European Journal of Mathematics
DOI 10.1007/s40879-017-0192-y

RESEARCH ARTICLE

Anosov subgroups: dynamical and geometric characterizations

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Received: 1 March 2017 / Revised: 4 October 2017 / Accepted: 8 October 2017

Abstract We study infinite covolume discrete subgroups of higher rank semisimple Lie groups, motivated by understanding basic properties of Anosov subgroups from various viewpoints (geometric, coarse geometric and dynamical). The class of Anosov subgroups constitutes a natural generalization of convex cocompact subgroups of rank one Lie groups to higher rank. Our main goal is to give several new equivalent characterizations for this important class of discrete subgroups. Our characterizations capture "rank one behavior" of Anosov subgroups and are direct generalizations of rank one equivalents to convex cocompactness. Along the way, we considerably simplify the original definition, avoiding the geodesic flow. We also show that the Anosov condition can be relaxed further by requiring only non-uniform unbounded expansion along the (quasi)geodesics in the group.

Keywords Discrete subgroups · Anosov subgroups · Symmetric spaces

The first author was partly supported by the NSF Grants DMS-12-05312 and DMS-16-04241. The first and second author are grateful to KIAS (the Korea Institute for Advanced Study) for its hospitality. The third author was partially supported by grant FEDER/Mineco MTM2015-66165-P.

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Published online: 30 October 2017

Mathematics Subject Classification 22E40 · 20F65 · 53C35

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1 Introduction

This paper is devoted to studying basic properties of Anosov subgroups of semisimple Lie groups from various viewpoints (geometric, coarse geometric and dynamical). The class of Anosov subgroups, introduced by Labourie [25] and further extended by Guichard and Wienhard [12], constitutes a natural generalization of convex cocompact subgroups of rank one Lie groups to higher rank. Our main goal here is to give several new equivalent characterizations for this important class of discrete subgroups, including a considerable simplification of their original definition. For convex cocompact subgroups as well as for word hyperbolic groups, it is very fruitful to have different viewpoints and alternative definitions, as they were developed by many authors starting with Ahlfors' work on geometric finiteness in the 60s, and later by Beardon, Maskit, Marden, Thurston, Sullivan, Bowditch and others. Besides a deeper understanding, it enables one to switch perspectives in a nontrivial way, adapted to the situation at hand. A main purpose of this paper is to demonstrate that much of this theory extends to Anosov subgroups, and we hope that the concepts and results presented here will be useful for their further study. In our related work, they lay the basis for the results on the Higher Rank Morse Lemma [21], compactifications of locally symmetric spaces for Anosov subgroups [15], the local-to-global principle and the construction of Morse–Schottky subgroups [20]. We refer to [22] for a survey of our work and to [16] for background discussion and many examples.

In rank one, among Kleinian groups and, more generally, among discrete subgroups of rank one Lie groups, one distinguishes *geometrically finite* subgroups. They form a large and flexible class of discrete subgroups which are strongly tied to the negatively curved symmetric spaces they act on. Therefore they have especially good geometric, topological and dynamical properties and one can prove many interesting results about them. The simplest are geometrically finite subgroups *without parabolics*, which lie at the root of this paper. They can be characterized in many (not obviously) equivalent ways: as convex cocompact subgroups, as undistorted subgroups, as subgroups with conical limit set, as subgroups which are expanding at their limit set, and as word hyperbolic subgroups with Gromov boundary equivariantly homeomorphic to their limit set, to name some.

In higher rank, a satisfying and sufficiently broad definition of geometric finiteness, with or without parabolics, remains yet to be found. Convex cocompactness turns out to be much too restrictive a condition: it was shown by Kleiner and the second author [24] that in higher rank only few subgroups are convex cocompact. Undistortion by itself, on the other hand, is way too weak: undistorted subgroups can even fail to be finitely presented. Thus, one is forced to look for suitable replacements of these notions in higher rank. It turns out that some of the other equivalent characterizations of convex cocompactness in rank one do admit useful modifications in higher rank, which lead to the class of Anosov subgroups. The Anosov condition is not too rigid and, at the same time, it imposes enough restrictions on the subgroups making it possible to analyze their geometric and dynamical properties. One way to think of Anosov subgroups is as geometrically finite subgroups without parabolics which exhibit some *rank one behavior*. Indeed, they are word hyperbolic and we will see that also extrinsically they display hyperbolic behavior in a variety of ways.

In this paper, we primarily consider four notions generalizing convex cocompactness to higher rank, all equivalent to the Anosov condition, see the Equivalence Theorem 1.1 below:

(i) asymptotic embeddedness,
(ii) expansivity,
(iii) conicality,
(iv) Morse property.

Whereas the Anosov condition and conditions (i) and (ii) are dynamical, (iii) is a condition on the asymptotic geometry of the subgroup, and (iv) is coarse geometric.

We now describe in more detail some of our concepts and results. Let $X = G/K$ be a symmetric space of noncompact type and, for simplicity, let the semisimple Lie group G be the connected component of its isometry group. Our approach to studying Anosov subgroups $\Gamma < G$ begins with the observation that they satisfy a strong form of discreteness which we call *regularity* and which is primarily responsible for their extrinsic "rank one behavior" alluded to above. Discreteness of a subgroup $\Gamma < G$ means that for sequences (γ_n) of distinct elements the distance $d(x, \gamma_n x)$ in X diverges to infinity. For higher rank symmetric spaces there is a natural vector-valued refinement d_Δ of the Riemannian distance d, which takes values in the euclidean Weyl chamber Δ of X. The regularity assumption on Γ, in its strongest form of σ_{mod}*-regularity*, means that $d_\Delta(x, \gamma_n x)$ diverges away from the boundary of Δ. We will work more generally with relaxations of this condition, called τ_{mod}-regularity, associated with a face τ_{mod} of the model spherical Weyl chamber σ_{mod}, where one only requires divergence of $d_\Delta(x, \gamma_n x)$ away from some of the faces of Δ, depending on τ_{mod}. To be precise, think of σ_{mod} as the visual boundary of the euclidean Weyl chamber, $\sigma_{mod} \cong \partial_\infty \Delta$. Given a face $\tau_{mod} \subseteq \sigma_{mod}$, we define τ_{mod}*-regularity* by requiring that $d_\Delta(x, \gamma_n x)$ diverges away from the faces of Δ whose visual boundaries do not contain τ_{mod}. We will also need the stronger notion of *uniform* τ_{mod}*-regularity* where one requires the divergence to be linear in terms of $d(x, \gamma_n x)$. Most of the discussion in this paper will take place within the framework of τ_{mod}-regular subgroups.

Classically, the asymptotic behavior of discrete subgroups $\Gamma < G$ is captured by their visual limit set $\Lambda(\Gamma)$ which is the accumulation set of their orbits $\Gamma x \subset X$ in the visual boundary $\partial_\infty X$. In our context of τ_{mod}-regular subgroups, the visual limit set is replaced by the τ_{mod}*-limit set* $\Lambda_{\tau_{mod}}(\Gamma)$ contained in the partial flag manifold $\mathrm{Flag}_{\tau_{mod}} = G/P_{\tau_{mod}}$ and defined as the accumulation set of Γ-orbits in the bordification $X \sqcup \mathrm{Flag}_{\tau_{mod}}$ of X, equipped with the topology of *flag convergence* (see Sect. 4.5). Here, $P_{\tau_{mod}}$ is a parabolic subgroup in the conjugacy class corresponding to τ_{mod}. The notion of τ_{mod}-limit set extends to arbitrary discrete subgroups.

We call a τ_{mod}-regular subgroup $\Gamma < G$ *nonelementary* if $|\Lambda_{\tau_{mod}}(\Gamma)| \geqslant 3$, and *antipodal* if it satisfies the visibility condition that any two distinct limit simplices in $\Lambda_{\tau_{mod}}(\Gamma)$ are antipodal. The latter means that they can be connected by a geodesic in X in the sense that the geodesic is asymptotic to interior points of the simplices. It is worth noting that the action of a τ_{mod}-regular antipodal subgroup on its τ_{mod}-limit set enjoys the classical convergence property, which is a typical rank one phenomenon.

Regularity, which is a condition on the asymptotic geometry of orbits in the symmetric space, can be converted into an equivalent dynamical condition about a certain contraction behavior of the subgroup on suitable flag manifolds (see Definition 4.1), allowing one to switch between geometry and dynamics. The contraction behavior here is a higher rank version of the classical convergence (dynamics) property in the theory of Kleinian groups. This yields an equivalent characterization of τ_{mod}-regular subgroups as τ_{mod}-*convergence subgroups* (see Definition 4.2). Also the limit sets, respectively, limit simplices can be defined purely dynamically as the possible limits of contracting sequences in Γ, i.e. of sequences converging to constants on suitable open and dense subsets of the flag manifolds, see Definition 4.25.

Much of the material in Sect. 4 can be found in some form already in the work of Benoist, see [3, Section 3], in the setting of Zariski dense subgroups of reductive algebraic groups over local fields, notably the notions of *regularity* and *contraction*, their essential equivalence, and the notion of *limit set*. For the sake of completeness we give independent proofs in our setting of discrete subgroups of semisimple Lie groups. Also our methods are rather different. We give here a geometric treatment and present the material in a form suitable to serve as a basis for the further development of our theory of discrete isometry groups acting on Riemannian symmetric spaces and euclidean buildings of higher rank, such as in our papers [15,19–21].

We now (mostly) restrict to the class of τ_{mod}-regular, equivalently, τ_{mod}-convergence subgroups and introduce various geometric and dynamical conditions in the spirit of geometric finiteness. We begin with three dynamical ones:

1. We say that a subgroup $\Gamma < G$ is τ_{mod}-*asymptotically embedded* if it is an antipodal τ_{mod}-convergence subgroup, Γ is word hyperbolic and there exists a Γ-equivariant homeomorphism

$$\alpha : \partial_\infty \Gamma \xrightarrow{\cong} \Lambda_{\tau_{\mathrm{mod}}}(\Gamma) \subset \mathrm{Flag}_{\tau_{\mathrm{mod}}}$$

from its Gromov boundary onto its τ_{mod}-limit set.

This condition can be understood as a continuity at infinity property for the orbit maps $o_x : \Gamma \to \Gamma x \subset X$: By extending an orbit map o_x to infinity by the boundary map α, one obtains a continuous map

$$o_x \sqcup \alpha : \Gamma \sqcup \partial_\infty \Gamma \to X \sqcup \mathrm{Flag}_{\tau_{\mathrm{mod}}}$$

from the Gromov compactification of Γ (see Proposition 5.16).

2. Our next condition is inspired by Sullivan's notion of *expanding actions* [29]. Following Sullivan, we call a subgroup $\Gamma < G$ *expanding at infinity* if its action on the appropriate partial flag manifold is expanding at the limit set. More precisely:

We call a τ_{mod}-convergence subgroup $\Gamma < G$ τ_{mod}-*expanding at the limit set* if for every limit flag in $\Lambda_{\tau_{\mathrm{mod}}}(\Gamma)$ there exist a neighborhood U in $\mathrm{Flag}_{\tau_{\mathrm{mod}}}$ and an element $\gamma \in \Gamma$ which is uniformly expanding on U, i.e. for some constant $c > 1$ and all $\tau_1, \tau_2 \in U$ it holds that

$$d(\gamma\tau_1, \gamma\tau_2) \geqslant c \cdot d(\tau_1, \tau_2).$$

Here, and in what follows the distance d is induced by a fixed Riemannian background metric on the flag manifold. Now we can formulate our second condition:

We say that a subgroup $\Gamma < G$ is τ_{mod}-*CEA* (*Convergence Expanding Antipodal*) if it is an antipodal τ_{mod}-convergence subgroup which is expanding at the limit set.

We note that the CEA condition does not a priori assume word hyperbolicity, not even finite generation.

3. The next condition is motivated by the original definition of Anosov subgroups. It is a hybrid of the previous two definitions, where we weaken asymptotic embeddedness (to boundary embeddedness) and strengthen expansivity. We drop the regularity/convergence assumption and, accordingly, make no use of the limit set in our definition. Compared to asymptotic embeddedness, we keep the word hyperbolicity of the subgroup but, instead of identifying its Gromov boundary with the limit set as in asymptotic embeddedness, we only require a boundary map embedding the Gromov boundary into the flag manifold. Compared to CEA, we require a stronger form of expansivity, now at the image of the boundary map.

We call a subgroup $\Gamma < G$ τ_{mod}*-boundary embedded* if Γ is word hyperbolic and there exists a Γ-equivariant continuous embedding

$$\beta\colon \partial_\infty\Gamma \to \mathrm{Flag}_{\tau_{\mathrm{mod}}}$$

sending distinct visual boundary points to antipodal simplices. If Γ is virtually cyclic, we require in addition that it is discrete in G. (Otherwise, discreteness is a consequence.) We will refer to β as a *boundary embedding*. In general, boundary embeddings are not unique.

The *infinitesimal expansion factor* of an element $g \in G$ at a simplex $\tau \in \mathrm{Flag}_{\tau_{\mathrm{mod}}}$ is

$$\epsilon(g, \tau) = \min_u |dg(u)|$$

where the minimum is taken over all unit tangent vectors $u \in T_\tau \mathrm{Flag}_{\tau_{\mathrm{mod}}}$, again using the Riemannian background metric.

Now we can formulate our version of the Anosov condition:

We say that a subgroup $\Gamma < G$ is τ_{mod}*-Anosov* if it is τ_{mod}-boundary embedded with boundary embedding β and satisfies the following expansivity condition: For every ideal point $\zeta \in \partial_\infty\Gamma$ and every normalized (by $r(0) = e \in \Gamma$) discrete geodesic ray $r\colon \mathbb{N} \to \Gamma$ asymptotic to ζ, the action $\Gamma \curvearrowright \mathrm{Flag}_{\tau_{\mathrm{mod}}}$ satisfies

$$\epsilon(r(n)^{-1}, \beta(\zeta)) \geqslant Ae^{Cn}$$

for $n \geqslant 0$ with constants $A, C > 0$ independent of r. (Here, we fix a word metric on Γ.)

The uniformity of expansion in this definition can be significantly weakened:

We say that a subgroup $\Gamma < G$ is *non-uniformly* τ_{mod}*-Anosov* if it is τ_{mod}-boundary embedded with boundary embedding β and, for every ideal point $\zeta \in \partial_\infty\Gamma$ and every discrete geodesic ray $r\colon \mathbb{N} \to \Gamma$ asymptotic to ζ, the action $\Gamma \curvearrowright \mathrm{Flag}_{\tau_{\mathrm{mod}}}$ satisfies

$$\sup_{n\in\mathbb{N}} \epsilon(r(n)^{-1}, \beta(\zeta)) = +\infty.$$

The original definition of Anosov subgroups in [12,25] is rather involved. It is based on geodesic flows for word hyperbolic groups and formulated in terms of expansion/contraction properties for lifted flows on associated bundles over the geodesic flow spaces (see Sect. 5.11). Our definition requires only an expansion property for the group action on a suitable flag manifold and avoids using the geodesic flow, whose construction is highly technical for word hyperbolic groups which do not arise as the fundamental group of a closed negatively curved Riemannian manifold. The geodesic flow is replaced by a simpler coarse geometric object, the space of quasigeodesics.

Now we come to the geometric notions.

4. The first geometric condition concerns the *orbit asymptotics*. The notion of conicality of limit simplices, due to Albuquerque [1, Definition 5.2], generalizes a well-known condition from the theory of Kleinian groups: In the case $\tau_{\mathrm{mod}} = \sigma_{\mathrm{mod}}$, a limit chamber $\sigma \in \Lambda_{\sigma_{\mathrm{mod}}}(\Gamma)$ of a σ_{mod}-regular subgroup $\Gamma < G$ is called *conical* if there exists a sequence $\gamma_n \to \infty$ in Γ such that for a(ny) point $x \in X$ the sequence of orbit points $\gamma_n x$ is contained in a tubular neighborhood of the euclidean Weyl chamber $V(x, \sigma)$ with tip x and asymptotic to σ. For general τ_{mod} and limit simplices $\tau \in \Lambda_{\tau_{\mathrm{mod}}}(\Gamma)$ of τ_{mod}-regular subgroups $\Gamma < G$, one replaces the euclidean Weyl chamber with the Weyl cone $V(x, \mathrm{st}(\tau))$ over the star of τ, that is, by the union of the euclidean Weyl chambers $V(x, \sigma)$ for all spherical Weyl chambers $\sigma \supset \tau$. A τ_{mod}-regular subgroup $\Gamma < G$ is called *conical* if all limit simplices are conical. Here is our fourth condition:

We say that a subgroup $\Gamma < G$ is τ_{mod}-*RCA* if it is τ_{mod}-regular, conical and antipodal.

For nonelementary τ_{mod}-regular antipodal subgroups, this extrinsic notion of conicality is equivalent to an intrinsic one defined in terms of the dynamics on the τ_{mod}-limit set (Proposition 5.34), which enables one to relate it to the dynamical notions above.

5. The last set of definitions concerns the *coarse extrinsic geometry*. We recall that a finitely generated subgroup $\Gamma < G$ is *undistorted* if the orbit maps $\Gamma \to X$ are quasiisometric embeddings. They then send discrete geodesics in Γ (with respect to a fixed word metric) to uniform quasigeodesics in X. Undistortion by itself is too weak a restriction, compared with the other notions defined previously. We will strengthen it in two ways. The first is by adding uniform regularity:

We say that a subgroup $\Gamma < G$ is τ_{mod}-*URU* if it is uniformly τ_{mod}-regular and undistorted.

According to the classical *Morse Lemma* in negative curvature, quasigeodesic segments in rank one symmetric spaces are uniformly Hausdorff close to geodesic segments with the same endpoints. This is no longer true in higher rank because it fails already in euclidean plane. Another way of strengthening undistortion is therefore by imposing a "Morse" type property on the quasigeodesics arising as orbit map images of the discrete geodesics in Γ.

As in the case of conicality above, where one replaces rays with Weyl cones when passing from rank one to higher rank, it is natural to replace geodesic segments with "diamonds" in a higher rank version of the Morse property. (This is suggested, for instance, by the geometry of free Anosov subgroups, see our examples of Morse–Schottky subgroups in [20] and [16, Section 3.10].) We define diamonds as follows: If $\tau_{\mathrm{mod}} = \sigma_{\mathrm{mod}}$ and xy is a σ_{mod}-regular segment, then the σ_{mod}-*diamond* with tips x, y is the intersection

$$\diamondsuit(x, y) = V(x, \sigma) \cap V(y, \widehat{\sigma})$$

of the euclidean Weyl chambers with tips at x and y containing xy. In the case of general τ_{mod}, the euclidean Weyl chambers are replaced with τ_{mod}-Weyl cones (see Sect. 2.5.3).

We say that a subgroup $\Gamma < G$ is τ_{mod}-*Morse* if it is τ_{mod}-regular, Γ is word hyperbolic and an(y) orbit map $o_x : \Gamma \to \Gamma x \subset X$ satisfies the following Morse condition: The images $o_x \circ s$ of discrete geodesic segments $s : [n_-, n_+] \cap \mathbb{Z} \to \Gamma$ are contained in uniform tubular neighborhoods of τ_{mod}-diamonds with tips uniformly close to the endpoints of $o_x \circ s$ (see Definition 5.21).

The definition does not a priori assume undistortion, but we show in this paper that Morse implies URU. That, conversely, URU implies Morse may seem unexpected at first but follows from our Higher Rank Morse Lemma for regular quasigeodesics [21].

We now arrive at our main result on the equivalence of various conditions introduced above. We state it for nonelementary subgroups because we use this assumption in some of our proofs.

Equivalence Theorem 1.1 *The following properties for subgroups $\Gamma < G$ are equivalent in the nonelementary case:*

(i) τ_{mod}-*asymptotically embedded,*
(ii) τ_{mod}-*CEA,*
(iii) τ_{mod}-*Anosov,*
(iv) *non-uniformly* τ_{mod}-*Anosov,*
(v) τ_{mod}-*RCA,*
(vi) τ_{mod}-*Morse.*

These properties imply τ_{mod}-*URU. Moreover, the boundary maps for properties* (i)*,* (iii) *and* (iv) *coincide.*

Here, "nonelementary" means $|\partial_\infty \Gamma| \geqslant 3$ in the Anosov conditions (iii) and (iv), which assume word hyperbolicity but no τ_{mod}-regularity, and means $|\Lambda_{\tau_{\mathrm{mod}}}(\Gamma)| \geqslant 3$ in all other cases.

Remark 1.2 (i) We prove in [21] that, conversely, τ_{mod}-URU implies τ_{mod}-Morse (without assuming nonelementary).

(ii) All implications between properties (i)–(vi) hold without assuming nonelementary, with the exception of (ii) ⇒ (v) ⇒ (i). In particular, the properties (i), (iii), (iv), (vi) and τ_{mod}-URU are equivalent in general.

(iii) The implication Anosov ⇒ URU had been known before [12].

(iv) Some of the implications in the theorem can be regarded as a description of geometric and dynamical properties of Anosov subgroups. Different characterizations of Anosov subgroups are useful in different contexts. For example:

Expansivity (ii) is used in [18, 19] to establish the cocompactness of Γ-actions on suitable domains of discontinuity in flag manifolds. Asymptotic embeddedness is used in [15] to construct Finsler compactifications of locally symmetric spaces for Anosov subgroups. The Morse property is used in [20] to prove a *local-to-global principle* for Anosov subgroups. The latter in turn leads to new proofs of openness and structural stability of Anosov representations, to a construction of free Anosov subgroups (Morse–Schottky subgroups), and to the semidecidability of Anosovness, see [20].

(v) In our paper [15] we establish two more characterizations of Anosov subgroups among uniformly regular subgroups, namely as *coarse retracts* and by *S-cocompactness*. The former property is a strengthening of undistortion. The latter means the existence of a certain kind of compactification of the corresponding locally symmetric space.

(vi) Other characterizations of Anosov subgroups can be found in [11].

Remark 1.3 Boundary embeddedness appears to be a considerable weakening of asymptotic embeddedness, even in the regular case. Nevertheless two results in this paper establish a close relation between the two concepts:

(i) For σ_{mod}-regular subgroups, boundary embeddedness, conversely, implies asymptotic embeddedness, while the boundary embedding may have to be modified (see Theorem 5.11).

(ii) For general τ_{mod}-regular subgroups, there is the following *dichotomy* for boundary embeddings (see Theorem 5.7) which is useful for verifying asymptotic embeddedness:
Either the image of the boundary embedding equals the τ_{mod}-limit set and the subgroup is asymptotically embedded. Or the image is disjoint from the limit set, and the limit set is not Zariski dense. The latter cannot happen for Zariski dense subgroups.

While the main results in this paper concern discrete subgroups of Lie groups, in Sect. 5.10, motivated by the Morse property, we discuss *Morse quasigeodesics* and *Finsler geodesics*. We characterize Morse subgroups as word hyperbolic subgroups whose intrinsic geodesics are extrinsically[1] uniform Morse quasigeodesics. Furthermore, we characterize Morse quasigeodesics as bounded perturbations of Finsler geodesics. Lastly, we analyze the Δ-distance along Finsler geodesics and Morse quasigeodesics. We show that, via the Δ-distance function, they project to Finsler geodesics and Morse quasigeodesics in Δ.

Most of the results in this paper were already contained in Chapters 1–6 of the preprint [20], however the presentation in this paper is more efficient. The further material on the Morse property in [20, Section 7] will appear elsewhere.

Organization of the paper. In Sect. 2 we first go through some standard material on symmetric spaces and spherical buildings and then, in Sects. 2.5–2.12, mostly develop concepts specific to our work, such as *stars at infinity*, *Weyl cones* and *diamonds*; these objects will appear frequently throughout the paper. In Sect. 3 we review several

[1] with respect to the ambient symmetric space

general notions from topological and smooth dynamics, notably *convergence actions* and *expansivity*. These notions will be applied in the context of dynamics of discrete isometry groups of symmetric spaces. In Sect. 4 we introduce two closely related notions, *regularity* and *contraction*, for sequences of isometries of symmetric spaces, specifying *divergence to infinity* in higher rank. Contraction is a higher rank analogue of the classical *convergence property*. Using these notions we define *limit sets* in flag manifolds. Section 5 is the brain of the paper. We introduce and analyze various notions generalizing the rank one concept of convex-cocompactness and prove their equivalence. In the Appendix to this section we verify that our simplified definition of Anosov subgroups is equivalent to the original one.

2 Geometry of symmetric spaces

In this section, we collect some material from the geometry of symmetric spaces and buildings. We explain the notions which are most important for the purposes of this paper, establish notation and give proofs for some of the less standard facts. No attempt of a complete review is made. For more detailed discussions, we refer the reader to [2,7,23] and [26].

We give a brief description of where various parts of this section are used in the paper:

Sections 2.2–2.5 are used essentially everywhere.

While the vector valued distance function d_Δ is used in many places in the paper, the rest of the material in Sects. 2.6 and 2.7 is used primarily in Sect. 2.9.1 on the separation of nested Weyl cones and in Sect. 5.10 where we analyze projections of Morse quasigeodesics to the euclidean model Weyl chamber Δ.

The material of Sect. 2.9 dealing with shadows at infinity is used in Sect. 4.4 when we prove the equivalence of *regularity* and *contractivity* for sequences of isometries of X. The main result of Sect. 2.9.1 on the separation of nested Weyl cones is used in Sect. 5.3 to prove that Morse subgroups are URU (Theorem 5.18).

The main results of Sects. 2.10 and 2.11 are Theorem 2.41 and Proposition 2.42 establishing estimates for the contraction and expansion of isometries of X acting on flag manifolds. (The other results are used only in Sects. 2.10 and 2.11). Theorem 2.41 and Proposition 2.42 are used in Sects. 5.7 and 5.8 while discussing discrete subgroups satisfying expansion properties (CEA and Anosov).

The material of Sect. 2.12 is used only in Sect. 5.10 where it is proven that Morse quasigeodesics are uniformly closed to Finsler geodesics and that Δ-distance projections of Finsler geodesics are again Finsler geodesics.

2.1 General metric space notation

We will use the notation $B(p,r)$ for the open r-ball with center p in a metric space, and $\overline{B}(p,r)$ for the closed r-ball.

A *tubular neighborhood* of a subset A in a metric space (Z,d) is a subset of the form

$$N_R(A) = \{z \in Z : d(z, A) < R\}$$

for some $R > 0$, its *radius*.

A *geodesic* in (Z, d) is an isometric embedding $I \to Z$ from a (possibly infinite) interval $I \subset \mathbb{R}$. In the context of finitely generated groups equipped with word metrics, we will also work with *discrete geodesics*; these are isometric embeddings from intervals $I \cap \mathbb{Z}$ in $\mathbb{Z}$. The notion of *discrete quasigeodesic* will be used similarly.

2.2 Spherical buildings

Spherical buildings occur in this paper as the visual boundaries of symmetric spaces of noncompact type, equipped with their structures of thick spherical Tits buildings.

2.2.1 Spherical geometry

Let S be a unit sphere in a euclidean space, and let $\sigma \subset S$ be a spherical simplex with dihedral angles $\leqslant \frac{\pi}{2}$. Then $\operatorname{diam}(\sigma) \leqslant \frac{\pi}{2}$.

For a face $\tau \subseteq \sigma$, we define the τ*-boundary* $\partial_\tau \sigma$ as the union of faces of σ which do not contain τ, and the τ*-interior* $\operatorname{int}_\tau(\sigma)$ as the union of open faces of σ whose closure contains τ. We obtain the decomposition

$$\sigma = \operatorname{int}_\tau(\sigma) \sqcup \partial_\tau \sigma.$$

If $\tau' \subset \tau$, then $\partial_{\tau'}\sigma \subset \partial_\tau \sigma$ and $\operatorname{int}_{\tau'}(\sigma) \supset \operatorname{int}_\tau(\sigma)$. Note that $\partial_\sigma \sigma = \partial \sigma$ and $\operatorname{int}_\tau(\sigma) = \operatorname{int}(\sigma)$.

We need the following fact about projections of spherical simplices to their faces:

Lemma 2.1 *The nearest point projection* $\operatorname{int}_\tau(\sigma) \to \operatorname{int}(\tau)$ *is well-defined.*

In other words, for every point $x \in \operatorname{int}_\tau(\sigma)$ there exists a point $p \in \operatorname{int}(\tau)$ such that $px \perp \tau$. In view of $\operatorname{diam}(\sigma) \leqslant \frac{\pi}{2}$, this point is necessarily unique.

Proof We argue by induction on the dimension of σ. Let $x \in \operatorname{int}_\tau(\sigma)$. We apply the induction assumption to the link $\Sigma_v\sigma$ at a vertex v of τ. Note that $\partial_{\Sigma_v\tau} \Sigma_v\sigma = \Sigma_v\partial_\tau\sigma$. Since $\overrightarrow{vx} \in \operatorname{int}_{\Sigma_v\tau}(\Sigma_v\sigma)$, the nearest point projection $\overline{\delta}$ of this direction to $\Sigma_v\tau$ is contained in $\operatorname{int}(\Sigma_v\tau)$ and has angle $< \frac{\pi}{2}$ with $\overrightarrow{vx}$. It follows that the nearest point projection p of x to τ is different from v and lies on the arc in direction $\overline{\delta}$, $\overrightarrow{vp} = \overline{\delta}$. In particular, it is not contained in a face of τ with vertex v. Letting run v through the vertices of τ, we conclude that $p \in \operatorname{int}(\tau)$. □

As a consequence of the lemma, the nearest point projection $\operatorname{int}_\tau(\sigma) \to \tau$ agrees with the nearest point projection $\operatorname{int}_\tau(\sigma) \to s$ to the geodesic sphere $s \subset S$ spanned by τ (i.e. containing τ as a top-dimensional subset), and its image equals $\operatorname{int}(\tau)$.

2.2.2 Spherical Coxeter complexes

A *spherical Coxeter complex* (a_{mod}, W) consists of a unit sphere (in a euclidean space) a_{mod} and a finite reflection group W acting isometrically on a_{mod}. We will refer to a_{mod} as the *model apartment* (because it will serve as the model for apartments in spherical buildings, see below).

A *wall* in a_{mod} is the fixed point set of a reflection in W. A *half-apartment* is a closed hemisphere in a_{mod} bounded by a wall. A *singular sphere* in a_{mod} is an intersection of walls.

A *chamber* in a_{mod} is the closure of a connected component of the complement of the union of the walls. The group W acts transitively on the set of chambers. The chambers are simplices with diameter $\leqslant \frac{\pi}{2}$ iff W fixes no point in a_{mod}, equivalently, the Coxeter complex does not split off a spherical join factor (in the category of Coxeter complexes). In this case, the collection of chambers defines on a_{mod} the structure of a simplicial complex, the simplices being intersections of chambers.

Every chamber is a fundamental domain for the action $W \curvearrowright a_{mod}$. The *spherical model chamber* can be defined as the quotient $\sigma_{mod} = a_{mod}/W$. We identify it with a chamber in the model apartment, $\sigma_{mod} \subset a_{mod}$, which we refer to as the *fundamental chamber*.

We call the natural projection

$$\theta \colon a_{mod} \to a_{mod}/W \cong \sigma_{mod}$$

the *type* map for a_{mod}. It restricts to an isometry on every chamber. A *face type* is a face of σ_{mod}. The type of a simplex $\overline{\tau} \subset a_{mod}$ is then defined as $\theta(\overline{\tau})$. Throughout the paper, we will use the notation $\tau_{mod}, \tau'_{mod}, \nu_{mod}, \nu'_{mod}, \ldots$ for face types. Furthermore, we will denote by $W_{\tau_{mod}} \leqslant W$ the stabilizer of the face type $\tau_{mod} \subseteq \sigma_{mod}$.

The *longest element* of the Weyl group is the unique element $w_0 \in W$ sending σ_{mod} to the opposite chamber $-\sigma_{mod}$. The *standard involution* (also known as the *opposition involution*) of the model chamber is given by

$$\iota := -w_0 \colon \sigma_{mod} \to \sigma_{mod}. \tag{1}$$

2.2.3 Spherical buildings

A *spherical building* modeled on a Coxeter complex (a_{mod}, W) is a CAT(1) metric space B equipped with a collection of isometric embeddings $\kappa \colon a_{mod} \to$ B, called *charts*. The image of a chart is an *apartment* in B. One requires that any two points are contained in an apartment and that the coordinate changes between charts are induced by isometries in W. (The precise axioms can be found e.g. in [23,26].) We will use the notation $\angle$ for the metric on B.

We assume that W fixes no point, equivalently, that σ_{mod} is a simplex with diameter $\leqslant \frac{\pi}{2}$.

Via the atlas of charts, the spherical building inherits from the spherical Coxeter complex a natural structure of a *simplicial complex* where the simplices are the images of the simplices in the model apartment. As already mentioned, the images of the charts are called *apartments*. Accordingly, the images of chambers (walls, half-apartments, singular spheres) in a_{mod} are called chambers (walls, half-apartments, singular spheres) in the building. The codimension one faces are called *panels*. The *interior* $\mathrm{int}(\tau)$ of a simplex τ is obtained by removing all proper faces; the interiors of simplices are called *open simplices*. The simplex *spanned* by a point is the smallest simplex containing it, equivalently, the simplex containing the point in its interior. We will sometimes denote the simplex spanned by ξ by τ_ξ.

A spherical building is *thick* if every wall is the bounds of at least three half-apartments, equivalently, if every panel is adjacent to (i.e. contained in the boundary of) at least three chambers. One can always pass to a thick spherical building structure by reducing the Weyl group, thereby coarsifying the simplicial structure.

The *space of directions* $\Sigma_\xi \mathrm{B}$ at a point $\xi \in \mathrm{B}$ is the space of germs $\overrightarrow{\xi\eta}$ of nondegenerate geodesic segments $\xi\eta \subset \mathrm{B}$, equipped with the natural angle metric $\angle_\xi$. Two segments $\xi\eta$ and $\xi\eta'$ represent the same direction in $\Sigma_\xi \mathrm{B}$, $\overrightarrow{\xi\eta} = \overrightarrow{\xi\eta'}$, iff they initially agree. The space of directions is again a spherical building.

A subset $C \subset \mathrm{B}$ is called (π-)*convex* if for any two points $\xi, \eta \in C$ with distance $\angle(\xi, \eta) < \pi$ the (unique) geodesic $\xi\eta$ connecting ξ and η in B is contained in C.

Due to the compatibility of charts, i.e. the property of the building atlas that the coordinate changes are induced by isometries in W, there is a well-defined *type* map

$$\theta\colon \mathrm{B} \to \sigma_{\mathrm{mod}}.$$

It is 1-Lipschitz and restricts to an isometry on every chamber $\sigma \subset \mathrm{B}$. We call the inverse $\kappa_\sigma = (\theta|_\sigma)^{-1}\colon \sigma_{\mathrm{mod}} \to \sigma$ the *chart* of the chamber σ. For a simplex $\tau \subset \mathrm{B}$, we call the face $\theta(\tau) \subseteq \sigma_{\mathrm{mod}}$ the *type* of the simplex and $\kappa_\tau = (\theta|_\tau)^{-1}\colon \theta(\tau) \to \tau$ its *chart*. We define the *type* of a point $\xi \in \mathrm{B}$ as its image $\theta(\xi) \in \sigma_{\mathrm{mod}}$. A point $\xi \in \mathrm{B}$ is called *regular* if its type is an interior point of σ_{mod}, $\xi \in \mathrm{int}(\sigma_{\mathrm{mod}})$, and *singular* otherwise.

We will sometimes say that a singular sphere has type τ_{mod} if it contains a top-dimensional simplex of type τ_{mod}. (A singular sphere has in general several types.)

For a singular sphere $s \subset \mathrm{B}$, we define $\mathrm{B}(s) \subset \mathrm{B}$ as the union of all apartments containing s. It is a convex subset and splits off s as a spherical join factor. Moreover, $\mathrm{B}(s)$ is a subbuilding, i.e. it inherits from B a spherical building structure modeled on the same Coxeter complex; the apartments of $\mathrm{B}(s)$ are precisely the apartments of B containing s. This building structure is however not thick, except in degenerate cases. In order to pass to a thick spherical building structure, take a maximal atlas of charts $\kappa\colon a_{\mathrm{mod}} \to \mathrm{B}(s)$ for which the maps $\kappa^{-1}|_s\colon s \to a_{\mathrm{mod}}$ coincide, and reduce the Weyl group to the pointwise stabilizer of s in W.

Two points $\xi, \widehat{\xi} \in \mathrm{B}$ are *antipodal* or *opposite* if $\angle(\xi, \widehat{\xi}) = \pi$, equivalently, if they are antipodal in one (every) apartment containing them. We then define the singular sphere $s(\xi, \widehat{\xi}) \subset \mathrm{B}$ *spanned* by the points $\xi, \widehat{\xi}$ as the smallest singular sphere containing them. Moreover, we define the *suspension* $\mathrm{B}(\xi, \widehat{\xi}) \subset \mathrm{B}$ of $\{\xi, \widehat{\xi}\}$ as the union of all geodesics connecting ξ and $\widehat{\xi}$, equivalently, as the union of all apartments containing ξ and $\widehat{\xi}$. Then $\mathrm{B}(\xi, \widehat{\xi}) = \mathrm{B}(s(\xi, \widehat{\xi}))$. As above, a thick spherical building structure on $\mathrm{B}(\xi, \widehat{\xi})$ is obtained by taking all charts $\kappa\colon a_{\mathrm{mod}} \to \mathrm{B}(\xi, \widehat{\xi})$ so that $\kappa^{-1}(\xi) = \theta(\xi) \in \sigma_{\mathrm{mod}}$, and reducing the Weyl group to the stabilizer of $\theta(\xi)$ in W.

Similarly, one defines *antipodal* or *opposite faces* $\tau, \hat{\tau} \subset B$ as faces which are antipodal in the apartments containing them both, equivalently, whose interiors contain a pair of antipodal points $\xi \in \mathrm{int}(\tau)$ and $\hat{\xi} \in \mathrm{int}(\hat{\tau})$. We define the singular sphere $s(\tau, \hat{\tau}) \subset B$ *spanned* by the simplices $\tau, \hat{\tau}$ again as the smallest singular sphere containing them, and the suspension $B(\tau, \hat{\tau})$ as the union of all apartments containing $\tau \cup \hat{\tau}$; then $s(\tau, \hat{\tau}) = s(\xi, \hat{\xi})$ and $B(\tau, \hat{\tau}) = B(\xi, \hat{\xi})$.

We will need some facts about antipodes. Recall that in a spherical building B every point $\xi \in B$ has an antipode in every apartment $a \subset B$, and hence for every simplex $\tau \subset B$ there exists an opposite simplex $\hat{\tau} \subset a$, cf. e.g. the first part of [23, Lemma 3.10.2]. We need the more precise statement that a point has *several* antipodes in an apartment unless it lies itself in this apartment:

Lemma 2.2 *Suppose that* $\xi \in B$ *has only one antipode in the apartment* $a \subset B$. *Then* $\xi \in a$.

Proof Suppose that $\xi \notin a$ and let $\hat{\xi} \in a$ be an antipode of ξ. We choose a "generic" segment $\xi\hat{\xi}$ of length π tangent to a at $\hat{\xi}$ as follows. The suspension $B(\xi, \hat{\xi})$ contains an apartment a' with the same unit tangent sphere at $\hat{\xi}$, $\Sigma_{\hat{\xi}} a' = \Sigma_{\hat{\xi}} a$. Inside a' there exists a segment $\xi\hat{\xi}$ whose interior does not intersect simplices of codimension $\geqslant 2$. Hence $\hat{\xi}\xi$ leaves a at an interior point $\eta \neq \xi, \hat{\xi}$ of a panel $\pi \subset a$, i.e. $a \cap \xi\hat{\xi} = \eta\hat{\xi}$ and $\pi \cap \xi\hat{\xi} = \eta$, and $\eta\xi$ initially lies in a chamber adjacent to π but not contained in a. Let $s \subset a$ be the wall containing π. By reflecting $\hat{\xi}$ at s, one obtains a second antipode for ξ in a. □

In thick buildings, simplices can be represented as intersections of apartments:

Lemma 2.3 *In a thick spherical building* B, *any simplex* $\tau \subset B$ *equals the intersection of the apartments containing it.*

Proof Since every simplex is an intersection of chambers, we are reduced to the case when τ is a chamber. Furthermore, since every chamber is an intersection of half-apartments, we are reduced to the corresponding assertion for half-apartments. The latter holds by thickness. □

2.3 Hadamard manifolds

In this section only, X denotes a Hadamard manifold, i.e. a simply connected complete Riemannian manifold with nonpositive sectional curvature. We will use the notation $\mathrm{Isom}(X)$ for the full isometry group of X.

Any two points in X are connected by a unique geodesic segment. We will use the notation xy for the oriented geodesic segment connecting x to y.

For points $x \neq y, z$ we denote by $\angle_x(y, z)$ the angle of the geodesic segments xy and xz. Furthermore, we denote by $\Sigma_x X$ the *space of directions* of X at x equipped with the angle metric $\angle_x$. It coincides with the unit tangent sphere at x.

A basic feature of Hadamard manifolds is the *convexity of the distance function*: Given any pair of geodesics $c_1(t), c_2(t)$ in X, the function $t \mapsto d(c_1(t), c_2(t))$ is convex.

Two geodesic rays $\rho_1, \rho_2 \colon [0, +\infty) \to X$ are called *asymptotic* if the convex function $t \mapsto d(\rho_1(t), \rho_2(t))$ on $[0, +\infty)$ is bounded, and they are called *strongly asymptotic* if $d(\rho_1(t), \rho_2(t)) \to 0$ as $t \to +\infty$.

Two geodesic lines $l_1, l_2 \subset X$ are *parallel* if they have finite Hausdorff distance. Equivalently, $l_1 \cup l_2$ bounds a flat strip in X.

The *ideal* or *visual boundary* $\partial_\infty X$ of X is the set of asymptote classes of geodesic rays in X. Points in $\partial_\infty X$ are called *ideal points*. For $x \in X$ and $\xi \in \partial_\infty X$ we denote by $x\xi$ the unique geodesic ray emanating from x and asymptotic to ξ, i.e. representing the ideal point ξ. There are natural identifications $\log_x \colon \partial_\infty X \to \Sigma_x X$ sending the ideal point ξ to the direction $\overrightarrow{x\xi}$.

The *cone* or *visual topology* on $\partial_\infty X$ is characterized by the property that the maps $\log_x$ are homeomorphisms with respect to it. Thus, $\partial_\infty X$ is homeomorphic to the sphere of dimension $\dim(X) - 1$. The visual topology has a natural extension to $\overline{X} = X \sqcup \partial_\infty X$ which can be described as follows in terms of sequential convergence: A sequence (x_n) in $\overline{X}$ converges to an ideal point $\xi \in \partial_\infty X$ iff, for some (any) base point $x \in X$, the sequence of geodesic segments or rays xx_n converges to the ray $x\xi$ (in the pointed Hausdorff topology with base points at x). This topology makes $\overline{X}$ into a closed ball. We define the *visual boundary* of a subset $A \subset X$ as the set $\partial_\infty A = \overline{A} \cap \partial_\infty X$ of its accumulation points at infinity.

The visual boundary $\partial_\infty X$ carries the natural *Tits angle metric* $\angle_{\text{Tits}}$ defined as

$$\angle_{\text{Tits}}(\xi, \eta) = \sup_{x \in X} \angle_x(\xi, \eta),$$

where $\angle_x(\xi, \eta)$ is the angle between the geodesic rays $x\xi$ and $x\eta$. The *Tits boundary* $\partial_{\text{Tits}} X$ is the metric space $(\partial_\infty X, \angle_{\text{Tits}})$. The Tits metric is lower semicontinuous with respect to the visual topology and, accordingly, the *Tits topology* induced by the Tits metric is finer than the visual topology. It is discrete if there is an upper negative curvature bound, and becomes nondiscrete if X contains nondegenerate flat sectors. For instance, the Tits boundary of flat r-space is the unit $(r-1)$-sphere, $\partial_{\text{Tits}} \mathbb{R}^r \cong S^{r-1}(1)$. An isometric embedding $X \to Y$ of Hadamard manifolds induces an isometric embedding $\partial_{\text{Tits}} X \to \partial_{\text{Tits}} Y$ of their Tits boundaries.

We will be using the visual topology on $\partial_\infty X$, unless explicitly said otherwise.

Let $\xi \in \partial_\infty X$ be an ideal point. For a geodesic ray $\rho \colon [0, +\infty) \to X$ asymptotic to ξ one defines the *Busemann function* b_ξ on X as the uniform monotonic limit

$$b_\xi(x) = \lim_{t \to +\infty} (d(x, \rho(t)) - t).$$

Along the ray, we have

$$b_\xi(\rho(t)) = -t.$$

Altering the ray ρ changes b_ξ by an additive constant. The point at infinity ξ thus determines b_ξ up to an additive constant. To remove this ambiguity, given $x \in X$, we define $b_{\xi,x}$ to be the Busemann function $b_{\xi,x}$ normalized at the point x by $b_{\xi,x}(x) = 0$.

The Busemann function b_ξ is convex, 1-Lipschitz and measures the relative distance from the ideal point ξ. The sublevel sets

$$\mathrm{Hb}_{\xi,x} := \{b_\xi \leqslant b_\xi(x)\} \subset X$$

are called (closed) *horoballs* centered at ξ. As sublevel sets of convex functions, they are convex. The visual boundaries of horoballs are $\frac{\pi}{2}$-balls at infinity with respect to the Tits metric,

$$\partial_\infty \mathrm{Hb}_{\xi,x} = \overline{B}\left(\xi, \frac{\pi}{2}\right) := \left\{\angle_{\mathrm{Tits}}(\xi, \cdot) \leqslant \frac{\pi}{2}\right\} \subset \partial_\infty X.$$

The level sets

$$\mathrm{Hs}_{\xi,x} := \{b_\xi = b_\xi(x)\} = \partial \mathrm{Hb}_{\xi,x}$$

are called *horospheres* centered at ξ.

As convex Lipschitz functions, Busemann functions are *asymptotically linear* along rays. If $\rho\colon [0, +\infty) \to X$ is a geodesic ray asymptotic to $\eta \in \partial_\infty X$, $\rho(+\infty) = \eta$, then

$$\lim_{t\to+\infty} \frac{b_\xi(\rho(t))}{t} = -\cos \angle_{\mathrm{Tits}}(\xi, \eta).$$

2.4 Symmetric spaces of noncompact type: basic concepts

In this section, we go through some well-known material and establish notation. Standard references are [2,7].

A symmetric space, denoted by X throughout this paper, is said to be of *noncompact type* if it is nonpositively curved and has no euclidean factor. In particular, it is a Hadamard manifold. We will write the symmetric space as

$$X = G/K$$

where G is a connected[2] semisimple Lie group with finite center acting isometrically and transitively on X, and $K < G$ is a maximal compact subgroup. The natural epimorphism $G \to \mathrm{Isom}(X)_o$ then has compact kernel. Every connected semisimple Lie group with finite center occurs in this way. The Lie group G carries a natural structure of a real algebraic group.

By the definition of symmetric spaces, in every point $x \in X$ there is a *point reflection* or *Cartan involution*, that is, an isometry σ_x which fixes x and has differential $-\mathrm{id}_{T_x X}$ in x.

A *transvection* of X is an isometry which is the product $\sigma_{x'}\sigma_x$ of two point reflections; it preserves the oriented geodesic through x and x' and the parallel vector fields

[2] What is really needed is a weaker property than connectedness, namely that G has finitely many connected components and acts on the Tits building of X by (type preserving) automorphisms. The latter is equivalent to the triviality of the G-action on the model chamber σ_{mod}, equivalently, on the Dynkin diagram. Under this assumption, the theory of discrete subgroups presented in this paper goes through unchanged.

along it. The transvections preserving a geodesic line $c(t)$ form a one parameter subgroup (T_t^c) of $\mathrm{Isom}(X)_o$ where T_t^c denotes the transvection mapping $c(s) \mapsto c(s+t)$.

An isometry ϕ of X is called *axial* if it preserves a geodesic l and does not fix l pointwise. Thus, ϕ acts as a nontrivial translation on l. (Note that an axial isometry need not be a transvection.) The geodesic l is called an *axis* of ϕ. Axes are in general not unique, but they are parallel to each other. For each axial isometry ϕ, the displacement function $x \mapsto d(x, \phi(x))$ on X attains its minimum on the convex subset of X which is the union of axes of ϕ. An isometry ϕ of X is *parabolic* if

$$\inf_{x \in X} d(x, \phi(x)) = 0$$

but g does not fix a point in X. Isometries fixing points are called *elliptic*.

A *flat* in X is a complete totally geodesic flat submanifold, equivalently, a convex subset isometric to a euclidean space. A maximal flat in X is a flat which is not contained in any larger flat; we will use the notation F for maximal flats. The group $\mathrm{Isom}(X)_o$ acts transitively on the set of maximal flats; the common dimension of maximal flats is called the *rank* of X. The space X has rank one if and only if it has strictly negative sectional curvature.

A maximal flat F is preserved by all transvections along geodesic lines contained in it. In general, there exist nontrivial isometries of X fixing F pointwise. The subgroup of isometries of F which are induced by elements of G is isomorphic to a semidirect product $W_{\mathrm{aff}} := \mathbb{R}^r \rtimes W$, the *affine Weyl group*, where r is the rank of X. The subgroup $\mathbb{R}^r$ acts simply transitively on F by translations. The linear part W is a finite reflection group, called the *Weyl group* of G and X. Since maximal flats are equivalent modulo G, the action $W_{\mathrm{aff}} \curvearrowright F$ is well-defined up to isometric conjugacy.

We will think of the Weyl group as acting on a *model flat* $F_{\mathrm{mod}} \cong \mathbb{R}^r$ fixing the origin $0 \in F_{\mathrm{mod}}$, and on its visual boundary sphere at infinity, the *model apartment* $a_{\mathrm{mod}} = \partial_{\mathrm{Tits}} F_{\mathrm{mod}} \cong S^{r-1}$. The pair (a_{mod}, W) is the *spherical Coxeter complex* associated to G. We identify the *euclidean model Weyl chamber* Δ with the complete cone $V(0, \sigma_{\mathrm{mod}}) \subset F_{\mathrm{mod}}$ with tip in the origin and visual boundary the *spherical model Weyl chamber* $\sigma_{\mathrm{mod}} \subset a_{\mathrm{mod}}$.

For every maximal flat $F \subset X$, we have an induced Tits isometric embedding $\partial_\infty F \subset \partial_\infty X$ of its visual boundary sphere. The natural identification $F \cong F_{\mathrm{mod}}$, unique up to the action of W_{aff}, induces a natural identification $\partial_\infty F \cong a_{\mathrm{mod}}$, unique up to the action of W.

The Coxeter complex structure on a_{mod} induces simplicial structures on the visual boundary spheres $\partial_\infty F$ of the maximal flats $F \subset X$. The spheres $\partial_\infty F$ cover $\partial_\infty X$, and their simplicial structures are compatible (i.e. the intersections are simplicial and the simplicial structures on the intersections agree). One thus obtains a G-invariant piecewise spherical *simplicial structure* on $\partial_\infty X$ which makes $\partial_\infty X$ into a *thick spherical building* and, also taking into account the visual topology, into a *topological* spherical building. It is called the *spherical* or *Tits building* $\partial_{\mathrm{Tits}} X$ associated to X. The Tits metric is the path metric with respect to the piecewise spherical structure, unless $\mathrm{rank}(X) = 1$, in which case $\partial_{\mathrm{Tits}} X$ is discrete with distance π between distinct points. We will sometimes refer to the simplices in $\partial_{\mathrm{Tits}} X$ also as *faces*. The visual boundaries of the maximal flats in X are precisely the *apartments* in $\partial_\infty X$, which in turn are precisely the convex subsets isometric, with respect to the Tits metric, to the unit sphere S^{r-1}.

We call a flat $f \subset X$ *singular* if it is the intersection of maximal flats. Its visual boundary $\partial_\infty f$ is then a *singular sphere* in $\partial_\infty X$.

We define the *Weyl sector* $V = V(x, \tau) \subset X$ with tip x and asymptotic to a simplex $\tau \subset \partial_\infty X$ as the union of rays $x\xi$ for the ideal points $\xi \in \tau$. Weyl sectors are contained in flats; they are isometric images of Weyl sectors $V(0, \tau_{\text{mod}}) \subset \Delta$ under charts $F_{\text{mod}} \to X$. These apartment charts restrict to canonical *sector charts* $\kappa_{x,\tau} = \kappa_{V(x,\tau)} : V(0, \tau_{\text{mod}}) \to V(x, \tau)$; at infinity, they induce simplex charts, $\partial_\infty \kappa_{x,\tau} = \kappa_\tau$.

If $\sigma \subset \partial_\infty X$ is a chamber, the sector $V(x, \sigma)$ is a *euclidean Weyl chamber*.

For a flat $f \subset X$, the *parallel set* $P(f) \subset X$ is the union of all flats $f' \subset X$ *parallel* to f, equivalently, with the same visual boundary sphere $\partial_\infty f' = \partial_\infty f$. The parallel set is a symmetric subspace and splits as the metric product

$$P(f) \cong f \times \mathrm{CS}(f) \tag{2}$$

of f and a symmetric space $\mathrm{CS}(f)$ called the *cross section*. The latter has no euclidean factor iff f is singular. Accordingly, the Tits boundary metrically decomposes as the spherical join

$$\partial_{\text{Tits}} P(f) \cong \partial_{\text{Tits}} f \circ \partial_{\text{Tits}} \mathrm{CS}(f).$$

It coincides with the subbuilding $(\partial_{\text{Tits}} X)(\partial_\infty f) \subset \partial_{\text{Tits}} X$ consisting of the union of all apartments in $\partial_\infty X$ containing $\partial_\infty f$, see Sect. 2.2.3.

For a singular sphere $s \subset \partial_\infty X$, we define the parallel set $P(s) \subset X$ as the union of the (necessarily singular) flats $f \subset X$ with visual boundary sphere $\partial_\infty f = s$, i.e. $P(s) = P(f)$; we denote its cross section by $\mathrm{CS}(s)$. For a pair of opposite points $\xi, \widehat{\xi} \in \partial_\infty X$, we define $P(\xi, \widehat{\xi}) \subset X$ as the parallel set of the singular sphere $s(\xi, \widehat{\xi}) \subset \partial_\infty X$ spanned by them, $P(\xi, \widehat{\xi}) = P(s(\xi, \widehat{\xi}))$. Similarly, for a pair of opposite simplices $\tau, \widehat{\tau} \subset \partial_\infty X$, we define $P(\tau_-, \tau_+) = P(s(\tau_-, \tau_+))$.

The action $G \curvearrowright \partial_\infty X$ on ideal points is not transitive if $\mathrm{rank}(X) \geqslant 2$. However, every G-orbit meets every chamber exactly once. The quotient is naturally identified with the spherical model chamber, and the projection

$$\theta : \partial_\infty X \to \partial_\infty X / G \cong \sigma_{\text{mod}}$$

is the *type* map, cf. Sect. 2.2.3.

A nondegenerate geodesic segment $xy \subset X$ is called *regular* if the unique geodesic ray $x\xi$ extending xy is asymptotic to a regular ideal point $\xi \in \partial_\infty X$.

Two ideal points $\xi, \eta \in \partial_\infty X$ are *antipodal*, $\angle_{\text{Tits}}(\xi, \eta) = \pi$, iff there exists a geodesic line $l \subset X$ asymptotic to them, $\partial_\infty l = \{\xi, \eta\}$. Their types are then related by $\theta(\xi_2) = \iota(\theta(\xi_1))$, where ι is the standard involution of σ_{mod}, see (1).

We say that two simplices $\tau_1, \tau_2 \subset \partial_\infty X$ are x-*antipodal* or x-*opposite* if $\tau_2 = \sigma_x \tau_1$, using the induced action of the point reflection σ_x on $\partial_\infty X$. Two simplices τ_1, τ_2 are opposite iff they are x-opposite for some point $x \in X$. Their types are then related by

$\theta(\tau_2) = \iota(\theta(\tau_1))$. We will frequently use the notation τ, $\hat{\tau}$ and $\tau_\pm$ for pairs of antipodal simplices. A pair of opposite chambers $\sigma_\pm$ is contained in a unique apartment, which we will denote by $a(\sigma_-, \sigma_+)$. It is the visual boundary of a unique maximal flat $F(\sigma_-, \sigma_+) \subset X$.

We will sometimes say that a singular flat $f \subset X$ has *type* τ_{mod} if its visual boundary $\partial_\infty f$ has type τ_{mod}, i.e. contains a top-dimensional simplex of type τ_{mod}. (A singular flat has in general several types.) The set $\mathcal{F}_{\tau_{mod}}$ of singular flats of type τ_{mod} is a homogeneous G-manifold. The flats of type σ_{mod} are the maximal flats and we denote $\mathcal{F} = \mathcal{F}_{\sigma_{mod}}$. A family of flats in $\mathcal{F}_{\tau_{mod}}$ is *bounded* if these flats intersect a fixed bounded subset of X.

Also, we will sometimes call the parallel set $P(s)$ of a singular sphere $\subset \partial_\infty X$ of *type* τ_{mod} or a τ_{mod}-*parallel set* if s has type τ_{mod}.

The stabilizers $P_\tau < G$ of the simplices $\tau \subset \partial_\infty X$ are the *parabolic subgroups* of G. The space $\mathrm{Flag}_{\tau_{mod}}$ of simplices of type τ_{mod} is called a (generalized) (*partial*) *flag manifold*. The action $G \curvearrowright \mathrm{Flag}_{\tau_{mod}}$ is transitive and we can write the flag manifold as a quotient $\mathrm{Flag}_{\tau_{mod}} \cong G/P_{\tau_{mod}}$, where $P_{\tau_{mod}}$ stands for a parabolic subgroup in the conjugacy class of parabolic subgroups P_τ of type $\theta(\tau) = \tau_{mod}$. Flag manifolds are compact smooth manifolds; they admit natural structures of projective real algebraic varieties (see e.g. [14, p. 160]). The topology on flag manifolds induced by the visual topology on $\partial_\infty X$ agrees with their manifold topology as homogeneous G-spaces. We will always use this topology. For ideal points $\xi \in \partial_\infty X$ with type $\theta(\xi) \in \mathrm{int}(\tau_{mod})$, there is a natural G-equivariant homeomorphic identification of the G-orbit $G\xi \subset \partial_\infty X$ with $\mathrm{Flag}_{\tau_{mod}}$ by assigning to the point $g\xi$ the (unique) simplex of type τ_{mod} containing it.

The flag manifolds $\mathrm{Flag}_{\tau_{mod}}$ and $\mathrm{Flag}_{\iota\tau_{mod}}$ are *opposite* in the sense that the simplices opposite to simplices of type τ_{mod} have type $\iota\tau_{mod}$. To ease notation, we will denote the pair of opposite flag manifolds also by $\mathrm{Flag}_{\pm\tau_{mod}}$ whenever convenient, i.e. we put $\mathrm{Flag}_{+\tau_{mod}} := \mathrm{Flag}_{\tau_{mod}}$ and $\mathrm{Flag}_{-\tau_{mod}} := \mathrm{Flag}_{\iota\tau_{mod}}$. The latter is also reasonable, because the simplices $-\tau_{mod}, \iota\tau_{mod} \subset a_{mod}$ lie in the same W-orbit, i.e. $-\tau_{mod}$ has type $\iota\tau_{mod}$. (Here we extend the notion of type to the model apartment, defining the type of a simplex in a_{mod} as its image under the natural quotient projection $a_{mod} \to a_{mod}/W \cong \sigma_{mod}$.) Similarly, we will use the notation $P_{\pm\tau_{mod}}$ for a pair of parabolic subgroups fixing opposite simplices in $\mathrm{Flag}_{\pm\tau_{mod}}$.

The stabilizers $B_\sigma < G$ of the chambers $\sigma \subset \partial_\infty X$ are the *minimal parabolic subgroups*[3] of G; they are conjugate. The space $\partial_{F\ddot{u}} X := \mathrm{Flag}_{\sigma_{mod}}$ of chambers is called the (generalized) *full flag manifold* or *Furstenberg boundary* of X, and we can write $\partial_{F\ddot{u}} X = G/B$, where again B stands for a minimal parabolic subgroup.

For a simplex $\hat{\tau} \in \mathrm{Flag}_{\iota\tau_{mod}}$ we define the *open Schubert stratum* $C(\hat{\tau}) \subset \mathrm{Flag}_{\tau_{mod}}$ as the subset of simplices opposite to $\hat{\tau}$; it is the open and *dense* $P_{\hat{\tau}}$-orbit. With respect to the algebraic structure on $\mathrm{Flag}_{\tau_{mod}}$, it is Zariski open, i.e. its complement is a proper subvariety.

We note that, if $\mathrm{rank}(X) = 1$, then there is only one flag manifold, namely $\partial_\infty X$, and the open Schubert strata are the complements of points.

[3] When the group G is complex, the minimal parabolic subgroups are the *Borel subgroups*, which is why we use the notation B for these subgroups.

2.5 Stars, cones and diamonds

2.5.1 Stars and suspensions

We first work inside the spherical model chamber σ_{mod}. We recall from Sect. 2.2.1 that, for a face type $\tau_{mod} \subseteq \sigma_{mod}$, the τ_{mod}-*boundary* $\partial_{\tau_{mod}}\sigma_{mod}$ of σ_{mod} is the union of the faces of σ_{mod} which do not contain τ_{mod}. The τ_{mod}-*interior* $\mathrm{int}_{\tau_{mod}}(\sigma_{mod})$ of σ_{mod} is the union of the open faces of σ_{mod} whose closure contains τ_{mod}. There is the decomposition

$$\sigma_{mod} = \mathrm{int}_{\tau_{mod}}(\sigma_{mod}) \sqcup \partial_{\tau_{mod}}\sigma_{mod}.$$

In particular, $\mathrm{int}_{\sigma_{mod}}(\sigma_{mod}) = \mathrm{int}\,\sigma_{mod}$ and $\partial_{\sigma_{mod}}\sigma_{mod} = \partial\sigma_{mod}$.

We say that a type in σ_{mod} is τ_{mod}-*regular* if it lies in $\mathrm{int}_{\tau_{mod}}(\sigma_{mod})$.

Now let B be a spherical building. As before, we assume that $\mathrm{diam}(\sigma_{mod}) \leqslant \frac{\pi}{2}$. A point $\xi \in B$ is called τ_{mod}-*regular* if its type is, $\theta(\xi) \in \mathrm{int}_{\tau_{mod}}(\sigma_{mod})$. We will *quantify* τ_{mod}-regularity as follows: Given a compact subset $\Theta \subset \mathrm{int}_{\tau_{mod}}(\sigma_{mod})$, we will say that a τ_{mod}-regular point $\xi \in B$ is Θ-*regular* if $\theta(\xi) \in \Theta$.

It will often be natural to impose a convexity property on Θ:

Definition 2.4 (*Weyl convex*) A subset $\Theta \subseteq \sigma_{mod}$ is τ_{mod}-*Weyl convex* if its symmetrization $W_{\tau_{mod}}\Theta \subset a_{mod}$ is convex.

Let $\tau \subset B$ be a simplex of type τ_{mod}. The τ_{mod}-*star* $\mathrm{st}(\tau) \subset B$ is the union of all chambers containing τ. Its boundary $\partial\,\mathrm{st}(\tau)$ is the union of all simplices in $\mathrm{st}(\tau)$ which do not contain τ; it consists of the points in $\mathrm{st}(\tau)$ with type in $\partial_{\tau_{mod}}\sigma_{mod}$. The *open* τ_{mod}-*star* $\mathrm{ost}(\tau)$ is the complement $\mathrm{ost}(\tau) = \mathrm{st}(\tau) - \partial\,\mathrm{st}(\tau)$; it consists of the τ_{mod}-regular points in $\mathrm{st}(\tau)$ and is open in B. For any simplex $\hat\tau$ opposite to τ, the star $\mathrm{st}(\tau)$ is contained in the suspension $B(\tau, \hat\tau)$.

Furthermore, we define the Θ-*star* $\mathrm{st}_\Theta(\tau) \subset \mathrm{ost}(\tau)$ as the subset of points with type Θ, that is, $\mathrm{st}_\Theta(\tau) = \mathrm{st}(\tau) \cap \theta^{-1}(\Theta)$.

We will use the following *separation* property: If $\angle(\Theta, \partial_{\tau_{mod}}\sigma_{mod}) \geqslant \epsilon > 0$, then $\mathrm{ost}(\tau)$ contains the open ϵ-neighborhood of $\mathrm{st}_\Theta(\tau)$.

Note that for chambers σ we have $\mathrm{st}(\sigma) = \sigma$ and $\mathrm{ost}(\sigma) = \mathrm{int}(\sigma)$.

The next result implies that stars are *convex*:

Lemma 2.5 (Convexity of stars)

(i) $\mathrm{st}(\tau)$ *is an intersection of simplicial* $\frac{\pi}{2}$*-balls.*

(ii) *For any simplex* $\hat\tau$ *opposite to* τ*, the star* $\mathrm{st}(\tau)$ *is an intersection of the suspension* $B(\tau, \hat\tau)$ *with simplicial* $\frac{\pi}{2}$*-balls containing* $\mathrm{st}(\tau)$ *and centered at points in* $B(\tau, \hat\tau)$.

Proof (i) Let $\sigma \not\subset \mathrm{st}(\tau)$ be a chamber, and let a be an apartment containing σ and τ. We can separate σ and $\mathrm{st}(\tau) \cap a$ by a wall in a, i.e. there exists a half-apartment $h \subset a$ which contains $\mathrm{st}(\tau) \cap a$ but not σ. Indeed, choose points $\xi \in \mathrm{int}(\tau)$ and $\eta \in \mathrm{int}(\sigma)$ such that the segment $\xi\eta$ intersects $\partial\sigma$ in a panel, and take the wall containing this panel. The simplicial $\frac{\pi}{2}$-ball with the same center as h then contains $\mathrm{st}(\tau)$ but not σ.

(ii) Note first that $\mathrm{st}(\tau) \subset \mathrm{B}(\tau, \widehat{\tau})$. Then we argue as in part (i), observing that if $\sigma \subset \mathrm{B}(\tau, \widehat{\tau})$ then a can be chosen inside $\mathrm{B}(\tau, \widehat{\tau})$. □

We extend convexity to Θ-stars:

Lemma 2.6 (Convexity of Θ-stars) *Let $\Theta \subseteq \sigma_{mod}$ be τ_{mod}-Weyl convex, and let τ be a simplex of type τ_{mod}. Then $\mathrm{st}_\Theta(\tau)$ is an intersection of $\frac{\pi}{2}$-balls.*

Proof For any apartment $a \supset \tau$, the intersection $\mathrm{st}_\Theta(\tau) \cap a$ is convex, as a consequence of the Weyl convexity of Θ.

Let $\zeta \in \mathrm{B}$. Every point in $\mathrm{st}_\Theta(\tau)$ lies in an apartment $a \supset \tau, \zeta$.

For any two apartments $a, a' \supset \tau, \zeta$ there exists an isometry $a \to a'$ fixing τ and ζ. (This follows from the compatibility of apartment charts axiom in the definition of spherical buildings.) It carries $\mathrm{st}_\Theta(\tau) \cap a$ to $\mathrm{st}_\Theta(\tau) \cap a'$. Hence, $\overline{B}(\zeta, \frac{\pi}{2})$ contains the first intersection iff it contains the second. Letting a' vary, it follows that $\overline{B}(\zeta, \frac{\pi}{2})$ contains $\mathrm{st}_\Theta(\tau)$ iff it contains $\mathrm{st}_\Theta(\tau) \cap a$.

Let $\xi \notin \mathrm{st}_\Theta(\tau)$. Then there is an apartment $a \supset \tau, \xi$ and, due to the convexity of $\mathrm{st}_\Theta(\tau) \cap a$, a point $\zeta \in a$ such that $\overline{B}(\zeta, \frac{\pi}{2})$ contains $\mathrm{st}_\Theta(\tau) \cap a$ but not ξ. By the above, $\mathrm{st}_\Theta(\tau) \subseteq \overline{B}(\zeta, \frac{\pi}{2})$. □

In the following, we restrict ourselves to the case $\mathrm{B} = \partial_\infty X$ and, besides the metric, also take into account the visual topology on the flag manifolds $\mathrm{Flag}_{\tau_{mod}}$. The discussion readily generalizes to arbitrary *topological* spherical buildings.

The τ_{mod}*-regular part* $\partial_\infty^{\tau_{mod}\text{-reg}} X$ of the visual boundary equals the union of the open τ_{mod}-stars. The natural projection

$$\partial_\infty^{\tau_{mod}\text{-reg}} X = \bigcup_{\tau \in \mathrm{Flag}_{\tau_{mod}}} \mathrm{ost}(\tau) \to \mathrm{Flag}_{\tau_{mod}} \tag{3}$$

assigns to every τ_{mod}-regular point $\xi \in \partial_\infty X$ the unique simplex $\tau \in \mathrm{Flag}_{\tau_{mod}}$ so that $\xi \in \mathrm{ost}(\tau)$.

Lemma 2.7 *The projection* (3) *is continuous.*

Proof Since both domain and target are manifolds, and thus metrizable, it suffices to verify sequential continuity. Suppose that $\xi_n \to \xi$ is a convergent sequence in $\partial_\infty^{\tau_{mod}\text{-reg}} X$, and let $\tau_n, \tau \in \mathrm{Flag}_{\tau_{mod}}$ be the images under the projection, i.e. the (unique) simplices so that $\xi_n \in \mathrm{ost}(\tau_n)$ and $\xi \in \mathrm{ost}(\tau)$. We must show that $\tau_n \to \tau$.

Let $\sigma_n \in \mathrm{Flag}_{\sigma_{mod}}$ be chambers (in general non-unique) containing the ξ_n. Then $\xi_n \in \sigma_n \supseteq \tau_n$. Due to the compactness of flag manifolds, we may assume after extraction that we have convergence $\tau_n \to \tau'$ and $\sigma_n \to \sigma'$. Then $\xi \in \sigma' \supseteq \tau'$, i.e. $\xi \in \mathrm{ost}(\tau')$, and hence $\tau' = \tau$. Again by compactness of $\mathrm{Flag}_{\tau_{mod}}$, it follows that $\tau_n \to \tau$ also before extraction. □

One can show that the projection (3) is a fiber bundle, but this fact will not be needed.

Let $\tau \in \mathrm{Flag}_{\tau_{mod}}$ and let $\widehat{\tau}$ be opposite to τ. Then τ is the *only* simplex in $\mathrm{B}(\tau, \widehat{\tau})$ which is opposite to $\widehat{\tau}$. In other words, the closed subset

$$\{\tau' \in \mathrm{Flag}_{\tau_{mod}} : \tau' \subset \mathrm{B}(\tau, \widehat{\tau})\} \tag{4}$$

intersects the open Schubert stratum $C(\widehat{\tau})$ in the single point τ, which is therefore an *isolated* point of this subset.

We know that $\mathrm{ost}(\tau)$ is an open subset of $\mathrm{B}(\tau, \widehat{\tau})$ with respect to the (Tits) metric.

Lemma 2.8 (Open stars) $\mathrm{ost}(\tau)$ *is open in* $\mathrm{B}(\tau, \widehat{\tau})$ *also with respect to the visual topology.*

Proof Consider the fiber bundle (3). The union U of the open τ_{mod}-stars over the simplices in $C(\widehat{\tau})$ is open in $\partial_\infty X$. Since τ is an isolated point of (4), the suspension $\mathrm{B}(\tau, \widehat{\tau})$ intersects U precisely in $\mathrm{ost}(\tau)$, which is therefore open in the suspension. □

2.5.2 Cones and parallel sets

We transfer notions about stars by coning off. Our discussion takes place in X and F_{mod}.

Consider first the euclidean model chamber $\Delta = V(0, \sigma_{\mathrm{mod}})$. Its τ_{mod}-*boundary*

$$\partial_{\tau_{\mathrm{mod}}} \Delta := V(0, \partial_{\tau_{\mathrm{mod}}} \sigma_{\mathrm{mod}}) \subseteq \partial \Delta$$

is the union of the faces which do not contain the face $V(0, \tau_{\mathrm{mod}})$. In particular $\partial_{\sigma_{\mathrm{mod}}} \Delta = \partial \Delta$.

In the symmetric space X, we define for a point $x \in X$ and a subset $A \subset \partial_\infty X$ the *cone* $V(x, A) \subset X$ as the union of the rays $x\xi$ for $\xi \in A$. We put $V(x, \emptyset) := \{x\}$.

Let $\tau \subset \partial_\infty X$ be a simplex of type τ_{mod}. The *Weyl cone* $V(x, \mathrm{st}(\tau))$ with tip at $x \in X$ is the union of the euclidean Weyl chambers $V(x, \sigma)$ for all chambers $\sigma \subseteq \mathrm{st}(\tau)$, equivalently, $\sigma \supseteq \tau$. Its *boundary* is given by $\partial V(x, \mathrm{st}(\tau)) = V(x, \partial \mathrm{st}(\tau))$, and its *interior* by $V(x, \mathrm{ost}(\tau)) - \{x\}$. We call the Weyl sector $V(x, \tau)$ the *central sector* of the Weyl cone $V(x, \mathrm{st}(\tau))$. Similarly, we will refer to $V(0, \tau_{\mathrm{mod}}) \subseteq \Delta$ as the *central sector* of the cone $W_{\tau_{\mathrm{mod}}} \Delta = V(0, W_{\tau_{\mathrm{mod}}} \sigma_{\mathrm{mod}}) \subset F_{\mathrm{mod}}$.

For the unique simplex $\widehat{\tau}$ x-opposite to τ, the Weyl cone $V(x, \mathrm{st}(\tau))$ is contained in the parallel set $P(\tau, \widehat{\tau})$. We say that the cone *spans* the parallel set.

Furthermore, for a compact subset $\Theta \subset \mathrm{int}_{\tau_{\mathrm{mod}}}(\sigma_{\mathrm{mod}})$, we define the Θ-*cone* $V(x, \mathrm{st}_\Theta(\tau))$.

Note that for chambers $\sigma \subset \partial_\infty X$ we have $V(x, \mathrm{st}(\sigma)) = V(x, \sigma)$.

We will call two Weyl cones or Θ-cones *asymptotic* if their visual boundary stars coincide.

The *Hausdorff distance* of asymptotic Weyl cones $V(y, \mathrm{st}(\tau))$ and $V(y', \mathrm{st}(\tau))$ is finite and bounded by the distance $d(y, y')$ of their tips. This follows immediately from the corresponding fact for rays.

The distance between boundaries of Weyl cones will be discussed later in Sect. 2.9.1.

We will need a fact about *projections*. Let

$$\pi_{x,\tau} = \pi_{V(x,\tau)} \colon V(x, \mathrm{st}(\tau)) \to V(x, \tau) \tag{5}$$

denote the nearest point projection of the Weyl cone to its central sector.

Lemma 2.9 $\pi_{x,\tau}$ *maps the interior of the Weyl cone to the interior of its central sector.*

In other words, for every point y in the interior of the Weyl cone there exists a point p in the interior of its central sector such that $py \perp V(x, \tau)$.

Proof This is a consequence of the general Lemma 2.1 on projections of spherical simplices to their faces. It yields at infinity that, for every chamber $\sigma \supseteq \tau$, the nearest point projection $\mathrm{int}_\tau(\sigma) \to \mathrm{int}(\tau)$ is well-defined. Equivalently, the nearest point projection $\mathrm{ost}(\tau) \to \mathrm{int}(\tau)$ is well-defined. The assertion follows by coning off. $\square$

As a consequence of the lemma, $\pi_{x,\tau}$ agrees with the nearest point projection of the Weyl cone to the singular flat spanned by the sector $V(x, \tau)$, because it does so on the interior.

Now we address *convexity*. We will see that the results on stars carry over to cones. First of all, by the definition of Weyl convexity, the cone $V(0, W_{\tau_{\mathrm{mod}}}\Theta) = W_{\tau_{\mathrm{mod}}} V(0, \Theta) \subset F_{\mathrm{mod}}$ is convex iff Θ is τ_{mod}-Weyl convex.

Proposition 2.10 (Convexity of cones)

(i) *The cones $V(x, \mathrm{st}(\tau))$ are convex.*
(ii) *If Θ is τ_{mod}-Weyl convex, then also the cones $V(x, \mathrm{st}_\Theta(\tau))$ are convex.*

Proof It suffices to verify (ii). We show that cones are intersections of horoballs. The horoball $\mathrm{Hb}_{\zeta,x}$ contains the cone $V(x, \mathrm{st}_\Theta(\tau))$ iff $\mathrm{st}_\Theta(\tau) \subseteq \overline{B}(\zeta, \frac{\pi}{2})$ in $\partial_\infty X$.

Let $y \neq x$ be a point and let $x\xi$ be a ray extending xy. Then $y \notin V(x, \mathrm{st}_\Theta(\tau))$ iff $\xi \notin \mathrm{st}_\Theta(\tau)$. Let $F \subset X$ be a maximal flat such that $xy \subset F$ and $\tau \subset \partial_\infty F$. According to the proof of Lemma 2.6, there exists a point $\zeta \in \partial_\infty F$ such that $\overline{B}(\zeta, \frac{\pi}{2})$ contains $\mathrm{st}_\Theta(\tau)$ but not ξ. Since $\mathrm{Hb}_{\zeta,x} \cap F$ is a half-space containing x in its boundary, it follows that also $y \notin \mathrm{Hb}_{\zeta,x}$. $\square$

The convexity of cones implies their nestedness:

Corollary 2.11 (Nestedness of cones)

(i) *If $y \in V(x, \mathrm{st}(\tau))$, then $V(y, \mathrm{st}(\tau)) \subseteq V(x, \mathrm{st}(\tau))$.*
(ii) *If $y \in V(x, \mathrm{st}_\Theta(\tau))$, then $V(y, \mathrm{st}_\Theta(\tau)) \subseteq V(x, \mathrm{st}_\Theta(\tau))$.*

Next we show an *openness* property for Weyl cones in the parallel sets spanned by them:

Lemma 2.12 (Open cones) *Let $x \in P(\tau, \hat{\tau})$. Then the boundary $\partial V(x, \mathrm{st}(\tau))$ of the Weyl cone $V(x, \mathrm{st}(\tau))$ disconnects the parallel set, and its interior $V(x, \mathrm{ost}(\tau)) - \{x\}$ is one of the connected components.*

Proof Since parallel sets are cones over their visual boundaries, i.e. $P(\tau, \hat{\tau}) = V(x, \partial_\infty X(\tau, \hat{\tau}))$, this follows from the visual openness of stars, cf. Lemma 2.8. $\square$

2.5.3 Diamonds

We say that a nondegenerate oriented geodesic segment $xy \subset X$ is τ_{mod}-*regular* if the unique geodesic ray $x\xi$ extending xy is asymptotic to a τ_{mod}-regular ideal point

$\xi \in \partial_\infty X$. In this case, we denote by $\tau(xy) \in \mathrm{Flag}_{\tau_{\mathrm{mod}}}$ the unique simplex such that $\xi \in \mathrm{ost}(\tau)$. Furthermore, we say that xy is Θ-*regular* with $\Theta \in \mathrm{int}_{\tau_{\mathrm{mod}}}(\sigma_{\mathrm{mod}})$ if $\theta(\xi) \in \Theta$.

Note that xy is τ_{mod}-regular if and only if yx is $\iota\tau_{\mathrm{mod}}$-regular, and Θ-regular iff yx is $\iota\Theta$-regular. The types of the simplices $\tau(xy)$ and $\tau(yx) \in \mathrm{Flag}_{\iota\tau_{\mathrm{mod}}}$ are then related by

$$\theta(\tau(yx)) = \iota\theta(\tau(xy)).$$

Let xy be a τ_{mod}-regular segment. We define its τ_{mod}-*diamond* as the intersection of Weyl cones

$$\diamondsuit_{\tau_{\mathrm{mod}}}(x, y) = V(x, \mathrm{st}(\tau_+)) \cap V(y, \mathrm{st}(\tau_-)) \subset P(\tau_-, \tau_+)$$

where $\tau_+ = \tau(xy)$ and $\tau_- = \tau(yx)$. The points x, y are the *tips* of the diamond. Furthermore, if xy is Θ-regular, we define its Θ-*diamond*

$$\diamondsuit_\Theta(x, y) = V(x, \mathrm{st}_\Theta(\tau_+)) \cap V(y, \mathrm{st}_\Theta(\tau_-)) \subset \diamondsuit_{\tau_{\mathrm{mod}}}(x, y).$$

The convexity of cones (Proposition 2.10) implies:

Proposition 2.13 (Convexity of diamonds)

(i) $\diamondsuit_{\tau_{\mathrm{mod}}}(x, y)$ *is convex.*
(ii) *If* Θ *is* τ_{mod}*-Weyl convex, then also* $\diamondsuit_\Theta(x, y)$ *is convex.*

And furthermore:

Corollary 2.14 (Nestedness of diamonds) *Suppose that* xy *and* $x'y'$ *are* τ_{mod}*-regular segments such that* $\tau(x'y') = \tau(xy)$, $\tau(y'x') = \tau(yx)$ *and* $x'y' \subset \diamondsuit_{\tau_{\mathrm{mod}}}(x, y)$. *Then:*

(i) $\diamondsuit_{\tau_{\mathrm{mod}}}(x', y') \subseteq \diamondsuit_{\tau_{\mathrm{mod}}}(x, y)$.
(ii) *If* xy *and* $x'y'$ *are* Θ*-regular, where* Θ *is* τ_{mod}*-Weyl convex, and if* $x'y' \subset \diamondsuit_\Theta(x, y)$, *then* $\diamondsuit_\Theta(x', y') \subseteq \diamondsuit_\Theta(x, y)$.

2.6 Vector valued distances

The Riemannian distance is not the complete two-point invariant on the symmetric space X, if $\mathrm{rank}(X) \geqslant 2$. In view of the natural identifications $X \times X/G \cong X/K \cong \Delta$, the full invariant is given by the quotient map

$$d_\Delta \colon X \times X \to \Delta$$

arising from dividing out the G-action, which we refer to as the Δ-*distance*. We will think of the elements of $\Delta \subset F_{\mathrm{mod}}$ as vectors and of d_Δ as a *vector-valued distance*. It relates to the Riemannian distance d on X by

$$d = \|d_\Delta\|,$$

where $\|\cdot\|$ is the euclidean norm on F_{mod}. See [16, Example 2.12] for the case $G = SL(n, \mathbb{R})$.

For the model flat, there are corresponding identifications $F_{mod} \times F_{mod}/W_{aff} \cong F_{mod}/W \cong \Delta$ and a Δ-distance

$$d_\Delta : F_{mod} \times F_{mod} \to \Delta.$$

It is compatible with the Δ-distance on X in that the charts $F_{mod} \to X$ are d_Δ-isometries. Similarly, one defines the Δ-distance on *euclidean buildings* via apartment charts, see [17].

The distance d_Δ is not symmetric, but satisfies

$$d_\Delta(y, x) = \iota d_\Delta(x, y).$$

We refer the reader to [17] and [28] for the detailed discussion of *metric properties* (such as "triangle inequalities" and "nonpositive curvature behavior") of d_Δ.

We note that a geodesic segment $xy \subset X$ is regular iff $d_\Delta(x, y) \in \operatorname{int}(\Delta)$. Similarly, xy is Θ-regular iff $d_\Delta(x, y) \in V(0, \Theta)$.

We define certain coarsifications of d_Δ by composing it with linear maps: For a face type τ_{mod}, let

$$\pi^\Delta_{\tau_{mod}} : \Delta \to V(0, \tau_{mod})$$

denote the nearest point projection. The composition

$$d_{\tau_{mod}} := \pi^\Delta_{\tau_{mod}} \circ d_\Delta \tag{6}$$

can also be regarded as a vector-valued distance on X, with values in the Weyl sector $V(0, \tau_{mod}) \subset \Delta$. Note that $d_{\sigma_{mod}} = d_\Delta$. Obviously,

$$\|d_{\tau_{mod}}\| \leqslant d$$

because $\pi^\Delta_{\tau_{mod}}$ is 1-Lipschitz.

Given a compact subset $\Theta \subset \operatorname{int}_{\tau_{mod}}(\sigma_{mod})$, for Θ-regular segments $xy \subset X$ it holds that

$$\|d_{\tau_{mod}}(x, y)\| \geqslant \epsilon(\Theta) \cdot d(x, y) \tag{7}$$

with a constant $\epsilon(\Theta) > 0$, where $\|\cdot\|$ denotes the euclidean norm. For the constant $\epsilon(\Theta)$ one can take the sine of the angular distance $\angle(\Theta, \partial_{\tau_{mod}}\sigma_{mod})$.

2.7 Refined side lengths of triangles

In this section, we assume more generally that X is a *CAT*(0) *model space*, i.e. a nonpositively curved Riemannian symmetric space or a thick euclidean building. We denote by

$$\mathcal{P}_3(X) \subset \Delta^3$$

the set of possible Δ-side lengths $(d_\Delta(x_1, x_2), d_\Delta(x_2, x_3), d_\Delta(x_3, x_1))$ of triangles $\Delta(x_1, x_2, x_3)$ in X. The following general result reduces the problem of determining $\mathcal{P}_3(X)$ from the symmetric space case to the euclidean building case:

Theorem 2.15 ([17, Theorem 1.2]) *$\mathcal{P}_3(X)$ depends only on the Weyl group W, and not on whether X is a Riemannian symmetric space or a thick euclidean building.*

In the paper [17], a detailed description of the set $\mathcal{P}_3(X)$ is given.

The next result concerns the Δ-side lengths of triangles $\Delta(x, y, z)$ in X such that the broken geodesic xyz is a Finsler geodesic (in the sense of Sect. 2.12 below):

Proposition 2.16 (i) *If $y \in V(x, \mathrm{st}(\tau))$ and $z \in V(y, \mathrm{st}(\tau))$ with $\tau \in \mathrm{Flag}_{\tau_{\mathrm{mod}}}$, then*

$$d_\Delta(x, z) \in V(d_\Delta(x, y), W_{\tau_{\mathrm{mod}}}\sigma_{\mathrm{mod}}) \cap \Delta.$$

(ii) *If $z \in V(y, \mathrm{st}_\Theta(\tau))$, where $\Theta \subset \mathrm{int}_{\tau_{\mathrm{mod}}}(\sigma_{\mathrm{mod}})$ is τ_{mod}-Weyl convex, then*

$$d_\Delta(x, z) \in V(d_\Delta(x, y), W_{\tau_{\mathrm{mod}}}\Theta) \cap \Delta.$$

Here, the cones $V(d_\Delta(x, y), \cdot)$ are to be understood as subsets of F_{mod}.

Proof We prove the stronger claim (ii).

The triangle $\Delta(x, y, z)$ lies in the parallel set $P = P(\widehat{\tau}, \tau)$ for the simplex $\widehat{\tau} \in \mathrm{Flag}_{\iota\tau_{\mathrm{mod}}}$ x-opposite to τ. The parallel set P is itself a symmetric space (with euclidean factor) with Weyl group $W' = W_{\tau_{\mathrm{mod}}} \subset W$. There is a natural inclusion $\sigma_{\mathrm{mod}} \subset \sigma'_{\mathrm{mod}} \subset a_{\mathrm{mod}}$ of spherical Weyl chambers such that σ'_{mod} equals the convex hull of σ_{mod} and the simplex $-\tau_{\mathrm{mod}}$ opposite to τ_{mod}, and a corresponding inclusion $\Delta \subset \Delta' \subset F_{\mathrm{mod}}$ of euclidean Weyl chambers such that Δ' is the convex hull of Δ and the sector $-V(0, \tau_{\mathrm{mod}})$.

Our claim is then a consequence of the following assertion on Δ'-side lengths: If $d_{\Delta'}(x, y) \in \Delta$ and $d_{\Delta'}(y, z) \in V(0, \Theta) \subset \Delta$, then

$$d_{\Delta'}(x, z) \in V(d_{\Delta'}(x, y), W_{\tau_{\mathrm{mod}}}\Theta) \cap \Delta.$$

Using Theorem 2.15, we may pass from symmetric spaces to euclidean buildings: The assertion is equivalent to the same assertion for any thick euclidean building $\widetilde{P}$ with the same Weyl group W'. (For instance, one can take $\widetilde{P}$ to be the complete euclidean cone over the spherical building $\partial_{\mathrm{Tits}} P$, which is a non-locally compact euclidean building with just one vertex.) It is easier to verify the statement in the building case due to the *locally conical* geometry of euclidean buildings.

Suppose therefore that $\Delta(\widetilde{x}, \widetilde{y}, \widetilde{z})$ is a triangle in a euclidean building $\widetilde{P}$ with Weyl group W', satisfying the same assumptions $d_{\Delta'}(\widetilde{x}, \widetilde{y}) \in \Delta$ and $d_{\Delta'}(\widetilde{y}, \widetilde{z}) \in V(0, \Theta)$. Taking advantage of the local conicality of buildings, we will do "induction along $\widetilde{y}\widetilde{z}$" and show that

$$d_{\Delta'}(\widetilde{x}, \widetilde{z}') \in V(d_{\Delta'}(\widetilde{x}, \widetilde{y}), W_{\tau_{mod}}\Theta) \cap \Delta \tag{8}$$

for all points $\widetilde{z}' \in \widetilde{y}\widetilde{z}$. Since this is a closed condition on $\widetilde{z}'$, it suffices to show that the subset of points satisfying it is half-open to the right. Moreover, since the points $\widetilde{z}' \in \widetilde{y}\widetilde{z}$ satisfying (8) also satisfy, like $\widetilde{y}$, the assumptions that $d_{\Delta'}(\widetilde{x}, \widetilde{z}') \in \Delta$ and $d_{\Delta'}(\widetilde{z}', \widetilde{z}) \in V(0, \Theta)$, it suffices to verify (8) for all points $\widetilde{z}' \in \widetilde{y}\widetilde{z}$ sufficiently close to $\widetilde{y}$.

This however reduces our claim to the *flat* case, because there exists a maximal flat $\widetilde{F} \subset \widetilde{P}$ which contains $\widetilde{x}\widetilde{y}$ along with a nondegenerate initial portion of the segment $\widetilde{y}\widetilde{z}$.[4] We may therefore assume that the triangle $\Delta(\widetilde{x}, \widetilde{y}, \widetilde{z})$ lies entirely in $\widetilde{F}$. Identifying $\widetilde{F} \cong F_{mod}$, we can once more reformulate our claim: If $\delta \in \Delta$ and $v \in V(0, W_{\tau_{mod}}\Theta)$, then

$$d_{\Delta'}(0, \delta + tv) \in V(\delta, W_{\tau_{mod}}\Theta) \cap \Delta$$

for all sufficiently small $t \geqslant 0$.

The stabilizer of δ in $W' = W_{\tau_{mod}}$ is a subgroup $W_{\nu_{mod}} \leqslant W_{\tau_{mod}}$ for a face type ν_{mod} with $\tau_{mod} \subseteq \nu_{mod} \subseteq \sigma_{mod}$ (namely, for the minimal face type $\nu_{mod} \supseteq \tau_{mod}$ such that $\delta \in V(0, \nu_{mod})$). We observe that the cone $\delta + V(0, W_{\tau_{mod}}\Theta)$ is $W_{\nu_{mod}}$-invariant and can be represented *locally* near δ as

$$\delta + V(0, W_{\tau_{mod}}\Theta) = W_{\nu_{mod}}\big((\delta + V(0, W_{\tau_{mod}}\Theta)) \cap \Delta\big).$$

The $W_{\tau_{mod}}$-invariance of $d_{\Delta'}(0, \cdot)$ yields the assertion. □

2.8 Strong asymptote classes

Let $\rho_1(t)$ and $\rho_2(t)$ be asymptotic geodesic rays in X, i.e. with the same ideal endpoint $\rho_1(+\infty) = \rho_2(+\infty) = \xi$. Equivalently, the convex function $t \mapsto d(\rho_1(t), \rho_2(t))$ on $[0, +\infty)$ is bounded. The rays are called *strongly asymptotic* if $d(\rho_1(t), \rho_2(t)) \to 0$ as $t \to +\infty$. One sees then using Jacobi fields that $d(\rho_1(t), \rho_2(t))$ decays exponentially with rate depending on the type of ξ (see [7]).

Strong asymptote classes are represented by rays in a parallel set:

Lemma 2.17 *Let $\xi, \widehat{\xi} \in \partial_\infty X$ be antipodal. Then every geodesic ray asymptotic to ξ is strongly asymptotic to a geodesic ray in the parallel set $P = P(\xi, \widehat{\xi})$.*

[4] This is clear for discrete euclidean buildings. (In particular, for buildings with only one vertex, like the complete euclidean cone over $\partial_{Tits} P$.) For the general case, see e.g. [23, Section 4.1.3].

Proof Let $c_1(t)$ be a geodesic line forward asymptotic to ξ (extending the given ray). Then the function $t \mapsto d(c_1(t), P)$ is convex and bounded on $[0, +\infty)$, and hence non-increasing. We claim that the limit

$$D := \lim_{t\to+\infty} d(c_1(t), P)$$

equals zero. To see this, we choose a geodesic line $c_2(t)$ in P forward asymptotic to ξ and use the transvections $T_t^{c_2}$ along c_2 to "pull back" c_1: The geodesics $c_1^s := T_{-s}^{c_2} c_1(\cdot + s)$ form a bounded family as $s \to +\infty$ and subconverge to a geodesic $c_1^{+\infty}$. Since the transvections $T_s^{c_2}$ preserve P, the distance functions $d(c_1^s(\cdot), P) = d(c_1(\cdot + s), P)$ converge locally uniformly on $\mathbb{R}$ and uniformly on $[0, +\infty)$ to the constant D. It follows that the limit geodesic $c_1^{+\infty}$ has distance $\equiv D$ from P. The same argument, applied to c_2 instead of the parallel set, implies that $c_1^{+\infty}$ is parallel to c_2. Thus, $c_1^{+\infty} \subset P(c_2) = P$ and, hence, $D = 0$.

Now we find a geodesic in P strongly asymptotic to c_1 as follows. Let $t_n \to +\infty$. We choose geodesics $c_n'(t)$ in P forward asymptotic to ξ by requiring that $c_n'(t_n) \in P$ is the nearest point projection of $c_1(t_n)$. Then $d(c_1(t_n), c_n'(t_n)) = d(c_1(t_n), P) \to 0$. The geodesics $c_n' \subset P$ are parallel, and their mutual Hausdorff distances d_{mn} are bounded above by the distances $d(c_m'(t), c_n'(t))$ independent of t. To estimate the Hausdorff distances, we observe that

$$\begin{aligned} d_{mn} &\leqslant d(c_m'(t), c_n'(t)) \leqslant d(c_m'(t), c_1(t)) + d(c_1(t), c_n'(t)) \\ &\leqslant d(c_m'(t_m), c_1(t_m)) + d(c_1(t_n), c_n'(t_n)) \end{aligned}$$

for $t \geqslant t_m, t_n$. The right-hand side converges $\to 0$ as $m, n \to +\infty$, and hence also d_{mn}. Thus, the geodesics c_n' form a Cauchy sequence and therefore converge to a geodesic in P. The limit geodesic is strongly asymptotic to c_1. □

We now derive a *criterion* for the *strong asymptoticity of rays*. Consider a geodesic line $c(t)$ asymptotic to $\xi \in \partial_\infty X$. We observe that for every $\eta \in \partial_\infty P(c)$ the restriction $b_\eta \circ c$ is *linear*, because there exists a flat f containing c with $\eta \in \partial_\infty f$.

As a consequence, for any two strongly asymptotic geodesic lines $c_1(t)$ and $c_2(t)$ asymptotic to ξ, the restricted Busemann functions $b_\eta \circ c_i$ *coincide* for every $\eta \in \mathrm{st}(\tau_\xi) \subset \partial_\infty P(c_1) \cap \partial_\infty P(c_2)$, where τ_ξ denotes the simplex spanned by ξ.

There is the following useful criterion for strong asymptoticity:

Lemma 2.18 *For geodesic lines $c_1(t)$ and $c_2(t)$ asymptotic to ξ the following are equivalent:*

(i) *$c_1(t)$ and $c_2(t)$ are strongly asymptotic.*
(ii) *$b_\eta \circ c_1 = b_\eta \circ c_2$ for every $\eta \in \mathrm{st}(\tau_\xi)$.*
(ii′) *$b_\eta \circ c_1 = b_\eta \circ c_2$ for every $\eta \in B(\xi, \epsilon)$ for some $\epsilon > 0$.*

Proof (i) ⇒ (ii) follows from the above discussion and (ii) ⇒ (ii′) is immediate.

In order to prove (ii′) ⇒ (i), we replace the geodesics c_i by a pair of parallel ones without changing their strong asymptote classes, applying Lemma 2.17. Using the implication (i) ⇒ (ii), which we already proved, we see that the c_i keep satisfying hypothesis (ii′). Since they now lie in a common flat, (ii′) immediately implies that they coincide, i.e. (i) follows. □

We generalize the discussion of strong asymptoticity to *sectors*.

Two Weyl sectors in X are *asymptotic* iff their visual boundary simplices coincide, equivalently, iff they have finite Hausdorff distance.

Fix a simplex $\tau \in \mathrm{Flag}_{\tau_{mod}}$ and consider two asymptotic sectors $V(x_1, \tau)$ and $V(x_2, \tau)$. The function $V(0, \tau_{mod}) \to [0, +\infty)$ given by

$$y \mapsto d(\kappa_{x_1,\tau}(y), \kappa_{x_2,\tau}(y)), \tag{9}$$

where $\kappa_{x_i,\tau}$ are the sector charts, is convex and bounded. We denote its infimum by $d_\tau(x_1, x_2)$. This defines a pseudo-metric d_τ on X, viewed as the set of (tips of) sectors asymptotic to τ.[5]

We say that the sectors $V(x_1, \tau)$ and $V(x_2, \tau)$ are *strongly asymptotic* if $d_\tau(x_1, x_2) = 0$. For any ideal point $\xi \in \mathrm{int}(\tau)$ this is equivalent to the rays $x_1\xi$ and $x_2\xi$ being strongly asymptotic. We denote by[6]

$$X_\tau^{par} = X/\sim_{d_\tau}$$

the *space of strong asymptote classes* of Weyl sectors asymptotic to τ.

We show now that, also in the case of sectors, parallel sets represent strong asymptote classes. For a simplex $\hat{\tau}$ opposite to τ we consider the restriction

$$P(\tau, \hat{\tau}) \to X_\tau^{par} \tag{10}$$

of the natural projection $X \to X_\tau^{par}$.

Proposition 2.19 *The map* (10) *is an isometry.*

Proof For points $x_1, x_2 \in P(\tau, \hat{\tau})$ the function (9) is constant $\equiv d(x_1, x_2)$. Hence (10) is an isometric embedding. To see that it is also surjective, we need to verify that every sector $V(x, \tau)$ is strongly asymptotic to a sector $V(x', \tau) \subset P(\tau, \hat{\tau})$. This follows from the corresponding fact for geodesic rays, see Lemma 2.17. □

2.9 Asymptotic Weyl cones

2.9.1 Separation of nested Weyl cones

Suppose that $y \in V(x, \mathrm{st}(\tau))$ with $\tau \in \mathrm{Flag}_{\tau_{mod}}$. By nestedness (Corollary 2.11), we have the inclusion of Weyl cones $V(y, \mathrm{st}(\tau)) \subseteq V(x, \mathrm{st}(\tau))$. We now determine the separation of their boundaries:

[5] Observe that $d_\tau(x_1, x_2)$ depends only on the strong asymptote classes of the sectors $V(x_i, \tau)$, and hence d_τ descends to $X_\tau^{par} \times X_\tau^{par}$. The triangle inequality is a consequence of Proposition 2.19 below. One can also verify the triangle inequality for d_τ directly, using the fact that, for bounded convex functions $\phi, \psi\colon V(0, \tau_{mod}) \to [0, +\infty)$, it holds that $\inf \phi + \inf \psi = \inf(\phi + \psi)$.

[6] Here, *par* stands for *parametrized*.

Proposition 2.20 (Separation) *The nearest point distance of the boundaries* $\partial V(x, \mathrm{st}(\tau))$ *and* $\partial V(y, \mathrm{st}(\tau))$ *equals* $d(\delta, \partial_{\tau_{\mathrm{mod}}}\Delta) = d(y, \partial V(x, \mathrm{st}(\tau)))$*, where* $\delta = d_\Delta(x, y)$.

Proof The natural submersion

$$d_\Delta(x, \cdot)\colon X \to \Delta$$

is 1-Lipschitz and restricts to an isometry on every euclidean Weyl chamber with tip at x. By restricting it to the Weyl cone $V(x, \mathrm{st}(\tau))$, one sees that

$$d(\cdot, \partial V(x, \mathrm{st}(\tau))) = d(d_\Delta(x, \cdot), \partial_{\tau_{\mathrm{mod}}}\Delta)$$

on $V(x, \mathrm{st}(\tau))$. According to Proposition 2.16 (i), the values of $d_\Delta(x, \cdot)$ on $V(y, \mathrm{st}(\tau))$ are contained in

$$V(\delta, W_{\tau_{\mathrm{mod}}}\sigma_{\mathrm{mod}}) \cap \Delta,$$

and clearly all these values are attained (on a euclidean Weyl chamber with tip at x and containing y). It follows that the nearest point distance of $V(y, \mathrm{st}(\tau))$ and $\partial V(x, \mathrm{st}(\tau))$ equals the nearest point distance of $V(\delta, W_{\tau_{\mathrm{mod}}}\sigma_{\mathrm{mod}}) \cap \Delta$ and $\partial_{\tau_{\mathrm{mod}}}\Delta$.

In order to see that the latter is given by $d(\delta, \partial_{\tau_{\mathrm{mod}}}\Delta)$, note that $d(\cdot, \partial_{\tau_{\mathrm{mod}}}\Delta)$ is the minimum of finitely many root functionals on Δ, namely of those corresponding to the walls of Δ not containing the sector $V(0, \tau_{\mathrm{mod}})$, equivalently, of those which are nonnegative on $W_{\tau_{\mathrm{mod}}}\Delta$. Each of these functionals attains its minimum on the cone $V(\delta, W_{\tau_{\mathrm{mod}}}\sigma_{\mathrm{mod}})$ at its tip δ. □

2.9.2 Shadows at infinity and strong asymptoticity of Weyl cones

For a simplex $\tau_- \in \mathrm{Flag}_{\iota\tau_{\mathrm{mod}}}$ and a point $x \in X$, we consider the function

$$\tau \mapsto d(x, P(\tau_-, \tau)) \tag{11}$$

on the open Schubert stratum $C(\tau_-) \subset \mathrm{Flag}_{\tau_{\mathrm{mod}}}$. We denote by $\tau_+ \in C(\tau_-)$ the simplex x-opposite to τ_-.

Lemma 2.21 *The function* (11) *is continuous and proper.*

Proof This follows from the fact that $C(\tau_-)$ and X are homogeneous spaces for the parabolic subgroup P_{τ_-}. Indeed, continuity follows from the continuity of the function

$$g \mapsto d(x, P(\tau_-, g\tau_+)) = d(g^{-1}x, P(\tau_-, \tau_+))$$

on P_{τ_-} which factors through the orbit map $P_{\tau_-} \to C(\tau_-)$, $g \mapsto g\tau_+$.

Regarding properness, note that a simplex $\tau \in C(\tau_-)$ is determined by any point y contained in the parallel set $P(\tau_-, \tau)$, namely as the simplex y-opposite to τ_-. Thus, if $P(\tau_-, \tau) \cap B(x, R) \neq \emptyset$ for some fixed $R > 0$, then there exists $g \in P_{\tau_-}$ such that $\tau = g\tau_+$ and $d(x, gx) < R$. In particular, g lies in a compact subset. This implies properness. □

Moreover, the function (11) has a unique minimum zero in τ_+.

We define the following open subsets of $C(\tau_-)$ which can be regarded as *shadows* of balls in X with respect to τ_-. For $x \in X$ and $r > 0$, we put

$$U_{\tau_-,x,r} := \{\tau \in C(\tau_-) : d(x, P(\tau_-, \tau)) < r\}. \tag{12}$$

The next fact expresses the *strong asymptoticity* of asymptotic Weyl cones:

Lemma 2.22 *For $r, R > 0$ there exists $d = d(r, R) > 0$ such that:*
If $y \in V(x, \mathrm{st}(\tau_-))$ with $d(y, \partial V(x, \mathrm{st}(\tau_-))) \geqslant d(r, R)$, then $U_{\tau_-,x,R} \subset U_{\tau_-,y,r}$.

Proof If $U_{\tau_-,x,R} \not\subset U_{\tau_-,y,r}$ then there exists $x' \in B(x, R)$ such that

$$d(y, V(x', \mathrm{st}(\tau_-))) \geqslant r.$$

Thus, if the assertion is wrong, there exist a sequence $x_n \to x_\infty$ in $\overline{B}(x, R)$ and an $\iota\tau_{\mathrm{mod}}$-regular sequence (y_n) in $V(x, \mathrm{st}(\tau_-))$ such that $d(y_n, V(x_n, \mathrm{st}(\tau_-))) \geqslant r$.

Let $\rho\colon [0, +\infty) \to V(x, \tau_-)$ be a geodesic ray with initial point x and asymptotic to an interior point of τ_-. By $\iota\tau_{\mathrm{mod}}$-regularity, the sequence (y_n) eventually enters every Weyl cone $V(\rho(t), \mathrm{st}(\tau_-))$. Since the distance function $d(\cdot, V(x_n, \mathrm{st}(\tau_-)))$ is convex and bounded, and hence non-increasing along rays asymptotic to $\mathrm{st}(\tau_-)$, we have that

$$R \geqslant d(x, V(x_n, \mathrm{st}(\tau_-))) \geqslant d(\rho(t), V(x_n, \mathrm{st}(\tau_-))) \geqslant d(y_n, V(x_n, \mathrm{st}(\tau_-))) \geqslant r$$

for $n \geqslant n(t)$. It follows that

$$R \geqslant d(\rho(t), V(x_\infty, \mathrm{st}(\tau_-))) \geqslant r$$

for all $t \geqslant 0$. However, the ray ρ is strongly asymptotic to $V(x_\infty, \mathrm{st}(\tau_-))$, cf. Proposition 2.19, a contradiction. ☐

2.10 Horocycles

We discuss various foliations of X naturally associated to a simplex $\tau \subset \partial_\infty X$.

We begin with foliations by flats and parallel sets: First, we denote by $\mathcal{F}_\tau$ the partition of X into the singular flats $f \subset X$ such that $\tau \subset \partial_\infty f$ is a top-dimensional simplex. Second, we consider the partition $\mathcal{P}_\tau$ of X into the parallel sets $P(\tau, \widehat{\tau})$ for the simplices $\widehat{\tau}$ opposite to τ. Note that $\mathcal{P}_\tau$ is a coarsening[7] of $\mathcal{F}_\tau$, and coincides with it iff τ is a chamber. The parabolic subgroup P_τ preserves both partitions and acts transitively on their leaves. This implies that these partitions are *smooth foliations*.

We will now show that there exist complementary orthogonal foliations. To do so, we describe preferred mutual identifications between the leaves of $\mathcal{F}_\tau$ as well as of $\mathcal{P}_\tau$ by the actions of certain subgroups of P_τ. Their orbits will be submanifolds orthogonal

[7] Meaning that the leaves of $\mathcal{P}_\tau$ are unions of leaves of $\mathcal{F}_\tau$.

and complementary to the foliations, i.e. the integral submanifolds of the distributions normal to them.

The tuple $(b_\xi)_{\xi\in\mathrm{Vert}(\tau)}$ of Busemann functions for the vertices ξ of τ (well-defined up to additive constants) provides affine coordinates simultaneously for each flat $f\in\mathcal{F}_\tau$. The Busemann functions at the other ideal points in τ are linear combinations of these. The group P_τ preserves the family of horospheres at every $\xi\in\tau$, and the action on it yields a natural "shift" homomorphism $\phi_\xi : P_\tau\to\mathbb{R}$. The intersection of their kernels forms the normal subgroup

$$\bigcap_{\xi\in\mathrm{Vert}(\tau)}\mathrm{Stab}(b_\xi)=\bigcap_{\xi\in\tau}\mathrm{Stab}(b_\xi)\lhd P_\tau. \tag{13}$$

It acts transitively on the set $\mathcal{F}_\tau$ of flats and preserves the coordinates; it thus provides consistent identifications between these flats. The level sets of $(b_\xi)_{\xi\in\mathrm{Vert}(\tau)}$ are submanifolds orthogonal and complementary to these flats, because the gradient directions of the Busemann functions b_ξ at a point $x\in f\in\mathcal{F}_\tau$ constitute a basis of the tangent space T_xf. These level sets form a smooth foliation $\mathcal{F}_\tau^\perp$ and are the orbits of the subgroup (13).

In order to describe the foliation normal to $\mathcal{P}_\tau$, we define the *horocyclic subgroup* at τ as the (smaller) normal subgroup $N_\tau\lhd P_\tau$ given by

$$N_\tau=\bigcap_{\xi\in\mathrm{st}(\tau)}\mathrm{Stab}(b_\xi)\lhd\mathrm{Fix}(\mathrm{st}(\tau))\lhd P_\tau.$$

It is the kernel of the P_τ-action on the set of all (unnormalized) Busemann functions centered at ideal points in $\mathrm{st}(\tau)$.

Note that as a consequence of Lemma 2.18, N_τ preserves the strong asymptote classes of geodesic rays at all ideal points $\xi\in\mathrm{ost}(\tau)$.

We now give a method for constructing isometries in N_τ. Let $\xi\in\mathrm{int}(\tau)$, and let $c(t)$ be a geodesic line forward asymptotic to it, $c(+\infty)=\xi$. Consider the one parameter group $(T_t^c)_{t\in\mathbb{R}}$ of transvections along c. The transvections T_t^c fix $\partial_\infty P(c)$ pointwise and shift the Busemann functions b_η centered at ideal points $\eta\in\partial_\infty P(c)$ by additive constants:

$$b_\eta\circ T_t^c-b_\eta\equiv -t\cdot\cos\angle_{\mathrm{Tits}}(\eta,\xi).$$

Note that $\mathrm{st}(\tau)\subset\partial_\infty P(c)$.

Lemma 2.23 *Let $c_1(t)$ and $c_2(t)$ be geodesic lines forward asymptotic to $\xi\in\mathrm{int}(\tau)$, which are strongly asymptotic. Then there exists an isometry*[8] *$n\in G$ with the properties:*

(i) $n\circ c_1=c_2$.
(ii) *n fixes $\partial_\infty P(c_1)\cap\partial_\infty P(c_2)$ pointwise.*
(iii) *$b_\eta\circ n\equiv b_\eta$ for all $\eta\in\partial_\infty P(c_1)\cap\partial_\infty P(c_2)$.*

In particular, $n\in N_\tau$.

[8] This isometry is unipotent but we will not need this fact.

Proof By our observation above, the isometries $T^{c_2}_{-t} \circ T^{c_1}_t$ fix $\partial_\infty P(c_1) \cap \partial_\infty P(c_2) \supseteq \mathrm{st}(\tau)$ pointwise and preserve the Busemann functions b_η for all $\eta \in \partial_\infty P(c_1) \cap \partial_\infty P(c_2)$. Thus, they belong to N_τ. Moreover, they form a bounded family. Therefore, as $t \to +\infty$, they subconverge to an isometry $n \in N_\tau$ which maps c_1 to c_2 while preserving parameterizations. □

Corollary 2.24 *N_τ acts transitively on*

(i) *every strong asymptote class of geodesic rays at every ideal point $\xi \in \mathrm{int}(\tau)$;*
(ii) *the set of leaves of $\mathcal{P}_\tau$.*

Proof Part (i) is a direct consequence of the lemma.

Also (ii) follows because every parallel set in $\mathcal{P}_\tau$ contains a (in fact, exactly one) geodesic ray of every strong asymptote class at any point $\xi \in \mathrm{int}(\tau)$, cf. Proposition 2.19. □

Remark 2.25 One also obtains that every geodesic asymptotic to an ideal point $\xi \in \partial\tau$ can be carried by an isometry in N_τ to any other strongly asymptotic geodesic. However, N_τ does not preserve strong asymptote classes at ξ in that case.

Lemma 2.26 *If $n \in N_\tau$ preserves a parallel set $P(\tau, \widehat{\tau})$, $n\widehat{\tau} = \widehat{\tau}$, then it acts trivially on it.*

Proof The hypothesis implies that n fixes $\mathrm{st}(\tau)$ and $\widehat{\tau}$ pointwise, and hence also their convex hull $\partial_\infty P(\tau, \widehat{\tau})$ in $\partial_{\mathrm{Tits}} X$. Thus n preserves every maximal flat $F \subset P(\tau, \widehat{\tau})$. Moreover it preserves all Busemann functions b_ξ centered at points $\xi \in \partial_\infty F \cap \mathrm{st}(\tau)$, and therefore must fix F pointwise, compare Lemma 2.18. □

Corollary 2.27 *The stabilizer of $P(\tau, \widehat{\tau})$ in N_τ is its pointwise fixator $K_{\tau,\widehat{\tau}} < G$.*

Proof The claim follows from the obvious inclusion $K_{\tau,\widehat{\tau}} \subset N_\tau$ together with the lemma. □

Remark 2.28 The subgroup N_τ decomposes as the semidirect product $U_\tau \rtimes K_{\tau,\widehat{\tau}}$, where $U_\tau \lhd P_\tau$ is the *unipotent radical* of P_τ.

By the above, N_τ provides consistent identifications between the parallel sets $P(\tau, \widehat{\tau})$. The N_τ-orbits are submanifolds orthogonal to the parallel sets and must have complementary dimension. They form a smooth foliation

$$\mathcal{H}_\tau = \mathcal{P}_\tau^\perp \tag{14}$$

refining $\mathcal{F}_\tau^\perp$, which we call the *horocyclic foliation* and its leaves the *horocycles* at τ. We denote the horocycle at τ through the point x by $\mathrm{Hc}_{\tau,x}$, i.e. $\mathrm{Hc}_{\tau,x} = N_\tau x$.

For incident faces, the associated subgroups and foliations are contained in each other: If $\upsilon \subset \tau$, then $\mathrm{st}(\upsilon) \supset \mathrm{st}(\tau)$ and $N_\upsilon < N_\tau$. Therefore, e.g. $\mathcal{H}_\upsilon$ refines $\mathcal{H}_\tau$.

Note that in rank one, horocycles are horospheres.

We also see how horocycles and strong asymptote classes relate; by Corollary 2.24 (i):

Corollary 2.29 (Strong asymptote classes are horocycles) *The sectors* $V(x_1, \tau)$ *and* $V(x_2, \tau)$ *are strongly asymptotic if and only if* x_1 *and* x_2 *lie in the same horocycle at* τ.

Moreover, the discussion shows that for the stabilizer $P_\tau \cap P_{\widehat{\tau}}$ of $P(\tau, \widehat{\tau})$ in P_τ it holds that $N_\tau(P_\tau \cap P_{\widehat{\tau}}) = P_\tau$ and $P_\tau \cap P_{\widehat{\tau}} \cap N_\tau = K_{\tau,\widehat{\tau}}$, and so the sequence

$$1 \to N_\tau \to P_\tau \to \mathrm{Isom}(X_\tau^{\mathrm{par}})$$

is exact.

Remark 2.30 Note that the homomorphism $P_\tau \to \mathrm{Isom}(X_\tau^{\mathrm{par}})$ is in general not surjective. Namely, let $X_\tau^{\mathrm{par}} =: f_\tau \times \mathrm{CS}(\tau)$ denote the decomposition (2) of $X_\tau^{\mathrm{par}} \cong P(\tau, \widehat{\tau})$. Then P_τ acts on the flat factor f_τ only by the group A_τ of translations. On the cross section, it acts by a subgroup $M_\tau \leqslant \mathrm{Isom}(\mathrm{CS}(\tau))$ containing the identity component. The above exact sequence is then a part of the *Langlands decomposition* of P_τ,

$$1 \to N_\tau \to P_\tau \to A_\tau \times M_\tau \to 1,$$

which, on the level of Lie algebras, is a split exact sequence.

We return now to Lemma 2.23. For later use, we elaborate on the special case when the geodesics c_i are contained in the parallel set of a singular flat of dimension rank minus one.

Consider a half-apartment $h \subset \partial_\infty X$; it is a simplicial $\frac{\pi}{2}$-ball in $\partial_\infty X$. We call its center ζ the *pole* of h. We define the *star* $\mathrm{st}(h)$ as the union of the stars $\mathrm{st}(\tau)$ where τ runs through all simplices with $\mathrm{int}(\tau) \subset \mathrm{int}(h)$, equivalently, which are spanned by interior points of h. Similarly, we define the *open star* $\mathrm{ost}(h)$ as the union of the corresponding open stars $\mathrm{ost}(\tau)$. Note that $\mathrm{int}(h) \subset \mathrm{ost}(h)$. Furthermore, we define the subgroup $N_h < G$ as the intersection of the horocyclic subgroups N_τ at these simplices τ,

$$N_h = \bigcap_{\mathrm{int}(\tau) \subset \mathrm{int}(h)} N_\tau.$$

We observe that N_h preserves the strong asymptote classes of geodesic rays at all ideal points $\xi \in \mathrm{ost}(h)$, and it preserves the family of maximal flats F with $\partial_\infty F \supset h$. The action on this set of flats is transitive. Indeed, parallel to Lemma 2.23, we have:

Lemma 2.31 *Let* $F_1, F_2 \subset P(\partial h)$ *be maximal flats with* $\partial_\infty F_i \supset h$. *Then there exists an isometry* $n \in N_h$ *with the properties:*

(i) $nF_1 = F_2$.
(ii) n *fixes* $\mathrm{st}(h)$ *pointwise.*
(iii) $b_\eta \circ n \equiv b_\eta$ *for all* $\eta \in \mathrm{st}(h)$.

Proof The parallel set $P(\partial h)$ splits as the product $f \times \mathrm{CS}(\partial h)$, see (2), where $f \subset X$ is a singular flat with $\partial_\infty f = \partial h$, and the cross section $\mathrm{CS}(\partial h)$ is a rank one symmetric space. Accordingly, the maximal flats F_i split as products $f \times \overline{c}_i$ with geodesics $\overline{c}_i \subset CS(\partial h)$ asymptotic to the pole $\zeta \in CS(\partial h)$ of h.

Let $\xi \in \mathrm{int}(h)$. We choose geodesics $c_1(t), c_2(t)$ in F_1, F_2 asymptotic to ξ. Their f-components are parallel geodesics in f, and their $CS(\partial h)$-components are geodesics in $CS(\partial h)$ asymptotic to ζ, equal to $\overline{c}_1, \overline{c}_2$ up to reparametrization. The geodesics c_1, c_2 are strongly asymptotic iff they have the same f-component and their $CS(\partial h)$-components are strongly asymptotic. We choose them in this way, using the fact that any two asymptotic geodesics in a rank one symmetric space become strongly asymptotic after suitable reparameterization.

We then can apply the limiting argument (in the proof of Lemma 2.23) to the compositions $T_{-t}^{c_2} \circ T_t^{c_1}$ and obtain an isometry $n \in N_{\tau_\xi}$ where $\tau_\xi \subset h$ denotes the simplex spanned by ξ. The isometry n carries F_1 to F_2, fixes $\mathrm{st}(\tau_\xi)$ pointwise and satisfies (iii) for all $\eta \in \mathrm{st}(\tau_\xi)$.

We observe that the isometries $T_{-t}^{c_2} \circ T_t^{c_1}$ act trivially on f and the limiting isometry n depends only on the $CS(\partial h)$-components of the geodesics c_i. Thus, by replacing the f-component of the c_i, we are not affecting n, but we can change the ideal endpoint ξ of the c_i to any other ideal point $\xi' \in \mathrm{int}(h)$. (We work here with constant speed parametrizations $c_i(t)$.) It follows that n fixes also $\mathrm{st}(\tau_{\xi'})$ pointwise and satisfies (iii) also for all $\eta \in \mathrm{st}(\tau_{\xi'})$. Varying ξ', we let $\tau_{\xi'}$ run through all simplices with $\mathrm{int}(\tau) \subset \mathrm{int}(h)$ and conclude also parts (ii)–(iii) of the assertion. $\square$

We obtain an analogue of Corollary 2.24:

Corollary 2.32 *N_h acts transitively on*

(i) *every strong asymptote class of geodesic rays at every ideal point $\xi \in \mathrm{int}(h)$;*
(ii) *the set of maximal flats F with $\partial_\infty F \supset h$.*

We describe a consequence of our discussion for the horocyclic foliations. The maximal flats F with $\partial_\infty F \supset h$ are contained in the parallel set $P(\partial h) \cong f \times CS(\partial h)$ and form the leaves of a smooth foliation $\mathcal{P}_h$ of $P(\partial h)$. This foliation is the pullback (via the natural projection $P(\partial h) \to CS(\partial h)$) of the one-dimensional foliation of the rank one symmetric space $CS(\partial h)$ by the geodesics asymptotic to the ideal point $\zeta \in \partial_\infty CS(\partial h)$, the center of h. There exists a foliation $\mathcal{H}_h$ of $P(\partial h)$ whose leaves are normal (orthogonal and complementary) to those of $\mathcal{P}_h$. The leaves of $\mathcal{H}_h$ have the form $\{y\} \times \mathrm{Hs}_{\zeta,z}$, where $y \in f$ and $\mathrm{Hs}_{\zeta,z} \subset CS(\partial h)$ is the horosphere centered at ζ and passing through $z \in CS(\partial h)$. We call the leaves of $\mathcal{H}_h$ the *horocycles at h* and the foliation $\mathcal{H}_h$ the *horocyclic* foliation. The leaf of $\mathcal{H}_h$ passing through $x \in P(\partial h)$ will be denoted $\mathrm{Hc}_{h,x}$. Corollary 2.32 implies that $\mathrm{Hc}_{h,x} = N_h x$.

Let τ be a simplex so that $\mathrm{int}(\tau) \subset \mathrm{int}(h)$. Then the foliation $\mathcal{P}_\tau$ of X by parallel sets restricts on $P(\partial h)$ to the foliation $\mathcal{P}_h$ by maximal flats, and the horocyclic foliation $\mathcal{H}_\tau$ restricts to the horocyclic foliation $\mathcal{H}_h$. (This follows from the fact that the foliations $\mathcal{P}_\tau$ and $\mathcal{H}_\tau$ are normal to each other, cf. (14).) In other words, the horocyclic foliations $\mathcal{H}_\tau$ for the various simplices τ with $\mathrm{int}(\tau) \subset \mathrm{int}(h)$ *coincide* on the parallel set $P(\partial h)$.

2.11 Contraction at infinity

2.11.1 *Identifications of horocycles*

We fix a simplex $\tau \subset \partial_\infty X$. Since every horocycle at τ intersects every parallel set $P(\tau, \widehat{\tau})$, $\widehat{\tau} \in C(\tau)$, exactly once, there are N_τ-equivariant diffeomorphisms

$$\mathrm{Hc}_{\tau,x} \xrightarrow{\cong} C(\tau) \tag{15}$$

sending a point $y \in \mathrm{Hc}_{\tau,x}$ to the unique simplex $\widehat{\tau} \in C(\tau)$ such that $\mathrm{Hc}_{\tau,x} \cap P(\tau, \widehat{\tau}) = \{y\}$. (The smoothness of these identifications follows from their N_τ-equivariance.) Composing the maps (15) and their inverses, we obtain N_τ-equivariant diffeomorphisms

$$\pi^{\tau}_{x'x} \colon \mathrm{Hc}_{\tau,x} \to \mathrm{Hc}_{\tau,x'}, \tag{16}$$

sending the intersection point $\mathrm{Hc}_{\tau,x} \cap P(\tau, \widehat{\tau})$ to the intersection $\mathrm{Hc}_{\tau,x'} \cap P(\tau, \widehat{\tau})$ for $\widehat{\tau} \in C(\tau)$.

Let $h \subset \partial_\infty X$ be a half-apartment such that $\mathrm{int}(\tau) \subset \mathrm{int}(h)$. Then, as discussed in the end of the previous section, the horocycles at τ intersect the parallel set $P(\partial h)$ in the horocycles at h. The latter are homogeneous spaces for the subgroup $N_h < N_\tau$. Thus, for $x, x' \in P(\partial h)$, the diffeomorphisms (16) restrict to N_h-equivariant diffeomorphisms

$$\pi^{h}_{x'x} \colon \mathrm{Hc}_{h,x} \xrightarrow{\cong} \mathrm{Hc}_{h,x'}$$

between the horocycles at h, while the diffeomorphisms (15) restrict to N_h-equivariant diffeomorphisms

$$\mathrm{Hc}_{h,x} \xrightarrow{\cong} C(h)$$

between the horocycles at h and the N_h-orbit $C(h) \subset C(\tau)$ consisting of the simplices which are contained in $\partial_\infty P(\partial h)$.

We estimate now the *contraction-expansion* of the identifications $\pi^{h}_{x'x}$. We build on the discussion at the end of the previous section. As we saw, the horocycles $\mathrm{Hc}_{h,x}$ in $P(\partial h) \cong f \times \mathrm{CS}(\partial h)$ are horospheres in the cross sections $pt \times \mathrm{CS}(\partial h)$. They therefore project isometrically onto the horospheres $\mathrm{Hs}_{\zeta,\overline{x}}$ in $\mathrm{CS}(\partial h)$, where $\overline{x}$ denotes the projection of x. Under these projections, the identifications $\pi^{h}_{x'x}$ correspond to the identifications

$$\pi^{\zeta}_{\overline{x}'\overline{x}} \colon \mathrm{Hs}_{\zeta,\overline{x}} \xrightarrow{\cong} \mathrm{Hs}_{\zeta,\overline{x}'} \tag{17}$$

of horospheres, i.e. for $x, x' \in P(\partial h)$, we have the commutative diagram:

$$\begin{tikzcd}
\mathrm{Hc}_{h,x} \arrow[r, "\pi^{h}_{x'x}"] \arrow[d] & \mathrm{Hc}_{h,x'} \arrow[d] \\
\mathrm{Hs}_{\zeta,\overline{x}} \arrow[r, "\pi^{\zeta}_{\overline{x}'\overline{x}}"] & \mathrm{Hs}_{\zeta,\overline{x}'}.
\end{tikzcd}$$

Estimating the contraction rate of $\pi^h_{x'x}$ therefore reduces to estimating it for $\pi^{\zeta}_{\bar{x}'\bar{x}}$ in the rank one symmetric space $\mathrm{CS}(\partial h)$.

We estimate the *infinitesimal* contraction. We assume that $\bar{x}'$ is closer to ζ than $\bar{x}$, $b_\zeta(\bar{x}) \geqslant b_\zeta(\bar{x}')$. Then there is actual contraction, at a uniform rate in terms of the distance between the horospheres. For the differential $d\pi^{\zeta}_{\bar{x}'\bar{x}}$ of $\pi^{\zeta}_{\bar{x}'\bar{x}}$, one has the estimate

$$e^{-c_1(b_\zeta(\bar{x})-b_\zeta(\bar{x}'))}\|\bar{v}\| \leqslant \|(d\pi^{\zeta}_{\bar{x}'\bar{x}})\bar{v}\| \leqslant e^{-c_2(b_\zeta(\bar{x})-b_\zeta(\bar{x}'))}\|\bar{v}\|$$

for all tangent vectors $\bar{v} \in T\mathrm{Hs}_{\zeta,\bar{x}}$, with constants $c_1 \geqslant c_2 > 0$ depending only on the rank one symmetric space $\mathrm{CS}(\partial h)$, in fact, depending only on X, because there are only finitely many isometry types of rank one symmetric spaces occurring as cross sections of parallel sets in X. The estimate follows from the standard fact that the exponential decay rate of decaying Jacobi fields along geodesic rays in $\mathrm{CS}(\partial h)$ is bounded below and above (in terms of the eigenvalues of the curvature tensor).

In view of $b_\zeta(x) - b_\zeta(x') = b_\zeta(\bar{x}) - b_\zeta(\bar{x}')$, we obtain for $\pi^h_{x'x}$:

Lemma 2.33 (Infinitesimal contraction of horocycle identifications) *If $b_\zeta(x) \geqslant b_\zeta(x')$, then*

$$e^{-c_1(b_\zeta(x)-b_\zeta(x'))}\|v\| \leqslant \|(d\pi^h_{x'x})v\| \leqslant e^{-c_2(b_\zeta(x)-b_\zeta(x'))}\|v\| \tag{18}$$

for all tangent vectors v to $\mathrm{Hc}_{h,x}$, with constants $c_1, c_2 > 0$ depending only on X.

2.11.2 Infinitesimal contraction of transvections

We now focus on transvections and their action at infinity. Suppose that $x, x' \in P(\tau, \hat{\tau})$ are distinct points. Let $\vartheta_{xx'}$ denote the transvection with axis $l = l_{xx'}$ through x and x' mapping $x' \mapsto x$; we orient the geodesic $l_{xx'}$ from x' to x, i.e. so that $\vartheta_{xx'}$ translates along it in the positive direction. The transvection $\vartheta_{xx'}$ preserves the parallel set $P(\tau, \hat{\tau})$ and fixes the simplices $\tau, \hat{\tau}$ at infinity.

We consider the action of $\vartheta_{xx'}$ on $C(\tau)$ and its differential at the fixed point $\hat{\tau}$. Modulo the identifications (15) and (16), the action of $\vartheta_{xx'}$ on $C(\tau)$ corresponds to the action of $\vartheta_{xx'} \circ \pi^{\tau}_{x'x}$ on $\mathrm{Hc}_{\tau,x}$, and the differential $(d\vartheta_{xx'})_{\hat{\tau}}$ of $\vartheta_{xx'}$ at $\hat{\tau}$ to the differential of $\vartheta_{xx'} \circ \pi^{\tau}_{x'x}$ at x.

We first consider the case when $\vartheta_{xx'}$ when $\xi := l_{xx'}(-\infty) \in \mathrm{ost}(\tau)$, equivalently, when x' lies in the interior of the Weyl cone $V(x, \mathrm{st}(\tau))$. Then $(d\vartheta_{xx'})_{\hat{\tau}}$ strictly contracts:

Lemma 2.34 *If $\xi \in \mathrm{ost}(\tau)$, then $(d\vartheta_{xx'})_{\hat{\tau}}$ is diagonalizable with eigenvalues in $(0, 1)$.*

Proof Since $\xi \in \mathrm{ost}(\tau)$, the group N_τ preserves the strong asymptote classes of geodesic rays at ξ,[9] cf. Sect. 2.10, i.e. the geodesics $nl_{xx'}$ for $n \in N_\tau$ are strongly backward asymptotic to $l_{xx'}$. Thus, by assigning to $n\hat{\tau} \in C(\tau)$ the geodesic $nl_{xx'}$, which is the unique geodesic in the parallel set $P(\tau, n\hat{\tau})$ strongly backward asymptotic to

[9] However, N_τ does not act transitively on it, unless $\xi \in \mathrm{int}(\tau)$.

$l_{xx'}$, we obtain a smooth family of geodesics in the strong backward asymptote class of $l_{xx'}$, parametrized by the manifold $C(\tau)$.

By differentiating this family, we obtain a linear embedding of the tangent space $T_{\hat{\tau}}C(\tau)$ into the vector space $\mathrm{Jac}_{l_{xx'},\xi}$ of Jacobi fields along $l_{xx'}$ which decay to zero at ξ. The effect of the differential $(d\vartheta_{xx'})_{\hat{\tau}}$ on $C(\tau)$ is given, in terms of these Jacobi fields, by the push-forward

$$J \mapsto (\vartheta_{xx'})_*(J) = d\vartheta_{xx'} \circ J \circ \vartheta_{x'x}.$$

The Jacobi fields in $\mathrm{Jac}_{l,\xi}$, which are of the form of a decaying exponential function times a parallel vector field along $l_{xx'}$, correspond to the eigenvectors of $(d\vartheta_{xx'})_{\hat{\tau}}$ with eigenvalues in (0, 1). It is a standard fact from the Riemannian geometry of symmetric spaces that the vector space $\mathrm{Jac}_{l_{xx'},\xi}$ has a basis consisting of such special Jacobi fields.[10] The same then follows for the linear subspace $L \subseteq \mathrm{Jac}_{l_{xx'},\xi}$ corresponding to $T_{\hat{\tau}}C(\tau)$. Thus the eigenvectors of $(d\vartheta_{xx'})_{\hat{\tau}}$ for positive eigenvalues span $T_{\hat{\tau}}C(\tau)$. □

We now give a uniform estimate for the contraction of $(d\vartheta_{xx'})_{\hat{\tau}}$:

Lemma 2.35 *If $\xi \in \mathrm{ost}(\tau)$, then the eigenvalues λ of $(d\vartheta_{xx'})_{\hat{\tau}}$ satisfy an estimate*

$$-\log\lambda \geqslant c\cdot d(x', \partial V(x, \mathrm{st}(\tau))) \tag{19}$$

with a constant $c > 0$ depending only on X.

Proof We continue the argument in the previous proof. Let $F \supset l_{xx'}$ be a maximal flat. Then $F \subset P(\tau, \hat{\tau})$. The smooth family $n\hat{\tau} \mapsto nl_{xx'}$ of geodesics parametrized by $C(\tau)$ embeds into the smooth family of maximal flats $n\hat{\tau} \mapsto nF$. They are all asymptotic to $\mathrm{st}(\tau) \cap \partial_\infty F$, i.e. $\partial_\infty(nF) \supset \mathrm{st}(\tau) \cap \partial_\infty F$. Accordingly, each Jacobi field $J \in L \subseteq \mathrm{Jac}_{l_{xx'},\xi}$ extends to a Jacobi field $\widehat{J}$ along F which decays to zero at all ideal points in $\mathrm{ost}(\tau)\cap\partial_\infty F$. (Here we use again that N_τ preserves the strong asymptote classes of geodesic rays at all points in $\mathrm{ost}(\tau)$.) Thus, we obtain a natural identification of $T_{\hat{\tau}}C(\tau)$ and L with a linear subspace $\widehat{L}$ of the vector space $\mathrm{Jac}_{F,\mathrm{ost}(\tau)\cap\partial_\infty F}$ of Jacobi fields along F which decay to zero at all ideal points in $\mathrm{ost}(\tau) \cap \partial_\infty F$.

The decomposition of Jacobi fields mentioned in the previous proof works in the same way along flats.[11] The vector space $\mathrm{Jac}_{F,\mathrm{ost}(\tau)\cap\partial_\infty F}$ has a basis consisting of Jacobi fields of the form $e^{-\alpha}V$ with an affine linear form α on F and a parallel vector field V along F. Furthermore, since G acts transitively on maximal flats, only *finitely* many affine linear forms α occur for these basis elements, independently of F. (The possible forms are determined by the root system of G, but we do not need this fact here.)

The decay condition on the forms α occurring in our decomposition is equivalent to the property that $\alpha \geqslant \alpha(x)$ on $V(x, \mathrm{st}(\tau)\cap\partial_\infty F) \subset F$ and $\alpha > \alpha(x)$ on the interior of this cone. It implies an estimate

[10] A transvection along a geodesic acts on the space of Jacobi fields along this geodesic as a diagonalizable transformation, see [7,13].

[11] As in the case of geodesics, a transvection along a flat acts on the space of Jacobi fields along this flat as a diagonalizable transformation, see [7,13].

$$\alpha(x') - \alpha(x) \geqslant c \cdot \underbrace{d(x', \partial V(x, \mathrm{st}(\tau) \cap \partial_\infty F))}_{=d(x', \partial V(x, \mathrm{st}(\tau)))}$$

with a constant $c = c(\alpha) > 0$. (The equality of distances follows from Proposition 2.20.) Since there are only finitely many forms α involved, the constant c can be taken *independent* of α.

Notice that the eigenvalues λ of $(d\vartheta_{xx'})_{\widehat{\tau}}$ are of the form

$$e^{-(\alpha(x') - \alpha(x))}.$$

The claimed upper bound for the eigenvalues follows. □

By continuity, the result extends to the case when x' lies in the boundary of the Weyl cone $V(x, \mathrm{st}(\tau))$. We obtain:

Corollary 2.36 *If $x' \in V(x, \mathrm{st}(\tau))$, then $(d\vartheta_{xx'})_{\widehat{\tau}}$ is diagonalizable with eigenvalues in $(0, 1]$ satisfying an estimate (19). In particular, the eigenvalues lie in $(0, 1)$, if x' lies in the interior of $V(x, \mathrm{st}(\tau))$.*

If x' lies outside the Weyl cone $V(x, \mathrm{st}(\tau))$, then $d(\vartheta_{xx'})_{\widehat{\tau}}$ has expanding directions. In order to see this, we consider the action of $\vartheta_{xx'}$ on certain invariant submanifolds of $C(\tau)$ corresponding to parallel sets of singular hyperplanes.

Again, there exists a maximal flat F with $l_{xx'} \subset F \subset P(\tau, \widehat{\tau})$. Let $h \subset \partial_\infty F$ be a half-apartment such that $\mathrm{int}(\tau) \subset \mathrm{int}(h)$. Then $l_{xx'} \subset F \subset P(\partial h)$. The transvection $\vartheta_{xx'}$ fixes $\partial_\infty F$ pointwise. Hence it preserves the parallel set $P(\partial h)$ and the submanifold $C(h) = N_h \widehat{\tau} \subset C(\tau)$.

If $l_{xx'}$ is parallel to the euclidean factor of $P(\partial h)$, equivalently, if $\partial_\infty l_{xx'} \subset \partial h$, then $\vartheta_{xx'}$ acts trivially on $\partial_\infty P(\partial h)$. Hence, $\vartheta_{xx'}$ acts also trivially on $C(h)$, because the latter consists of simplices contained in $\partial_\infty P(\partial h)$.

In the general case, the action of $\vartheta_{xx'}$ on $C(h)$ corresponds to the restriction of the action of $\vartheta_{xx'} \circ \pi^{\tau}_{x'x}$ to $\mathrm{Hc}_{h,x} = \mathrm{Hc}_{\tau,x} \cap P(\partial h)$. When projecting to $\mathrm{CS}(\partial h)$, the latter action in turn corresponds to the action of $\vartheta_{\overline{x}\overline{x}'} \circ \pi^{\zeta}_{\overline{x}'\overline{x}}$ on the horosphere $\mathrm{Hs}_{\zeta,\overline{x}}$. Here, $\vartheta_{\overline{x}\overline{x}'}$ denotes the transvection on $\mathrm{CS}(\partial h)$ with axis $l_{\overline{x}\overline{x}'}$ through $\overline{x}$ and $\overline{x}'$ mapping $\overline{x}' \mapsto \overline{x}$, and $\pi^{\zeta}_{\overline{x}'\overline{x}}$ is the natural identification (17). The axis $l_{\overline{x}\overline{x}'}$ is the image of F under the projection (if $\overline{x} = \overline{x}'$, we define it in this way). It is asymptotic to ζ and another ideal point $\widehat{\zeta} \in C(\zeta) = \partial_\infty \mathrm{CS}(\partial h) - \{\zeta\}$. The simplex $\widehat{\tau}$ corresponds to $\widehat{\zeta}$ under the natural N_h-equivariant identification $C(h) \cong C(\zeta)$, and the action of $\vartheta_{xx'}$ on $C(h)$ corresponds to the action of $\vartheta_{\overline{x}\overline{x}'}$ on $C(\zeta)$.

We now obtain analogues of Lemmas 2.34 and 2.35. Recall that $\xi = l_{xx'}(-\infty)$.

Lemma 2.37 *If $\xi \in \mathrm{int}(h)$, then $(d\vartheta_{xx'})_{\widehat{\tau}}|_{T_{\widehat{\tau}}C(h)}$ is diagonalizable with eigenvalues $\lambda \in (0, 1)$ satisfying an estimate*

$$c_2 \leqslant \frac{-\log \lambda}{b_\zeta(x) - b_\zeta(x')} \leqslant c_1 \tag{20}$$

with constants $c_1, c_2 > 0$ depending only on X.

Proof The diagonalizablility follows by applying Lemma 2.34 to $\mathrm{CS}(\partial h)$ and $(d\vartheta_{\overline{xx'}})_{\widehat{\zeta}}$. Since $\xi \in \mathrm{int}(h)$, we have that $b_\zeta(x) - b_\zeta(x') = b_\zeta(\overline{x}) - b_\zeta(\overline{x}') > 0$, and the eigenvalue estimate follows from the contraction estimate (18). □

Corollary 2.38 *If* $x' \in P(\tau, \widehat{\tau}) - V(x, \mathrm{st}(\tau))$, *then* $(d\vartheta_{xx'})_{\widehat{\tau}}$ *has some eigenvalues in* $(1, +\infty)$.

Proof By our assumption, we have that $\xi \notin \mathrm{st}(\tau)$. Therefore, the half-apartment $h \subset \partial_\infty F$ can be chosen so that its interior contains, besides $\mathrm{int}(\tau)$, also $l_{xx'}(+\infty)$. (Recall that the convex subcomplex $\mathrm{st}(\tau) \cap \partial_\infty F$ is an intersection of half-apartments in $\partial_\infty F$, cf. Lemma 2.3.) Then the estimate (20) applied to $\vartheta_{x'x} = \vartheta_{xx'}^{-1}$ yields that $(d\vartheta_{xx'})_{\widehat{\tau}}^{-1}$ has some eigenvalues in $(0, 1)$. □

Complementing Corollary 2.36, we bound the contraction rate from above, if $x' \in V(x, \mathrm{st}(\tau))$:

Lemma 2.39 *If* $\xi \in \mathrm{st}(\tau)$, *then* $(d\vartheta_{xx'})_{\widehat{\tau}}$ *has some eigenvalue* $\lambda \in (0, 1]$ *satisfying an estimate*

$$-\log \lambda \leqslant c_1 \cdot d(x', \partial V(x, \mathrm{st}(\tau)))$$

with a constant $c_1 > 0$ *depending only on* X.

Proof Since $xx' \subset F$, some nearest point y' to x' on $\partial V(x, \mathrm{st}(\tau))$ lies in F, cf. Proposition 2.20. Hence we can choose the half-apartment $h \subset \partial_\infty F$ such that $b_\zeta(y') = b_\zeta(x)$ and

$$d(x', \partial V(x, \mathrm{st}(\tau))) = b_\zeta(x) - b_\zeta(x').$$

Now let λ be an eigenvalue of $(d\vartheta_{xx'})_{\widehat{\tau}}|_{T_{\widehat{\tau}}C(h)}$ and apply the upper bound in (20). □

Putting the information (Corollaries 2.36, 2.38 and Lemmas 2.37, 2.39) together, we obtain:

Proposition 2.40 (Infinitesimal contraction of transvections at infinity) *Let* $\tau, \widehat{\tau} \subset \partial_\infty X$ *be opposite simplices, and let* ϑ *be a nontrivial transvection with an axis* $l \subset P(\tau, \widehat{\tau})$ *through the point* x. *Then the following hold for the differential* $d\vartheta_{\widehat{\tau}}$ *of* ϑ *on* $C(\tau)$ *at the fixed point* $\widehat{\tau}$:

(i) $d\vartheta_{\widehat{\tau}}$ *is diagonalizable with eigenvalues in* $(0, 1]$ *iff* $\vartheta^{-1}x \in V(x, \mathrm{st}(\tau))$, *and diagonalizable with eigenvalues in* $(0, 1)$ *iff* $\vartheta^{-1}x \in V(x, \mathrm{ost}(\tau))$.

(ii) *If* $\vartheta^{-1}x \in V(x, \mathrm{st}(\tau))$, *then the eigenvalues* λ *of* $d\vartheta_{\widehat{\tau}}$ *satisfy an estimate*

$$c_2 \cdot d(\vartheta^{-1}x, \partial V(x, \mathrm{st}(\tau))) \leqslant -\log \lambda \leqslant c_1 \cdot d(\vartheta^{-1}x, \partial V(x, \mathrm{st}(\tau)))$$

with constants $c_1, c_2 > 0$ *depending only on* X.

We deduce a consequence for the action of general isometries in G. For later use, we will formulate it in terms of expansion (of their inverses) rather than contraction.

We need the following notion: For a diffeomorphism Φ of a Riemannian manifold M, we define the *expansion factor* at $x \in M$ as

$$\epsilon(\Phi, x) = \inf_{v \in T_x M - \{0\}} \frac{\|d\Phi(v)\|}{\|v\|} = \|(d\Phi_x)^{-1}\|^{-1}, \tag{21}$$

compare (25) in Sect. 3.1 below.

We equip the flag manifolds $\mathrm{Flag}_{\tau_{mod}}$ with auxiliary Riemannian metrics.

Theorem 2.41 (Infinitesimal expansion of isometries at infinity) *Let* $\tau \in \mathrm{Flag}_{\tau_{mod}}$, $x \in X$, *and* $g \in G$ *be such that* $d(gx, V(x, \mathrm{st}(\tau))) \leqslant r$. *Then for the action of* g^{-1} *on* $\mathrm{Flag}_{\tau_{mod}}$ *we have the estimate*

$$C^{-1} \cdot d(gx, \partial V(x, \mathrm{st}(\tau))) - A \leqslant \log \epsilon(g^{-1}, \tau) \leqslant C \cdot d(gx, \partial V(x, \mathrm{st}(\tau))) + A$$

with constants $C, A > 0$ *depending only on* x, r *and the chosen Riemannian metric on* $\mathrm{Flag}_{\tau_{mod}}$.[12]

Proof We write g as a product $g = tb$ of a transvection t along a geodesic l through x with $l(+\infty) \in \mathrm{st}(\tau)$ and an isometry $b \in G$ such that $d(x, bx) \leqslant r$. Then t fixes τ on $\mathrm{Flag}_{\tau_{mod}}$, and the expansion factor $\epsilon(g^{-1}, \tau)$ equals $\epsilon(t^{-1}, \tau)$ up to a multiplicative constant depending on r and the chosen Riemannian metric on $\mathrm{Flag}_{\tau_{mod}}$.

When replacing the metric, $\epsilon(t^{-1}, \tau)$ changes at most by another multiplicative constant, and we may therefore assume that the Riemannian metric is invariant under the maximal compact subgroup $K_x < G$ fixing x. Now the eigenspace decomposition of dt_τ on $T_\tau \mathrm{Flag}_{\tau_{mod}}$ is orthogonal. Consequently,

$$\epsilon(t^{-1}, \tau) = \lambda_{max}^{-1}$$

where λ_{max} denotes the maximal eigenvalue of dt_τ.

Let $\hat{\tau}$ denote the simplex x-opposite to τ. Applying Proposition 2.40(ii) to $\vartheta = t$ while exchanging the roles of τ and $\hat{\tau}$, we obtain the estimate

$$c_2 \cdot d(t^{-1}x, \partial V(x, \mathrm{st}(\hat{\tau}))) \leqslant -\log \lambda \leqslant c_1 \cdot \underbrace{d(t^{-1}x, \partial V(x, \mathrm{st}(\hat{\tau})))}_{=d(tx, \partial V(x, \mathrm{st}(\tau)))}$$

for the eigenvalues λ of dt_τ, and so

$$c_2 \cdot d(tx, \partial V(x, \mathrm{st}(\tau))) \leqslant \log \epsilon(t^{-1}, \tau) \leqslant c_1 \cdot d(tx, \partial V(x, \mathrm{st}(\tau))),$$

which is the desired estimate. □

Let us now consider sequences (g_n) in G. The theorem can be used to draw conclusions from the expansion behavior at infinity of the sequence of inverses (g_n^{-1}) on the geometry of an orbit sequence $(g_n x)$ in X: If $(g_n x)$ lies in a tubular neighborhood of the Weyl cone $V(x, \mathrm{st}(\tau))$, then the expansion factors $\epsilon(g_n^{-1}, \tau)$ on $\mathrm{Flag}_{\tau_{mod}}$ are bounded below, and their logarithms measure the distance of $(g_n x)$ to the boundary of the Weyl cone. In particular, if the expansion factors diverge, $\epsilon(g_n^{-1}, \tau) \to +\infty$, then (the projection of) $(g_n x)$ enters deep into the cone $V(x, \mathrm{st}(\tau))$.

[12] The estimate depends also on the point x because the choice of the auxiliary metric on $\mathrm{Flag}_{\tau_{mod}}$ reduces the symmetry: The action of a compact subgroup of G on $\mathrm{Flag}_{\tau_{mod}}$ is uniformly bilipschitz, but not the G-action.

The next result shows how to recognize from expansion whether the orbit sequence $(g_n x)$ remains in a tubular neighborhood of the Weyl cone $V(x, \mathrm{st}(\tau))$, once it stays close to the parallel set spanned by it:

Proposition 2.42 *Let $\tau, \widehat{\tau} \subset \partial_\infty X$ be opposite simplices. Suppose that (g_n) is a sequence in G such that, for some point $x \in X$, the sequence $(g_n x)$ is contained in a tubular neighborhood of the parallel set $P(\tau, \widehat{\tau})$, but drifts away from the Weyl cone $V(x, \mathrm{st}(\tau))$,*

$$d(g_n x, V(x, \mathrm{st}(\tau))) \to +\infty$$

as $n \to +\infty$. Then $\epsilon(g_n^{-1}, \tau) \to 0$.

Proof We may assume that $x \in P = P(\tau, \widehat{\tau})$. As in the proof of Theorem 2.41, we can reduce to the case that the g_n are transvections along geodesics l_n in P through the point x. We need to show that the differentials $(dg_n^{-1})_\tau$ on $\mathrm{Flag}_{\tau_{\mathrm{mod}}}$ have (some) small eigenvalues, i.e. that their minimal eigenvalue goes $\to 0$.

We proceed as in the proof of Corollary 2.38. Let $F_n \subset P$ be a maximal flat containing l_n. Then also

$$d(g_n x, V(x, \mathrm{st}(\tau)) \cap F_n) \to +\infty,$$

cf. Proposition 2.20. There exist half-apartments $h_n \subset \partial_\infty F_n$ with centers ζ_n, so that $b_{\zeta_n} \leqslant b_{\zeta_n}(x)$ on $V(x, \mathrm{st}(\tau)) \cap F_n$ (and hence also on $V(x, \mathrm{st}(\tau))$) and $b_{\zeta_n}(g_n x) - b_{\zeta_n}(x) \to +\infty$. Let $\widehat{h}_n \subset \partial_\infty F_n$ denote the complementary half-apartments, $\partial \widehat{h}_n = \partial h_n$, and $\widehat{\zeta}_n$ their centers. Then $b_{\zeta_n} + b_{\widehat{\zeta}_n} \equiv \mathrm{const}$ on F_n. It suffices to show that the differentials $(dg_n^{-1})_\tau$ are contracting on the invariant subspaces $T_\tau C(\widehat{h}_n) \subseteq T_\tau C(\widehat{\tau})$ with norms going $\to 0$. According to Lemma 2.37, the eigenvalues of $(dg_n^{-1})_\tau|_{T_\tau C(\widehat{h}_n)}$ are positive and bounded above by

$$e^{-c_2\left(b_{\hat{\zeta}_n}(x) - b_{\hat{\zeta}_n}(g_n x)\right)} = e^{-c_2(b_{\zeta_n}(g_n x) - b_{\zeta_n}(x))} \to 0.$$

This finishes the proof. $\square$

2.12 Finsler geodesics

We will work with the following notion of Finsler geodesic:

Definition 2.43 (*Finsler geodesics*) A continuous path $c : I \to X$ is a τ_{mod}-*Finsler geodesic* if it is contained in a parallel set $P(\tau_-, \tau_+)$ with $\tau_\pm \in \mathrm{Flag}_{\pm\tau_{\mathrm{mod}}}$ such that[13]

$$c(t_+) \in V(c(t_-), \mathrm{st}(\tau_+)) \tag{22}$$

for all subintervals $[t_-, t_+] \subseteq I$. It is Θ-*regular* if, moreover,

$$c(t_+) \in V(c(t_-), \mathrm{st}_\Theta(\tau_+)). \tag{23}$$

[13] The notation is to be understood here in the formula $\tau_\pm \in \mathrm{Flag}_{\pm\tau_{\mathrm{mod}}}$ and in other formulas later that either both signs are plus or both signs are minus.

We call a τ_{mod}-Finsler geodesic *uniformly* τ_{mod}*-regular* if it is Θ-regular for some $W_{\tau_{mod}}$-convex compact subset $\Theta \subset \mathrm{int}_{\tau_{mod}}(\sigma_{mod})$.

Note that we do not require the parameterization of Finsler geodesics to be by arc length. The terminology is justified by the fact that τ_{mod}-Finsler geodesics are (up to parameterization) the geodesics for certain G-invariant "polyhedral" Finsler metrics, see [15, Section 5.1.3].

The condition (22) is equivalent to $c(t_-) \in V(c(t_+), \mathrm{st}(\tau_-))$, and it follows that the subpaths $c|_{[t_-,t_+]}$ are contained in the diamonds $\diamondsuit_{\tau_{mod}}(c(t_-), c(t_+))$. Similarly, (23) is equivalent to $c(t_-) \in V(c(t_+), \mathrm{st}_\Theta(\tau_-))$, because Θ is assumed ι-invariant, and in the Θ-regular case $c|_{[t_-,t_+]}$ is contained in $\diamondsuit_\Theta(c(t_-), c(t_+))$.

It is worth mentioning the following Finsler geometric interpretation of *diamonds*: They are Finsler versions of Riemannian geodesic segments in the sense that the union of all τ_{mod}-Finsler geodesic segments with endpoints $x_\pm$ fills out $\diamondsuit_{\tau_{mod}}(x_-, x_+)$, see also [15, Section 5.1.3].

We now discuss the "drift" component of τ_{mod} Finsler geodesics. We work with the vector valued distance $d_{\tau_{mod}} = \pi^\Delta_{\tau_{mod}} \circ d_\Delta$, introduced in (6). We first consider the case of broken geodesics xyz which are τ_{mod}-Finsler geodesics:

Lemma 2.44 (Additivity) *Let* $\tau \in \mathrm{Flag}_{\tau_{mod}}$. *If* $y \in V(x, \mathrm{st}(\tau))$ *and* $z \in V(y, \mathrm{st}(\tau))$*, then*

$$d_{\tau_{mod}}(x, y) + d_{\tau_{mod}}(y, z) = d_{\tau_{mod}}(x, z).$$

Proof The τ_{mod}-distance can be expressed in terms of the projections of Weyl cones to their central sectors. Consider the nearest point projection

$$\pi_{x,\tau} : V(x, \mathrm{st}(\tau)) \to V(x, \tau),$$

cf. (5). Note that it coincides with the nearest point projection from $V(x, \mathrm{st}(\tau))$ to the singular flat spanned by the sector $V(x, \tau)$, compare Lemma 2.9 and the comment thereafter. Then

$$d_{\tau_{mod}}(x, \cdot) = d_\Delta(x, \pi_{x,\tau}(\cdot))$$

on $V(x, \mathrm{st}(\tau))$.

In order to relate $d_{\tau_{mod}}(y, z)$ to $d_{\tau_{mod}}(x, y)$ and $d_{\tau_{mod}}(x, z)$, we observe that the sectors $V(y, \tau)$ and $V(\pi_{x,\tau}(y), \tau) \subseteq V(x, \tau)$ are parallel and isometrically identified by $\pi_{x,\tau}$. Moreover,

$$\pi_{x,\tau}|_{V(y,\mathrm{st}(\tau))} = (\pi_{x,\tau}|_{V(y,\tau)}) \circ \pi_{y,\tau}.$$

Therefore,

$$d_{\tau_{mod}}(y, z) = d_\Delta(y, \pi_{y,\tau}(z)) = d_\Delta(\pi_{x,\tau}(y), \pi_{x,\tau}(z)).$$

The additivity formula follows in view of the nestedness $\pi_{x,\tau}(z) \in V(\pi_{x,\tau}(y), \tau)$. $\square$

Applying the lemma to τ_{mod}-Finsler geodesics yields:

Proposition 2.45 (Additivity of τ_{mod}-distance along Finsler geodesics) *If $c: I \to X$ is a τ_{mod}-Finsler geodesic, then*

$$d_{\tau_{\mathrm{mod}}}(c(t_0), c(t_1)) + d_{\tau_{\mathrm{mod}}}(c(t_1), c(t_2)) = d_{\tau_{\mathrm{mod}}}(c(t_0), c(t_2))$$

for all $t_0 \leqslant t_1 \leqslant t_2$ in I.

We reformulate this as:

Proposition 2.46 (τ_{mod}-projection of Finsler geodesics) *If $c: [0, T] \to X$ is a τ_{mod}-Finsler geodesic, then so is*

$$\bar{c}_{\tau_{\mathrm{mod}}} := d_{\tau_{\mathrm{mod}}}(c(0), c): [0, T] \to V(0, \tau_{\mathrm{mod}}),$$

and

$$\bar{c}_{\tau_{\mathrm{mod}}}(t_2) = \bar{c}_{\tau_{\mathrm{mod}}}(t_1) + d_{\tau_{\mathrm{mod}}}(c(t_1), c(t_2))$$

for all $0 \leqslant t_1 \leqslant t_2 \leqslant T$.

Note that the equality in the last proposition implies:

$$d(\bar{c}_{\tau_{\mathrm{mod}}}(t_1), \bar{c}_{\tau_{\mathrm{mod}}}(t_2)) = \|d_{\tau_{\mathrm{mod}}}(c(t_1), c(t_2))\|. \tag{24}$$

We now study the Δ-*distance along Finsler geodesics*. This is based on Proposition 2.16 which concerns the Δ-side lengths of triangles $\Delta(x, y, z)$ in X such that the broken geodesic xyz is a Finsler geodesic. Applying this proposition to Finsler geodesics, we obtain our main result concerning their geometry:

Theorem 2.47 (Δ-projection of Finsler geodesics)

(i) *If $c: [0, T] \to X$ is a τ_{mod}-Finsler geodesic, then so is*

$$\bar{c}_{\Delta} := d_{\Delta}(c(0), c): [0, T] \to \Delta.$$

(ii) *If c is also Θ-regular, with $\Theta \subset \mathrm{int}_{\tau_{\mathrm{mod}}}(\sigma_{\mathrm{mod}})$ compact and τ_{mod}-Weyl convex, then so is $\bar{c}_{\Delta}$. Moreover, the distances between points on c and $\bar{c}_{\Delta}$ are comparable:*

$$d(\bar{c}_{\Delta}(t_1), \bar{c}_{\Delta}(t_2)) \geqslant \epsilon(\Theta) \cdot d(c(t_1), c(t_2))$$

for $0 \leqslant t_1 \leqslant t_2 \leqslant T$ with a constant $\epsilon(\Theta) > 0$.

We note that $d(\bar{c}_{\Delta}(t_1), \bar{c}_{\Delta}(t_2)) \leqslant d(c(t_1), c(t_2))$, because $d_{\Delta}(c(0), \cdot)$ is 1-Lipschitz.

Proof (i) Applying Proposition 2.16 to the triangles $\Delta(c(0), c(t_1), c(t_2))$, $0 \leqslant t_1 \leqslant t_2 \leqslant T$, yields

$$\bar{c}_\Delta(t_2) \in V(\bar{c}_\Delta(t_1), W_{\tau_{\mathrm{mod}}}\Delta),$$

the cone being understood as a subset of F_{mod}, which means that $\bar{c}_\Delta$ is a τ_{mod}-Finsler geodesic.

(ii) That $\bar{c}_\Delta$ is now Θ-regular, follows similarly. The comparability of distances we deduce using our earlier discussion of τ_{mod}-distances along Finsler geodesics. We estimate:

$$\begin{aligned} d(\bar{c}_\Delta(t_1), \bar{c}_\Delta(t_2)) &\geqslant d(\bar{c}_{\tau_{\mathrm{mod}}}(t_1), \bar{c}_{\tau_{\mathrm{mod}}}(t_2)) \\ &= \|d_{\tau_{\mathrm{mod}}}(c(t_1), c(t_2))\| \geqslant \epsilon(\Theta)\cdot d(c(t_1), c(t_2)) \end{aligned}$$

The first inequality holds, because $\bar{c}_{\tau_{\mathrm{mod}}} = \pi^{\Delta}_{\tau_{\mathrm{mod}}} \circ \bar{c}_\Delta$ and $\pi^{\Delta}_{\tau_{\mathrm{mod}}}$ is 1-Lipschitz. The equality follows from (24). The last inequality comes from the lower bound for the length of the τ_{mod}-component of Θ-regular segments, cf. (7). □

3 Topological dynamics

3.1 Expansion

Let first Z be a metric space and let $\Gamma \curvearrowright Z$ be a continuous action by a discrete group. We will use the following notions of *metric expansion*, compare [29, Section 9]:

Definition 3.1 (*Metric expansion*)

- A homeomorphism h of Z is *expanding* at a point $z \in Z$ if there exists a neighborhood U of z and a constant $c > 1$ such that $h|_U$ is c-expanding in the sense that
$$d(hz_1, hz_2) \geqslant c \cdot d(z_1, z_2).$$
for all points $z_1, z_2 \in U$.
- A sequence of homeomorphisms h_n of Z has *diverging expansion* at the point $z \in Z$ if there exists a sequence of neighborhoods U_n of z and numbers $c_n \to +\infty$ such that $h_n|_{U_n}$ is c_n-expanding.
- The action $\Gamma \curvearrowright Z$ is *expanding* at $z \in Z$ if there exists an element $\gamma \in \Gamma$ which is expanding at z. The action has *diverging expansion* at $z \in Z$ if Γ contains a sequence which has diverging expansion at z.
- The action $\Gamma \curvearrowright Z$ is *expanding* at a compact Γ-invariant *subset* $E \subset Z$ if it is expanding at all points $z \in E$.

We observe that the properties of diverging expansion depend only on the bilipschitz class of the metric. Furthermore, if an action is expanding at an invariant compact subset then, due to iteration, it has diverging expansion at every point of the subset.

Now let M be a Riemannian manifold and let $\Gamma \curvearrowright M$ be a smooth action. There are infinitesimal analogs of the above expansion conditions.

We recall from (21) that, for a diffeomorphism Φ of M, the *expansion factor* $\epsilon(\Phi, x)$ at a point $x \in M$ is defined as:

$$\epsilon(\Phi, x) = \inf_{v \in T_x M - \{0\}} \frac{\|d\Phi(v)\|}{\|v\|} = \|(d\Phi_x)^{-1}\|^{-1}. \tag{25}$$

Definition 3.2 (*Infinitesimal expansion*)

- A diffeomorphism Φ of M is *infinitesimally expanding* at a point $x \in M$ if $\epsilon(\Phi, x) > 1$.
- A sequence of diffeomorphisms Φ_n of M has *diverging infinitesimal expansion* at x if $\epsilon(\Phi_n, x) \to +\infty$ as $n \to +\infty$.
- The action $\Gamma \curvearrowright M$ is *infinitesimally expanding* at x if there exists an element $\gamma \in \Gamma$ which is infinitesimally expanding at x. The action has *diverging infinitesimal expansion* at x if Γ contains a sequence which has diverging infinitesimal expansion at x.
- The action $\Gamma \curvearrowright M$ is *infinitesimally expanding* at a compact Γ-invariant *subset* $E \subset M$ if it is infinitesimally expanding at all points $x \in M$.

If the manifold M is compact, the properties of diverging infinitesimal expansion are independent of the Riemannian metric. In the general case, if an action is infinitesimally expanding at an invariant compact subset then it has diverging infinitesimal expansion at every point of the subset.

We note that for smooth actions on Riemannian manifolds infinitesimal and metric expansion are equivalent.

3.2 Discontinuity and dynamical relation

Let Z be a compact metrizable space, and let $\Gamma < \mathrm{Homeo}(Z)$ be a countably infinite subgroup (although in the definition of a proper action below we allow for subsemigroups). We consider the action $\Gamma \curvearrowright Z$.

Definition 3.3 (*Discontinuous*) A point $z \in Z$ is called *wandering* with respect to the Γ-action if the action is *discontinuous* at z, i.e. if z has a neighborhood U such that $U \cap \gamma U \neq \emptyset$ for at most finitely many $\gamma \in \Gamma$.

Nonwandering points are called *recurrent*.

Definition 3.4 (*Domain of discontinuity*) We call the set

$$\Omega_{\mathrm{disc}} \subset Z$$

of wandering points the *wandering set* or *domain of discontinuity* for the action $\Gamma \curvearrowright Z$.

Note that Ω_{disc} is open and Γ-invariant.

Definition 3.5 (*Proper*) The action of a *subsemigroup* $\Gamma < \mathrm{Homeo}(X)$ on an open subset $U \subset Z$ is called *proper* if for every compact subset $K \subset U$, $K \cap \gamma K \neq \emptyset$ for at most finitely many $\gamma \in \Gamma$.

If Γ is a subgroup of $\mathrm{Homeo}(X)$ acting properly discontinuously on $U \subset X$ then the action of Γ on U is then discontinuous, $U \subseteq \Omega_{\mathrm{disc}}$, and therefore is called *properly discontinuous*.

Definition 3.6 (*Domain of proper discontinuity*) If $\Gamma < \mathrm{Homeo}(X)$ is a subgroup, we call a Γ-invariant open subset $\Omega \subseteq \Omega_{\mathrm{disc}}$ on which Γ acts properly a *domain of proper discontinuity* for Γ.

The orbit space Ω/Γ is then Hausdorff. Note that in general there is *no unique* maximal *proper* domain of discontinuity.

Discontinuity and proper discontinuity can be nicely expressed using the notion of dynamical relation. The following definition is due to Frances [8, Definition 1]:

Definition 3.7 (*Dynamically related*) Two points $z, z' \in Z$ are called *dynamically related* with respect to a sequence (h_n) in $\mathrm{Homeo}(Z)$,

$$z \overset{(h_n)}{\sim} z'$$

if there exists a sequence $z_n \to z$ in Z such that $h_n z_n \to z'$.

The points z, z' are called *dynamically related* with respect to the Γ-action,

$$z \overset{\Gamma}{\sim} z'$$

if there exists a sequence $\gamma_n \to \infty$ in Γ such that $z \overset{(\gamma_n)}{\sim} z'$.

Here, for a sequence (γ_n) in Γ we write $\gamma_n \to \infty$ if every element of Γ occurs at most finitely many times in the sequence.

One verifies (see e.g. [16, Lemma 4.22] and the preceding discussion):

(i) Dynamical relation is a closed relation in $Z \times Z$.
(ii) Points in different Γ-orbits are dynamically related if and only if their orbits cannot be separated by disjoint Γ-invariant open subsets.

The concept of dynamical relation is useful for our discussion of discontinuity, because:

(i) A point is nonwandering if and only if it is dynamically related to itself.
(ii) The action is proper on an open subset $U \subset Z$ if and only if no two points in U are dynamically related.

3.3 Convergence groups

Let Z be a compact metrizable space with at least three points. A sequence (h_n) in $\mathrm{Homeo}(Z)$ is *contracting* if there exist points $z_\pm \in Z$ such that

$$h_n|_{Z-\{z_-\}} \to z_+ \tag{26}$$

uniformly on compacts as $n \to +\infty$. Equivalently, there is no dynamical relation $z \overset{(h_n)}{\sim} z'$ between points $z \neq z_-$ and $z' \neq z_+$. This condition is clearly symmetric, i.e. (26) is equivalent to the dual condition that

$$h_n^{-1}|_{Z-\{z_+\}} \to z_-$$

uniformly on compacts as $n \to +\infty$. The points $z_\pm$ are uniquely determined, since $|Z| \geqslant 3$.

A sequence (h_n) in $\mathrm{Homeo}(Z)$ is said to *converge* to a point $z \in Z$,

$$h_n \to z \tag{27}$$

if every subsequence contains a contracting subsequence which, outside its exceptional point, converges to the constant map $\equiv z$.

One considers the following stronger form of convergence:

Definition 3.8 (*Conical convergence*) A converging sequence $h_n \to z$ converges *conically*,

$$h_n \xrightarrow{\text{con}} z$$

if for some relatively compact sequence $(\widehat{z}_n)$ in $Z - \{z\}$, the sequence of pairs of distinct points $h_n^{-1}(\widehat{z}_n, z)$ is relatively compact in $(Z \times Z)^{\text{dist}}$.

Here, $(Z \times Z)^{\text{dist}} \subset Z \times Z$ denotes the complement of the diagonal.

Lemma 3.9 *If $h_n \xrightarrow{\text{con}} z$, then the condition in the definition holds for all relatively compact sequences $(\widehat{z}_n)$ in $Z - \{z\}$.*

Proof Let $(\widehat{z}_n)$ be a relatively compact sequence in $Z - \{z\}$. For every contracting subsequence (h_{n_k}) there exists a point $\widehat{z} \in Z$ such that

$$h_{n_k}^{-1}|_{Z-\{z\}} \to \widehat{z}$$

uniformly on compacts. In particular, $h_{n_k}^{-1}\widehat{z}_{n_k} \to \widehat{z}$ and the relative compactness of $(h_{n_k}^{-1}(\widehat{z}_{n_k}, z))$ in $(Z \times Z)^{\text{dist}}$ becomes equivalent to the condition that the sequence $(h_{n_k}^{-1} z)$ does not accumulate at $\widehat{z}$. The latter condition is independent of the sequence $(\widehat{z}_n)$. ◻

The following criterion for being a conical limit point of a subsequence is immediate:[14]

Lemma 3.10 *A sequence (h_n) in $\mathrm{Homeo}(Z)$ has a subsequence conically converging to $z \in Z$ iff there exists a subsequence (h_{n_k}) and a point $z_- \in Z$ such that the following conditions are satisfied:*

[14] Here it suffices that $|Z| \geqslant 2$.

(i) $h_{n_k}^{-1}|_{Z-\{z\}} \to z_-$ *uniformly on compacts.*
(ii) $(h_{n_k}^{-1}z)$ *converges to a point different from* z_-.

Now we pass to group actions. A continuous action $\Gamma \curvearrowright Z$ of a discrete group Γ is a *convergence action* if every sequence (γ_n) of pairwise distinct elements in Γ contains a subsequence converging to a point, equivalently, a contracting subsequence. The kernel of a convergence action is finite, and we will identify Γ with its image in $\mathrm{Homeo}(Z)$ which we will call a *convergence group*.

The *limit set* $\Lambda \subset Z$ of a convergence group $\Gamma < \mathrm{Homeo}(Z)$ is the subset of all points which occur as limits z_+ as in (26), equivalently, as limits z as in (27) for sequences $\gamma_n \to \infty$ in Γ. The limit set is Γ-invariant and compact. A limit point $\lambda \in \Lambda$ is *conical* if it occurs as the limit of a conically converging sequence. A convergence group is said to have *conical limit set* if all limits points are conical, and to be *non-elementary* if $|\Lambda| \geqslant 3$. Tukia [30, Theorem 2S] has shown that in the non-elementary case the limit set is perfect and the Γ-action on it is minimal.

If the limit set is conical, then Γ and its action on Λ are very special:

Theorem 3.11 (Bowditch [4]) *Suppose that* $\Gamma < \mathrm{Homeo}(Z)$ *is a non-elementary convergence group with conical limit set* Λ. *Then* Γ *is word hyperbolic and* $\Lambda \cong \partial_\infty \Gamma$ *equivariantly.*

The converse is easier to see:

Theorem 3.12 ([9,10,30]) *The natural action of a non-virtually cyclic word hyperbolic group on its Gromov boundary is a minimal conical convergence action.*

3.4 Expanding convergence groups

The following result connects expansion with convergence dynamics.

Lemma 3.13 *If* $\Gamma \curvearrowright Z$ *is an expanding convergence action on a perfect compact metric space, then all points in* Z *are conical limit points.*

Proof We start with a general remark concerning expanding actions. For every point $z \in Z$ there exist an element $\gamma \in \Gamma$ and constants $r > 0$ and $c > 1$ such that γ is a c-expansion on the ball $B(z, r)$ and $\gamma(B(z, r')) \supset B(\gamma z, cr')$ for all radii $r' \leqslant r$. To see this, suppose that c is a local expansion factor for γ at z and, by contradiction, that there exist sequences of radii $r_n \to 0$ and points $z_n \notin B(z, r_n)$ such that $\gamma z_n \in B(\gamma z, cr_n)$. Then $z_n \to z$ due to the continuity of γ^{-1} and, for large n, we obtain a contradiction to the local c-expansion of γ. Since Z is compact, the constants r and c can be chosen uniformly. It follows by iterating expanding maps that for every point z and every neighborhood V of z there exists $\gamma \in \Gamma$ such that $\gamma(V) \supset B(\gamma z, r)$, equivalently, $\gamma(Z - V) \subset Z - B(\gamma z, r)$.

To verify that a point z is conical, let V_n be a shrinking sequence of neighborhoods of z,

$$\bigcap_n V_n = \{z\},$$

and let $\gamma_n \in \Gamma$ be elements such that $\gamma_n^{-1}(Z - V_n) \subset Z - B(\gamma_n^{-1}z, r)$. Since V_n is shrinking and $\gamma_n^{-1}(V_n) \supset B(\gamma_n^{-1}z, r)$ contains balls of uniform radius r, it follows that the γ_n^{-1} do not subconverge uniformly on any neighborhood of z; here we use that Z is perfect. In particular, $\gamma_n \to \infty$. The convergence action property implies that, after passing to a subsequence, the γ_n^{-1} must converge locally uniformly on $Z - \{z\}$. Moreover, we can assume that the sequence of points $\gamma_n^{-1}z$ converges. By construction, its limit will be different (by distance $\geqslant r$) from the limit of the sequence of maps $\gamma_n^{-1}|_{Z-\{z\}}$. Hence the point z is conical. $\square$

Combining this with Bowditch's dynamical characterization of hyperbolic groups, we obtain:

Corollary 3.14 *If $\Gamma \curvearrowright Z$ is an expanding convergence action on a perfect compact metric space, then Γ is word hyperbolic and $Z \cong \partial_\infty \Gamma$ equivariantly.*

Note that, conversely, the natural action $\Gamma \curvearrowright \partial_\infty \Gamma$ of a word hyperbolic group Γ on its Gromov boundary is expanding with respect to a *visual* metric, see e.g. [6].

4 Regularity and contraction

In this section, we discuss a class of discrete subgroups of semisimple Lie groups which will be the framework for most of our investigations in this paper. In particular, it contains Anosov subgroups. The class of subgroups will be distinguished by an asymptotic *regularity* condition which in rank one just amounts to discreteness, but in higher rank is strictly stronger. The condition will be formulated in two equivalent ways. First dynamically in terms of the action on a flag manifold, then geometrically in terms of the orbits in the symmetric space.

4.1 Contraction

Consider the action

$$G \curvearrowright \mathrm{Flag}_{\tau_{mod}}$$

on the flag manifold of type τ_{mod}. Recall that for a simplex τ_- of type $\iota\tau_{mod}$ we denote by $C(\tau_-) \subset \mathrm{Flag}_{\tau_{mod}}$ the open dense P_{τ_-}-orbit; it consists of the simplices opposite to τ_-.

We introduce the following dynamical conditions for sequences and subgroups in G:

Definition 4.1 (*Contracting sequence*) A sequence (g_n) in G is τ_{mod}-*contracting* if there exist simplices $\tau_+ \in \mathrm{Flag}_{\tau_{mod}}$, $\tau_- \in \mathrm{Flag}_{\iota\tau_{mod}}$ such that

$$g_n|_{C(\tau_-)} \to \tau_+ \tag{28}$$

uniformly on compacts as $n \to +\infty$.

See [16, Example 2.56] for the case $G = SL(n, \mathbb{R})$ and σ_{mod}-contracting sequences.

Definition 4.2 (*Convergence type dynamics*) A subgroup $\Gamma < G$ is a τ_{mod}-*convergence subgroup* if every sequence (γ_n) of distinct elements in Γ contains a τ_{mod}-contracting subsequence.

Note that τ_{mod}-contracting sequences diverge to infinity and therefore τ_{mod}-convergence subgroups are necessarily *discrete*.

A notion for sequences in G equivalent to τ_{mod}-contraction had been introduced by Benoist in [3], see in particular part (5) of his Lemma 3.5.

The contraction property exhibits a symmetry:

Lemma 4.3 (Symmetry) *Property* (28) *is equivalent to the dual property that*

$$g_n^{-1}|_{C(\tau_+)} \to \tau_- \tag{29}$$

uniformly on compacts as $n \to +\infty$.

Proof Suppose that (28) holds but (29) fails. Equivalently, after extraction there exists a sequence $\xi_n \to \xi \neq \tau_-$ in $\mathrm{Flag}_{\iota\tau_{mod}}$ such that $g_n\xi_n \to \xi' \in C(\tau_+)$. Since $\xi \neq \tau_-$, there exists $\widehat{\tau}_- \in C(\tau_-)$ not opposite to ξ. (For instance, take an apartment in $\partial_\infty X$ containing τ_- and ξ, and let $\widehat{\tau}_-$ be the simplex opposite to τ_- in this apartment.) Hence there is a sequence $\tau_n \to \widehat{\tau}_-$ in $\mathrm{Flag}_{\tau_{mod}}$ such that τ_n is not opposite to ξ_n for all n. (It can be obtained e.g. by taking a sequence $h_n \to e$ in G such that $\xi_n = h_n\xi$ and putting $\tau_n = h_n\widehat{\tau}_-$.) Since $\widehat{\tau}_- \in C(\tau_-)$, condition (28) implies that $g_n\tau_n \to \tau_+$. It follows that τ_+ is not opposite to ξ', because $g_n\tau_n$ is not opposite to $g_n\xi_n$ and being opposite is an open condition. This contradicts $\xi' \in C(\tau_+)$. Therefore, condition (28) implies (29). The converse implication follows by replacing the sequence (g_n) with (g_n^{-1}). □

Lemma 4.4 (Uniqueness) *The simplices* $\tau_\pm$ *in* (28) *are uniquely determined.*

Proof Suppose that besides (28) we also have $g_n|_{C(\tau'_-)} \to \tau'_+$ with simplices $\tau'_\pm \in \mathrm{Flag}_{\pm\tau_{mod}}$. Since the subsets $C(\tau_-)$ and $C(\tau'_-)$ are open dense in $\mathrm{Flag}_{\tau_{mod}}$, their intersection is nonempty and hence $\tau'_+ = \tau_+$. Using the equivalent dual conditions (29) we similarly obtain that $\tau'_- = \tau_-$. □

4.2 Regularity

The second set of asymptotic properties concerns the geometry of the orbits in X. We first consider sequences in the euclidean model Weyl chamber Δ. Recall that $\partial_{\tau_{mod}}\Delta = V(0, \partial_{\tau_{mod}}\sigma_{mod}) \subset \Delta$ is the union of faces of Δ which do not contain the sector $V(0, \tau_{mod})$. Note that $\partial_{\tau_{mod}}\Delta \cap V(0, \tau_{mod}) = \partial V(0, \tau_{mod}) = V(0, \partial\tau_{mod})$.

Definition 4.5 A sequence (δ_n) in Δ is

- τ_{mod}-*regular* if it drifts away from $\partial_{\tau_{mod}}\Delta$,

$$d(\delta_n, \partial_{\tau_{mod}}\Delta) \to +\infty.$$

- τ_{mod}-*pure* if it is contained in a tubular neighborhood of the sector $V(0, \tau_{\mathrm{mod}})$ and drifts away from its boundary,

$$d(\delta_n, \partial V(0, \tau_{\mathrm{mod}})) \to +\infty.$$

Note that (δ_n) is τ_{mod}-regular/pure iff $(\iota\delta_n)$ is $\iota\tau_{\mathrm{mod}}$-regular/pure. We extend these notions to sequences in X and G:

Definition 4.6 (*Regular and pure*)

- A sequence (x_n) in X is τ_{mod}-*regular*, respectively, τ_{mod}-*pure* if for some (any) base point $o \in X$ the sequence of Δ-distances $d_\Delta(o, x_n)$ in Δ has this property.
- A sequence (g_n) in G is τ_{mod}-*regular*, respectively, τ_{mod}-*pure* if for some (any) point $x \in X$ the orbit sequence $(g_n x)$ in X has this property.
- A subgroup $\Gamma < G$ is τ_{mod}-*regular* if all sequences of distinct elements in Γ have this property.

That these properties are independent of the base point and stable under bounded perturbation of the sequences, is due to the triangle inequality $|d_\Delta(x, y) - d_\Delta(x', y')| \leqslant d(x, x') + d(y, y')$.

Subsequences of τ_{mod}-regular/pure sequences are again τ_{mod}-regular/pure.

Clearly, τ_{mod}-pureness is a strengthening of τ_{mod}-regularity; a sequence in Δ is τ_{mod}-pure iff it is τ_{mod}-regular and contained in a tubular neighborhood of $V(0, \tau_{\mathrm{mod}})$.

The face type of a pure sequence is uniquely determined. Moreover, a τ_{mod}-regular sequence is τ'_{mod}-regular for every face type $\tau'_{\mathrm{mod}} \subset \tau_{\mathrm{mod}}$, because $\partial_{\tau'_{\mathrm{mod}}} \Delta \subset \partial_{\tau_{\mathrm{mod}}} \Delta$.

A sequence (g_n) is τ_{mod}-regular/pure iff the inverse sequence (g_n^{-1}) is $\iota\tau_{\mathrm{mod}}$-regular/pure, because $d_\Delta(x, g_n^{-1} x) = d_\Delta(g_n x, x) = \iota d_\Delta(x, g_n x)$.

Note that τ_{mod}-regular subgroups are in particular *discrete*. If $\mathrm{rank}(X) = 1$, then discreteness is equivalent to (σ_{mod}-)regularity. In higher rank, *regularity* can be considered as a *strengthening of discreteness*: A discrete subgroup $\Gamma < G$ may not be τ_{mod}-regular for any face type τ_{mod}; this can happen e.g. for free abelian subgroups of transvections of rank $\geqslant 2$.

A property for sequences in G equivalent to regularity had appeared in [3, Lemma 3.5 (1)].

See [16, Example 2.24] for σ_{mod}-regularity in the case $G = \mathrm{SL}(n, \mathbb{R})$.

Lemma 4.7 (Pure subsequences) *Every sequence, which diverges to infinity, contains a τ_{mod}-pure subsequence for some face type $\tau_{\mathrm{mod}} \subseteq \sigma_{\mathrm{mod}}$.*

Proof In the case of sequences in Δ, take τ_{mod} to be a minimal face type so that a subsequence is contained in a tubular neighborhood of $V(0, \tau_{\mathrm{mod}})$. ◻

Note also that a sequence, which diverges to infinity, is τ_{mod}-regular iff it contains ν_{mod}-pure subsequences only for face types $\nu_{\mathrm{mod}} \supseteq \tau_{\mathrm{mod}}$.

The lemma implies in particular, that every sequence $\gamma_n \to \infty$ in a discrete subgroup $\Gamma < G$ contains a subsequence which is τ_{mod}-regular, even τ_{mod}-pure, for some face type τ_{mod}.

Remark 4.8 Regularity has a natural Finsler geometric interpretation, cf. [15]: A sequence in X is τ_{mod}-regular iff, in the Finsler compactification $\overline{X}^{Fins} = X \sqcup \partial_{\infty}^{Fins} X$ of X, it accumulates at the closure of the stratum $S_{\tau_{mod}} \subset \partial_{\infty}^{Fins} X$ at infinity.

4.3 Contraction implies regularity

In this section and the next, we relate contractivity and regularity for sequences and, as a consequence, establish the equivalence between τ_{mod}-regularity and the τ_{mod}-convergence property for discrete subgroups.

To relate contraction and regularity, it is useful to consider the G-action on flats. We recall that $\mathcal{F}_{\tau_{mod}}$ denotes the space of flats $f \subset X$ of type τ_{mod} (see Sect. 2.4). Two flats $f_{\pm} \in \mathcal{F}_{\tau_{mod}}$ are *dynamically related* with respect to a sequence (g_n) in G,

$$f_- \overset{(g_n)}{\sim} f_+,$$

if there exists a sequence of flats $f_n \to f_-$ in $\mathcal{F}_{\tau_{mod}}$ such that $g_n f_n \to f_+$. The action of (g_n) on $\mathcal{F}_{\tau_{mod}}$ is *proper* iff there are no dynamical relations with respect to subsequences, cf. Sect. 3.2.

Dynamical relations between singular flats yield dynamical relations between maximal ones:

Lemma 4.9 *If $f_{\pm} \in \mathcal{F}_{\tau_{mod}}$ are flats such that $f_- \overset{(g_n)}{\sim} f_+$, then for every maximal flat $F_+ \supseteq f_+$ there exist a maximal flat $F_- \supseteq f_-$ and a subsequence (g_{n_k}) such that $F_- \overset{(g_{n_k})}{\sim} F_+$.*

Proof Let $f_n \to f_-$ be a sequence in $\mathcal{F}_{\tau_{mod}}$ such that $g_n f_n \to f_+$. Then there exists a sequence of maximal flats $F_n \supseteq f_n$ such that $g_n F_n \to F_+$. The sequence (F_n) is bounded because the sequence (f_n) is, and hence (F_n) subconverges to a maximal flat $F_- \supseteq f_-$. □

For pure sequences there are dynamical relations between singular flats of the corresponding type with respect to suitable subsequences:

Lemma 4.10 *If (g_n) is τ_{mod}-pure, then the action of (g_n) on $\mathcal{F}_{\tau_{mod}}$ is not proper.*

More precisely, there exist simplices $\tau_{\pm} \in \mathrm{Flag}_{\tau_{mod}}$ such that for every flat $f_+ \in \mathcal{F}_{\tau_{mod}}$ asymptotic to τ_+ there exist a flat $f_- \in \mathcal{F}_{\tau_{mod}}$ asymptotic to τ_- and a subsequence (g_{n_k}) such that

$$f_- \overset{(g_{n_k})}{\sim} f_+.$$

Proof By pureness, there exists a sequence (τ_n) in $\mathrm{Flag}_{\tau_{mod}}$ such that

$$\sup_n d(g_n x, V(x, \tau_n)) < +\infty \tag{30}$$

for any point $x \in X$. There exists a subsequence (g_{n_k}) such that $\tau_{n_k} \to \tau_+$ and $g_{n_k}^{-1}\tau_{n_k} \to \tau_-$.

Let $f_+ \in \mathfrak{F}_{\tau_{\mathrm{mod}}}$ be asymptotic to τ_+. We choose $x \in f_+$ and consider the sequence of flats $f_k \in \mathfrak{F}_{\tau_{\mathrm{mod}}}$ through x asymptotic to τ_{n_k}. Then $f_k \to f_+$. The sequence of flats $(g_{n_k}^{-1} f_k)$ is bounded as a consequence of (30). Therefore, after further extraction, we obtain convergence $g_{n_k}^{-1} f_k \to f_-$. The limit flat f_- is asymptotic to τ_- because the f_k are asymptotic to $g_{n_k}^{-1}\tau_{n_k}$. □

By a diagonal argument one can also show that the subsequences (g_{n_k}) in the two previous lemmas can be made independent of the flats F_+ respectively f_+.

For contracting sequences, the possible dynamical relations between maximal flats are restricted as follows:

Lemma 4.11 *Suppose that* (g_n) *is* τ_{mod}-contracting *with* (28)*, and that* $F_- \overset{(g_n)}{\sim} F_+$ *for maximal flats* $F_\pm \in \mathfrak{F}$*. Then* $\tau_\pm \subset \partial_\infty F_\pm$*.*

Proof Suppose that $\tau_- \not\subset \partial_\infty F_-$. Then the visual boundary sphere $\partial_\infty F_-$ contains at least two different simplices $\hat\tau_-, \hat\tau'_-$ opposite to τ_-, cf. Lemma 2.2.

Let $F_n \to F_-$ be a sequence in $\mathfrak{F}$ such that $g_n F_n \to F_+$. Due to $F_n \to F_-$, there exist sequences of simplices $\tau_n, \tau'_n \subset \partial_\infty F_n$ such that $\tau_n \to \hat\tau_-$ and $\tau'_n \to \hat\tau'_-$. In particular, $\tau_n \neq \tau'_n$ for large n. After extraction, we also obtain convergence $g_n\tau_n \to \hat\tau_+$ and $g_n\tau'_n \to \hat\tau'_+$. Moreover, since $g_n F_n \to F_+$, it follows that the limits $\hat\tau_+, \hat\tau'_+$ are *different* simplices in $\partial_\infty F_+$.

This is however in conflict with the contraction property (28). In view of $\hat\tau_-, \hat\tau'_- \in C(\tau_-)$, the latter implies that $g_n\tau_n \to \tau_+$ and $g_n\tau'_n \to \tau_+$, convergence to the same simplex, a contradiction. Thus, $\tau_- \subset \partial_\infty F_-$.

Considering the inverse sequence (g_n^{-1}) yields that also $\tau_+ \subset \partial_\infty F_+$, cf. Lemma 4.3. □

Combining the previous lemmas, we obtain:

Lemma 4.12 *If a sequence in* G *is* τ_{mod}*-contracting and* ν_{mod}*-pure, then* $\tau_{\mathrm{mod}} \subseteq \nu_{\mathrm{mod}}$.

Proof We denote the sequence by (g_n) and assume (28). According to Lemmas 4.10 and 4.9, by ν_{mod}-purity, there exist simplices $\nu_\pm \in \mathrm{Flag}_{\nu_{\mathrm{mod}}}$ such that for *every* maximal flat F_+ with $\partial_\infty F_+ \supset \nu_+$ there exist a maximal flat F_- with $\partial_\infty F_- \supset \nu_-$ and a subsequence (g_{n_k}) such that

$$F_- \overset{(g_{n_k})}{\sim} F_+.$$

By Lemma 4.11, always $\tau_+ \subset \partial_\infty F_+$. Varying F_+, it follows that $\tau_+ \subseteq \nu_+$, cf. Lemma 2.3. □

From these observations, we conclude:

Proposition 4.13 (Contracting implies regular) *If a sequence in* G *is* τ_{mod}*-contracting, then it is* τ_{mod}*-regular.*

Proof Consider a sequence in G which is not τ_{mod}-regular. Then a subsequence is ν_{mod}-pure for some face type $\nu_{mod} \subseteq \partial_{\tau_{mod}} \sigma_{mod}$, compare Lemma 4.7. The condition on the face type is equivalent to $\nu_{mod} \not\supseteq \tau_{mod}$. By the last lemma, the subsequence cannot be τ_{mod}-contracting. $\square$

4.4 Regularity implies contraction

We now prove a converse to Proposition 4.13. Since contractivity involves a convergence condition, we can expect regular sequences to be contracting only after extraction.

Consider a τ_{mod}-regular sequence (g_n) in G. After fixing a point $x \in X$, there exist simplices $\tau_n^{\pm} \in \mathrm{Flag}_{\pm\tau_{mod}}$ (unique for large n) such that

$$g_n^{\pm 1} x \in V(x, \mathrm{st}(\tau_n^{\pm})). \tag{31}$$

Note that the sequence (g_n^{-1}) is $\iota\tau_{mod}$-regular, compare the comment after Definition 4.6.

Lemma 4.14 *If $\tau_n^{\pm} \to \tau_{\pm}$ in $\mathrm{Flag}_{\pm\tau_{mod}}$, then (g_n) is τ_{mod}-contracting with (28).*

Proof Since $x \in g_n V(x, \mathrm{st}(\tau_n^-)) = V(g_n x, \mathrm{st}(g_n \tau_n^-))$, it follows together with $g_n x \in V(x, \mathrm{st}(\tau_n^+))$ that the Weyl cones $V(g_n x, \mathrm{st}(g_n \tau_n^-))$ and $V(x, \mathrm{st}(\tau_n^+))$ lie in the same parallel set, namely in $P(g_n \tau_n^-, \tau_n^+)$, and face in opposite directions. In particular, the simplices $g_n \tau_n^-$ and τ_n^+ are x-opposite, and thus $g_n \tau_n^-$ converges to the simplex $\widehat{\tau}_+$ which is x-opposite to τ_+,

$$g_n \tau_n^- \to \widehat{\tau}_+.$$

Since the sequence $(g_n^{-1} x)$ is $\iota\tau_{mod}$-regular, it holds that

$$d(g_n^{-1} x, \partial V(x, \mathrm{st}(\tau_n^-))) \to +\infty.$$

By Lemma 2.22, for any $r, R > 0$, one has for $n \geqslant n(r, R)$ the inclusion of shadows (cf. (12))

$$U_{\tau_n^-, x, R} \subset U_{\tau_n^-, g_n^{-1} x, r}.$$

Consequently, there exist sequences of positive numbers $R_n \to +\infty$ and $r_n \to 0$ such that

$$U_{\tau_n^-, x, R_n} \subset U_{\tau_n^-, g_n^{-1} x, r_n}$$

for large n, equivalently

$$g_n U_{\tau_n^-, x, R_n} \subset U_{g_n \tau_n^-, x, r_n}. \tag{32}$$

Since $\tau_n^- \to \tau_-$ and $R_n \to +\infty$, the shadows $U_{\tau_n^-,x,R_n} \subset C(\tau_n^-) \subset \mathrm{Flag}_{\tau_{\mathrm{mod}}}$ *exhaust* $C(\tau_-)$ in the sense that every compact in $C(\tau_-)$ is contained in $U_{\tau_n^-,x,R_n}$ for large n.[15] On the other hand, since $g_n\tau_n^- \to \widehat{\tau}_+$ and $r_n \to 0$, the $U_{g_n\tau_n^-,x,r_n}$ *shrink*, i.e. Hausdorff converge to the point τ_+.[16] Therefore, (32) implies that

$$g_n|_{C(\tau_-)} \to \tau_+$$

uniformly on compacts, i.e. (g_n) is τ_{mod}-contracting. □

With the lemma, we can add the desired converse to Proposition 4.13 and obtain a characterization of regularity in terms of contraction:

Proposition 4.15 *The following properties are equivalent for sequences in G:*

(i) *Every subsequence contains a τ_{mod}-contracting subsequence.*
(ii) *The sequence is τ_{mod}-regular.*

Proof This is a direct consequence of the lemma. For the implication (ii)$\Rightarrow$(i) one uses the compactness of flag manifolds. The implication (i)$\Rightarrow$(ii) is obtained as follows, compare the proof of Proposition 4.13: If a sequence is not τ_{mod}-regular, then it contains a ν_{mod}-pure subsequence for some face type $\nu_{\mathrm{mod}} \not\supseteq \tau_{\mathrm{mod}}$. Every subsequence of this subsequence is again ν_{mod}-pure and hence not τ_{mod}-contracting by Lemma 4.12. □

A version of Proposition 4.15 had already been proven by Benoist in [3, Lemma 3.5].

We conclude for subgroups:

Theorem 4.16 *A subgroup $\Gamma < G$ is τ_{mod}-regular iff it is a τ_{mod}-convergence subgroup.*

Proof By definition, Γ is τ_{mod}-regular iff every sequence (γ_n) of distinct elements in Γ is τ_{mod}-regular, and τ_{mod}-convergence iff every such sequence (γ_n) has a τ_{mod}-contracting subsequence. According to the proposition, both conditions are equivalent. □

4.5 Convergence at infinity and limit sets

The discussion in the preceding two sections leads to a natural notion of convergence at infinity for regular sequences in X and G. As regularity, it can be expressed both in terms of orbit geometry in X and dynamics on flag manifolds.

15 Indeed, for fixed $R > 0$ we have Hausdorff convergence $U_{\tau_n^-,x,R} \to U_{\tau_-,x,R}$ in $\mathrm{Flag}_{\tau_{\mathrm{mod}}}$, which follows e.g. from the transitivity of the action $K_x \curvearrowright \mathrm{Flag}_{\iota\tau_{\mathrm{mod}}}$ of the maximal compact subgroup $K_x < G$ fixing x. Furthermore, the shadows $U_{\tau_-,x,R}$ exhaust $C(\tau_-)$ as $R \to +\infty$, cf. the continuity part of Lemma 2.21.

16 Indeed, $U_{g_n\tau_n^-,x,r} \to U_{\hat{\tau}_+,x,r}$ in $\mathrm{Flag}_{\tau_{\mathrm{mod}}}$ for fixed $r > 0$, and $U_{\hat{\tau}_+,x,r} \to \tau_+$ as $r \to 0$, using again the continuity part of Lemma 2.21 and the fact that the function (11) assumes the value zero only in τ_+.

We first consider a τ_{mod}-regular sequence (g_n) in G. Flexibilizing condition (31), we choose points $x, x' \in X$ and consider a sequence (τ_n) in $\mathrm{Flag}_{\tau_{\mathrm{mod}}}$ such that

$$\sup_n d(g_n x, V(x', \mathrm{st}(\tau_n))) < +\infty. \tag{33}$$

Note that the condition is independent of the choice of the points x and x'.[17]

Lemma 4.17 *The accumulation set of* (τ_n) *in* $\mathrm{Flag}_{\tau_{\mathrm{mod}}}$ *depends only on* (g_n).

Proof Let (τ'_n) be another sequence in $\mathrm{Flag}_{\tau_{\mathrm{mod}}}$ such that $d(g_n x, V(x', \mathrm{st}(\tau'_n)))$ is uniformly bounded. Assume that after extraction $\tau_n \to \tau$ and $\tau'_n \to \tau'$. We must show that $\tau = \tau'$.

We may suppose that $x' = x$. There exist bounded sequences (b_n) and (b'_n) in G such that

$$g_n b_n x \in V(x, \mathrm{st}(\tau_n)) \qquad \text{and} \qquad g_n b'_n x \in V(x, \mathrm{st}(\tau'_n))$$

for all n. Note that the sequences $(g_n b_n)$ and $(g_n b'_n)$ in G are again τ_{mod}-regular. By Lemma 4.14, after further extraction, they are τ_{mod}-contracting with

$$g_n b_n|_{C(\tau_-)} \to \tau \qquad \text{and} \qquad g_n b'_n|_{C(\tau'_-)} \to \tau'$$

uniformly on compacts for some $\tau_-, \tau'_- \in \mathrm{Flag}_{\iota\tau_{\mathrm{mod}}}$. Moreover, we may assume convergence $b_n \to b$ and $b'_n \to b'$. Then

$$g_n|_{C(b\tau_-)} \to \tau \qquad \text{and} \qquad g_n|_{C(b'\tau'_-)} \to \tau'$$

uniformly on compacts. With Lemma 4.4 it follows that $\tau = \tau'$. ☐

In view of the lemma, we can define the following notion of convergence:

Definition 4.18 (*Flag convergence of sequences in G*) A τ_{mod}-regular sequence (g_n) in G τ_{mod}-*flag converges* to a simplex $\tau \in \mathrm{Flag}_{\tau_{\mathrm{mod}}}$,

$$g_n \to \tau,$$

if $\tau_n \to \tau$ in $\mathrm{Flag}_{\tau_{\mathrm{mod}}}$ for some sequence (τ_n) in $\mathrm{Flag}_{\tau_{\mathrm{mod}}}$ satisfying (33).

We can now characterize contraction in terms of flag convergence. We rephrase Lemma 4.14 and show that its converse holds as well:

Lemma 4.19 *For a sequence* (g_n) *in* G *and simplices* $\tau_\pm \in \mathrm{Flag}_{\pm\tau_{\mathrm{mod}}}$, *the following are equivalent:*

[17] Recall that the Hausdorff distance of asymptotic Weyl cones $V(y, \mathrm{st}(\tau))$ and $V(y', \mathrm{st}(\tau))$ is bounded by the distance $d(y, y')$ of their tips.

(i) (g_n) *is* τ_{mod}*-contracting with* $g_n|_{C(\tau_-)} \to \tau_+$ *uniformly on compacts.*
(ii) (g_n) *is* τ_{mod}*-regular and* $g_n^{\pm 1} \to \tau_\pm$.

In part (ii), the sequence (g_n^{-1}) is $\iota\tau_{\mathrm{mod}}$-regular and $g_n^{-1} \to \tau_-$ means $\iota\tau_{\mathrm{mod}}$-flag convergence.

Proof The implication (ii) ⇒ (i) is Lemma 4.14.

Conversely, suppose that (i) holds. Since the sequence (g_n) is τ_{mod}-contracting, it is τ_{mod}-regular by Proposition 4.13. Let $(\tau_n^\pm)$ be sequences satisfying (31). We must show that $\tau_n^\pm \to \tau_\pm$. Otherwise, after extraction we obtain that $\tau_n^\pm \to \tau'_\pm$ with $\tau'_+ \neq \tau_+$ or $\tau'_- \neq \tau_-$. Then also $g_n|_{C(\tau'_-)} \to \tau'_+$ by Lemma 4.14, and Lemma 4.4 implies that $\tau'_\pm = \tau_\pm$, a contradiction. □

Vice versa, we can characterize flag convergence in terms of contraction and thus give an alternative *dynamical definition* of it:

Lemma 4.20 *For a sequence* (g_n) *in G, the following are equivalent:*

(i) (g_n) *is* τ_{mod}*-regular and* $g_n \to \tau$.
(ii) *There exist a bounded sequence* (b_n) *in G and* $\tau_- \in \mathrm{Flag}_{\iota\tau_{\mathrm{mod}}}$ *such that* $g_n b_n|_{C(\tau_-)} \to \tau$ *uniformly on compacts.*
(iii) *There exists a bounded sequence* (b'_n) *in G such that* $b'_n g_n^{-1}|_{C(\tau)}$ *converges to a constant map uniformly on compacts.*

Proof (ii) ⇒ (i): According to the previous lemma the sequence $(g_n b_n)$ is τ_{mod}-regular and τ_{mod}-flag converges, $g_n b_n \to \tau$. Since $d(g_n x, g_n b_n x)$ is uniformly bounded, this is equivalent to (g_n) being τ_{mod}-regular and $g_n \to \tau$.

(i) ⇒ (ii): The sequence (g_n^{-1}) is $\iota\tau_{\mathrm{mod}}$-regular. There exists a bounded sequence (b'_n) in G such that $(b'_n g_n^{-1})$ $\iota\tau_{\mathrm{mod}}$-flag converges, $b'_n g_n^{-1} \to \tau_- \in \mathrm{Flag}_{\iota\tau_{\mathrm{mod}}}$. We put $b_n = {b'_n}^{-1}$. Since also $(g_n b_n)$ is τ_{mod}-regular and $g_n b_n \to \tau$, it follows from the previous lemma that $g_n b_n|_{C(\tau_-)} \to \tau$ uniformly on compacts.

The equivalence (ii) ⇔ (iii) with $b'_n = b_n^{-1}$ follows from Lemma 4.3. □

We carry over the notion of flag convergence to sequences in X.

Consider now a τ_{mod}-regular sequence (x_n) in X. We choose again a base point $x \in X$ and consider a sequence (τ_n) in $\mathrm{Flag}_{\tau_{\mathrm{mod}}}$ such that

$$\sup_n d(x_n, V(x, \mathrm{st}(\tau_n))) < +\infty, \tag{34}$$

analogous to (33). As before, the condition is independent of the choice of the point x, and we obtain a version of Lemma 4.17:

Lemma 4.21 *The accumulation set of* (τ_n) *in* $\mathrm{Flag}_{\tau_{\mathrm{mod}}}$ *depends only on* (x_n).

Proof Let (g_n) be a sequence in G such that the sequence $(g_n^{-1} x_n)$ in X is bounded. Then (g_n) is τ_{mod}-regular and (34) becomes equivalent to (33). This reduces the claim to Lemma 4.17. □

We therefore can define, analogous to Definition 4.18 above:

Definition 4.22 (*Flag convergence of sequences in X*) A τ_{mod}-regular sequence (x_n) in X τ_{mod}*-flag converges* to a simplex $\tau \in \mathrm{Flag}_{\tau_{\mathrm{mod}}}$,

$$x_n \to \tau,$$

if $\tau_n \to \tau$ in $\mathrm{Flag}_{\tau_{\mathrm{mod}}}$ for some sequence (τ_n) in $\mathrm{Flag}_{\tau_{\mathrm{mod}}}$ satisfying (34).

For any τ_{mod}-regular sequence (g_n) in G and any point $x \in X$, we have $g_n \to \tau$ iff $g_n x \to \tau$.

Flag convergence and flag limits are stable under bounded perturbations of sequences:

Lemma 4.23 (i) *For any τ_{mod}-regular sequence (g_n) and any bounded sequence (b_n) in G, the sequences (g_n) and $(g_n b_n)$ have the same τ_{mod}-flag accumulation sets in* $\mathrm{Flag}_{\tau_{\mathrm{mod}}}$.

(ii) *If (x_n) and (x'_n) are τ_{mod}-regular sequences in X such that $d(x_n, x'_n)$ is uniformly bounded, then both sequences have the same τ_{mod}-flag accumulation set in* $\mathrm{Flag}_{\tau_{\mathrm{mod}}}$.

Proof (i) The sequence $(g_n b_n)$ is also τ_{mod}-regular and satisfies condition (33) iff (g_n) does. (ii) The sequence (x'_n) satisfies condition (34) iff (x'_n) does. $\square$

Remark 4.24 There is a natural topology on the bordification $X \sqcup \mathrm{Flag}_{\tau_{\mathrm{mod}}}$ which induces τ_{mod}-flag convergence. Moreover, the bordification embeds into a natural Finsler compactification of X, compare Remark 4.8.

Flag convergence leads to a notion of limit sets in flag manifolds for subgroups:

Definition 4.25 (*Flag limit set*) For a subgroup $\Gamma < G$, the τ_{mod}*-limit set*

$$\Lambda_{\tau_{\mathrm{mod}}}(\Gamma) \subset \mathrm{Flag}_{\tau_{\mathrm{mod}}}$$

is the set of possible limit simplices of τ_{mod}-flag converging τ_{mod}-regular sequences in Γ, equivalently, the set of simplices τ_+ as in (28) for all τ_{mod}-contracting sequences in Γ.

The limit set is Γ-invariant and closed, as a diagonal argument shows.

Remark 4.26 Benoist introduced in [3, Section 3.6] a notion of limit set Λ_Γ for Zariski dense subgroups Γ of reductive algebraic groups over local fields which in the case of real semisimple Lie groups is equivalent to (the dynamical version of) our concept of σ_{mod}-limit set $\Lambda_{\sigma_{\mathrm{mod}}}$.[18] What we call the τ_{mod}-limit set $\Lambda_{\tau_{\mathrm{mod}}}$ for other face types $\tau_{\mathrm{mod}} \subsetneq \sigma_{\mathrm{mod}}$ is mentioned in his Remark 3.6(3), and his work implies that, in the Zariski dense case, $\Lambda_{\tau_{\mathrm{mod}}}$ is the image of $\Lambda_{\sigma_{\mathrm{mod}}}$ under the natural projection $\mathrm{Flag}_{\sigma_{\mathrm{mod}}} \to \mathrm{Flag}_{\tau_{\mathrm{mod}}}$ of flag manifolds.

See [16, Example 3.8] where a specific example has been worked out.

[18] Benoist's limit set Λ_Γ is contained in the flag manifold Y_Γ which in the case of real Lie groups is the full flag manifold G/B, see the beginning of Section 3 of his paper. It consists of the limit points of sequences contracting on G/B, cf. his Definitions 3.5 and 3.6.

4.6 Uniform regularity

In this section we introduce stronger forms of the regularity conditions discussed in Sect. 4.2. We first consider sequences in the euclidean model Weyl chamber Δ.

Definition 4.27 A sequence $\delta_n \to \infty$ in Δ is *uniformly* τ_{mod}*-regular* if it drifts away from $\partial_{\tau_{\text{mod}}}\Delta$ at a linear rate with respect to its norm,

$$\liminf_{n\to+\infty} \frac{d(\delta_n, \partial_{\tau_{\text{mod}}}\Delta)}{\|\delta_n\|} > 0.$$

We extend these notions to sequences in X and G, compare Definition 4.6:

Definition 4.28 (*Uniformly regular*)

- A sequence (x_n) in X is *uniformly* τ_{mod}*-regular* if for some (any) base point $o \in X$ the sequence of Δ-distances $d_\Delta(o, x_n)$ in Δ has this property.
- A sequence (g_n) in G is *uniformly* τ_{mod}*-regular* if for some (any) point $x \in X$ the orbit sequence $(g_n x)$ in X has this property.
- A subgroup $\Gamma < G$ is *uniformly* τ_{mod}*-regular* if all sequences of distinct elements in Γ have this property.

For a subgroup $\Gamma < G$, uniform τ_{mod}-regularity is equivalent to the visual limit set $\Lambda(\Gamma) \subset \partial_\infty X$ being contained in the union of the open τ_{mod}-stars.

A subgroup $\Gamma < G$ is uniformly τ_{mod}-regular iff it is uniformly $\iota\tau_{\text{mod}}$-regular.

5 Asymptotic and coarse properties of discrete subgroups

This section is the core of the paper. In Sect. 5.2, motivated by the boundary map part of the original Anosov notion, we study equivariant embeddings of the Gromov boundaries of word hyperbolic subgroups into flag manifolds. We show how these boundary embeddings can be used, especially for regular subgroups, to control the geometry of the orbits in the symmetric space: Intrinsic[19] geodesic lines in the subgroup are uniformly close to parallel sets in the symmetric space. Moreover, in the generic case, for instance for Zariski dense subgroups, intrinsic rays in the subgroup are close to Weyl cones. This conicality property implies in particular that the boundary map continuously extends the orbit maps to infinity and identifies the Gromov boundary with the limit set. This leads us to notion of asymptotically embedded subgroups discussed in Sect. 5.3. We find that asymptotic embeddedness has strong implications for the coarse extrinsic geometry of subgroups: They are undistorted, and moreover their intrinsic geodesics satisfy a higher rank version of the "Morse property"; they are uniformly close to diamonds. This motivates the notion of Morse subgroups studied in Sect. 5.4. The higher rank Morse property immediately implies that the limit set is conical and antipodal. We call regular subgroups with the latter properties RCA and study them in Sect. 5.5. Using Bowditch's dynamical characterization of hyperbolic

[19] I.e. with respect to the word metric.

groups, we show that RCA subgroups are asymptotically embedded, closing part of the circle. In Sect. 5.7, we observe that conicality implies expansive dynamics at the limit set, which yields another equivalent property for subgroups, this time formulated purely in terms of the dynamics on flag manifolds. In Sects. 5.8 and 5.11, we discuss different (uniform and non-uniform) versions of our Anosov condition and show that it is equivalent to the previous conditions as well as to the original definition of Anosov subgroups. In Sect. 5.10 we take up the discussion of the Morse property. Leaving the context of discrete subgroups, we study the geometry of Morse quasigeodesics in symmetric spaces. We characterize them as bounded perturbations of Finsler quasigeodesics and study the behavior of the Δ-distance along them: we prove that via the Δ-distance they project to Morse quasigeodesics in Δ. We also obtain another characterization of Morse subgroups by the quasiconvexity property that their intrinsic geodesics are extrinsically Morse quasigeodesics, equivalently, are uniformly close to Finsler geodesics.

5.1 Antipodality

If X has rank one, then G acts transitively on pairs of distinct points in $\partial_\infty X$. Thus there are only two possibilities for the *relative position* of two points in the visual boundary: They can coincide or be different. In higher rank, the G-actions on the associated flag manifolds are in general not two point transitive and there are more possibilities for the relative position.

We recall (see Sect. 2.4) that two simplices $\tau, \tau' \subset \partial_\infty X$ are called *opposite* or *antipodal* if they are opposite simplices in the apartments $a \subset \partial_\infty X$ containing them both. Their types are then related by $\theta(\tau') = \iota\theta(\tau)$. In particular, if three simplices are pairwise opposite, their types must be equal and ι-invariant.

Definition 5.1 (*Antipodal*) Suppose that τ_{mod} is ι-invariant.

- A subset of $\mathrm{Flag}_{\tau_{mod}}$ is *antipodal* if it consists of pairwise opposite simplices.
- A map into $\mathrm{Flag}_{\tau_{mod}}$ is *antipodal* if it sends different elements to opposite simplices.
- A subgroup $\Gamma < G$ is τ_{mod}-*antipodal* if $\Lambda_{\tau_{mod}}(\Gamma)$ is antipodal.

Being antipodal is an *open* condition for pairs of points in flag manifolds. It is the *generic* relative position. Antipodal maps are in particular *injective*.

We note that for a τ_{mod}-antipodal τ_{mod}-convergence subgroup $\Gamma < G$ the action

$$\Gamma \curvearrowright \Lambda_{\tau_{mod}}(\Gamma)$$

has *convergence dynamics* in the usual sense, see Sect. 3.3: If (γ_n) is a sequence in Γ such that $\gamma_n|_{C(\tau_-)} \to \tau_+$, then $\tau_\pm \in \Lambda_{\tau_{mod}}(\Gamma)$. Due to antipodality, $\Lambda_{\tau_{mod}}(\Gamma) - \{\tau_-\} \subset C(\tau_-)$ and we obtain the desired convergence property.

5.2 Boundary embeddings and limit sets

In this section, we study embeddings of word hyperbolic groups into semisimple Lie groups which admit a certain kind of continuous boundary map. We will assume that τ_{mod} is ι-invariant.

Definition 5.2 (*Boundary embedded*) A subgroup $\Gamma < G$ is τ_{mod}*-boundary embedded* if it is word hyperbolic and there exists an antipodal Γ-equivariant continuous embedding

$$\beta\colon \partial_\infty \Gamma \to \mathrm{Flag}_{\tau_{\mathrm{mod}}}$$

of the Gromov boundary $\partial_\infty \Gamma$ of Γ. The map β is called a *boundary embedding*. If $|\partial_\infty \Gamma| \leqslant 2$, we require in addition that Γ is discrete in G.

Thus, τ_{mod}-boundary embedded subgroups are necessarily *discrete*, since Γ acts on $\beta(\partial_\infty \Gamma)$ as a discrete convergence group if $|\partial_\infty \Gamma| \geqslant 3$.[20]

Boundary embeddings are in general not unique. This is so by trivial reasons if $|\partial_\infty \Gamma| = 2$, cf. below, but it also happens if $|\partial_\infty \Gamma| \geqslant 3$, see [20, Example 6.20].

In order to understand the implications of a boundary embedding, we will first use it to obtain control on the geometry of the Γ-orbits in X.

We fix a word metric on Γ. Via the antipodal boundary embedding β one can assign to every discrete geodesic *line*[21] $l\colon \mathbb{Z} \to \Gamma$ a parallel set in X. Namely, let $\zeta_\pm := l(\pm\infty) \in \partial_\infty \Gamma$ denote the ideal endpoints of the line. Their image simplices $\beta(\zeta_\pm) \in \mathrm{Flag}_{\tau_{\mathrm{mod}}}$ are opposite and determine the parallel set

$$P(\beta(\zeta_-), \beta(\zeta_+)) \subset X.$$

We consider the images of the discrete geodesic lines l in Γ under the orbit map $o_x : \Gamma \to \Gamma x \subset X$ for a point $x \in X$ (fixed throughout the discussion) and claim that the discrete paths $lx : \mathbb{Z} \to X$ are uniformly close to the corresponding parallel sets:[22]

Lemma 5.3 (Lines go close to parallel sets) *The discrete path lx is contained in a tubular neighborhood of the parallel set $P(\beta(\zeta_-), \beta(\zeta_+))$ with uniform radius $\rho = \rho(\Gamma, x)$.*

Here and below, we mean by the dependence of a constant on Γ that it depends on Γ as a subgroup of G and also on the chosen word metric on Γ.

Proof This can be seen by a simple compactness argument: Let

$$(\mathrm{Flag}_{\tau_{\mathrm{mod}}} \times \mathrm{Flag}_{\tau_{\mathrm{mod}}})^{\mathrm{opp}} \subset \mathrm{Flag}_{\tau_{\mathrm{mod}}} \times \mathrm{Flag}_{\tau_{\mathrm{mod}}} \tag{35}$$

[20] Note that boundary embedded subgroups are not required to be regular, although they frequently are, see [16, Theorem 3.11].

[21] Recall that by a discrete geodesic line, we mean an isometric embedding of $\mathbb{Z}$, cf. Sect. 2.1.

[22] For a map $\phi\colon N \to \Gamma$ and a point $x \in X$ we denote by $\phi x\colon N \to X$ the map sending $n \in N$ to $\phi(n)x \in X$.

denote the subspace of pairs of opposite simplices. It is the open and dense G-orbit and in particular a homogeneous G-space. The latter implies that the function on $(\mathrm{Flag}_{\tau_{\mathrm{mod}}} \times \mathrm{Flag}_{\tau_{\mathrm{mod}}})^{\mathrm{opp}} \times X$ assigning

$$(\tau_-, \tau_+, x') \mapsto d(x', P(\tau_-, \tau_+)) \tag{36}$$

is continuous, because $d(gx', P(h\tau_-, h\tau_+)) = d(h^{-1}gx', P(\tau_-, \tau_+))$ for $g, h \in G$. Also the map

$$\mathcal{L} \to (\mathrm{Flag}_{\tau_{\mathrm{mod}}} \times \mathrm{Flag}_{\tau_{\mathrm{mod}}})^{\mathrm{opp}} \times X$$

from the space $\mathcal{L}$ of discrete geodesic lines $l \colon \mathbb{Z} \to \Gamma$.[23] sending $l \mapsto (\beta(l(-\infty)), \beta(l(+\infty)), l(0)x)$ is continuous. Composing both, we see that the map

$$l \mapsto d\big(l(0)x, P(\beta(l(-\infty)), \beta(l(+\infty)))\big)$$

is continuous. Since it is also Γ-periodic, the cocompactness of the action $\Gamma \curvearrowright \mathcal{L}$ implies that it is bounded, whence the assertion. ◻

From now on, we *assume* that the subgroup $\Gamma < G$ is, in addition to being τ_{mod}-boundary embedded, also τ_{mod}-*regular*. This assumption will enable us to further restrict the orbit geometry and will lead to information on the relation between the boundary embedding and the limit set.

We now analyze the position of the images of *rays* in Γ along the parallel sets. Let $r \colon \mathbb{N}_0 \to \Gamma$ be a discrete geodesic ray with ideal endpoint $\zeta := r(+\infty) \in \partial_\infty \Gamma$. There is a dichotomy for the position of the orbit path $rx \colon \mathbb{N}_0 \to X$ relative to the Weyl cone $V(r(0)x, \mathrm{st}(\beta(\zeta)))$ with tip at its initial point, namely the path must either drift away from the cone or dive deep into it:

Lemma 5.4 (Rays dive into Weyl cones or drift away) *There exist constants $\rho' = \rho'(\Gamma, x) > 0$ and for all $R > 0$ numbers $n_0 = n_0(\Gamma, x, R) \in \mathbb{N}$ such that the following holds: For all $n \in \mathbb{N}$ with $n \geqslant n_0$, the point $r(n)x$ either has*

(i) *distance $\geqslant R$ from the Weyl cone $V(r(0)x, \mathrm{st}(\beta(\zeta)))$, or has*
(ii) *distance $\leqslant \rho'$ from this Weyl cone and distance $\geqslant R$ from its boundary.*

Proof In a word hyperbolic group, discrete geodesic rays are contained in uniformly bounded neighborhoods of discrete geodesic lines. Thus, r is contained in a tubular neighborhood with uniform radius $c(\Gamma)$ of a line $l \colon \mathbb{Z} \to \Gamma$ asymptotic to $\zeta = r(+\infty)$ and some $\widehat{\zeta} \in \partial_\infty \Gamma - \{\zeta\}$.

It follows from the previous lemma that the path rx is contained in a tubular neighborhood of the parallel set $P = P(\beta(\widehat{\zeta}), \beta(\zeta))$ with uniform radius $\rho''(\Gamma, x)$. Let $x_0 \in P$ be a point with $d(x_0, r(0)x) \leqslant \rho''$. The Weyl cone $V(r(0)x, \mathrm{st}(\beta(\zeta)))$ is then ρ''-Hausdorff close to the asymptotic Weyl cone $V(x_0, \mathrm{st}(\beta(\zeta))) \subset P$.

[23] The space $\mathcal{L}$ of discrete geodesic lines $l \colon \mathbb{Z} \to \Gamma$ is equipped with the topology of pointwise convergence. It is a locally compact Hausdorff space on which Γ acts properly discontinuously and cocompactly.

Now we use that the interior of the Weyl cone $V(x_0, \mathrm{st}(\beta(\zeta)))$ is *open* in the parallel set P and the boundary $\partial V(x_0, \mathrm{st}(\beta(\zeta)))$ of the cone *disconnects* the parallel set, see Lemma 2.12. The τ_{mod}-regularity of Γ implies (along with the triangle inequality for Δ-lengths) that the path rx drifts away from $\partial V(x_0, \mathrm{st}(\beta(\zeta)))$ at a *uniform* rate,

$$d\big(r(n)x, \partial V(x_0, \mathrm{st}(\beta(\zeta)))\big) \geqslant \phi(n)$$

with a function $\phi(n) \to +\infty$ as $n \to +\infty$ independent of the ray r. The assertion follows. □

For all rays in Γ the same of the two alternatives must occur:

Lemma 5.5 (Dichotomy) *For all discrete geodesic rays $r: \mathbb{N}_0 \to \Gamma$, either*

(i) *rx drifts away from the Weyl cone $V(r(0)x, \mathrm{st}(\beta(\zeta)))$, $\zeta = r(+\infty)$, at a uniform rate,*

$$d\big(r(n)x, V(r(0)x, \mathrm{st}(\beta(\zeta)))\big) \to +\infty$$

uniformly as $n \to +\infty$, or

(ii) *rx is contained in the tubular $\rho'(\Gamma, x)$-neighborhood of $V(r(0)x, \mathrm{st}(\beta(\zeta)))$ and drifts away from its boundary at a uniform rate,*

$$d\big(r(n)x, \partial V(r(0)x, \mathrm{st}(\beta(\zeta)))\big) \to +\infty$$

uniformly as $n \to +\infty$.

Proof We give two arguments. The first one is restricted to the nonelementary case: As a consequence of the previous lemma, for every ray r one of the alternatives (i) and (ii) occurs with growth rates independent of the ray. Which alternative occurs, depends only on the asymptote class $\zeta = r(+\infty)$ of the ray, and depends on it continuously, i.e. the subsets of endpoints for either alternative are open in $\partial_\infty\Gamma$. Since they are also Γ-invariant, if $|\partial_\infty\Gamma| \geqslant 3$, the minimality of the action $\Gamma \curvearrowright \partial_\infty\Gamma$ implies that one of the subsets must be empty.

The second argument works in the general case: Again we use that it depends only on the asymptote class of the ray, which alternative occurs. We show that the same alternative occurs for any two distinct asymptote classes $\zeta, \widehat{\zeta} \in \partial_\infty\Gamma$. After replacing a ray r asymptotic to ζ with a subray, we may assume that we are in the situation of the proof of the previous lemma (whose notation we adopt), i.e. that r lies in a uniform tubular neighborhood of a line $l: \mathbb{Z} \to \Gamma$ asymptotic to $\widehat{\zeta}$ and ζ. Moreover, we assume that alternative (ii) holds for ζ and claim that it holds for $\widehat{\zeta}$, as well.

To see this, fix $R \gg \rho', \rho''$ and $n \gg n_0$. Let $x_n \in P = P(\beta(\widehat{\zeta}), \beta(\zeta))$ be a point with $d(x_n, r(n)x) \leqslant \rho''$. Since (ii) holds for r, the point x_n must lie deep inside the cone $V(x_0, \mathrm{st}(\beta(\zeta))) \subset P$. This is equivalent to x_0 lying deep inside the cone $V(x_n, \mathrm{st}(\beta(\widehat{\zeta}))) \subset P$ opening towards the opposite direction. This however implies that $r(0)x$ is uniformly close (with distance $\leqslant 2\rho'' \ll R$) to the cone $V(r(n)x, \mathrm{st}(\beta(\widehat{\zeta})))$. Thus alternative (ii) holds for the subray $l|_{(-\infty,n]\cap\mathbb{Z}}$ of l, and hence also for its ideal endpoint $\widehat{\zeta}$. □

On the other hand, in the nonelementary case, the ray images always drift away (at non-uniform rates) from "opposite" Weyl cones:

Lemma 5.6 (Drifting away from opposite cones) *Suppose that* $|\partial_\infty\Gamma| \geqslant 3$. *Then for every discrete geodesic ray* $r\colon \mathbb{N}_0 \to \Gamma$ *and ideal point* $\widehat{\zeta} \in \partial_\infty\Gamma - \{\zeta\}$, $\zeta = r(+\infty)$, *it holds that*

$$d\big(r(n)x, V(r(0)x, \mathrm{st}(\beta(\widehat{\zeta})))\big) \to +\infty$$

as $n \to +\infty$.

Proof The ray r is contained in a (non-uniform) tubular neighborhood of a line $l\colon \mathbb{Z} \to \Gamma$ asymptotic to $\widehat{\zeta}$ and ζ. The line image lx, and therefore also the ray image rx is contained in a tubular neighborhood of the parallel set $P = P(\beta(\widehat{\zeta}), \beta(\zeta))$.

It follows that the accumulation set $\mathrm{acc}_{\tau_{mod}}(r) \subset \mathrm{Flag}_{\tau_{mod}}$ of r (with respect to τ_{mod}-flag convergence, compare Sect. 4.5) consists of simplices contained in $\partial_\infty P$: Indeed, the nearest point projections $x_n \in P$ of $r(n)x$ lie in euclidean Weyl chambers $V(x_0, \sigma_n) \subset P$. Therefore, in view of Lemma 4.17, $\mathrm{acc}_{\tau_{mod}}(r)$ equals the accumulation set of the sequence (τ_n) in $\mathrm{Flag}_{\tau_{mod}}$ consisting of the type τ_{mod} faces $\tau_n \subseteq \sigma_n \subset \partial_\infty P$.

Now we use nonelementarity and vary the ideal point opposite to ζ. Since $|\partial_\infty\Gamma| \geqslant 3$, there exists a third ideal point $\widehat{\zeta}' \in \partial_\infty\Gamma - \{\zeta, \widehat{\zeta}\}$. It determines another parallel set $P' = P(\beta(\widehat{\zeta}'), \beta(\zeta))$, and the simplices in $\mathrm{acc}_{\tau_{mod}}(r)$ must also be contained in $\partial_\infty P'$. In view of $\beta(\widehat{\zeta}) \not\subset \partial_\infty P'$, it follows that $\beta(\widehat{\zeta}) \notin \mathrm{acc}_{\tau_{mod}}(r)$.

Since rx is contained in a tubular neighborhood of P, we also again have the dichotomy, analogous to the previous lemma, that rx either drifts away from the Weyl cone $V(r(0)x, \mathrm{st}(\beta(\widehat{\zeta})))$ at a uniform rate, as claimed, or stays in a tubular neighborhood of it and drifts away only from its boundary. However, in the latter case, we would have (conical) flag convergence $r(n) \to \beta(\widehat{\zeta})$ as $n \to +\infty$, equivalently, $\mathrm{acc}_{\tau_{mod}}(r) = \{\beta(\widehat{\zeta})\}$, a contradiction. ◻

If Γ is virtually cyclic, i.e. if $|\partial_\infty\Gamma| = 2$, there is a trivial way of modifying the boundary embedding. Namely, then the action $\Gamma \curvearrowright \partial_\infty\Gamma$ commutes with the transposition $t\colon \partial_\infty\Gamma \to \partial_\infty\Gamma$ exchanging the points, and therefore $-\beta := \beta \circ t$ is a boundary embedding as well. Therefore the previous lemma may fail. However, if it fails for β, then it holds for $-\beta$, because case (ii) of the dichotomy in Lemma 5.5 arises.

From the above observations on the orbit geometry we will now deduce information about the limit set and its position relative to the image of the boundary embedding. Let

$$\bar{o}_x = o_x \sqcup \beta\colon \overline{\Gamma} = \Gamma \sqcup \partial_\infty\Gamma \to X \sqcup \mathrm{Flag}_{\tau_{mod}} \tag{37}$$

denote the extension of the orbit map $o_x\colon \Gamma \to \Gamma x \subset X$ to the Gromov compactification $\overline{\Gamma}$ of Γ by $\bar{o}_x|_{\partial_\infty\Gamma} := \beta$. We say that the extension $\bar{o}_x$ is *continuous at infinity* if for all sequences $\gamma_n \to \infty$ in Γ we have flag convergence $\gamma_n \to \beta(\zeta)$ whenever $\gamma_n \to \zeta \in \partial_\infty\Gamma$ in $\overline{\Gamma}$.

We obtain the following dichotomy corresponding to the one in Lemma 5.5:

Theorem 5.7 (Boundary embedding and limit set) *Let* $\Gamma < G$ *be a* τ_{mod}*-regular* τ_{mod}*-boundary embedded subgroup. Then for every boundary embedding* β *either*

(i) $\beta(\partial_\infty\Gamma)\cap\Lambda_{\tau_{\mathrm{mod}}}(\Gamma)=\emptyset$, *and no simplex in* $\beta(\partial_\infty\Gamma)$ *is opposite to a simplex in* $\Lambda_{\tau_{\mathrm{mod}}}(\Gamma)$,[24] *or*

(ii) $\beta(\partial_\infty\Gamma)=\Lambda_{\tau_{\mathrm{mod}}}(\Gamma)$. *Moreover, the extension* $\bar{o}_x$ *is continuous at infinity, after replacing* β *with* $-\beta$ *in the case* $|\partial_\infty\Gamma|=2$, *if necessary.*

Proof Assume first that case (ii) of Lemma 5.5 occurs. Consider a sequence $\gamma_n\to\infty$ in Γ. There exist rays $r_n:\mathbb{N}_0\to\Gamma$ starting in $r_n(0)=e$ and passing at uniformly bounded distance of γ_n. We denote their ideal endpoints by $\zeta_n:=r_n(+\infty)$. Then the orbit points $\gamma_n x$ lie in uniform tubular neighborhoods of the Weyl cones $V(x,\mathrm{st}(\beta(\zeta_n)))$. If $\gamma_n\to\zeta\in\partial_\infty\Gamma$ in $\overline{\Gamma}$, equivalently, $\zeta_n\to\zeta$ in $\partial_\infty\Gamma$, then $\beta(\zeta_n)\to\beta(\zeta)$ in $\mathrm{Flag}_{\tau_{\mathrm{mod}}}$, and it follows τ_{mod}-flag convergence $\gamma_n x\to\beta(\zeta)$. This shows that $\bar{o}_x$ is continuous at infinity and $\beta(\partial_\infty\Gamma)\subseteq\Lambda_{\tau_{\mathrm{mod}}}(\Gamma)$. To see the opposite inclusion, suppose that $\gamma_n x\to\lambda\in\Lambda_{\tau_{\mathrm{mod}}}(\Gamma)$. After extraction, we get convergence $\gamma_n\to\zeta\in\partial_\infty\Gamma$ and conclude from the above that $\lambda=\beta(\zeta)$. Thus also $\Lambda_{\tau_{\mathrm{mod}}}(\Gamma)\subseteq\beta(\partial_\infty\Gamma)$, and conclusion (ii) of the theorem is satisfied.

If $|\partial_\infty\Gamma|=2$ and case (ii) of Lemma 5.5 occurs for $-\beta$, we reach the same conclusion after replacing β with $-\beta$.

Assume now that we are in case (i) of Lemma 5.5. After replacing β with $-\beta$ in the case $|\partial_\infty\Gamma|=2$, if necessary, we may also assume that the conclusion of Lemma 5.6 holds. As before, we consider a sequence $\gamma_n\to\infty$ in Γ and rays r_n. Suppose that $\gamma_n\to\zeta\in\partial_\infty\Gamma$ and let $\widehat{\zeta}\in\partial_\infty\Gamma-\{\zeta\}$ be arbitrary. Since $\zeta_n\to\zeta$, there exist for all large n lines $l_n:\mathbb{Z}\to\Gamma$ with ideal endpoints $l_n(-\infty)=\widehat{\zeta}$ and $l_n(+\infty)=\zeta_n$. The lines l_n pass at uniformly bounded distance from e and γ_n, and they contain the rays r_n in uniform tubular neighborhoods. (For the rest of this argument, uniformity will mean that bounds are independent of n.)

By Lemma 5.3, the ray images $r_n x$ lie in uniform tubular neighborhoods of the parallel sets $P_n=P(\beta(\widehat{\zeta}),\beta(\zeta_n))$ and drift away from both Weyl cones $V(x,\mathrm{st}(\beta(\widehat{\zeta})))$ and $V(x,\mathrm{st}(\beta(\zeta_n)))$. The drift is uniform in the latter case by Lemma 5.5 (i), and also in the former case since $r_n(0)x=x$ and $d(x,P_n)$ is bounded.

The uniformity implies that the orbit points $\gamma_n x$ lie in uniform tubular neighborhoods of Weyl cones $V(x,\mathrm{st}(\tau_n))$ for simplices $\tau_n\in\mathrm{Flag}_{\tau_{\mathrm{mod}}}$ with $\tau_n\subset\partial_\infty P_n$ but $\tau_n\neq\beta(\widehat{\zeta}),\beta(\zeta_n)$. (Indeed, as in the proof of the previous lemma, $\gamma_n x$ is uniformly close to a euclidean Weyl chamber $V(x,\sigma_n)$ with visual boundary chamber $\sigma_n\subset\partial_\infty P_n$ but $\sigma_n\not\subset\mathrm{st}(\beta(\widehat{\zeta}))\cup\mathrm{st}(\beta(\zeta_n))$, and we let $\tau_n\subseteq\sigma_n$ be the type τ_{mod} face.) In particular, τ_n is not opposite to both $\beta(\widehat{\zeta})$ and $\beta(\zeta_n)$. The accumulation set of the sequence (τ_n) in $\mathrm{Flag}_{\tau_{\mathrm{mod}}}$, which coincides with the τ_{mod}-flag accumulation set of the sequence (γ_n), therefore consists of simplices which are not opposite to both $\beta(\widehat{\zeta})$ and $\beta(\zeta)$, because oppositeness is an open property. Letting $\widehat{\zeta}$ run through $\partial_\infty\Gamma-\{\zeta\}$, it follows that these simplices are not opposite to *any* simplex in $\beta(\partial_\infty\Gamma)$.

Every limit simplex in $\Lambda_{\tau_{\mathrm{mod}}}(\Gamma)$ arises as the τ_{mod}-flag limit of a sequence (γ_n) which converges at infinity in $\overline{\Gamma}$. We obtain that no simplex in $\Lambda_{\tau_{\mathrm{mod}}}(\Gamma)$ is opposite to a simplex in $\beta(\partial_\infty\Gamma)$. In particular, $\Lambda_{\tau_{\mathrm{mod}}}(\Gamma)\cap\beta(\partial_\infty\Gamma)=\emptyset$. Thus, conclusion (i) of the theorem holds. □

[24] Note that in view of the antipodality of β the second part of (i) implies the first part.

Consequently, as soon as a boundary embedding hits the limit set, it identifies it with the Gromov boundary of the subgroup and moreover continuously extends the orbit maps:

Corollary 5.8 *Let $\Gamma < G$ be a τ_{mod}-regular τ_{mod}-boundary embedded subgroup with boundary embedding β. If $\beta(\partial_\infty\Gamma) \cap \Lambda_{\tau_{mod}}(\Gamma) \neq \emptyset$, then $\beta(\partial_\infty\Gamma) = \Lambda_{\tau_{mod}}(\Gamma)$. Moreover, the extension $\bar{o}_x$ is continuous at infinity, after replacing β with $-\beta$ in the case $|\partial_\infty\Gamma| = 2$, if necessary.*

Otherwise, if the boundary embedding avoids the limit set, the image of the boundary embedding and the limit set must have special position:

Lemma 5.9 *In case* (i) *of Theorem 5.7, both $\beta(\partial_\infty\Gamma)$ and $\Lambda_{\tau_{mod}}(\Gamma)$ are not Zariski dense in* $\mathrm{Flag}_{\tau_{mod}}$. *In particular, Γ is not Zariski dense in G.*

Proof Since no simplex in $\beta(\partial_\infty\Gamma)$ is opposite to a simplex in $\Lambda_{\tau_{mod}}(\Gamma)$, it follows that $\beta(\partial_\infty\Gamma)$ is disjoint from the union of open Schubert strata $C(\lambda)$ over all limit simplices $\lambda \in \Lambda_{\tau_{mod}}(\Gamma)$. In other words, $\beta(\partial_\infty\Gamma)$ is contained in the intersection of the proper subvarieties $\partial C(\lambda) = \mathrm{Flag}_{\tau_{mod}} - C(\lambda)$. Similarly, $\Lambda_{\tau_{mod}}(\Gamma)$ lies in the intersection of the $\partial C(\tau)$ over all simplices $\tau \in \beta(\partial_\infty\Gamma)$. In particular, both are Γ-invariant proper subvarieties, which forces Γ to be non-Zariski dense. ◻

Therefore, the first alternative in the theorem cannot occur in the Zariski dense case, compare [12, Theorem 1.5]:

Corollary 5.10 *Let $\Gamma < G$ be a Zariski dense τ_{mod}-regular τ_{mod}-boundary embedded subgroup. Then it admits a unique boundary embedding β, and $\beta(\partial_\infty\Gamma) = \Lambda_{\tau_{mod}}(\Gamma)$.*

Proof By the lemma, for any boundary embedding β, only case (ii) in the theorem can occur. It follows that $\beta(\partial_\infty\Gamma) = \Lambda_{\tau_{mod}}(\Gamma)$. Moreover, β is uniquely determined because, due to the density of attractive fixed points of infinite order elements, there are no Γ-equivariant self homeomorphisms of $\partial_\infty\Gamma$ besides the identity. (Note that $|\partial_\infty\Gamma| \geqslant 3$ by Zariski density.) ◻

It is worth noting that in the case $\tau_{mod} = \sigma_{mod}$ the boundary embedding can always be modified so that it maps onto the limit set:

Theorem 5.11 *Let $\Gamma < G$ be a σ_{mod}-regular σ_{mod}-boundary embedded subgroup. Then there exists a boundary embedding β with $\beta(\partial_\infty\Gamma) = \Lambda_{\sigma_{mod}}(\Gamma)$.*

Proof In the case $\tau_{mod} = \sigma_{mod}$, the parallel sets considered above are maximal flats and the Weyl cones are euclidean Weyl chambers. What makes it possible to push the argument further, is the fact that the walls in a maximal flat through a fixed point disconnect the flat into euclidean Weyl chambers. Therefore, the above discussion now yields more precise information about the position of the paths rx:

Since the rx are uniformly close to maximal flats (provided by a boundary embedding β' for Γ, cf. Lemma 5.3), σ_{mod}-regularity forces them to dive into (uniform tubular neighborhoods of) Weyl chambers inside these flats. It follows that the paths rx are contained in uniform tubular neighborhoods of euclidean Weyl chambers with tips at the initial points $r(0)x$. Again by regularity, the asymptote class of the Weyl chamber depends only on the asymptote class of the ray r. We therefore obtain a new boundary map $\beta\colon \partial_\infty\Gamma \to \mathrm{Flag}_{\sigma_{mod}}$ such that rx is contained in the tubular $\rho'(\Gamma, x)$-neighborhood of the euclidean Weyl chamber $V(r(0)x, \beta(\zeta))$ for $\zeta = r(+\infty)$. Clearly, $\beta(\partial_\infty\Gamma) \subseteq \Lambda_{\sigma_{mod}}(\Gamma)$ and β is Γ-equivariant. An argument as in the last part of the proof of Lemma 5.5 shows that β is antipodal.

To verify that β is continuous, suppose that $\zeta_n \to \zeta$ in $\partial_\infty\Gamma$ and $\beta(\zeta_n) \to \sigma$ in $\mathrm{Flag}_{\sigma_{mod}}$. We must show that $\sigma = \beta(\zeta)$. Let $r_n, r\colon \mathbb{N}_0 \to \Gamma$ be rays starting in e and asymptotic to ζ_n, ζ. We note that for any sequence $m_n \to +\infty$ in $\mathbb{N}_0$, we have σ_{mod}-flag convergence $r_n(m_n) \to \sigma$, because $r_n(m_n)x$ lies in a uniform tubular neighborhood of $V(x, \mathrm{st}(\beta(\zeta_n)))$. On the other hand, if m_n grows sufficiently slowly, then the sequence $(r_n(m_n))$ in Γ is contained in a tubular neighborhood of r, and hence $r_n(m_n) \to \beta(\zeta)$. This shows that $\sigma = \beta(\zeta)$, as desired.

Thus, β is a boundary embedding. Since also $\beta(\partial_\infty\Gamma) \subseteq \Lambda_{\sigma_{mod}}(\Gamma)$, we conclude using Theorem 5.7 that $\beta(\partial_\infty\Gamma) = \Lambda_{\sigma_{mod}}(\Gamma)$. □

5.3 Asymptotic embeddings and coarse extrinsic geometry

The discussion in the previous section, notably part (ii) of the conclusion of Theorem 5.7, motivates the following strengthening of the notion of boundary embeddedness:

Definition 5.12 (*Asymptotically embedded*) A subgroup $\Gamma < G$ is τ_{mod}-*asymptotically embedded* if it is τ_{mod}-regular, τ_{mod}-antipodal, word hyperbolic and there is a Γ-equivariant homeomorphism

$$\alpha\colon \partial_\infty\Gamma \xrightarrow{\cong} \Lambda_{\tau_{mod}}(\Gamma) \subset \mathrm{Flag}_{\tau_{mod}}$$

from its Gromov boundary onto its τ_{mod}-limit set.

The definition can also be phrased purely *dynamically* in terms of the Γ-action on $\mathrm{Flag}_{\tau_{mod}}$, by replacing τ_{mod}-regularity with the τ_{mod}-convergence condition.

Note that τ_{mod}-asymptotically embedded subgroups are necessarily *discrete* by τ_{mod}-regularity. We also keep assuming that τ_{mod} is ι-*invariant*; this is implicit in τ_{mod}-antipodality.

We observe that the boundary map α is antipodal, because it is injective with antipodal image. It is therefore a boundary embedding for Γ, i.e. τ_{mod}-asymptotically embedded implies τ_{mod}-boundary embedded. According to Corollary 5.8, the extension

$$\bar{o}_x = o_x \sqcup \alpha\colon \overline{\Gamma} = \Gamma \sqcup \partial_\infty\Gamma \to X \sqcup \mathrm{Flag}_{\tau_{mod}} \tag{38}$$

cf. (37), is continuous, after replacing α with $-\alpha$ in the case $|\partial_\infty\Gamma| = 2$, if necessary. We will refer to α then as the *asymptotic embedding* for Γ.

We rephrase the *criteria* for asymptotic embeddedness obtained in the previous section (cf. Corollaries 5.8, 5.10 and Theorem 5.11):

Theorem 5.13 *Let $\Gamma < G$ be a τ_{mod}-regular τ_{mod}-boundary embedded subgroup with boundary embedding β. If $\beta(\partial_\infty\Gamma) \cap \Lambda_{\tau_{mod}}(\Gamma) \neq \emptyset$, then Γ is τ_{mod}-asymptotically embedded, and β is the asymptotic embedding, after replacing it with $-\beta$ in the case $|\partial_\infty\Gamma| = 2$, if necessary.*

Theorem 5.14 *Zariski dense τ_{mod}-regular τ_{mod}-boundary embedded subgroups are τ_{mod}-asymptotically embedded and admit no other boundary embedding besides their asymptotic embedding.*

Theorem 5.15 *σ_{mod}-Regular σ_{mod}-boundary embedded subgroups are σ_{mod}-asymptotically embedded.* (*But they may admit boundary embeddings different from the asymptotic embedding.*)

We also summarize what the discussion in the previous section yields for the *orbit geometry* of asymptotically embedded subgroups. In addition to the continuity at infinity (38) of the orbit maps o_x, $x \in X$, we obtained (cf. Lemmas 5.3 and 5.5):

Proposition 5.16 (Orbit geometry of asymptotically embedded subgroups) *Let $\Gamma < G$ be a τ_{mod}-asymptotically embedded subgroup with asymptotic embedding α. Then:*

(i) *For every discrete geodesic line $l : \mathbb{Z} \to \Gamma$, the path lx is contained in a tubular neighborhood of uniform radius $\rho(\Gamma, x)$ of the parallel set $P(\alpha(\zeta_-), \alpha(\zeta_+))$, where $\zeta_\pm := l(\pm\infty) \in \partial_\infty\Gamma$.*

(ii) *For every discrete geodesic ray $r : \mathbb{N}_0 \to \Gamma$, the path rx is contained in a tubular neighborhood of uniform radius $\rho'(\Gamma, x)$ of the Weyl cone $V(r(0)x, \mathrm{st}(\alpha(\zeta)))$, where $\zeta := r(+\infty) \in \partial_\infty\Gamma$, and drifts away from its boundary at a uniform rate,*

$$d\big(r(n)x, \partial V(r(0)x, \mathrm{st}(\alpha(\zeta)))\big) \to +\infty \tag{39}$$

uniformly as $n \to +\infty$.

These properties motivate the *Morse property* to be introduced and discussed below. Let us first draw some further immediate consequences for the coarse extrinsic geometry of subgroups and see how property (ii) leads to *undistortion* and *uniform regularity*.

We consider the orbit path rx for a discrete ray r. According to property (ii), the path rx must stay uniformly close to the Weyl cone $V(r(0)x, \mathrm{st}(\alpha(\zeta)))$ predicted by the boundary map and drift away from the boundary of the cone at a uniform rate. Since the same applies to all subrays of r, it follows that the cones $V(r(n)x, \mathrm{st}(\alpha(\zeta)))$ must, up to bounded perturbation, be *uniformly nested*. This forces the orbit path rx to have a *linear* drift away from the boundary of the Weyl cone and in particular towards infinity, i.e. rx is uniformly τ_{mod}-regular and undistorted. We combine these properties in the following notion:

Definition 5.17 (*URU*) A finitely generated subgroup $\Gamma < G$ is τ_{mod}-*URU*, if it is

- uniformly τ_{mod}-regular, and
- undistorted, i.e. the inclusion $\Gamma \subset G$, equivalently, the orbit maps $\Gamma \to \Gamma x \subset X$, are *quasiisometric embeddings* with respect to a word metric on Γ.

Note that URU subgroups cannot contain parabolic elements.

The above discussion before the definition thus leads to:

Theorem 5.18 *τ_{mod}-asymptotically embedded subgroups $\Gamma < G$ are τ_{mod}-URU.*

Proof We add some details to the discussion above: Let $x_n \in V(r(0)x, \mathrm{st}(\alpha(\zeta)))$ be the nearest point projections of the points $r(n)x$, $n \in \mathbb{N}_0$. Then $d(r(n)x, x_n) \leqslant \rho' = \rho'(\Gamma, x)$ by part (ii) of the proposition. We consider the sequence of Weyl cones $V(x_n, \mathrm{st}(\alpha(\zeta))) \subset V(r(0)x, \mathrm{st}(\alpha(\zeta)))$. Note that the cones $V(r(n)x, \mathrm{st}(\alpha(\zeta)))$ and $V(x_n, \mathrm{st}(\alpha(\zeta)))$ are asymptotic to each other and have Hausdorff distance $\leqslant d(r(n)x, x_n) \leqslant \rho'$, as do their boundaries. Applying (ii) to the subrays of r, it follows that the paths $m \mapsto r(n+m)x$ are contained in uniform neighborhoods of the cones $V(x_n, \mathrm{st}(\alpha(\zeta)))$ and drift away from their boundaries at uniform rates. Thus, for every $d_0 > 0$ there exists a number $m_0 = m_0(\Gamma, x, d_0) \in \mathbb{N}$ such that

$$x_{n+m} \in V(x_n, \mathrm{st}(\alpha(\zeta)))$$

and

$$d(x_{n+m}, \partial V(x_n, \mathrm{st}(\alpha(\zeta)))) \geqslant d_0$$

for all $n \geqslant 0$ and $m \geqslant m_0$. The latter inequality implies that the boundaries of the Weyl cones $V(x_n, \mathrm{st}(\alpha(\zeta)))$ and $V(x_{n+m}, \mathrm{st}(\alpha(\zeta)))$ have (nearest point) distance $\geqslant d_0$, cf. Proposition 2.20(ii). From the uniform nestedness of the cones $V(x_{km_0}, \mathrm{st}(\alpha(\zeta)))$ for $k \in \mathbb{N}_0$, it follows that the drift (39) away from the boundary of the Weyl cone is *uniformly linear*. Consequently, the ray images rx are uniformly undistorted and uniformly τ_{mod}-regular. Since any pair of elements in Γ lies in a uniform tubular neighborhood of some discrete geodesic ray, our assertion follows. $\square$

Remark 5.19 (i) That, conversely, URU implies asymptotic embeddedness is proven in [21]. In particular, URU subgroups are necessarily word hyperbolic.
(ii) In [15] we prove that URU subgroups $\Gamma < G$ satisfy the even stronger coarse geometric property of being *coarse Lipschitz retracts* of G.

Similarly, we also derive a version of Proposition 5.16 for discrete geodesic *segments* in Γ:

Consider a line $l\colon \mathbb{Z} \to \Gamma$ and denote $\zeta_\pm = l(\pm\infty)$. Let $x_n \in P(\alpha(\zeta_-), \alpha(\zeta_+))$ be the nearest point projections of the points $l(n)x$, $n \in \mathbb{Z}$. As in the proof of the previous theorem, we see using Proposition 5.16(i)–(ii), that for any $d_0 > 0$ there exists $m_0' = m_0'(\Gamma, x, d_0) \in \mathbb{N}$ such that

$$x_{n\pm m} \in V(x_n, \mathrm{st}(\alpha(\zeta_\pm)))$$

and

$$d(x_{n\pm m}, \partial V(x_n, \mathrm{st}(\alpha(\zeta_\pm)))) \geqslant d_0$$

for all n and $m \geqslant m_0'$. It follows that, for $n_\pm \in \mathbb{Z}$ with $n_+ - n_- \geqslant m_0'$, the diamond

$$\diamondsuit_{\tau_{\mathrm{mod}}}(x_{n_-}, x_{n_+}) = V(x_{n_-}, \mathrm{st}(\alpha(\zeta_+))) \cap V(x_{n_+}, \mathrm{st}(\alpha(\zeta_-))) \subset P(\alpha(\zeta_-), \alpha(\zeta_+))$$

is defined and, using Proposition 5.16 (ii), contains the finite subpath $l|_{[n_-,n_+]\cap\mathbb{Z}}x$ in a uniform tubular neighborhood.

Our discussion yields the following complement to, respectively, strengthening of Proposition 5.16, saying that the images of discrete geodesic segments in Γ are contained in uniform neighborhoods of diamonds with tips at uniform distance from the endpoints:

Proposition 5.20 (Segments go close to diamonds) *Let $\Gamma < G$ be a τ_{mod}-asymptotically embedded subgroup. Then for every discrete geodesic segment $s : [n_-, n_+] \cap \mathbb{Z} \to \Gamma$, the path sx is contained in a tubular neighborhood of uniform radius $\rho'' = \rho''(\Gamma, x)$ of a diamond $\diamondsuit_{\tau_{\mathrm{mod}}}(x_-, x_+)$ with $d(x_\pm, s(n_\pm)x) \leqslant \rho''$.*

Proof This is a consequence of the above discussion, because every discrete geodesic segment in Γ lies in a uniform neighborhood of a discrete geodesic line. □

5.4 Morse property

The *Morse Lemma* for Gromov hyperbolic spaces asserts that quasigeodesic segments are uniformly close to geodesic segments with the same endpoints. Proposition 5.20 along with Proposition 5.16 in the previous section can be interpreted as saying that, for asymptotically embedded subgroups $\Gamma < G$, the images of discrete geodesic segments, rays and lines in Γ under the orbit maps into X satisfy a higher rank version of the Morse Lemma, with geodesic segments replaced by diamonds.

This motivates the following notion (we keep assuming that τ_{mod} is ι-invariant):

Definition 5.21 (*Morse*) A subgroup $\Gamma < G$ is τ_{mod}-*Morse* if it is τ_{mod}-regular, word hyperbolic and satisfies the following property:

For every discrete geodesic segment $s : [n_-, n_+] \cap \mathbb{Z} \to \Gamma$, the path sx is contained in a tubular neighborhood of uniform radius $\rho'' = \rho''(\Gamma, x)$ of a diamond $\diamondsuit_{\tau_{\mathrm{mod}}}(x_-, x_+)$ with tips at distance $d(x_\pm, s(n_\pm)x) \leqslant \rho''$ from the endpoints.

Note that the definition does not a priori assume the existence of a boundary map, neither does it assume undistortion. These will be consequences.

As we saw, asymptotically embedded subgroups are Morse. We will now show that, conversely, asymptotic embeddedness follows from the Morse property, in fact from an a priori weaker version of it for rays in Γ (instead of segments):

Theorem 5.22 *For a subgroup $\Gamma < G$ the following properties are equivalent:*

(i) Γ *is τ_{mod}-asymptotically embedded.*
(ii) Γ *is τ_{mod}-Morse.*
(iii) Γ *is τ_{mod}-regular, word hyperbolic and satisfies the following property: For every discrete geodesic ray $r : \mathbb{N}_0 \to \Gamma$, the path rx is contained in a tubular neighborhood of uniform radius $\rho''' = \rho'''(\Gamma, x)$ of a τ_{mod}-Weyl cone with tip at the initial point $r(0)x$.*

The τ_{mod}*-Weyl cone in* (iii) *is then the cone* $V(r(0)x, \alpha(r(+\infty)))$ *where* α *is the asymptotic embedding for* Γ.

Proof The implication (i)$\Rightarrow$(ii) is Proposition 5.20. The implication (ii)$\Rightarrow$(iii) is immediate by a limiting argument. It remains to show that (iii)$\Rightarrow$(i).

We first observe that the τ_{mod}-Weyl cone $V(r(0)x, \mathrm{st}(\tau))$ containing the path rx in a tubular neighborhood is uniquely determined. This follows from the τ_{mod}-flag convergence $r(n) \to \tau$. Moreover, τ depends only on the asymptote class $r(+\infty)$ of the ray r. Hence there is a well-defined map at infinity

$$\check{\alpha}\colon \partial_\infty \Gamma \to \mathrm{Flag}_{\tau_{\mathrm{mod}}}$$

such that for every ray r the path rx is contained in a uniform tubular neighborhood of the Weyl cone $V(r(0)x, \mathrm{st}(\check{\alpha}(r(+\infty))))$. Our goal is to show that $\check{\alpha}$ is an asymptotic embedding.

Lemma 5.23 $\check{\alpha}$ *is continuous and continuously extends the orbit maps* o_x *at infinity.*

Proof We proceed as in the proof of Theorem 5.11 (continuity of β). Consider a converging sequence $\zeta_n \to \zeta$ in $\partial_\infty\Gamma$. Let $r_n, r\colon \mathbb{N}_0 \to \Gamma$ be rays starting in e and asymptotic to ζ_n, ζ. We note that for any sequence $m_n \to +\infty$ in $\mathbb{N}_0$, the flag accumulation set of the sequence $(r_n(m_n))$ in $\mathrm{Flag}_{\tau_{\mathrm{mod}}}$ equals the accumulation set of the sequence $(\check{\alpha}(\zeta_n))$ in $\mathrm{Flag}_{\tau_{\mathrm{mod}}}$, and in particular does not depend on the sequence (m_n). On the other hand, if (m_n) grows sufficiently slowly, then the sequence $(r_n(m_n))$ in Γ is contained in a tubular neighborhood of r, and hence flag converges to $\check{\alpha}(\zeta)$. It follows that $\check{\alpha}(\zeta_n) \to \check{\alpha}(\zeta)$. This shows that $\check{\alpha}$ is continuous.

Proceeding as in the first part of the proof of Theorem 5.7, we then see that, for a sequence $\gamma_n \to \infty$ in Γ, convergence $\gamma_n \to \zeta \in \partial_\infty\Gamma$ in $\overline{\Gamma}$ implies flag convergence $\gamma_n \to \check{\alpha}(\zeta)$, i.e. $\check{\alpha}$ continuously extends o_x at infinity. ∎

The continuous extension part of the lemma implies

Corollary 5.24 $\check{\alpha}(\partial_\infty\Gamma) = \Lambda_{\tau_{\mathrm{mod}}}(\Gamma)$.

In order to see that $\Lambda_{\tau_{\mathrm{mod}}}(\Gamma)$ is antipodal and $\check{\alpha}$ is an asymptotic embedding for Γ, it remains to verify:

Lemma 5.25 *The map* $\check{\alpha}$ *is antipodal.*

Proof Let $\zeta_\pm \in \partial_\infty\Gamma$ be distinct, and let $l\colon \mathbb{Z} \to \Gamma$ be a line with $l(\pm\infty) = \zeta_\pm$. Applying property (iii) to the subrays $l|_{[-n,+\infty)}$ for large $n \in \mathbb{N}$, we get that the point $l(0)x$ is uniformly close to the cones $V(l(-n)x, \mathrm{st}(\check{\alpha}(\zeta_+)))$, equivalently, there exists a bounded sequence of points $y_n \in V(l(-n)x, \mathrm{st}(\check{\alpha}(\zeta_+)))$. By τ_{mod}-regularity, $d(y_n, \partial V(l(-n)x, \mathrm{st}(\check{\alpha}(\zeta_+)))) \to +\infty$ as $n \to +\infty$. We denote by $\tau_n^- \in \mathrm{Flag}_{\tau_{\mathrm{mod}}}$ the simplex $l(-n)x$-opposite to y_n.[25] Then $l(-n)x \in V(y_n, \mathrm{st}(\tau_n^-))$, and hence $l(-n)x$ is uniformly close to $V(l(0)x, \mathrm{st}(\tau_n^-))$. In view of the flag convergence $l(-n) \to \check{\alpha}(\zeta_-)$,

[25] I.e. $l(-n)x \in V(y_n, \mathrm{st}(\tau_n^-))$. Then $l(-n)x, y_n \in P(\tau_n^-, \check{\alpha}(\zeta_+))$.

it follows that $\tau_n^- \to \check{\alpha}(\zeta_-)$ in $\text{Flag}_{\tau_{\text{mod}}}$. Since the parallel sets $P(\tau_n^-, \check{\alpha}(\zeta_+))$ lie at bounded distance from $l(0)x$, as they contain the points y_n, the sequence (τ_n^-) is relatively compact in the open Schubert stratum $C(\check{\alpha}(\zeta_+))$. Hence $\check{\alpha}(\zeta_-) \in C(\check{\alpha}(\zeta_+))$, i.e. $\check{\alpha}(\zeta_-)$ is opposite to $\check{\alpha}(\zeta_+)$. ■

This concludes the proof of the theorem. □

Note that the theorem implies in particular that τ_{mod}*-Morse subgroups are* τ_{mod}*-URU*, because asymptotically embedded subgroups are URU by Theorem 5.18.

Remark 5.26 We restricted our definition of the Morse property to *word hyperbolic* subgroups because, as shown in [21], URU subgroups are always word hyperbolic. This had been unknown at the time of writing the first version of [20].

5.5 Conicality

The condition for discrete subgroups which we study in this section concerns the asymptotic geometry of their orbits, i.e. how they approach infinity. To state it, we first need to elaborate on our discussion of convergence at infinity for sequences from Sect. 4.5.

For arbitrary τ_{mod}, consider a τ_{mod}-flag converging sequence (x_n) in X,

$$x_n \to \tau \in \text{Flag}_{\tau_{\text{mod}}}.$$

The following notion of going "straight" to the limit simplex generalizes *conical* or *radial* convergence at infinity in rank one symmetric spaces where one requires the sequence to stay in a tubular neighborhood of a geodesic ray. Working with rays also in higher rank turns out to be too restrictive,[26] and we replace the rays with Weyl cones, compare [1, Definition 5.2]:

Definition 5.27 (*Conical convergence*) A τ_{mod}-flag converging sequence $x_n \to \tau \in \text{Flag}_{\tau_{\text{mod}}}$ converges τ_{mod}*-conically*,

$$x_n \xrightarrow{\text{con}} \tau,$$

if it is contained in a tubular neighborhood of a Weyl cone $V(x, \text{st}(\tau))$ for some point $x \in X$. Accordingly, τ_{mod}-flag converging sequences in G are said to converge τ_{mod}*-conically* if their orbit sequences in X do.

Note that the Weyl cones $V(x, \text{st}(\tau))$ for different points $x \in X$ are Hausdorff close to each other, and the conical convergence condition is therefore independent of the choice of x.

The next result describes a situation for sequences close to parallel sets where flag convergence already implies the stronger form of conical convergence:

[26] From our construction of Anosov–Schottky subgroups, see [20], it immediately follows that in higher rank they are generically not ray conical, for instance never in the Zariski dense case. This implies furthermore that Zariski dense Anosov subgroups are never ray conical.

Lemma 5.28 *Suppose that a sequence (x_n) in X τ_{mod}-flag converges, $x_n \to \tau \in \mathrm{Flag}_{\tau_{\mathrm{mod}}}$.*

(i) *If (x_n) is contained in a tubular neighborhood of a parallel set $P(\widehat{\tau}, \tau)$ for some $\widehat{\tau} \in C(\tau)$, or*

(ii) *if, more generally, there exists a relatively compact sequence $(\widehat{\tau}_n)$ in $C(\tau)$ such that*

$$\sup_n d(x_n, P(\widehat{\tau}_n, \tau)) < +\infty,$$

then $x_n \xrightarrow{\mathrm{con}} \tau$.

Proof Suppose first that the stronger condition (i) holds and that $x_n \overset{\mathrm{con}}{\nrightarrow} \tau$. Let $x \in P(\widehat{\tau}, \tau)$. As in the proof of Lemma 5.4, it follows from the openness of the cone $V(x, \mathrm{st}(\tau))$ in the parallel set $P(\widehat{\tau}, \tau)$ that, after extraction, the sequence (x_n) drifts away from $V(x, \mathrm{st}(\tau))$. As in the proof of Theorem 5.7, the points x_n are then contained in uniform neighborhoods of cones $V(x, \mathrm{st}(\tau_n))$ with simplices $\tau_n \in \mathrm{Flag}_{\tau_{\mathrm{mod}}}$ satisfying $\tau_n \subset \partial_\infty P(\widehat{\tau}, \tau)$ but $\tau_n \neq \tau$. Since τ is the only simplex in $C(\widehat{\tau})$ which lies in $P(\widehat{\tau}, \tau)$, see (4) and the discussion preceding Lemma 2.8, the sequence (τ_n) is contained in the closed set $\mathrm{Flag}_{\tau_{\mathrm{mod}}} - C(\widehat{\tau})$, and hence so is its accumulation set. In particular, τ does not belong to the accumulation set of (τ_n) in $\mathrm{Flag}_{\tau_{\mathrm{mod}}}$. Since the latter set equals the flag accumulation set of the sequence (x_n) in $\mathrm{Flag}_{\tau_{\mathrm{mod}}}$, it follows in particular that $x_n \nrightarrow \tau$, a contradiction.

Suppose now that the weaker condition (ii) holds. Since $C(\tau)$ is a homogeneous P_τ-space, there exist $\widehat{\tau} \in C(\tau)$ and a bounded sequence (b_n) in P_τ such that $\widehat{\tau}_n = b_n \widehat{\tau}$. The sequence $(b_n^{-1} x_n)$ is then contained in a tubular neighborhood of $P(\widehat{\tau}, \tau)$, i.e. it satisfies condition (i). Moreover, we also have flag convergence $b_n^{-1} x_n \to \tau$.[27] Hence, by the above, it follows that $b_n^{-1} x_n \xrightarrow{\mathrm{con}} \tau$. By the definition of conical convergence, this means that the sequence $(b_n^{-1} x_n)$ lies in a tubular neighborhood of the cone $V(x, \mathrm{st}(\tau))$ for some point $x \in X$, equivalently, that

$$\sup_n d(x_n, V(b_n x, \mathrm{st}(\tau))) < +\infty.$$

Now the cones $V(b_n x, \mathrm{st}(\tau))$ are asymptotic to $V(x, \mathrm{st}(\tau))$ and have finite Hausdorff distance $\leqslant d(x, b_n x)$ from it. This Hausdorff distance is uniformly bounded and it also follows that the sequence (x_n) lies in a tubular neighborhood of $V(x, \mathrm{st}(\tau))$, i.e. $x_n \xrightarrow{\mathrm{con}} \tau$. □

As we did with regularity and flag convergence, we will now also rephrase conical convergence for sequences in G in terms of their dynamics on flag manifolds.

For a flag convergent sequence, conical convergence is reflected as follows by the dynamics on the space of parallel sets, equivalently, on the space of pairs of opposite simplices, cf. (35):

[27] Because the b_n are bounded and fix τ on $\mathrm{Flag}_{\tau_{\mathrm{mod}}}$.

Lemma 5.29 *Suppose that a sequence (g_n) in G τ_{mod}-flag converges, $g_n \to \tau \in \mathrm{Flag}_{\tau_{mod}}$. Then for a relatively compact sequence $(\widehat{\tau}_n)$ in $C(\tau)$, the following are equivalent:*

(i) $g_n \xrightarrow{\text{con}} \tau$.
(ii) *The parallel sets $g_n^{-1}P(\widehat{\tau}_n, \tau)$ all intersect a fixed bounded subset in X.*
(iii) *The sequence of pairs $g_n^{-1}(\widehat{\tau}_n, \tau)$ is relatively compact in $(\mathrm{Flag}_{\iota\tau_{mod}} \times \mathrm{Flag}_{\tau_{mod}})^{opp}$.*

Proof We first note that conditions (ii) and (ii′) are equivalent as a consequence of:

Sublemma 5.30 *A subset $A \subset (\mathrm{Flag}_{\iota\tau_{mod}} \times \mathrm{Flag}_{\tau_{mod}})^{opp}$ is relatively compact iff the corresponding parallel sets $P(\tau_-, \tau_+)$ for $(\tau_-, \tau_+) \in A$ all intersect a fixed bounded subset of X, i.e.*

$$\sup_{(\tau_-,\tau_+)\in A} d(x, P(\tau_-, \tau_+)) < +\infty$$

for a base point $x \in X$.

Proof The forward direction follows from the continuity of the function (36).[28]

For the converse direction we note that for a pair

$$(\tau_-, \tau_+) \in (\mathrm{Flag}_{\iota\tau_{mod}} \times \mathrm{Flag}_{\tau_{mod}})^{opp}$$

the intersection of parabolic subgroups $P_{\tau_-} \cap P_{\tau_+}$ preserves the parallel set $P(\tau_-, \tau_+)$ and acts transitively on it. Consequently, the set of triples

$$(\tau_-, \tau_+, x') \in (\mathrm{Flag}_{\iota\tau_{mod}} \times \mathrm{Flag}_{\tau_{mod}})^{opp} \times X$$

such that $x' \in P(\tau_-, \tau_+)$ is still a homogeneous G-space. Let us fix in it a reference triple (τ_0^-, τ_0^+, x). Then the parallel sets $P(\tau_-, \tau_+)$ intersecting a closed ball $\overline{B}(x, R)$ are of the form $gP(\tau_0^-, \tau_0^+)$ with $g \in G$ such that $d(x, gx) \leqslant R$. It follows that the set of these pairs $(\tau_-, \tau_+) = g(\tau_0^-, \tau_0^+)$ is compact. ∎

Continuing with the proof of the lemma, let $x \in X$ be a base point. In view of

$$d(x, g_n^{-1}P(\widehat{\tau}_n, \tau)) = d(g_n x, P(\widehat{\tau}_n, \tau))$$

condition (ii) is equivalent to

$$\sup_n d(g_n x, P(\widehat{\tau}_n, \tau)) < +\infty. \tag{40}$$

The implication (ii)⇒(i) thus follows from the previous lemma. The reverse implication (i)⇒(ii) is easy: Since $\sup_n d(x, P(\widehat{\tau}_n, \tau)) < +\infty$, compare the sublemma,

[28] Since here τ_{mod} is not required to be ι-invariant, we consider the function on $(\mathrm{Flag}_{\iota\tau_{mod}} \times \mathrm{Flag}_{\tau_{mod}})^{opp} \times X$.

the cone $V(x, \mathrm{st}(\tau))$ is contained in uniform tubular neighborhoods of all parallel sets $P(\widehat{\tau}_n, \tau)$, and conical convergence implies the same for the sequence $(g_n x)$, i.e. (40) is satisfied. $\square$

Combining the lemma with our earlier dynamical characterization of flag convergence, see Lemma 4.20, we obtain:

Proposition 5.31 (Dynamical characterization of conical convergence) *A sequence (g_n) in G is τ_{mod}-regular and $g_n \xrightarrow{\mathrm{con}} \tau \in \mathrm{Flag}_{\tau_{\mathrm{mod}}}$ iff there exists a bounded sequence (b_n) in G and a simplex $\tau_- \in \mathrm{Flag}_{\iota\tau_{\mathrm{mod}}}$ such that the following conditions are satisfied:*

(i) *$b_n g_n^{-1}|_{C(\tau)} \to \tau_-$ uniformly on compacts.*
(ii) *The accumulation set of the sequence $(b_n g_n^{-1}\tau)$ in $\mathrm{Flag}_{\tau_{\mathrm{mod}}}$ is contained in $C(\tau_-)$.*

Proof Suppose first that (g_n) is τ_{mod}-regular and $g_n \xrightarrow{\mathrm{con}} \tau \in \mathrm{Flag}_{\tau_{\mathrm{mod}}}$. Then we have in particular flag convergence $g_n \to \tau$, and Lemma 4.20 yields (b_n) and τ_- with (i). The conical convergence $g_n \xrightarrow{\mathrm{con}} \tau$ is equivalent to $g_n b_n^{-1} \xrightarrow{\mathrm{con}} \tau$, and so the previous lemma implies for any $\widehat{\tau} \in C(\tau)$ that the sequence $b_n g_n^{-1}(\widehat{\tau}, \tau)$ is relatively compact in $(\mathrm{Flag}_{\iota\tau_{\mathrm{mod}}} \times \mathrm{Flag}_{\tau_{\mathrm{mod}}})^{\mathrm{opp}}$. Since $b_n g_n^{-1}\widehat{\tau} \to \tau_-$ by (i), the sequence $(b_n g_n^{-1}\tau)$ therefore cannot accumulate at points outside $C(\tau_-)$.

Suppose now vice versa that (b_n) and τ_- with (i)–(ii) are given. By Lemma 4.20, (i) implies that (g_n) is τ_{mod}-regular and $g_n \to \tau$, and the same follows for the sequence $(g_n b_n^{-1})$. Furthermore, (i)–(ii) imply that for any $\widehat{\tau} \in C(\tau)$ the sequence $b_n g_n^{-1}(\widehat{\tau}, \tau)$ is relatively compact in $(\mathrm{Flag}_{\iota\tau_{\mathrm{mod}}} \times \mathrm{Flag}_{\tau_{\mathrm{mod}}})^{\mathrm{opp}}$. Thus $g_n b_n^{-1} \xrightarrow{\mathrm{con}} \tau$ by the previous lemma, and hence $g_n \xrightarrow{\mathrm{con}} \tau$. $\square$

We deduce the following criterion for being a conical limit simplex of a subsequence:

Corollary 5.32 *A sequence (g_n) in G has a τ_{mod}-regular subsequence τ_{mod}-conically converging to $\tau \in \mathrm{Flag}_{\tau_{\mathrm{mod}}}$ iff there exists a subsequence (g_{n_k}) and a simplex $\tau_- \in \mathrm{Flag}_{\iota\tau_{\mathrm{mod}}}$ such that the following conditions are satisfied:*

(i) *$g_{n_k}^{-1}|_{C(\tau)} \to \tau_-$ uniformly on compacts.*
(ii) *$(g_{n_k}^{-1}\tau)$ converges to a simplex in $C(\tau_-)$.*

Proof Suppose that there is a τ_{mod}-regular subsequence (g_{n_k}) with $g_{n_k} \xrightarrow{\mathrm{con}} \tau$. The proposition yields a bounded sequence (b_k) and τ_- such that properties (i)–(ii) in the proposition are satisfied for the sequence $(b_k g_{n_k}^{-1})$. After extraction, we obtain convergence $b_k \to b$ in G and $b_k g_{n_k}^{-1}\tau \to \widehat{\tau}_- \in C(\tau_-)$ in $\mathrm{Flag}_{\tau_{\mathrm{mod}}}$. The asserted properties (i)–(ii) then result from replacing τ_- with $b^{-1}\tau_-$. The converse is immediate in view of the proposition. $\square$

Now we turn to subgroups.

Definition 5.33 (*Conical limit set*) For a subgroup $\Gamma < G$, a limit simplex $\lambda \in \Lambda_{\tau_{\mathrm{mod}}}(\Gamma)$ is τ_{mod}-*conical* if there exists a τ_{mod}-regular sequence (γ_n) in Γ such that $\gamma_n \xrightarrow{\mathrm{con}} \lambda$. The *conical τ_{mod}-limit set* $\Lambda^{\mathrm{con}}_{\tau_{\mathrm{mod}}}(\Gamma) \subseteq \Lambda_{\tau_{\mathrm{mod}}}(\Gamma)$ is the subset of conical limit simplices. The subgroup Γ *has conical τ_{mod}-limit set* or *is τ_{mod}-conical* if all limit simplices are conical, $\Lambda^{\mathrm{con}}_{\tau_{\mathrm{mod}}}(\Gamma) = \Lambda_{\tau_{\mathrm{mod}}}(\Gamma)$.

We restrict ourselves to τ_{mod}-antipodal τ_{mod}-regular subgroups and assume in particular that τ_{mod} is ι-invariant. Recall that then the action

$$\Gamma \curvearrowright \Lambda_{\tau_{mod}}(\Gamma)$$

is a *convergence action*, see Sect. 5.1. This raises the question how the τ_{mod}-conicality of limit simplices compares to their intrinsic conicality with respect to this convergence action, cf. Sect. 3.3. We show that these properties are equivalent:

Proposition 5.34 (Conical vs. intrinsically conical limit simplex) *Let $\Gamma < G$ be a τ_{mod}-antipodal τ_{mod}-regular subgroup with $|\Lambda_{\tau_{mod}}(\Gamma)| \geqslant 3$. Then a limit simplex in $\Lambda_{\tau_{mod}}(\Gamma)$ is conical iff it is intrinsically conical for the convergence action $\Gamma \curvearrowright \Lambda_{\tau_{mod}}(\Gamma)$.*

Proof That conicality implies intrinsic conicality is, in view of the corollary, an immediate consequence of antipodality and Lemma 3.10.

Suppose that, conversely, $\lambda \in \Lambda_{\tau_{mod}}(\Gamma)$ is intrinsically conical. Again invoking Lemma 3.10, this means that there exist a sequence (γ_n) in Γ and a limit simplex $\lambda_- \in \Lambda_{\tau_{mod}}(\Gamma)$ such that $\gamma_n^{-1}|_{\Lambda_{\tau_{mod}}(\Gamma)-\{\lambda\}} \to \lambda_-$ uniformly on compacts and $\gamma_n^{-1}\lambda \to \widehat{\lambda}_- \in \Lambda_{\tau_{mod}}(\Gamma) - \{\lambda_-\} \subset C(\lambda_-)$. On the other hand, since Γ is a τ_{mod}-convergence subgroup, after extraction, the sequence (γ_n^{-1}) becomes τ_{mod}-contracting and there are limit simplices $\lambda', \lambda'_- \in \Lambda_{\tau_{mod}}(\Gamma)$ such that $\gamma_n^{-1}|_{C(\lambda')} \to \lambda'_-$ uniformly on compacts. In view of antipodality, $C(\lambda')$ contains $\Lambda_{\tau_{mod}}(\Gamma) - \{\lambda'\}$. Since $|\Lambda_{\tau_{mod}}(\Gamma)| \geqslant 3$, it follows that $C(\lambda')$ intersects $\Lambda_{\tau_{mod}}(\Gamma) - \{\lambda\}$ and therefore $\lambda'_- = \lambda_-$. Moreover, from $\gamma_n^{-1}\lambda \to \widehat{\lambda}_- \neq \lambda_-$ it follows that $\lambda \notin C(\lambda')$ and hence also $\lambda' = \lambda$. We conclude that $\gamma_n^{-1}|_{C(\lambda)} \to \lambda_-$ uniformly on compacts and $\gamma_n^{-1}\lambda \to \widehat{\lambda}_- \in C(\lambda_-)$. Corollary 5.32 now yields that the limit simplex λ is τ_{mod}-conical. $\square$

Corollary 5.35 (Conical vs. intrinsically conical subgroup) *Let $\Gamma < G$ be a τ_{mod}-antipodal τ_{mod}-regular subgroup with $|\Lambda_{\tau_{mod}}(\Gamma)| \geqslant 3$. Then Γ is τ_{mod}-conical iff all simplices in $\Lambda_{\tau_{mod}}(\Gamma)$ are conical limit points for the convergence action $\Gamma \curvearrowright \Lambda_{\tau_{mod}}(\Gamma)$.*

We introduce the following asymptotic condition on the orbit geometry of subgroups:

Definition 5.36 (*RCA*) A subgroup $\Gamma < G$ is τ_{mod}-*RCA* if it is τ_{mod}-regular, τ_{mod}-conical and τ_{mod}-antipodal.

From the corollary we deduce, using the dynamical characterization of word hyperbolic groups and their boundary actions, the following equivalence:

Theorem 5.37 *For a subgroup $\Gamma < G$ with $|\Lambda_{\tau_{mod}}(\Gamma)| \geqslant 3$ the following properties are equivalent:*

(i) τ_{mod}-*RCA.*
(ii) τ_{mod}-*asymptotically embedded.*

The implication (ii)$\Rightarrow$(i) *holds without restriction on the size of the limit set.*

Proof Since this is part of both conditions, we assume that Γ is τ_{mod}-regular and τ_{mod}-antipodal.

The implication (ii) $\Rightarrow$ (i) follows, without restriction on the size of $\Lambda_{\tau_{mod}}(\Gamma)$, from the implication (i) $\Rightarrow$ (iii) of Theorem 5.22.

Suppose now that $|\Lambda_{\tau_{mod}}(\Gamma)| \geqslant 3$. According to the previous corollary, the subgroup Γ is τ_{mod}-RCA if and only if the convergence action $\Gamma \curvearrowright \Lambda_{\tau_{mod}}(\Gamma)$ is (intrinsically) conical. In view of Theorems 3.11 and 3.12 this is equivalent to Γ being word hyperbolic and $\Lambda_{\tau_{mod}}(\Gamma)$ being Γ-equivariantly homeomorphic to $\partial_\infty\Gamma$, i.e. to Γ being τ_{mod}-asymptotically embedded. $\square$

5.6 Subgroups with two-point limit sets

For antipodal regular subgroups with two-point limit sets, some of our conditions are automatically satisfied:

Lemma 5.38 *Suppose that $\Gamma < G$ is τ_{mod}-antipodal τ_{mod}-regular with $|\Lambda_{\tau_{mod}}(\Gamma)| = 2$. Then:*

(i) *Γ is τ_{mod}-RCA.*
(ii) *Γ is virtually cyclic.*
(iii) *The orbit maps $o_x : \Gamma \to \Gamma x \subset X$ extend continuously to infinity by an asymptotic embedding. In particular, Γ is τ_{mod}-asymptotically embedded.*

Proof (i) By antipodality, $\Lambda_{\tau_{mod}}(\Gamma)$ consists of a pair of opposite simplices $\lambda_\pm \in \mathrm{Flag}_{\tau_{mod}}$. The subgroup Γ therefore preserves the parallel set $P(\lambda_-, \lambda_+)$. The limit simplices $\lambda_\pm$ must be conical by Lemma 5.28. Hence Γ is τ_{mod}-RCA.

(ii) Pick a point $x \in P(\lambda_-, \lambda_+)$. By conicality, there exists an element $\gamma_0 \in \Gamma$ which fixes $\lambda_\pm$ and so that $\gamma_0 x$ lies in the interior of the Weyl cone $V = V(x, \mathrm{st}(\lambda_+)) \subset P(\lambda_-, \lambda_+)$. We consider the biinfinite nested sequence of Weyl cones $\gamma_0^n V$ for $n \in \mathbb{Z}$. The cones $\gamma_0^n V$ cover $P(\lambda_-, \lambda_+)$, cf. Proposition 2.20. Moreover, $\gamma_0^{n+1} V$ is contained in the interior of $\gamma_0^n V$ and has finite Hausdorff distance from it. By regularity, the difference of cones $V - \gamma_0 V$ can only contain finitely many points of the orbit Γx. The corresponding elements in Γ form a set of representatives for the cosets of the infinite cyclic subgroup Γ_0 generated by γ_0 in Γ. Hence Γ is virtually cyclic.

(iii) Since $\gamma_0^{\pm n} \to \lambda_\pm$ as $n \to +\infty$, the restrictions of the orbit maps to Γ_0 extend continuously to $\partial_\infty\Gamma_0 \cong \partial_\infty\Gamma$ by an asymptotic embedding α. Since Γ_0 has finite index in Γ, the map α is a continuous extension also of the orbit maps of Γ itself. Moreover, it is Γ-equivariant. $\square$

5.7 Expansion

We define another purely dynamical condition for subgroups, inspired by Sullivan's notion of *expanding actions* [29], namely that their action on the appropriate flag manifold is expanding at the limit set in the sense of Definition 3.1. As before, we equip the flag manifolds with auxiliary Riemannian metrics.

Definition 5.39 (*CEA*) A subgroup $\Gamma < G$ is τ_{mod}-*CEA* (*convergence, expanding, antipodal*) if it is τ_{mod}-convergence, τ_{mod}-antipodal and the action $\Gamma \curvearrowright \text{Flag}_{\tau_{\text{mod}}}$ is expanding at $\Lambda_{\tau_{\text{mod}}}(\Gamma)$.

The next result relates conicality to infinitesimal expansion, cf. Definition 3.2. For smooth actions on Riemannian manifolds, metric and infinitesimal expansion are equivalent.

Lemma 5.40 (Expansion at conical limit simplices) *Let (g_n) be a τ_{mod}-regular sequence in G such that $g_n \xrightarrow{\text{con}} \tau \in \text{Flag}_{\tau_{\text{mod}}}$. Then the inverse sequence (g_n^{-1}) has diverging infinitesimal expansion on $\text{Flag}_{\tau_{\text{mod}}}$ at τ, i.e.*

$$\epsilon(g_n^{-1}, \tau) \to +\infty.$$

Proof This follows from the expansion estimate in Theorem 2.41. □

Applied to subgroups, the lemma yields:

Proposition 5.41 (Conical implies expansive) *Let $\Gamma < G$ be a subgroup. If $\lambda \in \Lambda^{\text{con}}_{\tau_{\text{mod}}}(\Gamma)$, then the action $\Gamma \curvearrowright \text{Flag}_{\tau_{\text{mod}}}$ has diverging infinitesimal expansion at λ. In particular, if Γ is τ_{mod}-conical, then $\Gamma \curvearrowright \text{Flag}_{\tau_{\text{mod}}}$ is expanding at $\Lambda_{\tau_{\text{mod}}}(\Gamma)$.*

Proof This is a direct consequence of the lemma, together with the fact that infinitesimal expansion implies metric expansion. □

We obtain the equivalence of conditions:

Theorem 5.42 *For a subgroup $\Gamma < G$ with $|\Lambda_{\tau_{\text{mod}}}(\Gamma)| \geqslant 2$, the following properties are equivalent:*

(i) τ_{mod}-*RCA*.
(ii) τ_{mod}-*CEA*.

The implication (i)$\Rightarrow$(ii) *holds without restriction on the size of the limit set.*

Proof We recall that τ_{mod}-regularity is equivalent to the τ_{mod}-convergence property, cf. Theorem 4.16. Thus either condition implies that Γ is τ_{mod}-regular and τ_{mod}-antipodal.

The implication (i)$\Rightarrow$(ii) is the previous proposition. (We do not need that $|\Lambda_{\tau_{\text{mod}}}(\Gamma)| \geqslant 2$.)

For the direction (ii)$\Rightarrow$(i) we first assume that $|\Lambda_{\tau_{\text{mod}}}(\Gamma)| \geqslant 3$ and consider the convergence action $\Gamma \curvearrowright \Lambda_{\tau_{\text{mod}}}(\Gamma)$. Since $\Lambda_{\tau_{\text{mod}}}(\Gamma)$ contains at least three points, it must be perfect[29] (see [30, Theorem 2S]). By assumption, the action $\Gamma \curvearrowright \Lambda_{\tau_{\text{mod}}}(\Gamma)$ is expanding. Therefore all points $\lambda \in \Lambda_{\tau_{\text{mod}}}(\Gamma)$ are intrinsically conical, cf. Lemma 3.13, and hence conical, i.e. Γ is τ_{mod}-conical, cf. Corollary 5.35.

In the case $|\Lambda_{\tau_{\text{mod}}}(\Gamma)| = 2$, the assertion follows from Lemma 5.38. □

[29] I.e. has no isolated points.

5.8 Anosov property

The Anosov condition combines boundary embeddedness with an infinitesimal expansion condition at the image of the boundary embedding:

Definition 5.43 (*Anosov*) A subgroup $\Gamma < G$ is τ_{mod}-*Anosov* if:

- Γ is τ_{mod}-boundary embedded with boundary embedding β.
- For every ideal point $\zeta \in \partial_\infty \Gamma$ and every normalized (by $r(0) = e \in \Gamma$) discrete geodesic ray $r\colon \mathbb{N} \to \Gamma$ asymptotic to ζ, the action $\Gamma \curvearrowright \mathrm{Flag}_{\tau_{\mathrm{mod}}}$ satisfies

$$\epsilon(r(n)^{-1}, \beta(\zeta)) \geqslant Ae^{Cn}$$

  for $n \geqslant 0$ with constants $A, C > 0$ independent of r.

We recall that boundary embedded subgroups are *discrete*.

Our notion of τ_{mod}-Anosov is equivalent to the notion of P-Anosov in [12] where $P < G$ is a parabolic subgroup in the conjugacy class corresponding to τ_{mod}, see Sect. 5.11. We note also that the study of (P_+, P_-)-Anosov subgroups quickly reduces to the case of P-Anosov subgroups by intersecting parabolic subgroups, cf. [12, Lemma 3.18].

In both our and the original definition uniform exponential expansion rates are required. We will see that the conditions can be relaxed without altering the class of subgroups. Uniformity can be dropped, and instead of exponential divergence the mere unboundedness of the expansion rate suffices.

Definition 5.44 (*Non-uniformly Anosov*) A subgroup $\Gamma < G$ is *non-uniformly* τ_{mod}-*Anosov* if:

- Γ is τ_{mod}-boundary embedded with boundary embedding β.
- For every ideal point $\zeta \in \partial_\infty \Gamma$ and every normalized[30] discrete geodesic ray $r\colon \mathbb{N}_0 \to \Gamma$ asymptotic to ζ, the action $\Gamma \curvearrowright \mathrm{Flag}_{\tau_{\mathrm{mod}}}$ satisfies

$$\sup_{n\in\mathbb{N}} \epsilon(r(n)^{-1}, \beta(\zeta)) = +\infty. \tag{41}$$

In other words, we require that for every ideal point $\zeta \in \partial_\infty \Gamma$ the expansion rate $\epsilon(\gamma_n^{-1}, \beta(\zeta))$ non-uniformly diverges along some sequence (γ_n) in Γ which converges to ζ *conically*.

We relate the Anosov to the Morse property, building on our discussion of the coarse extrinsic geometry of subgroups in Sects. 5.3 and 5.4.

Theorem 5.45 (Non-uniformly Anosov implies Morse) *Each non-uniformly τ_{mod}-Anosov subgroup $\Gamma < G$ is τ_{mod}-Morse. Moreover, the boundary embedding β of Γ sends $\partial_\infty \Gamma$ homeomorphically onto $\Lambda_{\tau_{\mathrm{mod}}}(\Gamma)$.*

[30] Here, the normalization can be dropped because no *uniform* growth is required.

Proof Let $\Gamma < G$ be non-uniformly τ_{mod}-Anosov. Since non-uniformly Anosov subgroups are boundary embedded by definition, discrete geodesic lines in Γ are mapped into uniform neighborhoods of τ_{mod}-parallel sets prescribed by the boundary embedding, see Lemma 5.3. The same follows for discrete geodesic rays in Γ because they lie in uniform neighborhoods of lines, compare the proof of Lemma 5.4: For every ray $r\colon \mathbb{N}_0 \to \Gamma$ asymptotic to $\zeta = r(+\infty)$ there exists an ideal point $\widehat{\zeta} \in \partial_\infty\Gamma - \{\zeta\}$ such that the path rx lies in the $\rho''(\Gamma, x)$-neighborhood of the parallel set $P = P(\beta(\widehat{\zeta}), \beta(\zeta))$. Here, as usual, $x \in X$ is some fixed base point.

The expansion condition (41) further restricts the position of the path rx along the parallel set: Let $x_n \in P$ denote points at distance $\leqslant \rho''$ from the points $r(n)x$, e.g. their nearest point projections to P. For a strictly increasing sequence $n_k \to +\infty$ with diverging expansion rate

$$\epsilon(r(n_k)^{-1}, \beta(\zeta)) \to +\infty$$

we have in view of Proposition 2.42 and Theorem 2.41 that $x_{n_k} \in V(x_0, \mathrm{st}(\beta(\zeta)))$ for large k and

$$d(x_{n_k}, \partial V(x_0, \mathrm{st}(\beta(\zeta)))) \to +\infty$$

(non-uniformly) as $k \to +\infty$. Fix a constant $d \gg \rho''$. It follows that there exists a smallest "entry time" $T = T(r) \in \mathbb{N}$ such that the point $r(T)x$ lies in the open $3\rho''$-neighborhood of the cone $V(r(0)x, \mathrm{st}(\beta(\zeta)))$ and has distance $> d$ from its boundary.

We observe next that $T(r') \leqslant T(r)$ for rays r' sufficiently close to r, because ζ varies continuously with r, and rays sufficiently close to r agree with r up to time $T(r)$. Thus, T is locally bounded above as a function of r. Since Γ acts cocompactly on rays, equivalently, since the space of rays with fixed initial point is compact, we conclude that T is *bounded above globally*, i.e. there exists a number $T_0 = T_0(\Gamma, x, d)$ such that $T(r) \leqslant T_0$ for all rays r.

As a consequence, for every ray r the above sequence of natural numbers (n_k) can be chosen with bounded increase $n_{k+1} - n_k \leqslant T_0$ and so that

$$x_{n_{k+1}} \in V(x_{n_k}, \mathrm{st}(\beta(\zeta)))$$

and

$$d(x_{n_{k+1}}, \partial V(x_{n_k}, \mathrm{st}(\beta(\zeta)))) > \frac{d}{2}$$

for all k, i.e. the sequence (n_k) increases uniformly linearly and the Weyl cones $V(x_{n_k}, \mathrm{st}(\beta(\zeta)))$ are uniformly nested, compare the proof of Theorem 5.18.

It follows that the paths rx are uniformly τ_{mod}-regular and undistorted, and are contained in uniform neighborhoods of the cones $V(r(0), \mathrm{st}(\beta(r(+\infty))))$. In particular, Γ satisfies property (iii) of Theorem 5.22, and therefore is τ_{mod}-Morse. It also follows that $\beta(\partial_\infty\Gamma) \subseteq \Lambda_{\tau_{mod}}(\Gamma)$. The equality $\beta(\partial_\infty\Gamma) = \Lambda_{\tau_{mod}}(\Gamma)$ follows from Theorem 5.7. ◻

A converse readily follows from our earlier results:

Theorem 5.46 *τ_{mod}-Morse subgroups $\Gamma < G$ are τ_{mod}-Anosov.*

Proof Let $\Gamma < G$ be τ_{mod}-Morse. By Theorems 5.22 and 5.18, Γ is then also τ_{mod}-asymptotically embedded and uniformly τ_{mod}-regular. Furthermore, denoting the asymptotic embedding by α and fixing a point $x \in X$, we know that for every ray $r : \mathbb{N}_0 \to \Gamma$ the path rx is contained in a uniform neighborhood of the Weyl cone $V(r(0)x, \alpha(r(+\infty)))$ and drifts away from its boundary at a uniform linear rate. With Theorem 2.41 it follows that the infinitesimal expansion factor $\epsilon(r(n)^{-1}, \alpha(r(+\infty)))$ for the action $\Gamma \curvearrowright \mathrm{Flag}_{\tau_{mod}}$ grows at a uniform exponential rate. Thus, Γ is τ_{mod}-Anosov. □

5.9 Equivalence of conditions

Combining our results comparing the various geometric and dynamical conditions for discrete subgroups, we obtain our main theorem (cf. Theorem 1.1):

Theorem 5.47 (Equivalence) *The following properties for subgroups $\Gamma < G$ are equivalent in the nonelementary*[31] *case:*

(i) *τ_{mod}-asymptotically embedded,*
(ii) *τ_{mod}-CEA,*
(iii) *τ_{mod}-Anosov,*
(iv) *non-uniformly τ_{mod}-Anosov,*
(v) *τ_{mod}-RCA,*
(vi) *τ_{mod}-Morse.*

These properties imply τ_{mod}-URU. Moreover, the boundary maps in (i), (iii) *and* (iv) *coincide.*

Proof By Theorem 5.22, (i) and (vi) are equivalent. By Theorems 5.45 and 5.46, conditions (iii), (iv) and (vi) are equivalent. The fact that the boundary maps in (i), (iii) and (iv) coincide follows from the second part of Theorem 5.45.

By Theorem 5.18, (i) implies τ_{mod}-URU. By Theorem 5.37, (i) and (v) are equivalent. By Theorem 5.42, (ii) and (v) are equivalent. □

Remark 5.48 (i) The equivalence of the conditions (i), (iii), (iv) and (vi), the fact that they imply τ_{mod}-URU, and the implications (i) ⇒ (v) ⇒ (ii) hold without restriction on the size of the limit set.
(ii) It is shown in [21] that, conversely, τ_{mod}-URU implies τ_{mod}-Morse.

For subgroups with small limit sets we have the following additional information, see Lemma 5.38:

Addendum 5.49 *For a τ_{mod}-antipodal τ_{mod}-regular subgroup $\Gamma < G$ with $|\Lambda_{\tau_{mod}}(\Gamma)| = 2$, properties* (i)–(vi) *and τ_{mod}-URU are always satisfied.*

[31] Meaning that $|\Lambda_{\tau_{mod}}(\Gamma)| \geqslant 3$ in (i), (ii), (v), (vi) and that Γ is word hyperbolic with $|\partial_\infty \Gamma| \geqslant 3$ in (iii), (iv).

We are unaware of examples of τ_{mod}-RCA or τ_{mod}-CEA subgroups with one limit point in higher rank. Note that such subgroups cannot be τ_{mod}-asymptotically embedded.

5.10 Morse quasigeodesics

When studying the coarse geometry of Anosov subgroups in Sects. 5.3 and 5.4, we were lead to the Morse and URU properties. We also saw that Morse implies URU. (The converse is true as well, but harder to prove, see [21].)

Thus, for Morse subgroups $\Gamma < G$, the images of the discrete geodesics in Γ under an orbit map are uniform quasigeodesics in X which are uniformly regular and satisfy a Morse type property involving closeness of subpaths to diamonds. Leaving the group-theoretic context, we will now make this class of quasigeodesics precise and study some of its geometric properties. (See also [20] for further discussion.) We will build in the uniform regularity into the Morse property by replacing the diamonds with smaller "uniformly regular" Θ-diamonds.

In the following, $\Theta \subset \mathrm{int}_{\tau_{mod}}(\sigma_{mod})$ denotes an ι-invariant τ_{mod}-Weyl convex compact subset which is used to quantify uniform regularity. We work with discrete paths; $I \subseteq \mathbb{R}$ denotes an interval and $n_\pm$ integers.

Definition 5.50 (*Morse quasigeodesic*) A quasigeodesic $q\colon I \cap \mathbb{Z} \to X$ is (Θ, ρ)-*Morse* if for every subinterval $[n_-, n_+] \subseteq I$ the subpath $q|_{[n_-,n_+]\cap\mathbb{Z}}$ is contained in the ρ-neighborhood of a diamond $\diamond_\Theta(x_-, x_+)$ with tips at distance $d(x_\pm, q(n_\pm)) \leqslant \rho$ from the endpoints.

We say that an infinite quasigeodesic is Θ-*Morse* if it is (Θ, ρ)-Morse for some ρ, and we say that it is τ_{mod}-*Morse* if it is Θ-Morse for some Θ.

The Θ-Morse property for quasigeodesics is clearly stable under bounded perturbation.

We say that some paths are *uniform* τ_{mod}-*Morse quasigeodesics* if they are uniform quasigeodesics[32] and (Θ, ρ)-Morse with the same Θ, ρ.

We can now interpret the Morse subgroup property in terms of Morse quasigeodesics:

Proposition 5.51 *A word hyperbolic subgroup $\Gamma < G$ is τ_{mod}-Morse if and only if an orbit map $o_x\colon \Gamma \to \Gamma x \subset X$ sends uniform quasigeodesics in Γ to uniform τ_{mod}-Morse quasigeodesics in X.*

Proof Suppose that Γ is τ_{mod}-Morse. We fix a word metric on Γ. In view of the Morse Lemma for word hyperbolic groups (Gromov hyperbolic spaces) it suffices to prove that o_x sends discrete geodesics in Γ to uniform τ_{mod}-Morse quasigeodesics in X.

First of all, since Morse subgroups are URU, we know that Γ is undistorted in G, i.e. o_x is a quasiisometric embedding. Equivalently, the o_x-images of discrete geodesics in Γ are uniform quasigeodesics. We need to show that they are uniformly τ_{mod}-Morse.

Consider a discrete geodesic segment $s\colon [n_-, n_+] \cap \mathbb{Z} \to \Gamma$. According to the Morse subgroup property of Γ, the image path $sx = o_x \circ s$ is contained in a tubular neighborhood of uniform radius $\rho'' = \rho''(\Gamma, x)$ of a diamond $\diamond_{\tau_{mod}}(x_-, x_+)$ with

[32] I.e. quasigeodesics with the same quasiisometry constants.

$d(x_\pm, s(n_\pm)x) \leqslant \rho''$. It will be enough to verify that sx is also contained in a uniform tubular neighborhood of the smaller Θ-diamond $\diamondsuit_\Theta(x_-, x_+)$ for some Θ independent of s.

For $n_- \leqslant n \leqslant n_+$, let $p_n \in \diamondsuit_{\tau_{\text{mod}}}(x_-, x_+)$ denote the nearest point projection of $s(n)x$. In view of the uniform upper bound ρ'' for the distances $d(x_\pm, s(n_\pm)x)$ and $d(p_n, s(n)x)$, the uniform regularity of Γ implies: If $n - n_-, n_+ - n \geqslant C_0$ (with a uniform constant C_0), then

$$d_\Delta(x_\pm, p_n) \in V(0, \Theta)$$

with a compact $\Theta \subset \text{int}_{\tau_{\text{mod}}}(\sigma_{\text{mod}})$ independent of s. Moreover, after enlarging Θ, we may assume that it is ι-invariant and τ_{mod}-Weyl convex. It follows that the diamond $\diamondsuit_\Theta(x_-, x_+)$ is defined and $p_n \in \diamondsuit_\Theta(x_-, x_+)$. Hence, sx is contained in a uniform tubular neighborhood of $\diamondsuit_\Theta(x_-, x_+)$.

Conversely, suppose that o_x sends discrete geodesics in Γ to uniform τ_{mod}-Morse quasigeodesics in X. Then Γ is undistorted and the geodesic segments with endpoints in the orbit Γx are uniformly close to Θ-regular segments, equivalently, the Δ-distances $d_\Delta(x, \gamma x)$ between orbit points are contained in a tubular neighborhood of the cone $V(0, \Theta)$. It follows that Γ is (uniformly) τ_{mod}-regular, and hence τ_{mod}-Morse. □

Next, we briefly discuss the *asymptotics* of infinite Morse quasigeodesics. There is much freedom for the asymptotic behavior of arbitrary quasigeodesics in euclidean spaces, and therefore also in symmetric spaces of higher rank. However, the asymptotic behavior of Morse quasigeodesics is as restricted as for quasigeodesics in rank one symmetric spaces.

Morse quasirays satisfy a version of the defining property for Morse quasigeodesic segments, with diamonds replaced by cones. As a consequence, although Morse quasirays in general do not converge at infinity in the visual compactification, they *flag converge*:

Lemma 5.52 (Conicality of Morse quasirays) *A (Θ, ρ)-Morse quasiray $q : \mathbb{N}_0 \to X$ is contained in the ρ-neighborhood of a Θ-cone $V(x, \text{st}_\Theta(\tau))$ with $d(x, q(0)) \leqslant \rho$ for a unique simplex $\tau \in \text{Flag}_{\tau_{\text{mod}}}$. Furthermore, $q(n) \to \tau$ conically.*

Proof The existence of the cone $V(x, \text{st}_\Theta(\tau))$ follows from the definition of Morse quasigeodesics by a limiting argument. Obviously, we have conical τ_{mod}-flag convergence $q(n) \to \tau$, which also implies the uniqueness of τ. □

Now we give a *Finsler geometric characterization* of Morse quasigeodesics. We show that they are the coarsification of (uniformly regular) Finsler geodesics (cf. Definition 2.43). Even though this is true in general, we will give the proof only in the infinite case (of rays and lines), since it is simpler and suffices for the purposes of this paper:

Theorem 5.53 (Morse quasigeodesics are uniformly close to Finsler geodesics) *Uniform τ_{mod}-Morse quasigeodesic rays and lines are uniformly Hausdorff close to uniformly τ_{mod}-regular τ_{mod}-Finsler geodesic rays and lines.*

Proof It suffices to treat the ray case. The line case follows by a limiting argument.

Let $q\colon \mathbb{N}_0 \to X$ be a (Θ, ρ)-Morse quasigeodesic ray. According to Lemma 5.52, q is contained in a uniform tubular neighborhood of a Weyl cone $V = V(q(0), \mathrm{st}(\tau))$. As in the proof that asymptotically embedded implies URU (Theorem 5.18), we consider the sequence of nearest point projections $x_n \in V$ of the points $q(n)$, $n \in \mathbb{N}_0$. Again by Lemma 5.52, the point x_{n+m} lies in a uniform tubular neighborhood of the Θ-cone $V(x_n, \mathrm{st}_\Theta(\tau)) \subset V$ for all $n, m \geqslant 0$.

We slightly enlarge Θ to Θ', such that $\Theta \subset \mathrm{int}(\Theta')$ as subsets of $\mathrm{int}_{\tau_{\mathrm{mod}}}(\sigma_{\mathrm{mod}})$. Then there exists $m_0 \in \mathbb{N}$ depending on Θ, Θ', ρ and the quasiisometry constants of q, such that

$$x_{n+m} \in V(x_n, \mathrm{st}_{\Theta'}(\tau))$$

for all $n \geqslant 0$ and $m \geqslant m_0$. The piecewise geodesic path

$$x_0 x_{m_0} x_{2m_0} x_{3m_0} \ldots$$

is then a Θ'-regular τ_{mod}-Finsler geodesic ray uniformly Hausdorff close to q. $\square$

We use the approximation of Morse quasigeodesics by Finsler geodesics to coarsify Theorem 2.47 and deduce an analogous result on the Δ-*distance along Morse quasigeodesics*. Again, we restrict ourselves to the infinite case of rays:

Theorem 5.54 (Δ-projection of Morse quasirays) *If $q\colon \mathbb{N}_0 \to X$ is a τ_{mod}-Morse quasiray, then so is*

$$\overline{q}_\Delta = d_\Delta(q(0), q)\colon \mathbb{N}_0 \to \Delta.$$

Moreover, uniform τ_{mod}-Morse quasirays q yield uniform τ_{mod}-Morse quasirays $\overline{q}_\Delta$.

Proof Suppose that q is a (Θ, ρ)-Morse quasiray. We enlarge Θ to Θ' such that $\Theta \subset \mathrm{int}(\Theta')$. According to the proof of Theorem 5.53, there exists a Θ'-regular τ_{mod}-Finsler geodesic ray $c\colon [0, +\infty) \to X$ which is uniformly close to q in terms of the data Θ, Θ', ρ and the quasiisometry constants, i.e. $d(c(n), q(n))$ is uniformly bounded. In particular, c is also a uniform quasiray.

For the Δ-projections $\overline{c}_\Delta = d_\Delta(c(0), c)$ and $\overline{q}_\Delta$, the pointwise distance $d(\overline{c}_\Delta(n), \overline{q}_\Delta(n))$ is also uniformly bounded. According to Theorem 2.47, $\overline{c}_\Delta$ is again a Θ'-regular τ_{mod}-Finsler geodesic ray and a uniform quasiray. It follows that $\overline{q}_\Delta$ is a (Θ', ρ')-Morse quasiray with uniform ρ' and uniform quasiisometry constants. $\square$

5.11 Appendix: The original Anosov definition

A notion of Anosov representations of surface groups into $\mathrm{PSL}(n, \mathbb{R})$ was introduced by Labourie in [25], and generalized to a notion of (P_+, P_-)-Anosov representations $\Gamma \to G$ of word hyperbolic groups into semisimple Lie groups by Guichard and Wienhard in [12]. The goal of this section is to review this definition of Anosov representations $\Gamma \to G$ using the language of *expanding and contracting flows* and then present a closely related and equivalent definition which avoids the language of flows.

Let Γ be a non-elementary (i.e. not virtually cyclic) word hyperbolic group with a fixed word metric d_Γ and Cayley graph C_Γ. Consider a *geodesic flow* $\widehat{\Gamma}$ of Γ; such a flow was originally constructed by Gromov [10] and then improved by Champetier [5] and Mineyev [27], resulting in definitions with different properties. We note that the exponential convergence of asymptotic geodesic rays will not be used in our discussion; as we will see, it is also irrelevant whether the trajectories of the geodesic flow are geodesics or uniform quasigeodesics in $\widehat{\Gamma}$. In particular, it will be irrelevant for us which definition of $\widehat{\Gamma}$ is used. Only the following properties of $\widehat{\Gamma}$ will be used in the sequel:

- $\widehat{\Gamma}$ is a proper metric space.
- There exists a properly discontinuous isometric action $\Gamma \curvearrowright \widehat{\Gamma}$.
- There exists a Γ-equivariant quasi-isometry $\pi : \widehat{\Gamma} \to \Gamma$; in particular, the fibers of π are relatively compact.
- There exists a continuous action $\mathbb{R} \curvearrowright \widehat{\Gamma}$, denoted ϕ_t and called the *geodesic flow*, whose trajectories are uniform quasigeodesics in $\widehat{\Gamma}$, i.e. for each $\widehat{m} \in \widehat{\Gamma}$ the *flow line*

$$t \to \widehat{m}_t := \phi_t(\widehat{m})$$

  is a uniform quasi-isometric embedding $\mathbb{R} \to \widehat{\Gamma}$.
- The flow ϕ_t commutes with the action of Γ.
- Each $\widehat{m} \in \widehat{\Gamma}$ defines a uniform quasigeodesic $m : t \mapsto m_t$ in Γ by the formula:

$$m_t = \pi(\widehat{m}_t).$$

  Following the notation in Sect. 3.3, we let $(\partial_\infty \Gamma \times \partial_\infty \Gamma)^{\mathrm{dist}}$ denote the subset of $\partial_\infty \Gamma \times \partial_\infty \Gamma$ consisting of pairs of distinct points. The natural map

$$e = (e_-, e_+) : \widehat{\Gamma} \to (\partial_\infty \Gamma \times \partial_\infty \Gamma)^{\mathrm{dist}}$$

  assigning to $\widehat{m}$ the pair of ideal endpoints $(m_{-\infty}, m_{+\infty})$ of m is continuous and surjective. In particular, every uniform quasigeodesic in $\widehat{\Gamma}$ is uniformly Hausdorff close to a flow line.

The reader can think of the elements of $\widehat{\Gamma}$ *as parameterized geodesics in* C_Γ, *so that* ϕ_t *acts on geodesics via reparameterization*. This was Gromov's original viewpoint, although not the one in [27].

We say that $\widehat{m} \in \widehat{\Gamma}$ is *normalized* if $\pi(\widehat{m}) = 1 \in \Gamma$. Similarly, maps $q : \mathbb{Z} \to \Gamma$, and $q : \mathbb{N} \to \Gamma$ will be called *normalized* if $q(0) = 1$. It is clear that every $\widehat{m} \in \widehat{\Gamma}$ can be sent to a normalized element of $\widehat{\Gamma}$ via the action of $m_0^{-1} \in \Gamma$.

Since trajectories of ϕ_t are uniform quasigeodesics, for each normalized $\widehat{m} \in \widehat{\Gamma}$ we have

$$C_1^{-1} t - C_2 \leqslant d_\Gamma(1, m_t) \leqslant C_1 t + C_2 \tag{42}$$

for some positive constants C_1, C_2.

Let $\mathrm{F}^{\pm} = \mathrm{Flag}_{\pm\tau_{\mathrm{mod}}}$ be a pair of *opposite* partial flag manifolds associated to the Lie group G, i.e. they are quotient manifolds of the form $\mathrm{F}^{\pm} = G/P_{\pm\tau_{\mathrm{mod}}}$, see Sect. 2.4. As usual, we will regard elements of $\mathrm{F}^{\pm}$ as simplices of type $\tau_{\mathrm{mod}}, \iota\tau_{\mathrm{mod}}$ in the Tits boundary of X.

Define the trivial bundles

$$E^{\pm} = \widehat{\Gamma} \times \mathrm{F}^{\pm} \to \widehat{\Gamma}.$$

For every representation $\rho\colon \Gamma \to G$, the group Γ acts on both bundles via its natural action on $\widehat{\Gamma}$ and via the representation ρ on $\mathrm{F}^{\pm}$. Put a Γ-invariant background Riemannian metric on the fibers of theses bundles, which varies continuously with respect to $\widehat{m} \in \widehat{\Gamma}$. We will use the notation $\mathrm{F}^{\pm}_{\widehat{m}}$ for the fiber above the point $\widehat{m}$ equipped with this Riemannian metric. Since the subspace of $\widehat{\Gamma}$ consisting of normalized elements is compact, it follows that for normalized $\widehat{m}, \widehat{m}'$ the identity map

$$\mathrm{F}^{\pm}_{\widehat{m}} \to \mathrm{F}^{\pm}_{\widehat{m}'}$$

is uniformly bilipschitz (with bilipschitz constant independent of $\widehat{m}, \widehat{m}'$). We will identify Γ-equivariant (continuous) sections of the bundles $E^{\pm}$ with equivariant maps $s_{\pm}\colon \widehat{\Gamma} \to \mathrm{F}^{\pm}$. These sections are said to be *parallel along flow lines* if

$$s_{\pm}(\widehat{m}) = s_{\pm}(\widehat{m}_t)$$

for all $t \in \mathbb{R}$ and $\widehat{m} \in \widehat{\Gamma}$.

Definition 5.55 Parallel sections $s_{\pm}$ are called *strongly parallel along flow lines* if for any two flow lines $\widehat{m}, \widehat{m}'$ with the same ideal endpoints, we have $s_{\pm}(\widehat{m}) = s_{\pm}(\widehat{m}')$.

Note that this property is automatic for the geodesic flows constructed by Champetier and Mineyev since (for their flows) any two flow lines which are at finite distance from each other are actually equal. Strongly parallel sections define Γ-equivariant *boundary maps*

$$\beta_{\pm}\colon \partial_{\infty}\Gamma \to \mathrm{F}^{\pm}$$

from the Gromov boundary $\partial_{\infty}\Gamma$ of the word hyperbolic group Γ by:

$$\beta_{\pm} \circ e_{\pm} = s_{\pm}. \tag{43}$$

Lemma 5.56 *The maps $\beta_{\pm}$ are continuous.*

Proof Let $(\xi_-^n, \xi_+^n) \to (\xi_-, \xi_+)$ be a converging sequence in $(\partial_{\infty}\Gamma \times \partial_{\infty}\Gamma)^{\mathrm{dist}}$. There exists a bounded sequence $(\widehat{m}^n)$ in $\widehat{\Gamma}$ such that $e_{\pm}(\widehat{m}^n) = \xi_{\pm}^n$. After extraction, the sequence $(\widehat{m}^n)$ converges to some $\widehat{m} \in \widehat{\Gamma}$. Continuity of $s_{\pm}$ implies that $\beta_{\pm}(\xi_{\pm}^n) = s_{\pm}(\widehat{m}^n) \to s_{\pm}(\widehat{m}) = \beta_{\pm}(\xi_{\pm})$. This shows that no subsequence of $(\beta_{\pm}(\xi_{\pm}^n))$ can have a limit $\neq \beta_{\pm}(\xi_{\pm})$, and the assertion follows from compactness of $\mathrm{F}^{\pm}$. ◻

Conversely, equivariant continuous maps $\beta_\pm$ define Γ-equivariant sections $s_\pm$ strongly parallel along flow lines, by the formula (43).

Consider the identity maps

$$\Phi_{\widehat{m},t} : \mathrm{F}^\pm_{\widehat{m}} \to \mathrm{F}^\pm_{\phi_t \widehat{m}}.$$

These maps distort the Riemannian metric on the fibers. Using (25), we define the *infinitesimal expansion factor* of the flow $\phi(t)$ on the fiber $\mathrm{F}^\pm_{\widehat{m}}$ at the point $s_\pm(\widehat{m})$ as:

$$\epsilon_\pm(\widehat{m}, t) := \epsilon(\Phi_{\widehat{m},t}, s_\pm(\widehat{m})).$$

Definition 5.57 The geodesic flow ϕ_t is said to be *uniformly exponentially expanding* on the bundles $E^\pm$ with respect to the sections $s_\pm$ if there exist constants $a, c > 0$ such that

$$\epsilon_\pm(\widehat{m}, \pm t) \geqslant a e^{ct}$$

for all $\widehat{m} \in \widehat{\Gamma}$ and $t \geqslant 0$.

Our next goal is to give an alternative interpretation for the uniform expansion in this definition. First of all, since the metrics on the fibers are Γ-invariant, it suffices to verify uniform exponential expansion only for normalized elements of $\widehat{\Gamma}$. For a normalized element $\widehat{m} \in \widehat{\Gamma}$ and $t \in \mathbb{R}$ consider the composition

$$m_t^{-1} \circ \Phi_{\widehat{m},t} : \mathrm{F}^\pm_{\widehat{m}} \to \mathrm{F}^\pm_{m_t^{-1}\widehat{m}_t}.$$

Note that $\pi(m_t^{-1}\widehat{m}_t) = m_t^{-1} m_t = 1$, i.e. both $\widehat{m}$ and $m_t^{-1}\widehat{m}_t$ are normalized. Since the group Γ acts isometrically on the fibers of the bundles $E^\pm$, the metric distortion of the above compositions is exactly the same as the distortion of $\Phi_{\widehat{m},t}$. Furthermore, since, as we noted above, the metrics on $\mathrm{F}^\pm_{\widehat{m}}$ and $\mathrm{F}^\pm_{m_t^{-1}\widehat{m}_t}$ are uniformly bilipschitz to each other (via the "identity" map), the rate of expansion for the above composition (up to a uniform multiplicative error) is the same as the expansion rate for the map

$$\rho(m_t^{-1}) : \mathrm{F}^\pm \to \mathrm{F}^\pm.$$

(Here we are using fixed background Riemannian metrics on $\mathrm{F}^\pm$.) Thus, we get the estimate

$$C_3^{-1}\epsilon(\rho(m_t^{-1}), \beta_\pm(m_{\pm\infty})) \leqslant \epsilon_\pm(\widehat{m}, t) \leqslant C_3\epsilon(\rho(m_t^{-1}), \beta_\pm(m_{\pm\infty}))$$

for some uniform constant $C_3 > 1$. By taking into account the Eq. (42), we obtain the following equivalent reformulation of Definition 5.57:

Lemma 5.58 *The geodesic flow is uniformly exponentially expanding with respect to the sections $s_\pm$ if and only if for every normalized uniform quasigeodesic $q : \mathbb{Z} \to \Gamma$,* *which is asymptotic to points* $\xi_\pm = q(\pm\infty) \in \partial_\infty\Gamma$*, the elements* $\rho(q(\pm n))^{-1}$ *act on* $T_{\beta_\pm(\xi_\pm)}\mathrm{F}^\pm$ *with uniform exponential expansion rate, i.e.*

$$\epsilon(\rho(q(\pm n))^{-1}, \beta_\pm(\xi_\pm)) \geqslant Ae^{Cn}$$

for all $\widehat{m} \in \widehat{\Gamma}$ *and* $n \geqslant 0$ *with some fixed constants* $A, C > 0$.

Proof There exists a normalized flow line $\widehat{m}$ uniformly close to q, i.e. $q(n)$ is uniformly close to m_{t_n} with $n \mapsto t_n$ being a uniform orientation-preserving quasiisometry $\mathbb{Z} \to \mathbb{Z}$. Then $m_{\pm\infty} = \xi_\pm$, and $\epsilon(\rho(q(\pm n))^{-1}, \beta_\pm(\xi_\pm))$ equals $\epsilon(\rho(m_{t_{\pm n}}^{-1}), \beta_\pm(m_{\pm\infty}))$ up to a uniform multiplicative error, and hence also $\epsilon_\pm(\widehat{m}, t_{\pm n})$. □

Since every uniform quasigeodesic ray in Γ extends to a uniform quasigeodesic line, and in view of Morse lemma for hyperbolic groups, in the above definition it suffices to consider only normalized discrete geodesic rays $r : \mathbb{N} \to \Gamma$.

We can now give the original and an alternative definition of Anosov representations.

Definition 5.59 A pair of continuous maps $\beta_\pm : \partial_\infty\Gamma \to \mathrm{F}^\pm$ is said to be *antipodal* if it satisfies the following conditions (called *compatibility* in [12]):

- For every pair of distinct ideal points $\zeta, \zeta' \in \partial_\infty\Gamma$, the simplices $\beta_+(\zeta), \beta_-(\zeta')$ in the Tits boundary of X are antipodal, equivalently, the corresponding parabolic subgroups of G are opposite. (In [12] this property is called *transversality*.)
- For every $\zeta \in \partial_\infty\Gamma$, the simplices $\beta_+(\zeta), \beta_-(\zeta)$ belong to the same spherical Weyl chamber, i.e. the intersection of the corresponding parabolic subgroups of G contains a minimal parabolic subgroup.

Note that, as a consequence, the maps $\beta_\pm$ are *embeddings*, because antipodal simplices cannot be faces of the same chamber.

Definition 5.60 [12] A representation $\rho : \Gamma \to G$ is said to be $(P_{+\tau_{\mathrm{mod}}}, P_{-\tau_{\mathrm{mod}}})$-Anosov if there exists an antipodal pair of continuous ρ-equivariant maps $\beta_\pm : \partial_\infty\Gamma \to \mathrm{F}^\pm$ such that the geodesic flow on the associated bundles $E^\pm$ satisfies the uniform expansion property with respect to the sections $s_\pm$ associated to the maps $\beta_\pm$.

The pair of maps (β_+, β_-) in this definition is called *compatible* with the Anosov representation ρ. Note that a $(P_{+\tau_{\mathrm{mod}}}, P_{-\tau_{\mathrm{mod}}})$-Anosov representation admits a unique compatible pair of maps. Indeed, the fixed points of infinite order elements $\gamma \in \Gamma$ are dense in $\partial_\infty\Gamma$. The maps $\beta_\pm$ send the attractive and repulsive fixed points of γ to fixed points of $\rho(\gamma)$ with contracting and expanding differentials, and these fixed points are unique. In particular, if $P_{+\tau_{\mathrm{mod}}}$ is conjugate to $P_{-\tau_{\mathrm{mod}}}$ (equivalently, $\iota\tau_{\mathrm{mod}} = \tau_{\mathrm{mod}}$) then $\beta_- = \beta_+$.

We note that Guichard and Wienhard in [12] use in their definition the uniform contraction property of the reverse flow ϕ_{-t} instead of the expansion property used above, but the two are clearly equivalent. Note also that in the definition, it suffices to verify the uniform exponential expansion property only for the bundle E_+. We thus obtain, as a corollary of Lemma 5.58, the following alternative definition of Anosov representations:

Proposition 5.61 (Alternative definition of Anosov representations) *A representation $\rho\colon \Gamma \to G$ is $(P_{+\tau_{\mathrm{mod}}}, P_{-\tau_{\mathrm{mod}}})$-Anosov if and only if there exists a pair of antipodal continuous ρ-equivariant maps $\beta_\pm\colon \partial_\infty\Gamma \to \mathrm{F}^\pm$ such that for every normalized discrete geodesic ray $r\colon \mathbb{N} \to \Gamma$ asymptotic to $\xi \in \partial_\infty\Gamma$, the elements $\rho(r(n))^{-1}$ act on $T_{\beta_+(\xi)}\mathrm{F}_+$ with uniform exponential expansion rate, i.e.*

$$\epsilon(\rho(r(n))^{-1}, \beta_+(\xi)) \geqslant Ae^{Cn}$$

for $n \geqslant 0$ with constants $A, C > 0$ which are independent of r.

We now restrict to the case that the parabolic subgroups $P_{\pm\tau_{\mathrm{mod}}}$ are conjugate to each other, i.e. the simplices $\iota\tau_{\mathrm{mod}} = \tau_{\mathrm{mod}}$. The $(P_{+\tau_{\mathrm{mod}}}, P_{-\tau_{\mathrm{mod}}})$-Anosov representations will in this case be called simply $P_{\tau_{\mathrm{mod}}}$*-Anosov*, where $P_{\tau_{\mathrm{mod}}} = P_{+\tau_{\mathrm{mod}}}$, or simply τ_{mod}-Anosov. Note that the study of general $(P_{+\tau_{\mathrm{mod}}}, P_{-\tau_{\mathrm{mod}}})$-Anosov representations quickly reduces to the case of P-Anosov representations by intersecting parabolic subgroups, cf. [12, Lemma 3.18]. Now,

$$\mathrm{F}^\pm = \mathrm{F} = G/P_{\tau_{\mathrm{mod}}} = \mathrm{Flag}_{\tau_{\mathrm{mod}}}$$

and

$$\beta_\pm = \beta\colon \partial_\infty\Gamma \to \mathrm{F}$$

is a single continuous embedding. The compatibility condition reduces to the *antipodality* condition: For any two distinct ideal points $\xi, \xi' \in \partial_\infty\Gamma$ the simplices $\beta(\xi)$ and $\beta(\xi')$ are antipodal to each other. In other words, β is a *boundary embedding* in the sense of Definition 5.2.

We thus arrive to our definition, compare Definition 5.43:

Definition 5.62 (*Anosov representation*) Let τ_{mod} be an ι-invariant face of σ_{mod}. We call a representation $\rho\colon \Gamma \to G$ $P_{\tau_{\mathrm{mod}}}$*-Anosov* or τ_{mod}-Anosov if it is τ_{mod}-boundary embedded with boundary embedding $\beta\colon \partial_\infty\Gamma \to \mathrm{F} = \mathrm{Flag}_{\tau_{\mathrm{mod}}}$ such that for every normalized discrete geodesic ray $r\colon \mathbb{N} \to \Gamma$ asymptotic to $\zeta \in \partial_\infty\Gamma$, the elements $\rho(r(n))^{-1}$ act on $T_{\beta(\zeta)}\mathrm{F}$ with uniform exponential expansion rate, i.e.

$$\epsilon(\rho(r(n))^{-1}, \beta(\zeta)) \geqslant Ae^{Cn}$$

for $n \geqslant 0$ with constants $A, C > 0$ independent of r.

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