Hyperbolic groups with low dimensional boundary

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Résumé

Soit G un groupe hyperbolique (en sens de Gromov) sans torsion. Si la dimension topologique du bord $\partial_\infty G$ est egale à un, et G n'est ni un produit amalgamé, ni une extension HNN sur un groupe cyclic, on montre que $\partial_\infty G$ est homéomorphe à l'éponge de Menger ou au tapis de Sierpinski. Si $\partial_\infty G$ est homéomorphe au tapis de Sierpinski, on montre que G est isomorphe à un sous-groupe quasi-convexe d'un groupe de dimension trois de dualité de Poincaré. On construit un exemple d'un groupe hyperbolique G qui est "topologiquement rigide": chaque homéomorphisme du bord $\partial_\infty G$ est induit par un élément $g \in G$.

Abstract

If a torsion-free hyperbolic group G has 1-dimensional boundary $\partial_\infty G$, then $\partial_\infty G$ is a Menger curve or a Sierpinski carpet provided G does not split over a cyclic group. When $\partial_\infty G$ is a Sierpinski carpet we show that G is a quasi-convex subgroup of a 3-dimensional hyperbolic Poincaré duality group. We also construct a "topologically rigid" hyperbolic group G: any homeomorphism of $\partial_\infty G$ is induced by an element of G.

1. Introduction

We recall that the boundary $\partial_\infty X$ of a locally compact Gromov hyperbolic space X is a compact metrizable topological space. Brian Bowditch observed that any compact metrizable space Z arises this way: view the unit ball B in Hilbert space as the Poincaré model of infinite dimensional hyperbolic space, topologically embed Z in the boundary of B, and then take the convex hull $CH(Z)$ to get a locally compact Gromov hyperbolic space with $\partial_\infty CH(Z) = Z$. On the other hand when X is the Cayley graph of a Gromov hyperbolic group G then the topology of $\partial_\infty X \simeq \partial_\infty G$ is quite restricted. It is known that $\partial_\infty G$ is finite dimensional, and either perfect, empty, or a two element set (in the last two cases the group G is **elementary**). It was shown recently by Bowditch and Swarup [B2, Sw1] that if $\partial_\infty G$ is connected then it does not have global cut-points, and thus is locally connected according to [BM]. The boundary

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of G necessarily has a "large" group of homeomorphisms: if G is nonelementary then its action on $\partial_\infty G$ is minimal, and G acts on $\partial_\infty G$ as a discrete uniform convergence group. It turns out that the last property gives a dynamical characterization of boundaries of hyperbolic groups, according to a theorem of Bowditch [B3]: if Z is a compact metrizable space with $|Z| \geq 3$ and $G \subset Homeo(Z)$ is a discrete uniform convergence subgroup, then G is hyperbolic and Z is G-equivariantly homeomorphic to $\partial_\infty G$. In general the action $G \curvearrowright \partial_\infty G$ is not effective, but if G is nonelementary, its ineffective kernel is a finite normal subgroup $N \lhd G$; moreover, every finite normal subgroup of G is contained in N. We let $\bar{G}$ denote the quotient G/N.

There are two questions which arise naturally:

Question A. *Which topological spaces are boundaries of hyperbolic groups?*

Question B. *Given a topological space Z, which hyperbolic groups have Z as the boundary?*

Regarding question A, all spheres, some homology spheres [DJ], the Sierpinski carpet, and the Menger curve [Be] arise as boundaries of hyperbolic groups. Moreover, according to Gromov and Champetier [Ch], "generic" finitely presentable groups are hyperbolic and have the Menger curve as boundary. On the other hand, as was noticed by Bestvina, it is unknown if higher-dimensional Universal Menger compacta [Bes1] appear as boundaries of hyperbolic groups (Dranishnikov has constructed hyperbolic groups with boundary homeomorphic to the 2-dimensional Menger compactum, [Dr]).

Considerably less is known about Question B. If $\partial_\infty G$ is zero-dimensional then G is a virtually free group [St, Gr, GH]. Recently it was proven in [Ga, CJ, T1] that any hyperbolic group whose boundary is homeomorphic to $\mathbb{S}^1$ acts discretely, cocompactly, and isometrically on the hyperbolic plane. We call such a group **virtually Fuchsian**. The case when $\partial_\infty G \simeq \mathbb{S}^2$ is a difficult open problem:

Conjecture 1. *(J. Cannon). If G is a hyperbolic group whose boundary is homeomorphic to $\mathbb{S}^2$, then G acts isometrically and properly discontinuously on hyperbolic 3-space $\mathbb{H}^3$.*

In section 7 we construct new examples of hyperbolic groups for which we answer Question B completely. These groups have a remarkable topological rigidity property:

Definition 2. A hyperbolic group G is said to be **topologically rigid** if every homeomorphism $f : \partial_\infty G \to \partial_\infty G$ is induced by an element of G.

Remark 3. Actually, the topologically rigid groups constructed in this paper are even *locally* topologically rigid in the following sense: if $U, V \subset \partial_\infty G$ are connected open subsets, then any homeomorphism $U \to V$ is induced by an element of G.

Our examples are the first known topologically rigid *nonelementary* hyperbolic groups (finite groups and groups G which fit into an exact sequence

$$1 \to \text{ finite group } \to G \to \mathbb{Z}/2 * \mathbb{Z}/2 \to 1$$

are topologically rigid for trivial reasons). The Cayley graph of a topologically rigid nonelementary hyperbolic group is a quasi-isometrically rigid metric space (every quasi-isometry is within bounded distance from an isometry), see Lemma 19. Previously known examples of quasi-isometrically rigid metric spaces include quaternionic hyperbolic spaces and the Cayley hyperbolic plane [Pa], higher rank symmetric spaces of noncompact type [KL], Cayley graphs of maximal non-arithmetic nonuniform lattices in isometry groups of rank 1 symmetric spaces of dimension > 2 [Sch], and universal covers of compact hyperbolic n-manifolds with nonempty totally geodesic boundary[1], $n \geq 3$. Topologically rigid groups have an even stronger rigidity property than quasi-isometrically rigid groups (see Lemma 20):

> If G' is a hyperbolic group whose boundary is homeomorphic to the boundary of a topologically rigid hyperbolic group G, then $\bar{G}'$ embeds in $\bar{G}$ as a finite index subgroup.

The topologically rigid groups mentioned above have 2-dimensional boundary; we prove in Corollary 18 that this is the minimal dimension for the boundary of a nonelementary topologically rigid group.

The remaining results of our paper concern hyperbolic groups with one-dimensional boundary.

Theorem 4. *Let G be a hyperbolic group which does not split over a finite or virtually cyclic subgroup, and suppose $\partial_\infty G$ is 1-dimensional. Then one of the following holds (see section 2 for definitions):*

1. $\partial_\infty G$ is a Menger curve.

2. $\partial_\infty G$ is a Sierpinski carpet.

3. $\partial_\infty G$ is homeomorphic to $\mathbb{S}^1$ and G maps onto a Schwartz triangle group with finite kernel.

It is probably impossible to classify hyperbolic groups whose boundaries are homeomorphic to the Menger curve (since this is the "generic" case), however it appears that a meaningful study is possible in the case of hyperbolic groups whose boundaries are homeomorphic to the Sierpinski carpet. Recall that the Sierpinski carpet $\mathcal{S}$ has a canonical collection of **peripheral circles** (see section 2).

Theorem 5. *Suppose that $\partial_\infty G \cong \mathcal{S}$. Then:*

1. There are only finitely many G-orbits of peripheral circles.

2. The stabilizer of each peripheral circle C is a quasi-convex virtually Fuchsian group which acts on C as a uniform convergence group. We call these subgroups **peripheral subgroups of** G.

3. If we "double" G along the collection of peripheral subgroups using amalgamated free product and iterated HNN-extension (see section 5), then the result is a hyperbolic group $\widehat{G}$ which contains G as a quasiconvex subgroup.

4. The boundary of $\widehat{G}$ is homeomorphic to $\mathbb{S}^2$. Hence by [BM], [Bes2], $\widehat{G}$ is a 3-dimensional Poincaré duality group in the torsion-free case.

[1]This was observed in a discussion Bernhard Leeb, Richard Schwartz, and the authors. The rigidity statement follows from a doubling construction and the technique of [Sch].

5. When G is torsion free, then $(G; H_1, \ldots, H_k)$ is a 3-dimensional Poincaré duality pair (see [DD] for the definition), where $H_1, \ldots, H_i$ are the peripheral subgroups of G.

A similar result holds in the case of higher dimensional analogs of the Sierpinski carpet, except that in part 2 one says that peripheral sphere stabilizers are hyperbolic groups with spherical boundary.

Known examples of groups with Sierpinski carpet boundary are consistent with the following:

Conjecture 6. *Let G be a hyperbolic group with Sierpinski carpet boundary. Then G acts discretely, cocompactly, and isometrically on a convex subset of $\mathbb{H}^3$ with nonempty totally geodesic boundary.*

There is now some evidence for this conjecture. It would follow from a positive solution of Cannon's conjecture together with Theorem 5, see section 5. Alternatively, in the torsion-free case, if one could show that (hyperbolic) 3-dimensional Poincaré duality groups are 3-manifold groups, then Thurston's Haken uniformization theorem could be applied to an irreducible 3-manifold with fundamental group isomorphic to the group $\widehat{G}$ produced in Theorem 5. Under extra conditions (such as coherence and the existence of a nontrivial splitting) it appears that one can show that a 3-dimensional Poincaré duality group is a 3-manifold group.

The conjecture above leads one to ask which hyperbolic groups have planar boundary. Concretely, one may ask if a torsion-free hyperbolic group with planar boundary has a finite index subgroup subgroup isomorphic to a discrete convex cocompact subgroup of $Isom(\mathbb{H}^3)$. Here is a cautionary example which shows that in general it is necessary to pass to a finite index subgroup: if one takes a surface of genus 1 with two boundary components and glues one boundary circle to the other by a degree 2 map, then the fundamental group G of the resulting complex K enjoys the following properties (see section 8):

1. G is torsion-free and hyperbolic.

2. G contains a finite index subgroup which is isomorphic to a discrete, convex cocompact subgroup of $Isom(\mathbb{H}^3)$ which does not act cocompactly on $\mathbb{H}^3$. In particular, the boundary of G is 1-dimensional and planar.

3. G is not a 3-manifold group.

2. Preliminaries

Properties of hyperbolic groups and spaces. For a proof of the following properties of hyperbolic groups, we refer the reader to [Gr, ABC+, GH, B3].

Let G be a nonelementary Gromov hyperbolic group, and suppose G acts discretely and cocompactly on a locally compact geodesic metric space X. Then the boundary of X is a compact metrizable space $\partial_\infty X$ on which $Isom(X)$ acts by homeomorphisms. For any $f \in Isom(X)$, we denote the corresponding homeomorphism of $\partial_\infty X$ by $\partial_\infty f$. The action of G on $\partial_\infty X$ is minimal, i.e. the G-orbit of every point is dense in $\partial_\infty X$. Let $\partial_\infty^2 X := \partial_\infty X \times \partial_\infty X - Diag$ be the space of distinct pairs in $\partial_\infty X$. Then the set of pairs of points $(x, y) \in \partial^2_\infty X$ which are fixed by an infinite cyclic subgroup of G is dense in $\partial^2_\infty X$. We let $\bar{\partial}^2_\infty X := \partial^2_\infty X/(x, y) \sim (y, x)$.

The group G acts cocompactly and properly discontinuously on $\partial^3 X := \{(x, y, z) \in (\partial_\infty X)^3 \mid x, y, z \text{ distinct}\}$. There is a natural topology on $X \cup \partial_\infty X$ which is a G-invariant compactification of X, and this is compatible with the topology on $\partial_\infty X$.

Recall that a subset S of a geodesic metric space is C-quasi-convex if every geodesic segment with endpoints in S is contained in the C-tubular neighborhood of S. Quasi-convex subsets of δ-hyperbolic metric spaces satisfy a **visibility property** (cf. [EbOn]):

> *Given R, C, $\delta \in (0, \infty)$ there is an R' with the following property (we may take $R' = R + 10\delta$). If X is a δ-hyperbolic metric space, $Y \subset X$ is C-quasi-convex, and $x \in X$ satisfies $d(x, Y) \geq R'$, then given any two unit speed geodesics γ_1, γ_2 starting at x and ending in Y, and any $t \in [0, R]$ we have $d(\gamma_1(t), Im(\gamma_2)) < \delta$ and $d(\gamma_2(t), Im(\gamma_1)) < \delta$.*

As a consequence of the visibility property, if $Y_k \subset X$ is a sequence of C-quasi-convex subsets of a δ-hyperbolic space X, and $d(x, Y_k) \to \infty$ as $k \to \infty$, then a subsequence of Y_k's converges to a single point $\xi \in \partial_\infty X$.

Sierpinski carpets and Menger curves. The classical construction of a Sierpinski carpet is analogous to the construction of a Cantor set: start with the unit square in the plane, subdivide it into nine equal subsquares, remove the middle open square, and then repeat this procedure inductively on the remaining squares. If we take a sequence $D_i \subset \mathbb{S}^2$ of disjoint closed 2-disks whose union is dense in $\mathbb{S}^2$ so that $Diam(D_i) \to 0$ as $i \to \infty$, then $\mathbb{S}^2 - \cup_i Interior(D_i)$ is a Sierpinski carpet; moreover any Sierpinski carpet embedded in $\mathbb{S}^2$ is obtained in this way [W]. Sierpinski carpets can also be characterized as follows [W]: a compact, 1-dimensional, planar, connected, locally connected space with no local cut points is a Sierpinski carpet.

We will use a few topological properties of Sierpinski carpets $\mathcal{S}$:

1. There is a unique embedding of $\mathcal{S}$ in $\mathbb{S}^2$ up to post-composition with a homeomorphism of $\mathbb{S}^2$.

2. There is a countable collection $\mathcal{C}$ of "peripheral circles" in $\mathcal{S}$, which are precisely the nonseparating topological circles in $\mathcal{S}$.

3. Given any metric d on $\mathcal{S}$ and any number $D > 0$, there are only finitely many peripheral circles in $\mathcal{S}$ of diameter $> D$.

The Menger curve may be constructed as follows. Start with the unit cube I^3 in $\mathbb{R}^3$. Consider the orthogonal projections $\pi_{ij} : I^3 \to F_{ij}$ of the unit cube onto the ij coordinate square, and let $\mathcal{S}_{ij} \subset F_{ij}$ be the Sierpinski carpet as constructed above. The Menger curve is the intersection $\cap_{i<j} \pi_{ij}^{-1}(\mathcal{S}_{ij})$. The Menger curve is universal among all compact metrizable 1-dimensional spaces: any such space can topologically embedded in the Menger curve. By [A1, A2], a compact, metrizable, connected, locally connected, 1-dimensional space is a Menger curve provided it has no local cut points, and no nonempty open subset is planar.

3. Proof of Theorem 4

The fact that G does not split over a finite group implies [St] that G is one-ended, and $\partial_\infty G$ is connected. Recall that by the results of [BM, B2, Sw1], the boundary of a one-ended hyperbolic group is locally connected and has no global cut points; furthermore, if $\partial_\infty G$ has local cut points then G splits over a virtually infinite cyclic subgroup unless $\partial_\infty G \simeq \mathbb{S}^1$ and G maps onto a Schwarz triangle group with finite kernel. Therefore from now on we will assume that $\partial_\infty G$ has no local cut points.

A 1-dimensional, compact, metrizable, connected, locally connected space Z with no local cut points is a Menger curve provided no point $z \in Z$ has a neighborhood which embeds in the plane (see section 2). Hence either $\partial_\infty G$ is a Menger curve or some $\xi \in \partial_\infty G$ has a planar neighborhood U; therefore we assume the latter holds.

Lemma 7. *Let $\Gamma \subset \partial_\infty G$ be a subset homeomorphic to a finite graph. Then Γ is a planar graph.*

Proof. Since the action of G on $\partial_\infty G$ is minimal, every G-orbit intersects the planar neighborhood U, and so every point of $\partial_\infty G$ has a planar neighborhood. Because $\partial_\infty G$ has no local cut points, we have $\partial_\infty G \setminus \Gamma \neq \emptyset$. So we can find a hyperbolic element $g \in G$ whose fixed point set $\{\eta_1, \eta_2\} \subset \partial_\infty G$ is disjoint from Γ (section 2). Hence for sufficiently large n, $g^n(\Gamma)$ is contained in a planar neighborhood of η_1 or η_2. □

We recall [C, M] that a compact, metrizable, connected, locally connected space X with no global cut points is planar as long as no nonplanar graph embeds in X. Therefore $\partial_\infty G$ is planar. Finally, by [W], $\partial_\infty G$ is Sierpinski carpet. □

4. Groups with Sierpinski carpet boundary

Let M be a compact hyperbolic manifold with nonempty totally geodesic boundary and let $G := \pi_1(M)$ be its fundamental group. The universal cover $\tilde{M}$ of M may be identified with a closed convex subset of $\mathbb{H}^3$ which is bounded by a countable disjoint collection $\mathcal{P}$ of totally geodesic planes. Each $P \in \mathcal{P}$ bounds an open half-space disjoint from $\tilde{M}$. $\tilde{M}$ is obtained from $\mathbb{H}^3$ by removing each of these open half-spaces, and $\partial_\infty \tilde{M} \subset \partial_\infty \mathbb{H}^3$ is obtained from $\partial_\infty \mathbb{H}^3 \simeq \mathbb{S}^2$ by deleting the open disks corresponding to these half-spaces. The closures of these disks are disjoint since the distance between distinct elements of $\mathcal{P}$ is bounded away from zero. As $\partial_\infty \tilde{M}$ has no interior points in $\mathbb{S}^2$, it is a Sierpinski carpet (see section 2). Note that the peripheral circles of $\partial_\infty \tilde{M}$ are in 1-1 correspondence with elements of $\mathcal{P}$, and therefore the conjugacy classes of G-stabilizers of peripheral circles are in 1-1 correspondence with $\mathcal{P}/G$, the set of boundary components of M. The stabilizer of a peripheral circle is the same as the stabilizer of the corresponding element of $\mathcal{P}$, so these stabilizers are quasi-convex in G.

The next theorem shows that similar conclusions hold for any hyperbolic group whose boundary is a Sierpinski carpet.

Theorem 8. *Let G be a hyperbolic group with boundary homeomorphic to the Sierpinski carpet $\mathcal{S}$. Then*

1. *There are finitely many G-orbits of peripheral circles in $\mathcal{S}$.*

2. The stabilizer of each peripheral circle C is a quasi-convex subgroup G whose boundary is C.

Proof. We recall that G acts cocompactly on the space $\partial^3 G := \{(x,y,z) \in (\partial_\infty G)^3 \mid x,\, y,\, z \text{ distinct}\}$. Therefore if $C_k \subset \partial_\infty G$ is a sequence of peripheral circles, $(x_k, y_k, z_k) \in \partial^3 G$ and $\{x_k, y_k, z_k\} \subset C_k$, then after passing to a subsequence we may find a sequence $g_k \in G$, $(x_\infty, y_\infty, z_\infty) \in \partial^3 G$ so that $(g_k x_k, g_k y_k, g_k z_k)$ converges to $(x_\infty, y_\infty, z_\infty)$. But this means that $Diam(g_k(C_k))$ is bounded away from zero, so $g_k(C_k)$ belongs to a finite collection of peripheral circles, and hence $g_k(C_k)$ is eventually constant. We conclude that there are only finitely many G-orbits of peripheral circles, and the stabilizer of any $C \in \mathcal{C}$ acts cocompactly on the space of distinct triples in C. By [B2] $Stab(C)$ is a quasi-convex subgroup of G, and $\partial_\infty Stab(C) = C$. From now on we will refer to stabilizers of peripheral circles as **peripheral subgroups**. By [Ga, CJ, T1] each peripheral subgroup is, modulo a finite normal subgroup, a cocompact Fuchsian group in $Isom(\mathbb{H}^2)$. □

5. Doubling Sierpinski carpet groups along peripheral subgroups

In this section we prove Theorem 5.

Let G be a hyperbolic group with $\partial_\infty G \simeq \mathcal{S}$, and let $H_1, \ldots, H_k$ be a set of representatives of conjugacy classes of peripheral subgroups of G. We define a graph of groups $\mathcal{G}$ as follows. The underlying graph has two vertices and k edges (no loops). Each vertex is labelled by a copy of G, the i^{th} edge is labelled by H_i, and the edge homomorphisms $H_i \to G$ are given by the inclusions. We let $\widehat{G}$ be the fundamental group of $\mathcal{G}$.

Next we construct a tree of spaces on which the group $\widehat{G}$ acts in a natural way. Let X_0 be a finite Cayley 2-complex for G, and let X_i be a finite Cayley 2-complex for the group H_i. The inclusion $H_i \hookrightarrow G$ is induced by a cellular map $h_i : X_i \to X_0$ between the 2-complexes. Let $h : \cup X_i \to X_0$ be the corresponding map from the disjoint union of the X_i's to X_0, and let X denote the mapping cylinder of h.

Let DX be the double of X along the collection of subcomplexes $X_i, i = 1, ..., k$. Consider now the universal cover $\widetilde{DX}$ of DX with the deck transformation group $\widehat{G}$. Let Y be the 1-skeleton of $\widetilde{DX}$. The 1-skeletons of the subcomplexes $X_i, i = 1, ..., k$ lift to disjoint *edge subspaces* of Y. A *vertex subspace* of Y is obtained as follows: take a connected component C of the complement of the edge spaces in Y, take the closure $\bar{C}$, and then add in all edge spaces which intersect $\bar{C}$. Each vertex space is a copy of the 1-skeleton of the universal cover of X. Let T be the graph corresponding to the decomposition of Y into vertex and edge subspaces: vertices v of T correspond to *vertex spaces* $Y_v \subset Y$, the edges e correspond to the *edge subspaces* $Y_e \subset Y$. An edge e is incident to a vertex v if and only if Y_e is contained in Y_v. It is standard that the graph T is actually a tree (compare [SW]). Let V and E denote the collections of vertices and edges in T respectively. If $v \in T$ we let E_v denote the collection of edges containing v.

Let $\sigma : DX \to DX$ be the natural involution of DX. A map $\tau : Y \to Y$ is a *reflection* if it is a lift of σ and it fixes some point; each reflection fixes some edge space in Y, and each edge space Y_e is the fixed point set of precisely one reflection r_e. Let Γ be the group generated by the reflections in Y. The group Γ is normalized by $\widehat{G}$ since conjugation of a reflection by an element of $\widehat{G}$ yields another reflection; likewise $\widehat{G}$ is normalized by Γ. Let $v \in T$ be any vertex. Then Γ is the free product of order two subgroups of the form $\langle r_e \rangle$ where $e \in E_v$. The vertex space Y_v is a fundamental domain for the action of Γ on Y. The group Γ preserves the tree structure of Y, so we have an induced action of Γ on T by tree automorphisms, each reflection r_e acting on T as an inversion of the edge e. The action of Γ on T naturally induces an action of Γ on $\partial_\infty T$. The space Y is a connected graph, and we give it the natural path-metric where each edge in Y has unit length.

Lemma 9. *1. The space Y is Gromov-hyperbolic.*

2. Edge and vertex spaces are all K-quasi-convex in Y for some K.

3. There is a function $C(R)$ such that for every R, the intersection of R-neighborhoods of any two distinct vertex or edge spaces has diameter at most $C(R)$ unless the spaces are incident.

Proof. The space Y is quasi-isometric to Cayley graph of $\widehat{G}$. The group $\widehat{G}$ is Gromov-hyperbolic by [BF2, BF3]. The assertions 2 and 3 follow from [Mi] and [Sw2]. ☐

We have a coarse Lipschitz projection $p : Y \to T$ which maps $(Y_v - \cup_{e \in E_v} Y_e)$ to v for each $v \in V$, and maps each edge space to the midpoint of the corresponding edge of T. If $\gamma : [0, \infty) \to Y$ is a unit speed geodesic ray, then $p \circ \gamma$ is a coarse Lipschitz path with the bounded backtracking property[2] by the quasi-convexity of vertex/edge spaces. Hence $p \circ \gamma$ maps into a finite tube around a geodesic ray τ in T. If $p \circ \gamma$ is unbounded in T, then the equivalence class of the ray τ is uniquely determined by γ and we label γ with the associated boundary point $[\tau] \in \partial_\infty T$. By the quasi-convexity of edge spaces, if γ hits an edge space for an unbounded sequence of times, then it remains in a quasi-convex tubular neighborhood of the edge space (of uniformly bounded thickness). In this case, we know that γ eventually remains in a bounded neighborhood of a unique edge space by property 3 in Lemma 9, and we label γ with this edge. If neither of the above two cases occurs, then for each edge e of the tree, we know that γ eventually lies in one of the two components of the complement of the edge space Y_e, and we label the edge with an arrow pointing in the direction of the corresponding subtree of T. There must be some (and at most one) vertex $v \in T$ such that all edges emanating from v have arrows pointing toward v; otherwise we could follow arrows and leave any bounded set. There must be an unbounded sequence of times t_k such that $\gamma(t_k)$ lies in the vertex space Y_v (by the construction of the edge labelling); by quasi-convexity of Y_v, this means that γ eventually lies in the R-neighborhood of Y_v; in this case we label γ by v. Equivalent geodesic rays are given the same label. We get a labelling map $\partial_\infty Label : \partial_\infty Y \to (T \cup \partial_\infty T)$ which is clearly Γ-equivariant.

We now examine the topology of $\partial_\infty Y$. This space is metrizable and we fix a metric d on $\partial_\infty Y$; in what follows we will implicitly use d when discussing metric properties of $\partial_\infty Y$. Recall that each vertex space Y_v is quasi-isometric to $G \simeq \tilde{X}$; since by

[2]A map $c : [0, \infty) \to T$ has the **bounded backtracking property** if for every $r \in (0, \infty)$ there is an $r' \in (0, \infty)$ such that if $t_1 < t_2$, and $d(c(t_1), c(t_2)) > r'$, then $d(c(t), c(t_1)) > r$ for every $t > t_2$.

Lemma 9 every subspace Y_v is quasi-convex in Y, we conclude that $\partial_\infty Y_v \subset \partial_\infty Y$ is a Sierpinski carpet. Similarly, the peripheral circles of the Sierpinski carpet $\partial_\infty Y_v$ are in 1-1 correspondence with the boundaries of edge spaces $Y_e \subset Y_v$. We note that the union $\cup_v \partial_\infty Y_v$ is dense in $\partial_\infty Y$, since this subset is $\widehat{G}$-invariant and $\widehat{G}$ is a nonelementary hyperbolic group.

By the visibility property of the uniformly quasi-convex edge spaces, there is at most one boundary point of $\partial_\infty Y$ labelled by any $\xi \in \partial_\infty T$. For each edge e in T, the set of points in $\partial_\infty Y$ labelled by e is the ideal boundary of the edge space Y_e, i.e. a circle. For each vertex $v \in T$, the set of points labelled by v is

$$\partial_\infty Y_v - \cup_{e \in E_v} \partial_\infty Y_e$$

i.e. the Sierpinski carpet $\partial_\infty Y_v$ minus the union of its peripheral circles.

Our next goal is to describe the topology of $\partial_\infty Y$ using the tree T. Choose $v \in T$. Every edge e of T separates T into two subtrees, and we let $T_{v,e} \subset T$ be the subtree disjoint from v. We define the **outward subset**, $Out_{v,e}$, for a pair $(v, e) \in V \times E$ to be the collection of points of $\partial_\infty Y$ labelled by elements of $T_{v,e} \cup \partial_\infty T_{v,e}$. The visibility property of Y implies that for a fixed $v \in T$ and any $\epsilon > 0$ there are only finitely many edges $e \subset T$ so that the diameter of $Out_{v,e}$ exceeds ϵ. Outward subsets of $\partial_\infty Y$ are open since a geodesic ray γ with $\partial_\infty \gamma \in Out_{v,e}$ will eventually leave any tubular neighborhood of the edge space Y_e, and so nearby boundary points correspond to rays which eventually lie in the same component of the complement of Y_e in Y. It follows that if $\xi \in \partial_\infty T$, and e_k is the sequence of edges occurring in the ray $\overline{v\xi}$, then the sequence of outward sets Out_{v,e_k} is a nested basis for the topology of $\partial_\infty Y$ at the point labelled by ξ. The closure of $Out_{v,e}$ is $Out_{v,e} \cup \partial_\infty Y_e$ because the complement to $Out_{v,e} \cup \partial_\infty Y_e$ is $Out_{w,e}$ where w is the endpoint of e furthest from v (obviously $\partial_\infty Y_e \subset \overline{Out_{v,e}}$).

Lemma 10. *Suppose $\xi_k \in \partial_\infty Y$ converges to $\xi_\infty \in \partial_\infty Y$. Then one of the following holds.*

1. ξ_∞ is labelled by a boundary point $Label(\xi_\infty) \in \partial_\infty T$. In this case $Label(\xi_k)$ converges to $Label(\xi_\infty)$ in the compact space $T \cup \partial_\infty T$.

2. ξ_∞ is labelled by a vertex $v \in T$. In this case, for any subset $\mathcal{E} \subseteq E_v$ containing all but finitely many elements of E_v, the sequence ξ_k eventually lies in

$$\partial_\infty Y_v \cup (\cup_{e \in \mathcal{E}} Out_{v,e}).$$

3. ξ_∞ is labelled by an edge e_0. In this case, if v, w are the endpoints of e_0 then for any subset $\mathcal{E} \subseteq E_v$ containing all but finitely many elements of E_v, and any subset $\mathcal{F} \subseteq E_w$ containing all but finitely many elements of E_w, the sequence ξ_k eventually lies in

$$\partial_\infty Y_v \cup \partial_\infty Y_w \cup (\cup_{e \in \mathcal{E}} Out_{v,e}) \cup (\cup_{e \in \mathcal{F}} Out_{w,e}).$$

Proof. Case 1. If v is any arbitrary vertex of T, and $e_1, e_2, \ldots$ is the sequence of edges comprising the geodesic ray $\overline{v\xi_\infty} \subset T$, then $Out_{v,e_j} \subset \partial_\infty Y$ is a neighborhood basis for ξ_∞. Therefore $Label(\xi_k)$ converges to $Label(\xi_\infty)$ by the definition of the topology on $T \cup \partial_\infty T$.

Case 2. If this weren't the case, then a subsequence of ξ_k would converge to an element of $\overline{Out_{v,e}} = Out_{v,e} \cup \partial_\infty Y_e$ for some $e \notin \mathcal{E}$. This contradicts the fact that ξ_∞ is labelled by v.

Case 3. Similar to case 2.

□

Proposition 11. *$\partial_\infty \widehat{G}$ is homeomorphic to $\mathbb{S}^2$.*

Proof. Let G' be the fundamental group of a compact hyperbolic 3-manifold M with nonempty totally geodesic boundary. Recall (see section 4) that $\partial_\infty G'$ is a Sierpinski carpet. Using the notation developed above (decorated with "primes"), $\widehat{G}'$ is the fundamental group of the double of M, so $\partial_\infty \widehat{G}'$ is homeomorphic to $\mathbb{S}^2$. We will construct a homeomorphism between $\partial_\infty \widehat{G}'$ and $\partial_\infty \widehat{G}$.

Choose vertices $v \in T$ and $v' \in T'$, and a bijection $E_v \to E_{v'}$. This induces an isomorphism between Coxeter groups $\Gamma \to \Gamma'$, which we will use to identify Γ with Γ'. There is a unique Γ-equivariant isomorphism $T \cup \partial_\infty T \to T' \cup \partial_\infty T'$ which induces the given bijection $E_v \to E_{v'}$; we will use primes to denote corresponding edges and vertices. Choose an enumeration $v = v_1, v_2, \dots$ of vertices of T so that $d(v_k, \cup_{j<k} v_j) = 1$. Choose a homeomorphism $f_1 : \partial_\infty Y_v \to \partial_\infty Y'_{v'}$. Using reflections from Γ we inductively extend f_1 to a homeomorphism $f_k : \cup_{i=1}^k \partial_\infty Y_{v_i} \to \cup_{i=1}^k \partial_\infty Y'_{v'_i}$ for each k, so that the resulting map $f_\infty : \cup_{i=1}^\infty \partial_\infty Y_{v_i} \to \cup_{i=1}^\infty \partial_\infty Y'_{v'_i}$ is Γ-equivariant. By construction, f_∞ is compatible with label maps, i.e. the following diagram commutes:

$$\begin{array}{ccc} \cup_{i=1}^\infty \partial_\infty Y_{v_i} & \xrightarrow{f_\infty} & \cup_{i=1}^\infty \partial_\infty Y'_{v'_i} \\ \mathit{Label}\big\downarrow & & \mathit{Label}\big\downarrow \\ T \cup \partial_\infty T & \xrightarrow{id} & T \cup \partial_\infty T \end{array}$$

We claim that f_∞ extends continuously to a homeomorphism $f : \partial_\infty Y \to \partial_\infty Y'$. In view of the naturality of our construction it is enough to show that f_∞ extends to a continuous map $f : \partial_\infty Y \to \partial_\infty Y' \simeq \partial_\infty \widehat{G}' \simeq \mathbb{S}^2$, since the inverse map may be produced by exchanging the roles of G and G'. Pick a sequence $\xi_k \in \partial_\infty Y$ which converges to some $\xi \in \partial_\infty Y$. We will show that $f_\infty(\xi_k)$ converges.

Case 1: ξ is labelled by some $\eta \in \partial_\infty T$. In this case there is a unique $\xi' \in \partial_\infty Y'$ which is labelled by $\eta' \in \partial_\infty T'$. We know that if e_i (resp e'_i) is the sequence of edges of the ray $\overline{v\eta}$ (resp $\overline{v'\eta'}$), then the outward sets Out_{v,e_i} (resp. Out_{v',e'_i}) form a basis for the topology of $\partial_\infty \widetilde{DX}$ (resp. $\partial_\infty Y'$) at ξ (resp. ξ'). Since f_∞ maps $Out_{v,e_i} \cap \cup_{i=1}^\infty \partial_\infty Y_{v_i}$ to $Out_{v',e'_i} \cap \cup_{i=1}^\infty \partial_\infty Y'_{v'_i}$, the sequence $f_\infty(\xi_k)$ converges to ξ'.

Case 2: ξ is labelled by a vertex $v \in T$. For each k either $\xi_k \in \partial_\infty Y_v$ or $\xi_k \in Out_{v,e_k}$ for a unique $e_k \in Edge_v$. By Lemma 10, in the latter case $Diam(Out_{v,e_k}) \to 0$ as $k \to \infty$. Construct a sequence $\zeta_k \in \partial_\infty Y_v$ so that $\zeta_k = \xi_k$ when $\xi_k \in \partial_\infty Y_v$, and $\zeta_k \in \partial_\infty Y_{e_k} = \overline{Out_{v,e_k}} \cap \partial_\infty Y_v$ otherwise. Note that $\lim_{k\to\infty} \zeta_k = \xi$ since $Diam(Out_{v,e_k}) \to 0$. The sequence $f_\infty(\zeta_k)$ converges to $f_\infty(\xi)$ since $f\big|_{\partial_\infty Y_v}$ is continuous. Observe that $d(f_\infty(\zeta_k), f_\infty(\xi_k))$ is zero when $\xi_k \in \partial_\infty Y_v$ and is at most $Diam(Out'_{v',e'_k})$ otherwise. Since each e_k occurs only finitely often, $Diam(Out'_{v',e'_k}) \to 0$ so

$$\lim_{k\to\infty} f_\infty(\xi_k) = \lim_{k\to\infty} f_\infty(\zeta_k) = f_\infty(\xi).$$

Case 3: ξ is labelled by an edge $e_0 \in T$. We leave this case to the reader, as it is similar to case 2.

□

Corollary 12. *Let G be a torsion-free hyperbolic group with Sierpinski carpet boundary and $H_1, ..., H_k$ be representatives of conjugacy classes of stabilizers of peripheral cicrles of the Sierpinski arpet. Then $\widehat{G}$ is a torsion-free hyperbolic group, and hence it is a 3-dimensional Poincaré duality group by [BM], [Bes2]. By [DD], if one splits a $PD(n)$ group over a $PD(n-1)$ subgroup, then the vertex groups (together with the incident edge subgroups) define $PD(n)$ pairs; therefore $(G; H_1, \ldots, H_k)$ is a Poincaré duality pair. In particular $\chi(G) = \frac{1}{2}\sum_i \chi(H_i) < 0$.*

Corollary 13. *Let G be a torsion-free hyperbolic group with Sierpinski carpet boundary. Suppose either*

A. Cannon's conjecture is true

or

B. Every 3-dimensional Poincaré duality group with a nontrivial splitting is the fundamental group of a closed 3-manifold.

Then G is the fundamental group of a compact hyperbolic 3-manifold with totally geodesic boundary.

Proof. Let $H_1, \ldots, H_k$, $\widehat{G}$, Γ, be as in the first part of this section. If A holds, then $\widehat{G}$ is the fundamental group of a closed hyperbolic 3-manifold M. Since $\widehat{G}$ splits nontrivially by its very definition, if B holds then $\widehat{G} = \pi_1(M)$ where M is a closed irreducible 3-manifold. M is Haken since its fundamental group splits, and so Thurston's uniformization theorem implies that M admits a hyperbolic structure. In either case we have $\widehat{G}$ acting on $\mathbb{H}^3$ discretely, cocompactly, and isometrically.

The reflection group Γ acts on $\widehat{G}$ by conjugation, with each reflection centralizing a unique quasi-convex edge subgroup of $\widehat{G}$. By Mostow rigidity, Γ acts isometrically on the universal cover of M normalizing the action $\widehat{G} \curvearrowright \mathbb{H}^3$. $G \subset \widehat{G}$ is a quasi-convex subgroup, and so it acts on $\mathbb{H}^3$ as a convex cocompact subgroup. The limit set of G in $\partial_\infty \mathbb{H}^3$ is a Sierpinski carpet, and because every peripheral subgroup of G is centralized by a unique reflection in $\Gamma \subset Isom(\mathbb{H}^3)$, the peripheral circles are fixed by reflections in Γ. Thus each peripheral circle of the limit set of G is a round circle, and so the convex hull of the limit set is a convex subset bounded by disjoint totally geodesic hyperbolic planes. It follows that G is the fundamental group of a compact hyperbolic manifold with totally geodesic boundary.

□

6. Examples

We now use Theorems 1 and 5 to see that some classes of hyperbolic groups have Menger curve boundary.

We first remark that a torsion-free hyperbolic group with Sierpinski carpet boundary has negative Euler characteristic by Corollary 12. So if G is a torsion-free hyperbolic group with 1-dimensional boundary, G doesn't split over a trivial or cyclic group, and $\chi(G) \geq 0$, then $\partial_\infty G$ is a Menger curve.

Theorem 14. *Let G be a torsion-free 2-dimensional hyperbolic group that does not split over trivial and cyclic subgroups and which fits into a short exact sequence:*

$$1 \to F \to G \to \mathbb{Z} \to 1$$

where F is finitely generated. Then $\partial_\infty G$ is the Menger curve.

Proof. In view of Theorem 1, it is enough to show that $\partial_\infty G$ cannot be a circle or a Sierpinski carpet. If $\partial_\infty G \simeq \mathbb{S}^1$, then G contains a finite index closed surface subgroup G'. But then we would have an exact sequence $1 \to F' \to G' \to Z \to 1$, where $F' = F \cap G'$ is finitely generated, which is absurd. Now suppose $\partial_\infty G$ is a Sierpinski carpet. Note that if F admits a finite Eilenberg-Maclane space, then it is easy to see that $\chi(G) = \chi(F)\chi(\mathbb{Z}) = 0$, so $\partial_\infty G$ cannot be a Sierpinski carpet by the remark above. However there are examples such that F is not a finitely presentable group (see [R]). We now consider the general case. Then $(G; H_1, \dots, H_k)$ is a *Poincare duality pair.* Let K_0 be a finite Eilenberg-Maclane space for the group G, let D be a disjoint union of finite Eilenberg-Maclane spaces for the groups $H_1, \dots, H_k$, and let K be the mapping cylinder for a map $D \to K_0$ which induces the given maps $H_i \hookrightarrow G$. We view D as a subcomplex of K. Consider the finite cyclic coverings

$$(K_n, D_n) \to (K, D)$$

which are induced by the homomorphisms $G \to \mathbb{Z} \to \mathbb{Z}_n$. Then each pair (K_n, D_n) again satisfies relative Poincare duality in dimension 3, so

$$H^*(K_n, D_n; \mathbb{Z}/2) \cong \tilde{H}^{3-*}(K_n; \mathbb{Z}/2)$$

We will use the notation $b_j(L)$ to denote the dimension (over $\mathbb{Z}/2$) of $H_j(L, \mathbb{Z}/2)$. Thus

$$\lim_{n\to\infty} b_1(D_n) = \infty \tag{15}$$

and $b_1(K_n) \le b_1(F) + 1 < \infty$. Consider the exact sequence of the pair (K_n, D_n):

$$\dots \to H^1(K_n; \mathbb{Z}/2) \to H^1(D_n; \mathbb{Z}/2) \to H^2(K_n, D_n; \mathbb{Z}/2) \to \dots$$

Since $b_1(K_n)$ is bounded by $b_1(F) + 1$, the equality (15) implies that

$$\lim_{n\to\infty} Dim_{\mathbb{Z}/2}(H^2(K_n, D_n; \mathbb{Z}/2)) = \infty.$$

This contradicts the fact that $H^2(K_n, D_n; \mathbb{Z}/2) \cong H_1(K_n; \mathbb{Z}/2)$. □

Now let F be a finitely generated free group and $\phi : F \to F$ be an irreducible hyperbolic automorphism (see [BF2] for the definition). Consider the extension

$$1 \to F \to G \to \mathbb{Z} \to 1$$

induced by ϕ. The group G is hyperbolic by [BF2]. The cohomological dimension of G is 2 by the Mayer-Vietoris sequence, thus the boundary of G is 1-dimensional by [BM].

Corollary 16. *$\partial_\infty G$ is the Menger curve.*

Proof. We will show that the group G does not split over a cyclic (possibly trivial) subgroup. Suppose that it does. Then we have the corresponding action of G on a minimal simplicial tree T with cyclic edge stabilizers. Consider the restriction of this action on the subgroup F. Let $T' \subset T$ be the minimal F-invariant subtree, then T' is $\mathbb{Z}$-invariant (since $\mathbb{Z}$ normalizes F), thus $T' = T$. By Grushko's theorem (in the case of trivial edge stabilizers) and the generalized accessibility theorem [BF1] (in the case of infinite cyclic stabilizers), the quotient T/F is a finite graph Γ. The action of $\mathbb{Z} = \langle z \rangle$ projects to action on Γ, after taking a finite iteration of ϕ (if necessary) we may assume that z acts trivially on Γ. Since G does not contain $\mathbb{Z}^2$-subgroups, the edge stabilizers for the action of F on T must be trivial. Thus we get a free product decomposition of F so that each factor is invariant under some iterate of z. This contradicts the assumption that the corresponding automorphism $\phi : F \to F$ is irreducible. □

Theorem 17. *Let $\mathcal{G}$ be a finite graph of groups. Suppose*

1. Each vertex group is a torsion-free hyperbolic group whose boundary is either a Menger curve or a Sierpinski carpet; and at least one vertex group has Menger curve boundary.

2. Each edge group is a finitely generated free group of rank at least 2, and includes as a quasi-convex subgroup of each of the corresponding vertex groups.

3. If T is the Bass-Serre tree for $\mathcal{G}$, and e_1, $e_2 \subset T$ are two edges emanating from the same vertex $v \in T$, then their stabilizers intersect trivially.

Then the fundamental group G of $\mathcal{G}$ is a hyperbolic group with Menger curve boundary.

Proof. Conditions 2 and 3 imply that G is hyperbolic by [BF2], and vertex groups are quasi-convex subgroup of G by [Mi, Sw2]. G is torsion-free since all vertex groups are torsion-free. G has cohomological dimension 2 by the Mayer-Vietoris sequence, so $\partial_\infty G$ has dimension 1 by [BM].

We claim that G does not split over trivial or infinite cyclic groups. To see this, let T be the Bass-Serre tree of $\mathcal{G}$, and let S be the Bass-Serre tree of a splitting of G over trivial and/or cyclic groups. Consider two adjacent vertices v_1, $v_2 \in T$, let $G_{v_i} \subset G$ be their stabilizers, and let G_e be the stabilizer of the edge joining them. Since G_{v_i} does not split over trivial or cyclic subgroups [B2], G_{v_i} has a nonempty fixed point set in S. If $s_i \in S$ is fixed by G_{v_i}, then the segment joining s_1 to s_2 will be fixed by G_e. Since G_e is free of rank at least 2, we see that $s_1 = s_2$. Therefore by induction we find that G has a global fixed point in S, which is a contradiction.

If the stabilizer of $v \in T$ has Menger curve boundary, then by the quasi-convexity of G_v in G, the Menger curve embeds in $\partial_\infty G$. This shows that $\partial_\infty G$ cannot be homeomorphic to $\mathbb{S}^1$ or the Sierpinski carpet. By Theorem 4 $\partial_\infty G$ is a Menger curve. □

7. Topologically rigid groups

In this section we will construct some examples of topologically rigid groups. Before proceeding, we first note a consequence of Theorem 4.

Corollary 18. *Let G be a nonelementary hyperbolic group with $Dim(\partial_\infty G) \le 1$. Then G is not topologically rigid.*

We will sketch a proof of the corollary, and leave the details to the reader.

Case I: G has more than one end. Then G splits as an amalgamated product or HNN extension over a finite group. Let $G \curvearrowright T$ be the action of G on the Bass-Serre tree associated to such a splitting, so there is only one edge orbit in T. Following along the same lines as in section 5, we construct a tree of spaces X, with vertex and edge spaces corresponding to vertices and edges in T. For each vertex $v \in T$, the vertex space $X_v \subset X$ is quasi-convex in X and as in section 5 we may label points in $\partial_\infty X$ with elements of $T \cup \partial_\infty T$. The outward sets (see section 5) are open and closed in $\partial_\infty X$. If e_1 and e_2 are incident to a vertex v then they lie in the same G_v-orbit (since G/T has only one edge). Out_{v,e_1} and Out_{v,e_2} are disjoint and homeomorphic, so we may define a homeomorphism of $\partial_\infty X$ by swapping them while holding everything else fixed. This construction yields a continuum of homeomorphisms of $\partial_\infty X$, so $G \to Homeo(\partial_\infty X)$ cannot be surjective.

Case II: G is 1-ended. If $\partial_\infty G$ is homeomorphic to $\mathbb{S}^1$, the Sierpinski carpet, or the Menger curve then G cannot be topologically rigid since each of these spaces has uncountable homeomorphism group. Therefore by Theorem 4 we may assume that G splits as an amalgamated free product or HNN extension over a virtually cyclic group. Let $G \curvearrowright T$ be the action of G on the Bass-Serre tree associated with such a splitting. If e is an edge in T, $e = \overline{v_1 v_2}$, then $Out_{v_1,e} - \partial_\infty X_e$ and $Out_{v_2,e} - \partial_\infty X_e$ are open and closed in $\partial_\infty X - \partial_\infty X_e$, and are preserved by G_e. Take an element $g \in G_e$ that fixes both points in $\partial_\infty G_e$, and define a homeomorphism $f : \partial_\infty X \to \partial_\infty X$ by $f\big|_{Out_{v_1,e}} = \partial_\infty g\big|_{Out_{v_1,e}}$ and $f\big|_{Out_{v_2,e}} = id\big|_{Out_{v_2,e}}$. This type of construction will give a continuum of homeomorphisms of $\partial_\infty X$, so again $G \to Homeo(\partial_\infty X)$ cannot be surjective. □

The following lemma relates topological rigidity of hyperbolic groups with quasi-isometric rigidity.

Lemma 19. *Suppose that G is a nonelementary Gromov-hyperbolic group, and X is a Cayley graph of G. Then there is a function $\phi(t, s)$ so that each (L, A)-quasi-isometry $f : X \to X$ which induces the identity mapping of $\partial_\infty X$, is $\phi(L, A)$-close to the identity. If G is topologically rigid then every (L, A)-quasi-isometry is $\phi(L, A)$-close to left translation by some $g \in G$.*

Proof. Suppose $f : X \to X$ is an (L, A)-quasi-isometry which induces the identity mapping on $\partial_\infty X$. Since G is nonelementary, $\partial_\infty X = \partial_\infty G$ contains infinitely many points. Let α, β be complete geodesics in X which are not asymptotic to each other in either direction. Therefore there exists a function $r(c)$ (which depends on X, α, β) such that the intersection between c-neighborhoods of α and β has diameter $\le r(c)$. Since X/G is compact, there is a constant C such that each point $x \in X$ is within distance $\le C$ from $g(\alpha)$ and from $g(\beta)$ for some $g \in G$. Stability of quasi-geodesics in Gromov-hyperbolic spaces implies that $d(g\alpha, f(g\alpha)) \le D, d(g\beta, f(g\beta)) \le D$ where D depends only only X, L, A and C. Thus $f(x) \in N_{C+D}(g\alpha) \cap N_{C+D}(g\beta)$, the diameter of the intersection is $\le r(C + D)$. Hence $d(x, f(x)) \le r(C + D) = \phi(L, A)$.

If G is topologically rigid and $f : X \to X$ is an (L, A) quasi-isometry, then $\partial_\infty f : \partial_\infty X \to \partial_\infty X$ is induced by some $g \in G$; hence by the argument above $d(g, f) = d(id, g^{-1} \circ f) < \phi(L, A)$. $\square$

Recall that for a hyperbolic group G, $\bar{G}$ denotes the quotient of G by the maximal normal finite subgroup.

Lemma 20. *If G' is a hyperbolic group whose boundary is homeomorphic to the boundary of a topologically rigid hyperbolic group G, then $\bar{G}'$ embeds in $\bar{G}$ as a finite index subgroup.*

Proof. We leave the case of elementary hyperbolic groups to the reader and assume that G (and hence G') is nonelementary. Recall that for a hyperbolic group G, $\partial^3 G$ denotes the collection of points in $(\partial_\infty G)^3$ where all three coordinates are distinct. Let $h : \partial_\infty G' \to \partial_\infty G$ be a homeomorphism. The kernels of the projections $G' \to Homeo(\partial_\infty G'), G \to Homeo(\partial_\infty G)$ are the maximal normal finite subgroups $N' \subset G', N \subset G$. Since G is topologically rigid, the conjugation by h determines an embedding $\iota : \bar{G}' \hookrightarrow \bar{G}$, where $\bar{G} := G/N, \bar{G}' := G'/N'$. The groups $\bar{G}', \bar{G}$ act properly discontinuously cocompactly on $\partial^3 G', \partial^3 G$. Hence $\iota(\bar{G}')$ also acts properly discontinuously cocompactly on $\partial^3 G$. It follows that $[\bar{G} : \iota(\bar{G}')] < \infty$. $\square$

Corollary 21. *If G' is a hyperbolic group quasi-isometric to a topologically rigid hyperbolic group G, then $\bar{G}'$ embeds in $\bar{G}$ as a finite index subgroup.*

Proof. A quasi-isometry between Gromov-hyperbolic metric spaces induces a homeomorphism between their boundaries. $\square$

Our construction of topologically rigid groups is based on the idea (realized precisely in Proposition 25) that a homeomorphism of $\mathbb{S}^2$ must be a Möbius transformation provided it preserves a sufficiently rich family of round circles. We begin with an analogous statement for homeomorphisms of $\mathbb{S}^1$.

Line configurations in $\mathbb{H}^2$. Let $\mathcal{L}$ be a locally finite collection of geodesics in $\mathbb{H}^2$ so that the complementary regions of $\cup_{L \in \mathcal{L}} L$ are bounded, and we assume that there is a cocompact lattice $\Gamma \subset Isom(\mathbb{H}^2)$ stabilizing $\mathcal{L}$. Let $\bar{\partial}^2_\infty \mathbb{H}^2$ be the space of unordered distinct pairs in $\partial_\infty \mathbb{H}^2$, and let $\partial_\infty \mathcal{L}$ be the collection of pairs of endpoints $\partial_\infty L$ for $L \in \mathcal{L}$, $\partial_\infty \mathcal{L} := \{\partial_\infty L \mid L \in \mathcal{L}\} \subset \bar{\partial}^2_\infty \mathbb{H}^2$. Note that if L_1, $L_2 \in \mathcal{L}$ and $\partial_\infty L_1 \cap \partial_\infty L_2 \neq \emptyset$ then $L_1 = L_2$. Let $Stab(\partial_\infty \mathcal{L}) \subset Homeo(\partial_\infty \mathbb{H}^2)$ be the group of homeomorphisms of $\partial_\infty \mathbb{H}^2$ which preserve $\partial_\infty \mathcal{L} \subset \partial^2_\infty \mathbb{H}^2$.

Lemma 22. *1. If L_1, $L_2 \in \mathcal{L}$ have nonempty intersection and $g \in Stab(\partial_\infty \mathcal{L})$ fixes $\partial_\infty L_1 \cup \partial_\infty L_2$ pointwise then $g = id$.*

2. $\{\partial_\infty \gamma \mid \gamma \in \Gamma\} \subset Homeo(\partial_\infty \mathbb{H}^2)$ is a finite index subgroup of $Stab(\partial_\infty \mathcal{L})$.

Proof. Our arguments essentially follow [CB, Proof of Theorem 2.7]. We will identify the space of geodesics in $\mathbb{H}^2$ with $\bar{\partial}^2_\infty \mathbb{H}^2$.

(1) Suppose L_1, $L_2 \in \mathcal{L}$ and $g \in Stab(\partial_\infty \mathcal{L})$ fixes $\partial_\infty L_1 \cup \partial_\infty L_2$ pointwise. If σ_1, σ_2 are the connected components of $\partial_\infty \mathbb{H}^2 - \partial_\infty L_1$, then $g(\sigma_i) = \sigma_i$ since $|\partial_\infty L_2 \cap \sigma_i| = 1$ and $\partial_\infty L_2$ is fixed by g. Observe that $\Sigma_i := \{\partial_\infty L \cap \sigma_i \mid L \in \mathcal{L} \text{ and } |L \cap L_1| = 1\} \subset \sigma_i$ is a discrete subset of σ_i with the order type (with respect to the ordering on $\sigma_i \simeq \mathbb{R}$) of the integers, and $g(\Sigma_i) = \Sigma_i$. But g fixes the point $\partial_\infty L_2 \cap \sigma_i \in \Sigma_i$ and is orientation

preserving, so $g\big|_{\Sigma_i} = id_{\Sigma_i}$. Therefore g fixes $\partial_\infty L$ for every $L \in \mathcal{L}$ with $L \cap L_1 \neq \emptyset$. The incidence graph of $\mathcal{L}$ is connected, so we may apply this argument inductively to see that g fixes $\partial_\infty L$ for every $L \in \mathcal{L}$. The set $\cup_{L\in\mathcal{L}}\partial_\infty L$ is dense in $\partial_\infty \mathbb{H}^2$, so $g = id$. This proves the first assertion of the lemma.

(2) We now show that every sequence $g_k \in Stab(\partial_\infty \mathcal{L})$ has a subsequence which is constant modulo Γ, which proves that $[Stab(\partial_\infty\mathcal{L}) : \Gamma] < \infty$. Pick $L_1, L_2 \in \mathcal{L}$ such that L_1 intersects L_2 in a point p. For each k let $g_{k*}L_i \in \mathcal{L}$ be the unique line with $\partial_\infty(g_{k*}L_i) = g_k(\partial_\infty L_i)$. Then $(g_{k*}L_1) \cap (g_{k*}L_2) = p_k$ for some $p_k \in \mathbb{H}^2$, and we may choose a sequence $\gamma_k \in \Gamma$ such that $\sup d(\underline{\gamma_k(p_k), p}) = R < \infty$. Then the lines $(\gamma_k \circ g_k)_* L_i$ lie in the finite set $\{L \in \mathcal{L} \mid \mathcal{L} \cap \overline{B(p,R)} \neq \emptyset\}$, so after passing to a subsequence we may assume that $(\gamma_k \circ g_k)\big|_{\partial_\infty L_i}$ independent of k for $i = 1, 2$. By the previous paragraph the sequence $\gamma_k \circ g_k \in Homeo(\partial_\infty \mathbb{H}^2)$ is constant. ☐

Plane configurations in $\mathbb{H}^3$. Below we prove an analog of Lemma 22 for a collection $\mathcal{H}$ of totally geodesic hyperplanes in $\mathbb{H}^3$.

Let $\mathcal{H}$ be a locally finite collection of totally geodesic planes in $\mathbb{H}^3$, with stabilizer $G := \{g \in Isom(\mathbb{H}^3) \mid g(H) \in \mathcal{H} \text{ for every } H \in \mathcal{H}\}$. Let $\partial_\infty\mathcal{H} := \{\partial_\infty H \mid H \in \mathcal{H}\}$. We assume that $\mathcal{H}$ satisfies the conditions:

1. G is a cocompact lattice in $Isom(\mathbb{H}^3)$.
2. The complementary regions of $\cup_{H\in\mathcal{H}} H$ are bounded.
3. If $H \in \mathcal{H}$, then the reflection in H does not preserve the collection $\mathcal{H}$.

Such examples will be constructed later in this section.

The local finiteness of $\mathcal{H}$ implies that there are finitely many G-orbits in $\mathcal{H}$, and that the stabilizer of each $H \in \mathcal{H}$ acts cocompactly on H.

Definition 23. We will say that three circles $\partial_\infty H_1$, $\partial_\infty H_2$, $\partial_\infty H_3$, where $H_i \in \mathcal{H}$, are in **standard position** if the three planes H_i intersect transversely in a single point $x \in \mathbb{H}^3$.

Note that if the circles $\partial_\infty H_1$, $\partial_\infty H_2$, $\partial_\infty H_3$ are in standard position and C_1, C_2, C_3 is another unordered triple of circles which bound elements of $\mathcal{H}$, then C_1, C_2, C_3 are in standard position if and only if there is a homeomorphism $f : \partial_\infty H_1 \cup \partial_\infty H_2 \cup \partial_\infty H_3 \to C_1 \cup C_2 \cup C_3$ which carries elements of $\mathcal{H}$ to elements of $\mathcal{H}$.

Let *Stand* denote the collection of unordered triples of circles in standard position. Thus the previous remark implies that *Stand* is invariant under the homeomorphisms $\mathbb{S}^2 \to \mathbb{S}^2$ which carry elements of $\mathcal{H}$ to elements of $\mathcal{H}$. We will say that two elements of *Stand* are **incident** if they have exactly two circles in common.

Lemma 24. *1. The incidence graph of Stand is connected.*

2. If $\gamma \subset \partial_\infty \mathbb{H}^3$ is homeomorphic to $\mathbb{S}^1$, then either $\gamma = \partial_\infty H$ for some $H \in \mathcal{H}$, or there is an $H \in \mathcal{H}$ so that $\partial_\infty H$ intersects both components of $\partial_\infty \mathbb{H}^3 - \gamma$.

Proof. The union $\cup_{H\in\mathcal{H}} H$ determines a polygonal subcomplex in $\mathbb{H}^3$ with connected 1-skeleton. Therefore the assertion 1 follows.

To prove the assertion 2, let U and U' denote the connected components of $\partial_\infty \mathbb{H}^3 - \gamma$. We may find $H, H' \in \mathcal{H}$ so that $\partial_\infty H \subset U, \partial_\infty H' \subset U'$. Since the incidence graph for $\mathcal{H}$ is connected we can find a chain of planes $H_0 = H, H_1, ..., H_n = H'$ in $\mathcal{H}$ so that consecutive planes intersect each other. We see that either $\gamma = \partial_\infty H_j$ for some H_j in this sequence or for some H_j the circle $\partial_\infty H_j$ intersects both U and U'. $\square$

Proposition 25. *Let $Stab(\partial_\infty \mathcal{H})$ be the group of homeomorphisms of $\partial_\infty \mathbb{H}^3$ which preserve $\partial_\infty \mathcal{H}$, $Stab(\partial_\infty \mathcal{H}) := \{g \in Homeo(\partial_\infty \mathbb{H}^3) \mid g(\partial_\infty H) \in \partial_\infty \mathcal{H} \text{ for all } H \in \mathcal{H}\}$. Then $Stab(\partial_\infty \mathcal{H}) = \{\partial_\infty g \mid g \in G\}$.*

Proof. Suppose $\{\partial_\infty H_1, \partial_\infty H_2, \partial_\infty H_3\} \in Stand$, $f \in Stab(\partial_\infty \mathcal{H})$, and $f(\partial_\infty H_i) = \partial_\infty H_i$ for $1 \leq i \leq 3$. Then for $1 \leq i \leq 3$ we may consider the collection $\mathcal{L}_i$ of geodesics in H_i of the form $H_i \cap H$ for $H \in \mathcal{H} - H_i$. Part 1 of Lemma 22 then implies that $f\big|_{\partial_\infty H_i} = id_{\partial_\infty H_i}$.

Now suppose $\{\partial_\infty H_1, \partial_\infty H_2, \partial_\infty H_3\}$, $\{\partial_\infty H_1, \partial_\infty H_2, \partial_\infty H_4\} \in Stand$ are incident, $f \in Stab(\partial_\infty \mathcal{H})$, and $f\big|_{\partial_\infty H_i} = id_{\partial_\infty H_i}$ for $1 \leq i \leq 3$. Then $f(\partial_\infty H_4) = \partial_\infty H_4$ since H_4 is the unique element of $\mathcal{H}$ whose boundary contains the 4-element set $\partial_\infty H_4 \cap (\partial_\infty H_1 \cup \partial_\infty H_2)$. Therefore by the previous paragraph we have

$$f\big|_{\partial_\infty H_4} = id_{H_4}.$$

Since the incidence graph of $Stand$ is connected we see by induction that $f\big|_{\partial_\infty H} = id_{\partial_\infty H}$ for all $H \in \mathcal{H}$, and this forces $f = id_{\partial_\infty \mathbb{H}^3}$.

Reasoning as in Lemma 22 we conclude that $[Stab(\partial_\infty \mathcal{H}) : G] < \infty$.

Let $G' \subset G$ be a finite index normal subgroup of $Stab(\partial_\infty \mathcal{H})$. Each $f \in Stab(\partial_\infty \mathcal{H})$ normalizes the action $G' \curvearrowright \partial_\infty \mathbb{H}^3$, so by Mostow rigidity each f is a Möbius transformation. Therefore for every $f \in Stab(\partial_\infty \mathcal{H})$ we have $f = \partial_\infty g$ for some $g \in G$. $\square$

Constructing topologically rigid groups. Let $G' \subset G$ be a finite index torsion-free subgroup of G so that for each $H \in \mathcal{H}$ the stabilizer of H in G' preserves the orientation on H. Let $\{H_1, \dots, H_k\}$ be a set of representatives of the G'-orbits in $\mathcal{H}$, and let $G_i := Stab_{G'}(H_i)$. For any $1 \leq i \leq k$, the set of geodesics

$$\{H \cap H_i \mid H \in \mathcal{H} - H_i,\ H \cap H_i \neq \emptyset\} \subset H_i$$

is finite modulo the action of G_i. Hence for each $1 \leq i \leq k$, there is a finite collection $\mathcal{Z}_i$ of conjugacy classes of maximal cyclic subgroups of G_i with the property that

(a) For any $g \in G' - G_i$, the intersection $gG_ig^{-1} \cap G_i$ is an element of $\mathcal{Z}_i$.

(b) For any $g \in G'$ and $i \neq j$, the intersection $gG_jg^{-1} \cap G_i$ is an element of $\mathcal{Z}_i$.

We now construct a double[3] of G' along the collection of subgroups $G_i := Stab(H_i)$, $1 \leq i \leq k$ as follows: construct a graph of groups $\mathcal{G}$ with two vertices v_1, v_2 and k edges $e_1, \dots, e_k$, where G_{v_i} is isomorphic to G' and G_{e_i} is isomorphic to G_i. Identify G_{v_i} with G'. We choose the embeddings $\iota_{ij} : G_{e_i} \to G_{v_j}$ so that the image coincides with the copy of $G_i \subset G'$ in G_{v_j} $(j = 1, 2)$, but so that the ι_{ij}'s satisfy the following condition:

$$\text{(Twisting)} \qquad \iota_{i1}^{-1}(\mathcal{Z}_i) \cap \iota_{i2}^{-1}(\mathcal{Z}_i) = \emptyset.$$

[3]If we double G' without "twisting" the edge inclusions then the resulting group $\widehat{G}$ is not hyperbolic. But it acts on a $CAT(0)$ space X so that $Homeo(\partial_\infty X)$ contains $\widehat{G}$ as a finite index subgroup.

To construct embeddings $\iota_{ij} : G_{e_i} \to G_{v_j}$ satisfying the twisting condition we first choose random embeddings $\varphi_{ij} : G_{e_i} \to G_{v_j}$ whose images are the copies of G_i, then let $\iota_{i1} := \varphi_{i1}$. Define ι_{i2} as the composition of φ_{i2} with a sufficiently high power of a pseudo-anosov automorphism of the surface group G_i $(i = 1, ..., k)$.

Let $\widehat{G} := \pi_1(\mathcal{G})$, let T be the Bass-Serre tree associated with $\mathcal{G}$, and let V and E denote the collections of vertices and edges in T respectively. $\widehat{G}$ acts (discretely, cocompactly) on a tree of spaces X constructed as in section 5, with vertex spaces X_v, $v \in V$ and edge spaces X_e, $e \in E$.

Lemma 26. *$\widehat{G}$ is a hyperbolic group. All vertex and edge groups G_x, $x \in V \cup T$ are quasi-convex subgroups of $\widehat{G}$.*

Proof. By [BF3], [Sw2], [Mi] it suffices to show that there is an upper bound on the length of essential annuli (see [BF3], section 1) in the graph of groups $\mathcal{G}$. Or equivalently, we need to show that there is an upper bound on the length of any segment in T which is fixed by a nontrivial element $g \in \widehat{G}$. We claim that if e_1, e_2, e_3 are 3 consecutive edges in the tree T, then $G_{e_1} \cap G_{e_2} \cap G_{e_3}$ is trivial; for the twisting condition implies that the intersections $G_{e_1} \cap G_{e_2}$ and $G_{e_2} \cap G_{e_3}$ are cyclic subgroups of G_{e_2} with trivial intersection. □

Lemma 27. *1. For every vertex $v \in V$, $\partial_\infty X_v \subset \partial_\infty X$ is a 2-sphere.*

2. For every edge $e \in E$, $\partial_\infty X_e \subset \partial_\infty X$ is a circle.

3. If $v_1 \neq v_2 \in V$ then $\partial_\infty X_{v_1} \cap \partial_\infty X_{v_2} \approx \mathbb{S}^1$ implies that v_1 and v_2 are the endpoints of an edge $e \in E$, and $\partial_\infty X_{v_1} \cap \partial_\infty X_{v_2} = \partial_\infty X_e$.

4. $\cup_{v \in V} \partial_\infty X_v$ is dense in $\partial_\infty X$.

5. Pick $e \in E$, and let T_1, $T_2 \subset T$ be the two subtrees that one gets by removing the interior of the edge e. Then $\partial_\infty X - \partial_\infty X_e$ has two connected components, namely the closures of $(\cup_{v \in T_i} \partial_\infty X_v) - \partial_\infty X_e$ in $\partial_\infty X - \partial_\infty X_e$ for $i = 1, 2$.

The proof of the lemma is similar to arguments from section 5, so we omit it.

Lemma 28. *If $\gamma \subset \partial_\infty X$ is homeomorphic to $\mathbb{S}^1$ and γ separates $\partial_\infty X$, then $\gamma = \partial_\infty X_e$ for some $e \in E$.*

Proof. We first claim that $\gamma \subset \partial_\infty X_v$ for some $v \in V$. Otherwise by Alexander duality $\partial_\infty X_v - \gamma$ is connected for every $v \in V$, and $(\partial_\infty X_{v_1} \cup \partial_\infty X_{v_2}) - \gamma$ is connected for any pair of adjacent vertices v_1, $v_2 \in V$. By induction this implies that $\cup_{v \in V} \partial_\infty X_v - \gamma$ is connected. By part 4 of Lemma 27 we conclude that $\partial_\infty X - \gamma$ is connected, a contradiction.

Hence we may assume that $\gamma \subset \partial_\infty X_v$ for some $v \in V$. Suppose $\gamma \neq \partial_\infty X_e$ for any $e \in E$ adjacent to v. Then any point $\xi \in \partial_\infty X - \gamma$ lies in the same component of $\partial_\infty X - \gamma$ as one of the two components of $\partial_\infty X_v - \gamma$. By Lemma 24 we can find an edge e adjacent to v so that $\partial_\infty X_e$ intersects both of the components U_1, U_2 of $\partial_\infty X_v - \gamma$. So we may connect U_1 to U_2 within $\partial_\infty X_w - \gamma$ where w is the other endpoint of e. This contradicts the assumption that γ separates $\partial_\infty X$. □

Thus, any homeomorphism $f : \partial_\infty X \to \partial_\infty X$ preserves the collection of circles $\{\partial_\infty X_e, e \in E\}$.

Let $\mathcal{C}$ denote the collection of unordered triples of circles $C_i = \partial_\infty X_{e_i}, e_i \in E$, which are **in standard position**, i.e. there exists a triple $H_1, H_2, H_3 \in \mathcal{H}$ which are in standard position and a homeomorphism $f : \partial_\infty H_1 \cup \partial_\infty H_2 \cup \partial_\infty H_3 \to C_1 \cup C_2 \cup C_3$ which carries each circle $\partial_\infty H_i$ to one of the circles $C_{j(i)}$. We define the incidence relation for elements of $\mathcal{C}$ the same way as before, let $\Gamma(\mathcal{C})$ denote the associated incidence graph. Thus $\mathcal{C}$ contains the subsets $\mathcal{S}_v$ where $\mathcal{S}_v$ consists of triples of circles in standard position which are contained in $\partial_\infty X_v$. Then the incidence graph $\Gamma(\mathcal{S}_v)$ is isomorphic to the incidence graph of $\mathcal{S}$, thus it is connected (see part 1 of Lemma 24). For each vertex $v \in V$ the union of triples of circles $\{C_1, C_2, C_3\} \in \mathcal{S}_v$ is dense in $\partial_\infty X_v$.

Lemma 29. *The subgraphs $\Gamma(\mathcal{S}_v)$ are the connected components of $\Gamma(\mathcal{C})$.*

Proof. It is enough to show that any $\{C_1, C_2, C_3\} \in \mathcal{C}$ is contained in $\partial_\infty X_v$ for some $v \in T$, since there is at most one $\partial_\infty X_v$ containing any given pair of circles.

Pick $\{C_1, C_2, C_3\} \in \mathcal{C}$, with $C_i = \partial_\infty X_{e_i}$ for $e_i \in E$. Note that $d(e_i, e_j) \leq 1$ for $1 \leq i, j \leq 3$ for otherwise we would have $C_i \cap C_j = \emptyset$. Also, observe that if two of the circles lie in some $\partial_\infty X_v$, then the third one must too (because $|\partial_\infty X_e \cap \partial_\infty X_v| \leq 2$ unless $\partial_\infty X_e \subset \partial_\infty X_v$). Clearly this forces the edges e_i to share a vertex. ☐

Define the incidence graph with the vertex set $\{\partial_\infty X_v,\ v \in T\}$, where the vertices v, w are connected by an edge if and only if $\partial_\infty X_v \cap \partial_\infty X_w \approx \mathbb{S}^1$. Lemma 18 implies that this graph is isomorphic to the tree T.

Proposition 30. *Any homeomorphism $f : \partial_\infty X \to \partial_\infty X$ preserves the collection of spheres $\{\partial_\infty X_v, v \in V\}$. In particular, f induces an isomorphism of the tree T.*

Proof. The homeomorphism f induces an automorphism $f_\#$ of the graph $\Gamma(\mathcal{C})$, thus it preserves its connected components. Therefore for each $v \in V$ there is $w = f_\#(v)$ such that $f_\# \Gamma(\mathcal{S}_v) = \Gamma(\mathcal{S}_w)$. However

$$\bigcup_{C \in \mathcal{S}_v} C$$

is dense in $\partial_\infty X_v$. Thus f preserves the collection of spheres $\{\partial_\infty X_v, v \in V\}$. The paragraph preceeding Proposition implies that f induces an automorphism of the tree T. ☐

Theorem 31. *The homeomorphism group of $\partial_\infty X$ contains $\widehat{G}$ as a subgroup of finite index. Therefore $Homeo(\partial_\infty X)$ is a topologically rigid hyperbolic group.*

Proof. For every $v \in V$, we identify $\partial_\infty X_v$ with $\partial_\infty \mathbb{H}^3$ via a homeomorphism which carries the collection $\{\partial_\infty X_e \mid e \in E,\ v \subset e\}$ to $\partial_\infty \mathcal{H}$; this homeomorphism is unique up to a Möbius transformation by Proposition 25.

Suppose $f \in Homeo(\partial_\infty X)$ and $f\big|_{\partial_\infty X_v} = id\big|_{\partial_\infty X_v}$ for some $v \in V$. Then f fixes $\partial_\infty X_e$ pointwise for every $e \in E$ containing v. Hence if $v' \in V$ is adjacent to v then $f(\partial_\infty X_{v'}) = \partial_\infty X_{v'}$. By Proposition 25 $f\big|_{\partial_\infty X_{v'}}$ is a Möbius transformation. Either $f\big|_{\partial_\infty X_{v'}} = id\big|_{\partial_\infty X_{v'}}$ or $f\big|_{\partial_\infty X_{v'}}$ is a reflection. But condition 3 on $\mathcal{H}$ rules out the latter possibility. Therefore by induction we conclude that f fixes $\partial_\infty X_w$ for every $w \in V$, and so $f = id$.

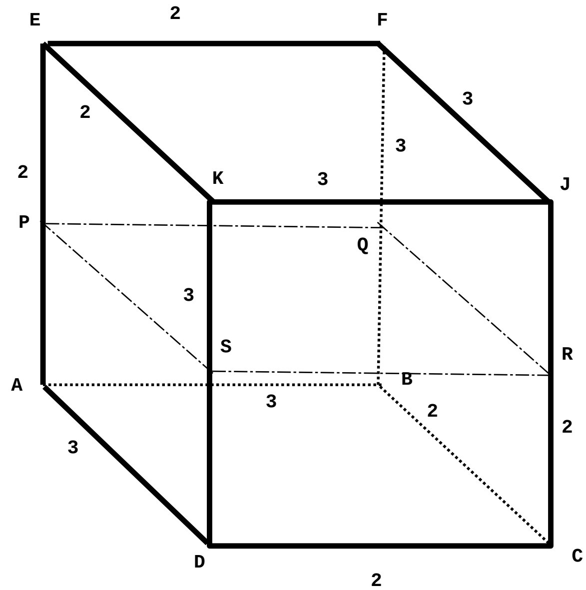

Figure 1: *The hyperbolic polyhedron* Φ.

Pick $v \in T$, and consider the possibilities for $f\big|_{\partial_\infty X_v}$ where $f \in Homeo(\partial_\infty X)$. There are clearly only finitely many such possibilities up to post-composition with elements of $\widehat{G}$; therefore by the preceding paragraph $\widehat{G}$ has finite index in $Homeo(\partial_\infty X)$. $\square$

An example of a plane configuration $\mathcal{H}$.

We now construct a specific example of a plane configuration $\mathcal{H}$ satisfying the three required conditions. We start with the 3-dimensional hyperbolic polyhedron Φ described in Figure 1: the edges of the polyhedron are labelled with 2 and 3, they indicate that the corresponding dihedral angles of the polyhedron are $\pi/2$ and $\pi/3$ respectively. Such a polyhedron exists by Andreev's theorem [An]. Note that Φ has an order 3 isometry θ which is a rotation around the geodesic segment $\overline{CE}$ and reflection symmetries in each of three quadrilaterals, two of which are depicted in Figure 2.

The polyhedron Φ contains three squares which "bisect" Φ; one of them $\beta_1 = PQRS$ which is indicated in Figure 1, the other two β_2, β_3 are obtained from β_1 by applying the rotation θ.

Lemma 32. *The bisectors $\beta_1, \beta_2, \beta_3$ are realized by totally-geodesic 2-dimensional polygons in Φ which are orthogonal to the boundary of Φ. More precisely, for each $1 \leq j \leq 3$ there is a totally geodesic plane $H_j \subset \mathbb{H}^3$ which intersects the same four edges of Φ as β_j and H_j intersects the faces of Φ orthogonally.*

Proof. It is enough to prove the assertion for β_1, the other two polygons are obtained via the rotation θ. The proof is similar to [Ka]: we first split open the cube Φ combinatorially along the bisector β_1 into two subcubes Φ_+ and Φ_-. Each polyhedron Φ_+, Φ_- has a face F_+, F_- which corresponds to the bisector β_1. We assign the label 2 to each edge of $\Phi_\pm$ is contained in $F_\pm$. Andreev's theorem again implies that Φ_+ and Φ_- can be realized by polyhedra in $\mathbb{H}^3$ (we retain the names $\Phi_\pm$ for these polyhedra). Our goal is to show that the homeomorphism $F_+ \to F_-$ (which is given by identification with the bisector β_1) is isotopic (rel. vertices) to an isometry of

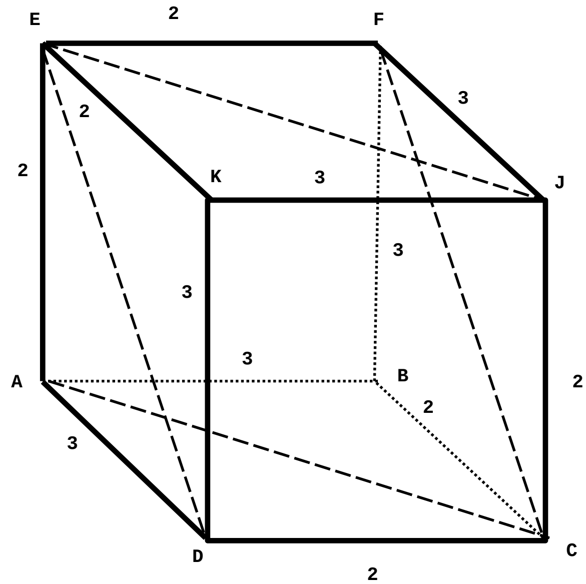

Figure 2: *Symmetries of the hyperbolic polyhedron* Φ.

the hyperbolic polygons. The polyhedron Φ admits a reflection symmetry which fixes the totally-geodesic rectangle $EJCA$; this symmetry also acts on the polyhedra Φ_+, Φ_- and quadrilaterals $F_\pm$ so that the fixed point sets are the geodesic segments corresponding to $\overline{PR}$. However it is clear that there exists a unique (up to vertex preserving isotopy) hyperbolic structure on quadrilateral $PQRS$ so that the edges are geodesic, angles are $\pi/2, \pi/3, \pi/2, \pi/3$ and the quadrilateral has an order 2 isometry fixing $\overline{PR}$. Thus we have a natural isometry $F_+ \to F_-$ and we can glue Φ_+ to Φ_- using this isometry. The result is a hyperbolic polyhedron Ψ which is combinatorially isomorphic to Φ this isomorphism preserves the angles. Thus by the uniqueness part of Andreev's theorem (alternatively one can use Mostow rigidity theorem) the polyhedra Φ, Ψ are isometric. On the other hand, the polyhedron Ψ contains totally geodesic 2-dimensional polygon $F_+ = F_-$ which is orthogonal to the boundary of Ψ. □

We retain the notation β_j ($j = 1,2,3$) for the totally-geodesic 2-dimensional hyperbolic polygons orthogonal to $\partial\Phi$ which realize the bisectors β_j. These polygons split Φ into 8 subpolyhedra $P_i, i = 1, ..., 8$, which are combinatorial cubes. Note that the dihedral angles between $\beta_j, j = 1,2,3$ are all equal and are different from $\pi/2$ (otherwise the combinatorial cube P_i which contains the vertex E would have all right angles which is impossible in hyperbolic space).

Now we construct the collection of planes $\mathcal{H}$ as follows: let $\mathcal{R} \subset Isom(\mathbb{H}^3)$ be the discrete group generated by reflections in the faces of Φ; the polyhedron Φ is a fundamental domain for $\mathcal{R}$. The 2-dimensional hyperbolic polygons $\beta_j = H_j \cap \Phi$ are orthogonal to $\partial\Phi$, the plane H_j is invariant under the subgroup $\mathcal{R}_j$ of $\mathcal{R}$ generated by reflections in the faces of Φ which are incident to β_j. The $\mathcal{R}$-orbit of these hyperplanes is $\mathcal{H}$. Note that

(0) If H is a member of $\mathcal{H}$ and the intersection $H \cap \Phi \neq \emptyset$ then $H \cap \Phi$ is equal to one of the bisectors β_j.

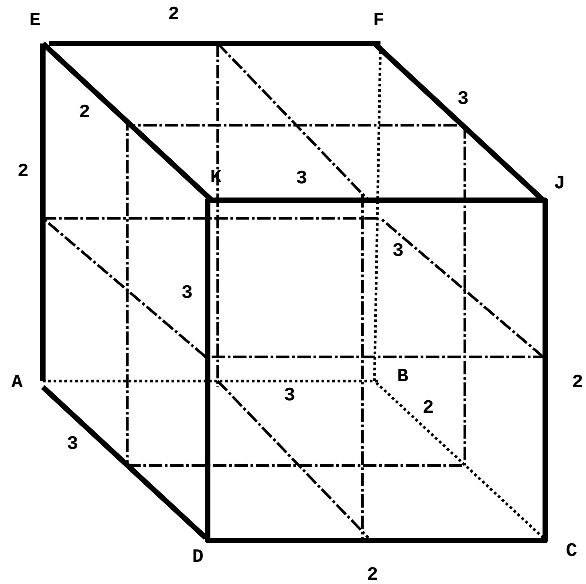

Figure 3: *"Bisectors" of the hyperbolic polyhedron* Φ.

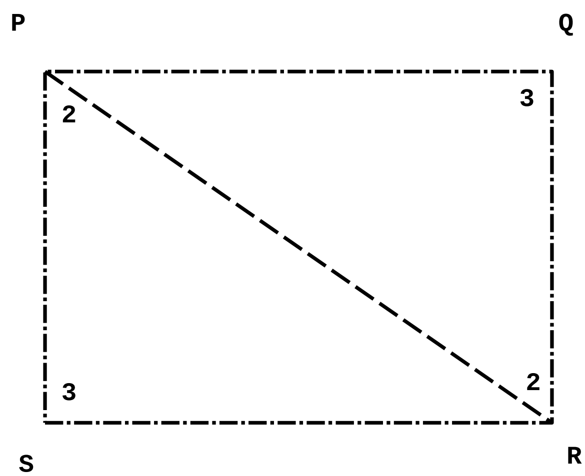

Figure 4: *Symmetry of the bisector* β_1.

We next check that $\mathcal{H}$ satisfies the required properties:

(1) The fundamental domain Φ for $\mathcal{R}$ is compact, hence the group $\mathcal{R}$ is a cocompact lattice.

(2) The complementary regions to $\mathcal{H}$ in $\mathbb{H}^3$ are finite unions of the polyhedra $P_i, i = 1, ..., 8$, thus they are bounded.

(3) Let ρ_j be the reflection in the plane H_j. Since the planes H_j, $1 \leq j \leq 3$ are not mutually orthogonal it follows that this reflection maps $H_i, i \neq j$, to a plane which does not belong to $\mathcal{H}$ (see Property (0) above); it follows that ρ does not preserve the configuration $\mathcal{H}$.

8. Groups with planar boundary

In this section we discuss the example mentioned at the end of the introduction.

Lemma 33. *Let S be a surface of genus 1 with two boundary components, C_1 and C_2. Let K be the complex obtained by gluing C_1 to C_2 by a degree 2 covering map $C_1 \to C_2$, and set $G := \pi_1(K)$. Then*

1. G is torsion-free and hyperbolic.

2. G contains a finite index subgroup which is isomorphic to a discrete, convex cocompact subgroup of $Isom(\mathbb{H}^3)$ which does not act cocompactly on $\mathbb{H}^3$. In particular, the boundary of G is 1-dimensional and planar.

3. G is not a 3-manifold group.

Proof. 1. The group G is torsion-free since it is an HNN-extension of a torsion-free group. The hyperbolicity of G follows from the Bestvina-Feighn combination theorem [BF2, BF3].

2. Our arguments are similar to [H]. We first construct a finite covering $p : F \to S$ such that

a. Each component of ∂F which covers C_1 does so with degree 1 and each component of ∂F which covers C_2 does so with degree 2.

b. There are twice as many circles in $p^{-1}(C_1)$ as there are in $p^{-1}(C_2)$.

To get the cover, consider the cone-type orbifold O obtained by attaching a disk D_1 to S along C_1, and a disk D_2 with a cone point of order 2 around C_2. Then O is an orbifold of hyperbolic type and hence admits a finite orbifold covering $p_0 : \tilde{O} \to O$ where $\tilde{O}$ is a manifold (see [Sc]). Now remove $p_0^{-1}(Interior(D_1) \cup Interior(D_2))$ from the surface $\tilde{O}$, and call the resulting surface F. Then $p := p_0\big|_F : F \to S$ is the covering with the required properties. Let m denote the number of boundary components of F which cover C_2. Now define a complex L by identifying each component of $p^{-1}(C_2)$ with precisely two components of $p^{-1}(C_1)$, so that the composition $F \to S \to K$ factors through a covering map $L \to K$. We claim that $\pi_1(L)$ is a 3-manifold group. Indeed, consider L as a graph of spaces where the vertex-spaces are F and m copies of the circle $\mathbb{S}^1$, the edge-spaces are $3m$ copies of $\mathbb{S}^1$ and the attaching maps are homeomorphisms. Replace the vertex space homeomorphic to F by $Y_v = F \times I$, $I = [0,1]$; replace each vertex space X_v homeomorphic to $\mathbb{S}^1$ by the solid torus $Y_v = \mathbb{S}^1 \times D^2$. The edge subspaces of $F \times I$ are the components of $\partial F \times I$; the edge subspaces Y_e incident to $\mathbb{S}^1 \times D^2 = Y_v$ are the annuli $\mathbb{S}^1 \times \alpha_i$ $(i = 1, 2, 3)$, where α_i are disjoint arcs of ∂D^2. The maps from edge-spaces to vertex-spaces are obvious inclusions. Then it is clear that the total space of the resulting graph of spaces $\{Y_v, Y_e\}$ is a 3-dimensional compact manifold with boundary, which we call N. The fundamental group of N is isomorphic to $\pi_1(L)$ since L is a deformation retract of N. The manifold N is clearly Haken, thus we apply Thurston's hyperbolization theorem to N and conclude that $\pi_1(L) = \pi_1(N)$ is isomorphic to a discrete, convex cocompact subgroup G of $Isom(\mathbb{H}^3)$. If $\mathbb{H}^3/G$ were compact then $\chi(N) = 0$ which is obviously false since $\chi(N)$ is a nonzero multiple of $\chi(K) = -2$.

3. Assume that G is a 3-manifold group, $G \cong \pi_1(M)$, M is a compact 3-manifold. We can assume that M is irreducible and, since $\pi_1(M) = G$ is the fundamental group of a graph of surface groups, it follows that M is Haken. Orient the loops C_i and let γ_i be the corresponding elements of G. Then for $i = 1, 2$, the group G splits over the subgroup $\langle \gamma_i \rangle$, $G = \pi_1(S) *_{\langle \gamma_i \rangle}$. Hence γ_i corresponds to an embedded essential annulus or a Moebius band A_i in M, $i = 1, 2$. On the other hand, γ_1^2 is

conjugate to $\gamma_2^{\pm 1}$ in G. This is impossible, see [JS], [J]. In [KK] we show that the group G cannot act discretely simplicially on a *coarse 3-dimensional Poincare duality space*; this gives another proof that G cannot act cocompactly on any contractible 3-dimensional manifold with boundary. □

References

[ABC+] Alonso, J.M., Brady, T., Cooper, D., Delzant, T., Ferlini, V., Lustig, M., Mihalik, M., Shapiro, M. and Short, H., *Notes on negatively curved groups*, Preprint, 1989.

[A1] Anderson, R. D. , *A characterization of the universal curve and a proof of its homogeneity,* Ann. of Math., vol. **67** (1958) p. 313–324.

[A2] Anderson, R.D., *One-dimensional continuous curves and a homogeneity theorem,* Ann. of Math., vol. **68** (1958) p. 1–16.

[An] Andreev, E., *On convex polyhedra in Lobachevsky space*, Math. USSR Sbornik, vol. **10** (1970) p. 413–440.

[Be] Benakli, N., *Groupes hyperboliques de bord la courbe de Menger ou la courbe de Sierpinski*, preprint, Paris, 1991.

[Bes1] Bestvina, M., *Characterizing k-dimensional universal Menger compacta*, Mem. Amer. Math. Soc., vol. **71** (1988), no. 380.

[Bes2] Bestvina, M., *Local homology properties of boundaries of groups*, Mich. Math. J., vol. **43**, (1996), no. 1, p. 123-139.

[BF1] Bestvina, M. and Feighn, M., *Bounding the complexity of group actions on trees*, Inv. Math., vol. **103** (1991) p. 449–469.

[BF2] Bestvina, M. and Feighn, M., *A combination theorem for hyperbolic groups,* Journ. of Diff. Geom., vol. **35** (1992) p. 85–101.

[BF3] Bestvina, M. and Feighn, M., *Addendum and correction to: "A combination theorem for negatively curved groups",* Journ. of Diff. Geom., vol. **43** (1996), p. 783–788.

[BM] Bestvina, M. and Mess, G., *The boundary of negatively curved groups*, Journal of AMS, vol. **4** (1991) p. 469–481.

[B1] Bowditch, B. H., *Notes on Gromov's hyperbolicity criterion for path-metric spaces*, In: "Group theory from a geometrical point of view"; ICTP, Trieste April 1990, editors: Ghys, E., Haefliger, A. and Verjovsky, A.; (1991) p. 64–167.

[B2] Bowditch, B. H., *Connectedness properties of limit sets*, Trans. of Amer. Math. Soc., vol. **351** (1999) p. 3673–3686.

[B3] Bowditch, B. H., *A topological characterisation of hyperbolic groups*, J. Amer. Math. Soc., vol. **11** (1998), no. 3, p. 643–667.

[B4] Bowditch, B. H., *Convergence groups and configuration spaces*, to appear in "Proceedings of the Canberra Group Theory Conference 1996".

[CB] Casson, A. and Bleiler, S., "Automorphisms of Surfaces After Nielsen and Thurston", Cambridge Univ. Press, 1983.

[CJ] Casson A. and Jungreis, D., *Convergence groups and Seifert fibered 3-manifolds,* Inventiones Math., vol. **118** (1994) F. 3, p. 441– 456.

[Ch] Champetier, C. , *Proprietes statistiques des groupes de presentation finie,* Adv. Math., vol. **116** (1995) p. 197–262.

[C] Claytor, S., *Topological immersion of Peanian continua in the spherical surface*, Ann. of Math., vol. **35** (1924) p. 809–835.

[DJ] Davis, M. and Januszkiewicz T., *Hyperbolization of polyhedra*, J. Diff. Geom., vol. **34** (1991) p. 347–388.

[Dr] Dranishnikov, A., *Private communication.*

[DD] Dicks W., and Dunwoody, M., "Groups acting on graphs", Cambridge Studies in Advanced Mathematics, vol. 17, 1989

[EbOn] Eberlein, P. and O'Neill, B., *Visibility manifolds*, Pacific J. Math., vol. **46** (1973) p. 45–109.

[Ga] Gabai, D., *Convergence groups are Fuchsian groups,* Annals of Mathematics, vol. **136** (1992), p. 447– 510.

[GH] Ghys, E. and de la Harpe, P., *Sur les groupes hyperbolic d'apres Mikhael Gromov*, Progress in Mathematics, vol. **83**, Birkhäuser, 1990.

[Gr] Gromov, M., *Hyperbolic groups*, In: "Essays in Group Theory", Publications of MSRI, vol. **8** (1987), p. 75– 264.

[H] Hempel, J., *Residual finiteness for* 3*-manifolds*, In: "Combinatorial group theory and topology" (Alta, Utah, 1984), p. 379–396, Annals of Math. Stud, vol. **111**, Princeton Univ. Press, 1987.

[JS] Jaco, W. and Shalen, P., "Seifert fibred spaces in 3-manifolds", Memoirs of AMS, vol. **220**, 1979.

[J] Johannson, K.,"Homotopy–equivalences of 3–manifolds with boundary", Lecture Notes in Math., Springer-Verlag, vol. **761**, 1979.

[Ka] Kapovich, M., *Deformations of representations of discrete subgroups of* $SO(3,1)$, Math. Annalen, vol. **299** (1994) p. 341– 354.

[KK] Kapovich, M. and Kleiner, B., *Coarse Alexander duality and duality groups*, Preprint (1999).

[KL] Kleiner, B. and Leeb, B., *Rigidity of quasi-isometries for symmetric spaces and Euclidean buildings*, Math. Publ. of IHES, vol. **86** (1997) p. 115-197.

[Mi] Mitra, M., *Height in splittings of hyperbolic groups*, Preprint (1997).

[M] Moise, E.E., *Remarks on the Claytor embedding theorem*, Duke Math. Journal, vol. **19** (1952) p. 199-202.

[Pa] Pansu, P., *Métriques de Carnot-Carathéodory et quasiisométries des espaces symétriques de rang un,* Ann. of Math., vol. **129** (1989), p. 1-60.

[R] Rips, E., *Subgroups of small cancellation groups,* Bull. of London Math. Soc., vol. **14** (1982) p. 45–47.

[Sch] Schwartz, R., *The quasi-isometry classification of rank one lattices,* Math. Publ. of IHES, vol. **82** (1995) 133– 168.

[Sc] Scott, P., *The geometries of* 3*-manifolds*, Bull. London Math. Soc., vol. **15** (1983) p. 401–487.

[SW] Scott, P. and Wall, T., *Topological methods in group theory,* In: "Homological Group Theory", London Math. Soc. Lecture Notes, vol. **36** (1979) p. 137–204.

[St] Stallings, J., *On torsion-free groups with infinitely many ends,* Ann. of Math., vol. **88** (1968) p. 312–334.

[Sw1] Swarup, G. A., *On the cut-point conjecture,* Electronic Research Announcements of AMS, vol. **2** (1996) p. 98–100.

[Sw2] Swarup, G. A., *Proof of a weak hyperbolization theorem,* Preprint (1998).

[T1] Tukia, P., *Homeomorphic conjugates of fuchsian groups*, J. Reine Angew. Math., vol. **391** (1988) p. 1–54.

[T2] Tukia, P., *Convergence groups and Gromov's metric hyperbolic spaces*, New Zealand Journal of Mathematics, vol. **23**, (1994), no. 2, p. 157–187.

[W] Whyburn, G. T.,*Topological characterization of the Sierpinski curve*, Fundamenta Math., vol. **45** (1958) p. 320–324.

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