List of problems collected at the workshop "Discrete subgroups of Lie groups" held at BIRS

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1 Background

Recent years have seen a great deal of progress in our understanding of "thin" subgroups, which are discrete matrix groups that have infinite covolume in their Zariski closure. (The subgroups of finite covolume are called "lattices" and, generally speaking, are much better understood.) Traditionally, thin subgroups are required to be contained in arithmetic lattices, which is natural in the context of number-theoretic and algorithmic problems but, from the geometric or dynamical viewpoint, is not necessary. Thin subgroups have deep connections with number theory (see e.g. [7, 8, 9, 19, 23]), geometry (e.g. [1, 15, 17, 42]), and dynamics (e.g. [4, 24, 28, 30]).

The well-known "Tits Alternative" [48] (based on the classical "ping-pong argument" of Felix Klein) constructs free subgroups of any matrix group that is not virtually solvable. (In most cases, it is easy to arrange that the resulting free group is thin.) Sharpening and refining this classical construction is a very active and fruitful area of research that has settled numerous old problems. For instance, Breuillard and Gelander [12] proved a quantitative form of the Tits Alternative, which shows that the generators of a free subgroup can be chosen to have small word length, with respect to any generating set of the ambient group. Kapovich, Leeb and Porti [27] provided a coarse-geometric proof of the existence of free subgroups that are Anosov. In a somewhat different vein, Margulis and Soifer [36] proved that $SL(n, \mathbb{Z}), n \geq 4$, contains free products of the form $\mathbb{Z}^2 \star F_k$ (where F_k is the free group of rank k) for all $k \geq 1$. (Answering a question of Platonov and Prasad, this implies that $SL(n, \mathbb{Z})$ has maximal subgroups of infinite index that are not free groups.) Also, it has been shown that $SL(n, \mathbb{Z})$ contains Coxeter groups (and, hence, right-angled Artin groups) when n is sufficiently large. Ping-pong type constructions have also emerged as an important technical tool in other contexts, such as for disproving the invariable generation property (by constructing a thin subgroup that intersects every conjugacy class [20]) and for proving the expander property for Cayley graphs of finite quotients of thin groups [23].

Conversely, there are also obstructions to the existence of thin subgroups. For example, no thin, Zariski-dense subgroup of $SL(n, \mathbb{Z})$ contains $SL(3, \mathbb{Z})$ [49]. Similarly, it has been shown in certain situations that thin, discrete, Zariski-dense subgroups cannot contain a lattice in a maximal unipotent subgroup of the ambient group [6, 39, 50].

The study of certain natural (and properly discontinuous) actions of thin groups is another important line of research. The famous Auslander Conjecture concerns actions of thin groups on affine spaces. (Namely, it is conjectured that if a group acts properly discontinuously and cocompactly on an affine space $\mathbb{R}^n$, then the group is virtually solvable.) Actions on more general geometric spaces (such as flag varieties) are also important. While it seems that nothing of interest can be said about such actions *in general*, a great deal of progress has been made in recent years analyzing actions of thin subgroups that satisfy further restrictions.

The Anosov property has been of particular interest (see, for example, [24], [28], and [53]), as well as other forms of strengthening of discreteness, e.g. *regularity*, which has its origin in [4], see also [28].

The following problem list collects some open problems for both lattices and thin subgroups of Lie groups.

2 Proper actions on affine spaces

Let Γ be a finitely generated group, $\rho : \Gamma \to GL(n, \mathbb{R})$ be the given representation, $Z^1(\Gamma, \rho)$ be the space of ρ-cocycles with values in $\mathbb{R}^n$, i.e. maps

$$u : \Gamma \to \mathbb{R}^n, u(\alpha\beta) = u(\alpha) + \rho(\alpha)u(\beta), \alpha, \beta \in \Gamma.$$

Note that $Z^1(\Gamma, \rho)$ is a finite-dimensional real vector space. Each cocycle $u \in Z^1(\Gamma, \rho)$ determines an affine action ρ_u of Γ on $V = \mathbb{R}^n$:

$$\rho_u(\gamma) : \mathbf{x} \mapsto \rho(\gamma)\mathbf{x} + u(\gamma).$$

Let $C = C_\rho \subset Z^1(\Gamma, \rho)$ denote the subset consisting of cocycles such that the action ρ_u of Γ on $\mathbb{R}^n$ is properly discontinuous.

Question 2.1 (N. Tholozan)**.** *Is C open in $Z^1(\Gamma, \rho)$? Is it convex?*

Note that the answer is positive for $n = 3$, this follows from the results of [22].

Question 2.2 (N. Tholozan)**.** *Does C_ρ depend continuously on ρ?*

Here one has to be careful with the topology used on the set of subsets of $Z^1(\Gamma, \rho)$. For closed subsets one uses Shabauty topology. For instance, if the subsets C are open, their complements are closed and, hence, one can interpret the question as of the continuity of the complement with respect to ρ.

For a finitely generated group Γ, let $P(\Gamma, n)$ denote the subset of $Hom(\Gamma, Aff(\mathbb{R}^n))$ consisting of representations defining proper actions of Γ on $\mathbb{R}^n$. Let $PA(\Gamma, n) \subset P(\Gamma, n)$ denote the subset consisting of actions ρ_u with P-Anosov linear part ρ (for some parabolic subgroup $P < GL(n, \mathbb{R})$).

Question 2.3 (G. Soifer)**.** *To which extent $P(\Gamma, n)$ is open?*

Note that in general $P(\Gamma, n)$ is not open even for $n = 3$, for instance, one can take a rank 2 free group Γ and a representation $\rho_u \in P(\Gamma, 3)$ whose linear part ρ contains unipotent elements. A small perturbation of ρ_u will yield a representation (with linear part in $SO(2, 1)$) with nondiscrete linear part, hence, a non-proper affine action. One can also perturb a representation so that the linear part is deformed to a Zariski dense subgroup of $SL(3, \mathbb{R})$, again resulting in a non-proper action.

It is known that

$$PA(\Gamma, 3) \cap Hom(\Gamma, SO(2, 1) \ltimes \mathbb{R}^3)$$

is open in

$$Hom(\Gamma, SO(2, 1) \ltimes \mathbb{R}^3)$$

(this follows from the results of [22] and stability of Anosov representations).

Question 2.4 (G. Soifer)**.** *To which extent $P(\Gamma, n)$ is open in general?*

Conjecture 2.5 (The Auslander conjecture)**.** *If $\Gamma < \mathrm{Aff}(\mathbb{R}^n)$ is a subgroup which acts properly discontinuously and co-compactly on $\mathbb{R}^n$, then Γ is virtually solvable.*

Abels, Margulis and Soifer proved the Auslander conjecture for the dimensions $n \leq 6$ and observed that the following problem is important for the further progress towards the Auslander conjecture:

Conjecture 2.6 (Abels, Margulis, Soifer)**.** *Let $\Gamma < O(4, 3) \ltimes \mathbb{R}^7 < \mathrm{Aff}(\mathbb{R}^7)$ be a subgroup acting properly discontinuously and co-compactly on $\mathbb{R}^7$. Then the linear part of Γ is not Zariski dense in $SO(4, 3)$.*

Question 2.7. *Does there exist a properly discontinuous cocompact group of affine transformations isomorphic to the fundamental group of a closed hyperbolic manifold?*

Partial progress towards a negative answer to this problem was described at the talk by Suhyoung Choi at the workshop: He proved that such an action cannot exist under certain assumptions on its linear part (strengthening the P-Anosov condition).

Question 2.8 (G. Mostow)**.** *Suppose that $M = \mathbb{R}^n/\Gamma$ is an affine manifold, such that the linear part of Γ is in $SL_n(\mathbb{R})$. Thus, Γ preserves the standard volume form on $\mathbb{R}^n$ and, hence, M has a canonical volume form as well. Does $Vol(M) < \infty$ imply that M is compact?*

This question is motivated by Mostow's theorem that lattices in solvable groups are cocompact.

Question 2.9 (G. Soifer)**.** *Study subgroups $\Gamma < \mathrm{Aff}(\mathbb{C}^n)$ acting properly discontinuously on $\mathbb{C}^n$.*

More specifically:

Question 2.10 (G. Soifer)**.** *Does there exist a free nonabelian subgroup $\Gamma < \mathrm{Aff}(\mathbb{C}^3)$ acting properly discontinuously on $\mathbb{C}^3$? For instance, consider the adjoint action of $SL_2(\mathbb{C})$ on $\mathbb{C}^3$ (identified with the Lie algebra of $SL_2(\mathbb{C})$). Consider a generic representation $\rho : F_2 = \langle a, b\rangle \to SL(2, \mathbb{C})$. Does there exist a cocycle $u \in Z^1(F_2, \mathbb{C}^3)$ and $m > 0$ such that the action on $\mathbb{C}^3$ of $\langle a^m, b^m\rangle$ given by ρ_u is properly discontinuous?*

Conjecture 2.11 (Markus Conjecture)**.** *Suppose that M is a compact n-dimensional affine manifold whose linear holonomy is in $SL(n, \mathbb{R})$. Is M complete?*

This conjecture is known when the linear holonomy of M has "discompacity 1" (Y. Carriere, [14]), e.g. when M is a flat Lorentzian manifold, and also for convex affine manifolds of dimension ≤ 5, [26].

Conjecture 2.12 (M. Kapovich)**.** *Suppose that M is a compact n-dimensional affine manifold whose linear holonomy is contained in a rank one subgroup of $SL(n, \mathbb{R})$. Is M complete?*

3 Discrete subgroups of $SL(3, \mathbb{R})$

Question 3.1 (M. Kapovich)**.**

1. *Does there exist a discrete a subgroup $\Gamma < SL(3, \mathbb{R})$ isomorphic to $\mathbb{Z}^2 \star \mathbb{Z}$ and containing only regular diagonalizable elements?*
2. *Does there exist a discrete subgroup $\Gamma < SL(3, \mathbb{Z})$ isomorphic to $\mathbb{Z}^2 \star \mathbb{Z}$?*

Note that there are known examples, [47], of discrete subgroups $\Gamma < SL(3, \mathbb{R})$ isomorphic to $\mathbb{Z}^2 \star \mathbb{Z}$ where $\mathbb{Z}^2$ is *super-singular*: It is generated by three singular diagonalizable matrices A, B, C satisfying $ABC = 1$.

Question 3.2 (K. Tsouvalas)**.** *Does there exist a discrete a subgroup $\Gamma < SL(3, \mathbb{R})$ isomorphic to $\pi_1(S) \star \mathbb{Z}$, where S is a closed hyperbolic surface ?*

Note that it is impossible to find an Anosov subgroup $\Gamma < SL(3, \mathbb{R})$ isomorphic to $\pi_1(S) \star \mathbb{Z}$ with this property, since every Anosov subgroup of $SL(3, \mathbb{R})$ is either virtually free or a virtually surface group. Note, furthermore, that $SL(4, \mathbb{Z})$ contains subgroups isomorphic to $\mathbb{Z}^2 \star \mathbb{Z}$ and $\pi_1(S) \star \mathbb{Z}$.

4 Subgroups of $SL(n, \mathbb{Z})$, $n \ge 3$

While many "exotic" finitely generated groups embed in $SL(n, \mathbb{Z})$ for large n, very few subgroups of $SL(3, \mathbb{Z})$ are known: All currently known finitely generated thin subgroups of $SL(3, \mathbb{Z})$ are either virtually free or are virtually isomorphic to surface groups.

Problem 4.1. *Construct thin subgroups of $SL(3, \mathbb{Z})$ which are neither virtually free nor are virtually isomorphic to surface groups.*

For the next questions, we will need some group-theoretic definitions:

Definition 4.1. *A group* Γ *is called* coherent *if every finitely generated subgroup of* Γ *is finitely presented. A group* Γ *is said to have the* Howson property *if the intersection of any two finitely generated subgroup of* Γ *is again finitely generated.*

It is known that $SL(2,\mathbb{Z})$ is coherent (and, moreover, every discrete subgroup of $SL(2,\mathbb{C})$ is coherent), while $SL(4,\mathbb{Z})$ is non-coherent. Every discrete subgroup of $SL(2,\mathbb{C})$ which is not a lattice has the Howson property. However, there are (even arithmetic) lattices in $SL(2,\mathbb{C})$ which do not have the Howson property. The reason for this is the existence of finitely generated geometrically infinite subgroups of such lattices.

Question 4.2 (J.-P. Serre)**.** *Is* $SL(3,\mathbb{Z})$ *coherent?*

The answer would be positive if every finitely generated thin subgroup of $SL(3,\mathbb{Z})$ were virtually isomorphic to either a free group or a surface group. While groups such as $\mathbb{Z}^2 \star \mathbb{Z}$ and $\pi_1(S) \star \mathbb{Z}$ (where S is a surface) are coherent, the existence of embeddings of such groups in $SL(3,\mathbb{Z})$ might help us to find embeddings of more complicated subgroups and, hopefully, address the coherence problem.

Problem 4.3 (J.-P. Serre)**.** *Is there a profinitely dense non-free subgroup in* $SL(3,\mathbb{Z})$*?*

Question 4.4 (A. Detinko)**.** *Does* $SL(3,\mathbb{Z})$ *have the Howson property?*

Question 4.5 (M. Kapovich)**.** *Suppose that* Γ_1, Γ_2 *are Anosov subgroups of* $SL(3,\mathbb{Z})$*. Is* $\Gamma_1 \cap \Gamma_2$ *finitely generated?*

Problem 4.6 (T. Gelander, C. Meiri)**.** *An element* $g \in SL(3,\mathbb{Z})$ *is called* complex *if for every* $m \geq 1$ *the matrix* g^m *has a non-real eigenvalue. Is it possible for a thin subgroup of* $SL(3,\mathbb{Z})$ *to contain a complex element?*

5 Algorithmic problems

Question 5.1 (A. Detinko)**.** *Is freeness decidable for finitely generated subgroups of arithmetic groups (e.g. of* $SL(n,\mathbb{Z}), n \geq 3$*)?*

Note that freeness is undecidable for subsemigroups. Freeness is decidable for subgroups of $SL(2,\mathbb{Z})$. It is also decidable for some special classes of subgroups of arithmetic groups:

(a) Anosov subgroups.

(b) Subgroups which admit finitely-sided Dirichlet domains in associated symmetric spaces.

Freeness is likely to be, at least effectively, undecidable. The reason is the existence of *badly distorted* finitely generated free subgroups of $SL(n,\mathbb{Z})$ for large n: These are free subgroups whose distortion function is comparable to the k-th Ackerman function (for any k), see [16, 11] for the description of embeddings of such free groups in free-by-cyclic groups and [25, 52] for embeddings to $SL(n,\mathbb{Z})$.

Question 5.2 (A. Detinko)**.** *Is arithmeticity decidable? More precisely, is there an algorithm that decides if a finitely generated Zariski dense subgroup* Λ *of an* irreducible *arithmetic group* Γ *(say,* $SL(n,\mathbb{Z})$*,* $n \geq 3$*) has finite index in* Γ*?*

Note that this problem is semidecidable: There is an algorithm which will terminate if $\Lambda < \Gamma$ has finite index. The problem is known to be decidable for subgroups of $SL(2,\mathbb{Z})$ and undecidable for subgroups of $SL(2,\mathbb{Z}) \times SL(2,\mathbb{Z})$.

Question 5.3 (M. Kapovich)**.** *Is the membership problem in finitely generated subgroups of* $SL(3,\mathbb{Z})$ *decidable?*

Note that all *known* finitely generated subgroups of $SL(3,\mathbb{Z})$ have at most exponential distortion, hence, have decidable membership problem. In contrast, the membership problem is undecidable for finitely generated subgroups of $SL(4,\mathbb{Z})$. The reason is that that group contains $SL(2,\mathbb{Z}) \times SL(2,\mathbb{Z})$, which, in turn, contains a direct product of two free groups of large ranks. The latter admits finitely generated normal subgroups with undecidable membership problem (Mikhailova subgroups, [37]). However, in this case, the ambient lattice is reducible.

Fact 5.4. *There exist irreducible arithmetic groups Γ such that for Zariski dense subgroups in Γ the membership problem is undecidable.*

Very likely, the subgroups Γ can be found in $SO(p,q)$ for suitable p,q. The existence of Γ is an application of the Rips construction of small cancellation groups with non-recursively distorted normal subgroups [46], combined with with the Cubulation Theorem of Dani Wise [51] and the embedability of cubulated groups in RACGs (Right-Angled Coxeter groups), see [52], which, in turn, admit Zariski dense representations in $O(p,q) \cap GL(p+q,\mathbb{Z})$, see [5].

Note that the membership problem is decidable for subgroups with recursive distortion function, e.g. for quasiisometrically embedded subgroups, such as Anosov subgroups.

6 Maximal subgroups

Recall that a subgroup M of a group Γ is said to be *maximal* if there is no *proper* subgroup $\Lambda < \Gamma$ containing M.

According to [35], every Zariski dense subgroup Γ in a semisimple Lie group G (of positive dimension) admits *maximal subgroups of infinite index*. However, very little is known about maximal subgroups in this setting. The construction of maximal subgroups in [35] is a two-step process: First, construct an infinite rank free profinitely dense subgroup $\Lambda < \Gamma$ (this step is essentially constructive) and then, use Zorn's Lemma to get a maximal subgroup M:

$$\Lambda < \mathrm{M} < \Gamma.$$

The second step is completely nonconstructive.

Question 6.1 (G. Margulis, G. Soifer)**.** *Suppose that $\Gamma < G$ is as above.*

1. *Is it true that for every maximal subgroup $\mathrm{M} < \Gamma$ is not finitely generated?*
2. *Is it true that Γ contains a free maximal subgroup?*

Note that Aka, Gelander and Soifer [2] proved that there exists a finitely generated profinitely dense subgroup Γ of $SL_n(\mathbb{Z})$ such that the number of generators of Γ does not depend on n.

Question 6.2 (G. Soifer)**.** *Does there exist a profinitely dense subgroup of $SL_n(\mathbb{Z})$ generated by two elements?*

7 Other problems on thin subgroups

Definition 7.1. *For a finitely generated group Γ with a finite generating subset S the* Kazhdan constant *$\kappa(\Gamma, S)$ is defined as*

$$\kappa(\Gamma, S) = \inf_{\pi, v} \max_{g \in S} \|v - \pi_g v\|,$$

where the infimum is taken over all unitary representations (H_π, π) of Γ without fixed unit vectors, and all unit vectors $v \in H_\pi$. Then Γ is said to have Property T iff $\kappa(\Gamma, S) > 0$ for some/every finite generating subset S. A group Γ is said to have uniform Property T*, if $\inf_S \kappa(\Gamma, S) > 0$ where the infimum is taken over all finite generating subsets S.*

The following question goes back to [32]:

Question 7.1 (A. Lubotzky)**.** *Does $SL(n,\mathbb{Z}), n \geq 3$, have the uniform Property T?*

Note that Lubotzky was asking the more general question whether Property T implies uniform Property T, which was answered in the negative independently by Gelander & Zuk [21], and Osin [40]. The problem is open for all n. In the case of many classes of higher rank uniform lattices, the answer is known to be negative.

Question 7.2 (Bekka, de la Harpe, Valette)**.** *Are there thin subgroups of $SL(n,\mathbb{Z}), n \geq 3$, satisfying Property T?*

The difficulty is that the known examples of (infinite discrete) groups satisfying Property T tend to be: (a) super-rigid arithmetic groups, or (b) some combinatorially defined groups for which all real-linear representations are nondiscrete or nonfaithful.

One can attempt to combine super-rigid lattices, but such combinations tend to destroy Property T. Alternatively, one can attempt to use polygons of groups where vertex groups have Property T, with suitable spectral conditions on links of vertices, but such constructions tend to produce relatively compact subgroups of $SL(n, \mathbb{R})$.

Note that the property τ (with respect to the family of finite index normal subgroups which are kernels of homomorphisms to $SL(n, \mathbb{Z}/q\mathbb{Z})$) holds for thin subgroups of $SL(n, \mathbb{Z}), n \geq 3$, see [10].

For the next question, recall that standard proofs of the Tits Alternative yield Zariski dense free subgroups of the given semisimple Lie group G.

Definition 7.2. *A free subgroup $\Gamma < G$ is* hereditarily Zariski dense *(or* strongly dense*, see [13]) if every noncyclic subgroup of Γ is Zariski dense in G.*

The following problem is raised in [13]:

Question 7.3 (Breuillard, Green, Guralnick, Tao)**.** *Is it true that every Zariski dense subgroup of a real semisimple Lie group G contains a hereditarily Zariski dense free subgroup? If so, is there a quantitative version of this result?*

It appears that the only case when the affirmative answer is known is when G is 3-dimensional (in which case it is an immediate corollary of the Tits Alternative).

8 Characterization of higher rank lattices

Definition 8.1 (Prasad–Raghunathan rank)**.** *Let Γ be a group. Let A_i denote the subset of Γ that consists of those elements whose centralizer contains a free abelian group of rank at most i as a subgroup of finite index. Thus, $A_0 \subset A_1 \subset \ldots$. The* Prasad–Raghunathan rank*, $\mathrm{prank}(\Gamma)$, of Γ is the minimal number i such that $\Gamma = \gamma_1 A_i \cup \cdots \cup \gamma_m A_i$ for some $\gamma_1, \ldots, \gamma_m \in \Gamma$.*

For instance, if Γ is a lattice in a semisimple Lie group of rank n, then $\mathrm{prank}(\Gamma) = n$. If M is a compact Riemannian manifold of nonpositive curvature with $\Gamma = \pi_1(M)$, then $\mathrm{prank}(\Gamma)$ equals the geometric rank of M, i.e. the largest n such that every geodesic in M is contained in an immersed n-dimensional flat.

Definition 8.2 (BGP, Bounded Generation Property)**.** *A group Γ is said to have BGP if there exist elements $\gamma_1, ..., \gamma_k$ such that every $\gamma \in \Gamma$ can be written as a product*

$$\gamma = \gamma_1^{n_1} \gamma_2^{n_2} \cdots \gamma_k^{n_k}$$

for some $n_1, ..., n_k \in \mathbb{Z}$. (Note that a power of each γ_i appears only once.)

Question 8.1 (G. Prasad)**.** *Does there exist a discrete Zariski dense subgroup $\Gamma < G$ (with G a simple real algebraic group) such that Γ is not a lattice but $\mathrm{prank}(\Gamma) = rank_{\mathbb{R}}(G)$?*

Question 8.2 (M. Kapovich)**.** *What algebraic properties distinguish higher rank (irreducible uniform) lattices?*

One such characterization was given by Lubotzky and Venkataramana [34], in terms of profinite completions.

Alternatively, notice that higher rank lattices Γ have Prasad–Raghunathan rank, $\mathrm{prank}(\Gamma) \geq 2$. Are there discrete linear groups Γ which are not virtually nontrivial direct products and are not lattices, satisfying $\mathrm{prank}(\Gamma) \geq 2$? In the case of groups Γ of integer points of split semisimple algebraic groups over $\mathbb{Z}$, a defining feature are the *Serre relators*. However, Serre relators are for unipotent elements, which do not exist in uniform lattices. Uniform higher rank lattices satisfy *approximate* Serre relators. Do these determine whether a discrete linear group is a higher rank lattice?

Notice that there are some indirect signs that an algebraic characterization of lattices is possible:

1. Higher rank lattices are quasiisometrically rigid (Kleiner & Leeb [29], Eskin [18]).
2. Higher rank lattices are rigid in the sense of the 1st order logic (Avni, Lubotzky, Mieri [3]).
3. Appearance of Serre relators in profinite completions, (Prasad, Rapinchuk [43]).

The situation is not entirely clear even for nonuniform lattices. Many classes of higher rank nonuniform lattices satisfy the BGP. Nonlinear groups that satisfy the BGP were constructed by A. Muranov [38].

Question 8.3 (M. Kapovich)**.** *Suppose that* Γ *is an abstract (infinite)* $\mathbb{R}$*-linear group satisfying the BGP. Is it isomorphic to a lattice in a Lie group?*

Problem 8.4 (M. Mj)**.** *Does* $SL(3,\mathbb{Z})$ *have the bounded generation property with respect to semisimple elements? I.e., is there a collection of* k *semisimple elements* $g_1, ..., g_k \in SL(3,\mathbb{Z})$ *such that every element of* $SL(3,\mathbb{Z})$ *has the form*

$$g = g_1^{n_1} ..., g_k^{n_k}?$$

Conjecturally, the answer is negative (for dynamical reasons related to ping-pong arguments) which should pave the way to prove that uniform lattices do not have bounded generation property.

9 Why are higher rank lattices super-rigid?

One way to say that an abstract group Γ is *super-rigid* is to require that for every field F and $n \in \mathbb{N}$, there are only finitely many conjugacy classes of representations $\Gamma \to GL(n,F)$. Of course, some groups do not admit any nontrivial linear representations, so it makes sense to restrict the discussion to finitely generated linear groups Γ.

Loosely speaking, such a group is (super) rigid if it satisfies some peculiar relators. There are many proofs of rigidity and super-rigidity of (higher rank irreducible) lattices, but none of these proofs (in the setting of uniform lattices) use relators satisfied by lattices, likely because such relators are simply unknown (see previous section). In contrast, there are known proofs of super-rigidity of some classes of higher rank non-uniform lattices (see [45] and references therein).

Question 9.1 (M. Kapovich)**.** *What are group-theoretic reasons that make higher rank uniform lattices (super)-rigid? Are the approximate Serre relators responsible for this? Or high Prasad-Raghunathan rank?*

One known result in this direction is that the BGP implies super-rigidity, see [41]. Another group-theoretic property implying super-rigidity is given by Lubotzky in [33].

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