epsilon$ to the complex plane. Since both

$$h_2^{-1} h_1 = f_2^{-1} \varphi_2^{-1} \varphi_1 f_1 \quad \text{and} \quad f_2$$

preserve $D_\epsilon$ we get:

$$\varphi_2^{-1} \varphi_1 f_1 : D_\epsilon \to D_\epsilon.$$

Thus the mapping $\psi = \varphi_2^{-1} \varphi_1 : D_\epsilon \to D_\epsilon$ is a conformal automorphism. It follows that $\psi$ is the identity (since it fixes three distinct boundary points $z_1, z_2, z_3$). We conclude that $\varphi_1 = \varphi_2$ and hence

$$f_1|\partial D_\epsilon = \varphi_1^{-1} h_1|\partial D_\epsilon = \varphi_2^{-1} h_2|\partial D_\epsilon = f_2|\partial D_\epsilon.$$

Recall that $V$ is chosen so that if $f_1|\partial D_\epsilon = f_2|\partial D_\epsilon$ then $f_1 = f_2$. This proves injectivity of the mapping $A$. □

We now proceed as in the standard proof [Ahl64] of the Ahlfors finiteness theorem: the mapping $A$ determines an embedding of the infinite-dimensional space $V$ to the finite-dimensional algebraic variety $Hom(G, PSL(2, \mathbb{C}))$, which is a contradiction.

**Corollary 6.** *The surface $\Omega(G)/G$ contains no disks and annuli.*

*Proof.* If a component $S_0 = \Omega_0/G_0$ is a disk or an annulus then $G_0$ is either trivial or cyclic. This implies that the complement to $\Omega_0$ in $\mathbb{S}^2$ is either empty or consists of one or two points. In any case it follows that $G$ is elementary which contradicts our assumptions. □

**Step 4.** Suppose that $G$ is a finitely generated Kleinian group and $\Omega_0$ is a component of $\Omega(G)$ with the stabilizer $G_0$ in $G$. Then $\Omega_0/G_0$ is conformally equivalent to the quotient $\mathbb{H}^2/\Gamma_0$ where $\Gamma_0 \subset PSL(2, \mathbb{R})$ is a subgroup whose limit set is the whole boundary circle of $\mathbb{H}^2$.

*Proof.* Let $x \in \Omega_0$ be a base-point and let $R : D \to \Omega_0$ be the Riemann mapping from the unit disk to $\Omega_0$. We will identify $D$ with the hyperbolic plane $\mathbb{H}^2$. Recall that $R$ has radial limits a.e. on the boundary of $D$. Let $\Gamma_0 := R^{-1}G_0R \subset Isom(\mathbb{H}^2)$. It suffices to show that $\Lambda(\Gamma_0) = \mathbb{S}^1$. Suppose that the limit set of $\Gamma_0$ is a proper subset of the unit circle. Let $\gamma \subset \Omega(\Gamma_0) \cap \mathbb{S}^1$ be a (nondegenerate) arc. Since the Riemann mapping has radial limits a.e. in $\gamma$ take a pair of "generic" distinct points $p, q \in \gamma$ so that the hyperbolic geodesic $\alpha \subset \mathbb{H}^2$ connecting them is mapped by $R$ to a smooth arc $R(\alpha) \subset \Omega_0$ which has limit points $a, b \in \partial\Omega_0 \cap \mathbb{C}$, so that

$$a = \lim_{z \to p, z \in \alpha} R(z), \quad b = \lim_{z \to q, z \in \alpha} R(z).$$

I will use the notation $\gamma$ for the part of $\gamma$ between $p$ and $q$ since we will not need the rest of this arc. Let $H$ denote the half-plane in $\mathbb{H}^2$ bounded by $\alpha$ which is adjacent to $\gamma$, by choosing $\gamma$ sufficiently small we get: $x \notin R(H)$. We note that if $a = b$ and the topological circle $L := R(\alpha) \cup a$ bounds the open disk $R(H) \subset \Omega_0$, then the function $R(z) - a$ tends to zero on $\gamma$; this contradicts Theorem 2.

*Remark 7.* Alternatively one can use F. and M. Rees theorem (see e.g. [Nar]) for this part of the proof.

Therefore we can assume that either $a \neq b$ (Case 1) or $a = b$ and the topological circle $L = R(\alpha) \cup a$ bounds an open disk which contains $R(H)$ and a nonempty part $E$ of the limit set of $G_0$ (Case 2). In the former case the arc $\alpha$ separates a part $E$ of $\partial\Omega_0$ from the base-point $x$ (if $\gamma$ is chosen sufficiently small). See Figure 1.

Since $E \subset \Lambda(G_0)$, there exists a sequence $g_n \in G_0$ such that $\lim_n g_n(x) \in E$. Therefore all but finitely many members of the sequence $g_n(x)$ belong to $R(H)$. Let $h_n \in \Gamma_0$ be the elements corresponding to $g_n$ under the isomorphism $\Gamma_0 \to G_0$ induced by $R$, let $y := R^{-1}(x)$. Since $\gamma \cap \Lambda(\Gamma_0) = \emptyset$, only finitely many members of the sequence $h_n(y)$ belong to the half-plane $H$. Contradiction. □

**Step 5.** Now there are several ways to argue. One can refer to [KS89] which gives a purely topological proof (under the assumption that $S(G) = \Omega(G)/G$ contains no disks and annuli) that $S(G)$ has finite **topological** type (i.e. it has finite number of components each of which is homeomorphic to a compact surface with a finite number of disks removed). Given this, we conclude that $S(G) = \Omega(G)/G$ has finite conformal type.

Alternatively, one can repeat the deformation-theoretic argument, however it cannot exclude the possibility that $S(G)$ contains infinitely many triply punctured

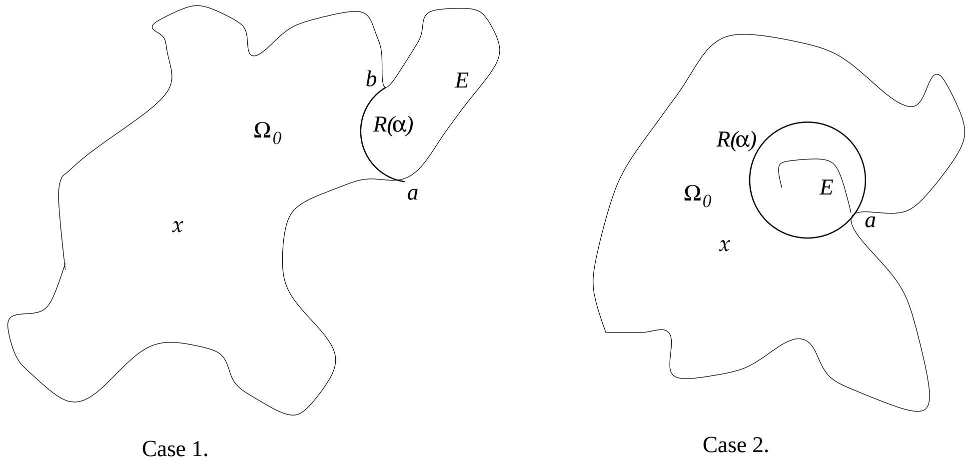


Figure 1:

spheres. To finish the proof one would have to use the algebraic trick of L. Greenberg [Gre67].

**The deformation-theoretic argument**. Suppose that there is a component $\Omega_0$ of $\Omega(G)$ such that $\Omega_0/G_0$ does not have finite topological type (where $G_0$ is the stabilizer of $\Omega_0$ in $G$). Then $\Omega_0/G_0$ is conformally equivalent to $S_0 = \mathbb{H}^2/\Gamma_0$ where $\Gamma_0$ is a non-finitely generated Kleinian group whose limit set is the whole boundary circle of $\Delta = \mathbb{H}^2$. The Teichmüller space $\mathcal{T}(S_0)$ of the surface $S_0$ is infinite-dimensional. Let $R : \mathbb{H}^2 \to \Omega_0$ be the Riemann mapping. This map has (distinct) radial limits $a_1, a_2, a_3$ at three distinct fixed points $b_1, b_2, b_3$ of hyperbolic elements of $\Gamma_0$ (since $R$ conjugates the Kleinian groups $\Gamma_0$ and $G_0$). We represent elements of $\mathcal{T}(S_0)$ by quasiconformal homeomorphisms $f : cl(\Delta) \to cl(\Delta)$ which fix the points $b_1, b_2, b_3$. These homeomorphisms form an infinite-dimensional Banach space $V$.

We now proceed as above, each $f \in V$ corresponds to a $G_0$-invariantBeltrami differential $\mu_f$. Extend $\mu_f$ to a $G$-invariant Beltrami differential $\nu_f$ on $\mathbb{S}^2$. Let $h_f$ denote the solution of the Beltrami equation $\bar{\partial} h_f = \nu_f \partial h_f$ normalized to fix the points $a_i, i = 1, 2, 3$. This determines a continuous mapping $A : V \to Hom(G, PSL(2, \mathbb{C}))$. We will show that the mapping $A$ is injective. Note that our normalization convention implies that each $h_f$ has the property:

$$R^{-1} \circ h_f \circ R = f$$

since the mappings $R^{-1} \circ h_f \circ R$ and $f$ differ by a conformal automorphism of $\mathbb{H}^2$ which fixes three distinct points $b_1, b_2, b_3$. Suppose $f_1, f_2 \in V$ are quasiconformal homeomorphisms such that $A(f_1) = A(f_2)$. Then the mapping $h := h_2^{-1} \circ h_1$ commutes with each element of $G$. Let $\theta : \Gamma_0 \to G_0$ denote the isomprphism induced by conjugation via $R$. Then the mapping $f := f_2^{-1} \circ f_1$ satisfies:

$$f = R^{-1} \circ h \circ R$$

and $f$ commutes with each element $\gamma$ of $\Gamma_0$:

$$f\gamma f^{-1} = (R^{-1}hR)\gamma(R^{-1}h^{-1}R) = R^{-1}h\theta(\gamma)h^{-1}R = R^{-1}\theta(\gamma)R = \gamma.$$

Hence $f_1, f_2$ represent the same point of the Teichmüller space $\mathcal{T}(S_0)$. This proves injectivity of $A$. Since $V$ is infinite-dimensional and $Hom(G, PSL(2,\mathbb{C}))$ is finite-dimenaional we get a contradiction.

This proves that each component of $S(G)$ has finite conformal type. Let $S(G)^*$ denote $S(G)$ with triply punctured spheres removed. To prove that $S(G)^*$ has finite number of components we have to repeat the same argument once again. If $S_0 = \Omega_0/G_0$ is not a triply-punctured sphere then the complex dimension of the Teichmüller space $\mathcal{T}(S_0)$ is at least 1. Let $S_i, i \in I$ denote the components of $S(G)$. Since (by the same arguments as above) the Bers mapping

$$A : \mathcal{T}(S(G)) \to \mathcal{T}(G) \subset Hom(G, PSL(2,\mathbb{C}))//PSL(2,\mathbb{C})$$

is injective we conclude that

$$\mathcal{T}(S(G)) = \prod_{i \in I} \mathcal{T}(S_i)$$

is finite-dimensional. Hence, $I$ is finite. □

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