

# THE WEAK HYPERBOLIZATION CONJECTURE FOR 3-DIMENSIONAL $CAT(0)$ GROUPS

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ABSTRACT. We prove a weak hyperbolization conjecture for  $CAT(0)$  3-dimensional Poincaré duality groups.

## 1. INTRODUCTION

For a variety of classes of groups, it is a well-known open problem whether the failure of Gromov hyperbolicity can be detected by the presence of special subgroups, e.g. rank 2 abelian groups or Baumslag-Solitar groups. This is of interest, for instance, for  $CAT(0)$  groups (even for the fundamental groups of finite 2-dimensional locally  $CAT(0)$  square complexes), for 1-relator groups, and 3-dimensional Poincaré duality groups. We say that a class of groups *satisfies the weak hyperbolization conjecture* if every group in the class is either Gromov hyperbolic, or contains a copy of  $\mathbb{Z}^2$ . We recall that the weak hyperbolization conjecture for 3-manifold groups was a part of the program for proving the Geometrization Conjecture for closed aspherical 3-manifolds, the other ingredient in the program being the Cannon conjecture. Although the work of Perelman has now resolved the full Geometrization Conjecture, the weak hyperbolization conjecture for  $PD(3)$  groups is a potential step in an approach to the following open question of C.T. Wall:

**Question 1** (Wall). Is every finitely presented  $PD(3)$  group over  $\mathbb{Z}$  isomorphic to the fundamental group of a closed aspherical 3-manifold?

Our main result is that weak hyperbolization conjecture holds for  $CAT(0)$  3-dimensional Poincaré duality groups over hereditary rings:

**Theorem 2.** *Let  $G$  be a 3-dimensional Poincaré duality group over a commutative hereditary ring  $\mathcal{R}$  with a unit. Suppose in addition that  $G$  is a  $CAT(0)$ -group, i.e., a group which admits a cocompact isometric properly discontinuous action  $G \curvearrowright X$  on a locally compact  $CAT(0)$  space  $X$ .*

*Then either  $G$  is Gromov-hyperbolic or  $G$  contains  $\mathbb{Z}^2$ .*

We note that special cases of this theorem were proven earlier by various people: S. Buyalo [8] and V. Schroeder [18] independently have proven that this theorem holds provided that  $X$  is the universal cover  $\tilde{M}$  of a closed 3-manifold  $M$ , the  $CAT(0)$ -structure on  $\tilde{M}$  is Riemannian and  $G = \pi_1(M)$  acts on  $X$  by deck-transformations. L. Mosher [16] proved that Theorem 2 holds provided that  $X = \tilde{M}$ ,  $G = \pi_1(M)$ , and the  $CAT(0)$  metric on is obtained by lifting a piecewise-Euclidean (locally)  $CAT(0)$ -cubulation from  $M$ . M. Bridson and L. Mosher also

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have an unpublished proof of Theorem 2 under the assumption that  $X = \tilde{M}$  has an arbitrary  $G$ -invariant  $CAT(0)$ -structure. Unlike all these proofs, our proof takes place on the ideal boundary of  $X$ ; this allows us to treat 3-dimensional Poincaré duality groups and relax the assumptions on the  $CAT(0)$  space.

*Outline of the proof of Theorem 2.* Assume that  $G$  is not Gromov hyperbolic, i.e., that  $X$  contains a 2-flat. By the work of Bestvina [2], the ideal boundary of  $X$  is homeomorphic to  $S^2$ . Our proof exploits the geometry of flats and parallel sets in  $X$ , and the pattern of their boundaries in the 2-sphere  $\partial_\infty X$ . The proof breaks into three cases.

**Case 1.  $X$  contains a 3-flat, Section 5.1.** This implies that  $X$  is at finite Hausdorff distance from the 3-flat, and we conclude that  $G$  is virtually  $\mathbb{Z}^3$ .

**Case 2.  $X$  contains no 3-flat but some parallel set  $P \subset X$  has full ideal boundary, i.e.  $\partial_\infty P = \partial_\infty X$ , Section 5.3.** We argue that  $P$  splits isometrically as  $\mathbb{R} \times Y$ , where  $\partial_\infty Y$  is a circle, and  $G$  acts as a convergence group on  $\partial_\infty Y$ . We then deduce that a finite index subgroup of  $G$  is isomorphic to the fundamental group of a 3-dimensional Seifert manifold.

**Case 3.  $X$  contains no 3-flat and no parallel set with full boundary, Section 5.4.** This is the main case. We show that every parallel set  $P$  in  $X$  is isometric to a product  $\mathbb{R} \times Y$ , where  $Y$  is Gromov hyperbolic. The ideal boundary of  $P$  is a suspension of the boundary  $\partial_\infty Y$ ; when  $P$  contains a 2-flat, we identify certain topological circles in  $\partial_\infty P$  which we call peripheral, and show that peripheral circles cannot cross one another in the 2-sphere  $\partial_\infty X$ .

Next, we choose a flat  $F \subset X$  whose boundary  $\partial_\infty F \subset \partial_\infty X$  is a peripheral circle, and consider its orbit  $\{g(F)\}_{g \in G}$ . Because the circles  $\{g(\partial_\infty F)\}_{g \in G}$  do not cross, we may use them to define a pretree  $\mathcal{T}$  on which  $G$  has a natural action. Using a Plante-type construction, we associate to  $\mathcal{T}$  an  $\mathbb{R}$ -tree  $T$ , which then inherits a nontrivial small stable  $G$ -action. By applying Rips' theory [3], we conclude that  $G$  admits a small nontrivial action on a simplicial tree. Using the fact that  $G$  is a  $PD(3)$  group, we deduce that the edge groups must be virtually  $\mathbb{Z}^2$ .

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## 2. GEOMETRIC PRELIMINARIES

In this section we briefly review several notions of metric geometry. We refer the reader to [1], [6] for the detailed discussion.

A *geodesic metric space* is a metric space  $(X, d)$  such that any two points  $x, y \in X$  in  $X$  are connected by geodesic, i.e. if  $D := d(x, y)$  then there exists an isometric embedding

$$\gamma : [0, D] \rightarrow X,$$

so that  $\gamma(0) = x, \gamma(D) = y$ .

Let  $X$  be a metric space and  $C \subset X$  be a subset. The  $r$ -neighborhood of  $C$  in  $X$  is defined as

$$N_r(C) := \{x \in X : d(x, C) < r\},$$

where  $d(x, C) := \inf\{d(x, c) : c \in C\}$ .

The *Hausdorff distance* between closed subsets of a metric space  $X$  is defined as

$$d_H(C_1, C_2) := \inf\{r : C_1 \subset N_r(C_2), \quad C_2 \subset N_r(C_1)\}.$$

Note that this distance is allowed to take infinite values. If  $X$  has finite diameter, the Hausdorff distance defines the *Hausdorff topology* on the set  $\mathcal{C}(X)$  of closed subsets of  $X$ . More generally, even for unbounded metric spaces  $X$  one defines the *Gromov–Hausdorff topology* on  $\mathcal{C}(X)$  as follows. We say that a sequence  $C_n \in \mathcal{C}(X)$  converges (in the Gromov–Hausdorff topology) to a closed set  $C \in \mathcal{C}(X)$  if for each closed metric ball  $B \subset X$  the intersections

$$C_n \cap B \in \mathcal{C}(B)$$

converge to  $C \cap B$  in the Hausdorff topology on  $\mathcal{C}(B)$ . Equivalently,  $C_n$ 's converge to  $C$  if the corresponding distance functions  $d(\cdot, C_n)$  converge to the distance function  $d(\cdot, C)$  uniformly on bounded subsets in  $X$ .

Given a number  $\kappa \in \mathbb{R}$  let  $M_\kappa$  denote the (unique up to isometry) complete simply-connected surface of the constant curvature  $\kappa$ . A geodesic metric space  $X$  is said to be a *CAT*( $\kappa$ ) space if  $X$  is complete as a metric space and geodesic triangles in  $X$  are “thinner” than triangles in  $M_\kappa$ . More precisely, consider a geodesic triangle  $T = [x, y, z] \subset X$  (with the vertices  $x, y, z$ ), in case when  $\kappa > 0$  (and  $M_\kappa$  is a sphere) we assume that the perimeter of this triangle is less than the circumference of the great circle in  $M_\kappa$ . Consider a triangle  $T' = [x', y', z'] \subset M_\kappa$  whose side-lengths are equal to the corresponding side-lengths of the triangle  $T$ . Let  $p$  be a point in the geodesic side  $\overline{xy}$  of  $T$  and let  $p' \in \overline{x'y'}$  be such that

$$d(x', p') = d(x, p).$$

Then we require

$$d(x, p) \leq d(x', p').$$

In this paper we will also need a generalization of the concept of a *CAT*(1) space to metric spaces  $X$  which are not geodesic. We assume that  $X$  is a disjoint union of geodesic metric spaces  $X_\alpha$ ,  $\alpha \in J$ , where each  $X_\alpha$  is a geodesic *CAT*(1) metric space and if  $\alpha \neq \beta$  the distance between any  $x \in X_\alpha, y \in X_\beta$  equals  $\pi$ . Then  $X$  will be also referred to as a *CAT*(1) space. An example of such a space is a space with discrete metric where distance between any pair of distinct points equals  $\pi$ .

If  $X$  is a *CAT*(1) space, we call points  $x, y \in X$  *antipodal* if  $d(x, y) = \pi$ .

Suppose that  $X$  is a *CAT*(0) space. Then the distance function on  $X$  is *convex*, i.e., its restriction to each geodesic in  $X$  is convex.

A space  $X$  is called *CAT*( $-\infty$ ) if it is *CAT*( $\kappa$ ) for each  $\kappa \in \mathbb{R}$ . A *metric tree* is a *CAT*( $-\infty$ ); in other words, it is a complete geodesic metric space where each geodesic triangle is isometric to a tripod.

A group  $G$  is called a *CAT*(0)-*group* if it admits an isometric properly discontinuous cocompact action on a locally compact *CAT*(0)-space.

Suppose that  $X$  is a *CAT*(0) space and  $F \subset X$  is a *k-flat*, i.e., an isometrically embedded copy of a Euclidean space  $\mathbb{R}^k$ . Then the *parallel set*  $P_F$  of  $F$  in  $X$  is the

union of all  $k$ -flats  $F' \subset X$  which are within finite distance from  $F$ . The parallel set  $P_F$  is closed, convex and is isometric to a product

$$F \times Y$$

where  $Y$  is a  $CAT(0)$  space, see for instance [6, Theorem II.2.14].

*Remark 3.* Theorem II.2.14 in [6] is stated in the case  $k = 1$ . The general case follows, for instance, by induction on the dimension of the flat.

We will say that a parallel set is *trivial* if  $k = 1$  and  $Y$  is bounded.

Given a  $CAT(0)$  space one defines the *ideal boundary* of  $X$  as the collection of equivalence classes of geodesic rays in  $X$ , where rays are equivalent if they are within finite Hausdorff distance from each other. This boundary has two (typically distinct) topologies:

1. The *visual topology*, in which case the ideal boundary is denoted  $\partial_\infty X$  and is called the *geometric boundary* of  $X$ .
2. The *Tits topology*, which is defined via the *Tits angular metric*, in which case the ideal boundary is denoted  $\partial_{Tits} X$ .

The second boundary is called *Tits boundary* of  $X$ ; this boundary is always a  $CAT(1)$  space.

For instance, in the case when  $X = \mathbb{H}^2$ ,  $\partial_\infty X$  is homeomorphic to  $S^1$ , while  $\partial_{Tits} X$  has discrete metric: the distance between distinct points equals  $\pi$ . A  $CAT(0)$  space is called a *visibility space* if any pair of distinct points in  $\partial_{Tits} X$  are antipodal.

A subset  $C \subset Z := \partial_{Tits} X$  is called *convex* if for any two non-antipodal points  $x, y \in Z$ , the geodesic segment  $\overline{xy}$  connecting  $x$  to  $y$ , is entirely contained in  $C$ . Intersection of two convex subsets of  $Z$  is also convex. If  $Y \subset X$  is a convex subset then  $\partial_{Tits} Y \subset Z$  is convex as well.

Let  $\delta \in [0, \infty)$  and consider a geodesic metric space  $X$ . A triangle  $T \subset X$  is called  $\delta$ -*thin* if there exists a point  $p \in X$  which is within distance  $\leq \delta$  from all three sides of  $T$ . A complete geodesic metric space  $X$  is called  $\delta$ -*hyperbolic* if each geodesic triangle  $T$  in  $X$  is  $\delta$ -thin. A space  $X$  is called *Gromov-hyperbolic* if it is  $\delta$ -hyperbolic for some  $\delta$ . A finitely generated group  $G$  is called *Gromov-hyperbolic* if its Cayley graph is Gromov-hyperbolic. One again defines the ideal boundary  $\partial_\infty X$  by looking at the equivalence classes of geodesic rays in  $X$ .

Suppose that  $G$  is a group acting isometrically, properly discontinuously and cocompactly on a  $CAT(0)$  space  $X$ . Then the group  $G$  is Gromov-hyperbolic iff  $X$  is a visibility space.

Let  $X$  be a Gromov-hyperbolic geodesic metric space which admits a cocompact isometric group action. We assume that the ideal boundary of  $X$  consists of more than 2 points; it then follows that  $\partial_\infty X$  has the cardinality of the continuum. The *displacement function* of an isometry  $g : X \rightarrow X$  is

$$dis(g) : x \rightarrow d(x, g(x)), x \in X.$$

**Lemma 4.** *Under the above assumptions there exists a constant  $D = D(X)$  such that for each  $g \in \text{Isom}(X)$  which fixes  $\partial_\infty X$  pointwise, the displacement of  $g$  is bounded from above by  $D$ .*

*Proof.* Let  $G \curvearrowright X$  be a cocompact isometric group action; pick a metric ball  $B = B(o, R) \subset X$  so that the  $G$ -orbit of  $B$  equals  $X$ . It then suffices to prove that there exists  $D < \infty$  such that for each isometry  $g$  of  $X$  fixing  $\partial_\infty X$  pointwise,

$$d(o, g(o)) \leq D.$$

Since the ideal boundary of  $X$  contains at least 4 points, there exists a pair of geodesics  $\gamma_1, \gamma_2 \subset X$  which have disjoint ideal boundaries. Without loss of generality we may assume that both  $\gamma_1, \gamma_2$  pass through the ball  $B$ .

Since  $X$  is  $\delta$ -hyperbolic, there exists a number  $r = r(\delta) < \infty$  such that if geodesics  $\alpha, \beta \subset X$  are within finite Hausdorff distance, then

$$d_H(\alpha, \beta) \leq r,$$

see for instance [6]. For every isometry  $g$  as above, the geodesics

$$\gamma_i, g(\gamma_i)$$

are within finite Hausdorff distance from each other; therefore

$$d_H(\gamma_i, g(\gamma_i)) \leq r, i = 1, 2.$$

Then

$$d(g(o), g(\gamma_i)) \leq R \Rightarrow d(g(o), \gamma_i) \leq R + r, \quad i = 1, 2.$$

However, since the geodesics  $\gamma_1, \gamma_2$  have disjoint ideal boundaries, the diameter of

$$S := N_{R+r}(\gamma_1) \cap N_{R+r}(\gamma_2)$$

is finite. Therefore, if we take  $D := \text{diam}(S)/2$ , the distance between  $o$  and  $g(o)$  is at most  $D$ .  $\square$

*Remark 5.* An analogue of Lemma 4 holds for quasi-isometries of  $X$  with uniformly bounded quasi-isometry constants.

### 3. PRETREES

In what follows we will need definitions and basic facts about pretrees; the definitions which we give follow [5].

A *pretree* is a set  $T$  together with a ternary relation (the *betweenness relation*)

$$\text{“}y \text{ is between } x \text{ and } z\text{”},$$

to be denoted  $\beta(xyz)$ , satisfying the following axioms:

Axiom 1.  $\beta(xyz)$  implies that  $x \neq y \neq z$ .

Axiom 2.  $\beta(xyz) \iff \beta(zyx)$ .

Axiom 3.  $\beta(xyz)$  and  $\beta(yxz)$  cannot hold simultaneously.

Axiom 4. If  $w \neq y$  then  $\beta(xyz)$  implies that either  $\beta(xyw)$  or  $\beta(wyz)$ .

Given a pretree  $T$  one can define *closed*, *open* and *half-open* intervals in  $T$  by

$$(x, z) := \{y \in T : \beta(xyz)\}, [x, z] := (x, z) \cup \{x, z\}, \text{ etc.}$$

Given an increasing union of intervals

$$[x_1, y_1] \subset [x_2, y_2] \subset \dots \subset [x_i, y_i] \subset \dots$$

we will also refer to the union of these intervals as a (possibly infinite) interval in  $T$ .

We note that  $\beta$  defines an order on each interval in  $T$ .

Define a “triangle” in  $T$  with vertices  $a, b, c$  to be the union of the segments (called “sides” of the triangle)  $[a, b], [b, c], [c, a]$ .

**Lemma 6.** *Each triangle  $\Delta$  in  $T$  is 0-thin, i.e., each side of  $\Delta$  is contained in the union of the two other sides.*

*Proof.* Follows immediately from Axiom 4. □

Suppose that  $T$  is a pretree which is given a measure  $\mu$  (without atoms) defined on closed intervals in  $T$  and the  $\sigma$ -algebra which these intervals generate. Define a function  $d(x, y)$  on  $T$  by  $d(x, y) := \mu([x, y])$ .

**Lemma 7.**  *$d$  is a pseudo-metric on  $T$ .*

*Proof.* It is clear that  $d$  is symmetric and  $d(x, x) = 0$  (since  $\mu$  has no atoms). The triangle inequality follows because for each triangle with the vertices  $a, b, c$  we have (see Lemma 6)

$$[a, b] \subset [a, c] \cup [b, c]. \quad \square$$

We note that if for each interval  $[a, b] \subset T$ , with  $a \neq b$ ,  $\mu(a, b) > 0$  then  $d$  is a metric. Moreover, it follows that  $(a, b) \neq \emptyset$  for each  $a \neq b$ . If the restriction of the metric  $d$  to each interval  $[x, y]$  is complete then  $[x, y]$  is order isomorphic to an interval in  $\mathbb{R}$  and moreover,  $([x, y], d)$  is isometric to an interval in  $\mathbb{R}$ . We thus get:

**Lemma 8.** *Suppose that for each interval  $[x, y] \subset T$ , with  $x \neq y$ ,  $\mu[x, y] > 0$ , and that the restriction of the metric  $d$  to each interval in  $T$  is complete. Then  $(T, d)$  is a metric tree.*

*Proof.* It is clear from the above discussion that  $T$  is a geodesic metric space. Since each triangle in  $T$  is 0-thin, it follows that each triangle in  $T$  is isometric to a tripod. Finally, let’s check completeness of  $T$ : Suppose that  $x_i, i \geq 0$ , is a Cauchy sequence in  $T$ . Then there exists an increasing sequence of intervals  $I_i \subset T$  such that

$$\lim_i \mu([x_0, x_i] \cap I_i) = \lim_i d(x_0, x_i).$$

Then completeness of  $d$  restricted to the union  $I$  of  $I_i$ ’s implies that  $(x_i)$  converges to a point in the interval  $I$ . □

#### 4. IDEAL BOUNDARIES OF $CAT(0)$ POINCARÉ DUALITY GROUPS

Let  $G \curvearrowright X$  be a discrete cocompact action of a  $PD(3)$  group  $G$  on a  $CAT(0)$ -space  $X$ . In this section we show that the ideal boundary of the  $CAT(0)$  space  $X$  is homeomorphic to  $S^2$ .

We refer the reader to [4], [7] for the background on the cohomology of groups. Recall [4], that an  $n$ -dimensional Poincaré duality group over a ring  $\mathcal{R}$  (for short,  $PD(n)$  group over  $\mathcal{R}$ ), is an  $FP$ -group over  $\mathcal{R}$  such that  $H^i(G, \mathcal{R}G)$  is isomorphic to  $\mathcal{R}$  as an  $\mathcal{R}$ -module when  $i = n$  and is trivial otherwise.

Let  $Z := \partial_\infty X$  be the ideal boundary of a locally compact  $CAT(0)$  space. M. Bestvina in [2] proved that the compactification

$$\bar{X} := X \cup Z$$

satisfies the axioms of the  $\mathcal{Z}$ -set compactification. Instead of listing all the axioms of the  $\mathcal{Z}$ -set compactification we mention only several properties:

1. If  $G \curvearrowright X$  is an isometric group action then this action extends to a topological action of  $G$  on  $\bar{X}$ .
2. There exists a natural isomorphism

$$H_c^*(X) \rightarrow \tilde{H}_c^{*-1}(Z),$$

which is compatible with inclusions of closed convex subsets  $X' \subset X$ .

3. We state the third property as a lemma:

**Lemma 9.** *If  $G$  is a  $PD(3)$  group acting isometrically, properly discontinuously and cocompactly on a  $CAT(0)$  space  $X$ , then the ideal boundary  $Z$  of  $X$  is homeomorphic to  $S^2$ .*

*Proof.* Bestvina proves, [2, Theorem 2.8], that if  $G$  is a  $PD(3)$  group over  $\mathcal{R}$ , then  $Z$  is homeomorphic to  $S^2$ . We note that Bestvina proves the latter theorem under more restrictive assumptions than we are working with (although, his class of groups  $G$  includes 3-manifold groups):

1. Bestvina assumes that the commutative ring  $\mathcal{R}$  is a PID. However this assumption is used only to apply the Universal Coefficients Theorem, which works for hereditary rings as well, see [9].

2. Bestvina's definition of an  $n$ -dimensional Poincaré duality group is more restrictive than the usual one: Instead of the  $FP$  property he assumes that a group  $G$  acts freely, properly discontinuously, cocompactly on a contractible cell complex  $Y$ . Note however that Bestvina in his proof uses only the fact that  $G \curvearrowright Y^{(i)}$  is cocompact on each  $i$ -skeleton of  $Y$ . Then existence of such an action for the  $CAT(0)$ -groups follows from a general construction described in [14]. Namely, if a group  $G$  admits a properly discontinuous cocompact action on a contractible space  $X$  (e.g. the  $CAT(0)$ -space in our case) then it also admits a free, properly discontinuous action on a contractible cell complex  $Y$  (possibly of infinite dimension) such that  $Y^{(i)}/G$  is compact for each  $i$ .

3. Bestvina assumes that the image of the orientation character  $\chi$  of the Poincaré duality group  $G$  is finite (he then passes to a finite index subgroup in  $G$  which is the kernel of  $\chi$ ). However this assumption can be omitted from his theorem using *twisting* of the action  $G \curvearrowright C_*(Y)$  by the character  $\chi$  as it is done in [14, Section 5.1].

With the above modifications, Bestvina's arguments apply in our case and it follows that  $\partial_\infty X$  is homeomorphic to the 2-sphere.  $\square$

## 5. PROOF OF THE MAIN THEOREM

**5.1. Case 1:  $X$  contains a 3-flat.** The main goal of this section is to show that, in case  $X$  contains a 3-flat, the group  $G$  contains a finite index subgroup isomorphic to  $\mathbb{Z}^3$ .

**Lemma 10.** *Suppose that  $S$  is a convex subset in  $X$  such that  $\partial_\infty S = \partial_\infty X$ . Then  $S$  is within finite Hausdorff distance from  $X$ .*

*Proof.* Pick a base-point  $o \in X$ . If  $S$  is not within finite Hausdorff distance from  $X$  then there exists a sequence of isometries  $g_i \in G$  such that  $d(o, g_i S)$  diverges to infinity. Consider the functions  $f_i := d(x, g_i S) - d(o, g_i S)$ . Then, according to Lemma 2.3 in [15], the sequence of functions  $f_i$  subconverges to a Busemann

function  $b$  on  $X$ . Clearly, the sublevel sets  $\{f_i \leq 0\}$  subconverge into the horoball  $U := \{b \leq 0\}$  in  $X$ . Since  $\partial_\infty \{f_i \leq 0\} = \partial_\infty g_i S = \partial_\infty X$ , it follows that  $\partial_\infty X = \partial_\infty U$ .

Let  $F$  be a 2-flat in  $X$ . Then  $\partial_\infty F \subset \partial_\infty U$  and convexity of horoballs in  $X$  imply that for each  $x \in F$

$$t = f(x) \Rightarrow F \subset \{z : b(z) \leq t\}.$$

It follows that the restriction  $b|_F$  is constant and thus  $F$  is contained in the horosphere  $\{x : b(x) = t\}$  for some  $t \in \mathbb{R}$ . Then Lemma 2.2 in [15] implies that  $X$  contains a half-space  $H := \mathbb{R}_+ \times F$ . Then, by taking an appropriate limit of the half-spaces  $h_j(H), h_j \in G$ , we see that  $X$  contains the 3-flat  $F' := F \times \mathbb{R}$ . By Lemma 9,  $\partial_\infty F' = \partial_\infty X$ . Suppose that  $F'$  is not within finite Hausdorff distance from  $X$ . Then, by repeating the same argument as above with  $S$  replaced with  $F'$  and then  $F$  replaced with  $F'$ , we see that  $X$  contains a 4-flat, which contradicts Lemma 9.

Therefore  $X$  is within finite Hausdorff distance from the 3-flat  $F'$ ; in particular, there are no horoballs in  $X$  which have the same ideal boundary as  $X$ . Contradiction.  $\square$

**Corollary 11.** *If  $X$  contains a 3-flat then the group  $G$  is virtually abelian, in particular, it contains  $\mathbb{Z} \times \mathbb{Z}$ .*

*Proof.* If  $F$  is a 3-flat in  $X$  then, by Lemma 9,  $\partial_\infty F = \partial_\infty X$  and, by Lemma 10,  $F$  is within finite Hausdorff distance from  $X$ . It follows that the group  $G$  is isomorphic to a lattice in  $Isom(\mathbb{R}^3)$  and hence it is virtually abelian and contains  $\mathbb{Z}^3$  as a subgroup of finite index.  $\square$

**Assumption 12.** From now on we will assume that  $X$  contains no 3-flats.

**5.2. Metric balls and parallel sets in  $X$ .** In this section we establish certain geometric properties of  $X$  which follow from Assumption 12.

**Lemma 13.** *There exists  $r_0 \in \mathbb{R}$  such that the following holds. For each ball  $B(x, r) \subset X$ , isometric to a disk of the radius  $r$  in  $\mathbb{R}^3$ , we have:  $r \leq r_0$ .*

*Proof.* If the assertion is false then there exists a sequence of balls  $B(x_i, r_i)$  with  $\lim_i r_i = \infty$ . Let  $g_i \in G$  be such that  $g_i(x_i)$  is a bounded sequence in  $X$ . Then the balls  $g_i(B(x_i, r_i))$  subconverge to a 3-flat in  $X$ . Contradiction.  $\square$

**Corollary 14.** *The set of 2-flats  $F' \subset X$  which are parallel to a flat  $F$  is compact in Gromov–Hausdorff topology.*

*Proof.* If not then  $X$  contains convex subsets isometric to  $[0, r] \times \mathbb{R}^2$  for arbitrarily large  $r$ . This contradicts the previous lemma.  $\square$

**Lemma 15.** *Suppose that  $Y \times \mathbb{R}$  is a parallel set in  $X$ . Then  $Y$  is Gromov–hyperbolic.*

*Proof.* We repeat the arguments in [6, Theorem 9.33]. If  $Y$  is not Gromov–hyperbolic then there exists a pair of points  $\xi, \eta \in \partial_\infty Y$  so that the Tits angle between  $\xi, \eta$  is positive but less than  $\pi$ . Pick a point  $o \in Y$  and consider a sequence of points  $y_i \in \overline{o\xi}$  which converge to  $\xi$  and the geodesic rays  $\overline{y_i\eta}$ . We identify the rays  $\overline{y_i\xi}, \overline{y_i\eta}$  with geodesic rays in  $Y \times \mathbb{R} \subset X$  (that share common point  $y_i$ ). Then, by



applying an appropriate sequence of elements  $g_i \in G$  (for which  $\{g_i(y_i)\}$  is bounded in  $X$ ) to  $Y \times \mathbb{R}$  and to the rays  $\overline{y_i\xi}, \overline{y_i\eta}$  and passing to the limit of a subsequence, we get:

1. The sets  $g_i(Y \times \mathbb{R})$  subconverge to a parallel set  $Y' \times \mathbb{R}$ .
2.  $Y'$  contains two geodesic rays  $\overline{y\xi'}, \overline{y\eta'}$  (limits of the sequences of rays  $g_i(\overline{y_i\xi}), g_i(\overline{y_i\eta})$ ) which bound a flat sector in  $Y'$ .

This contradicts Lemma 13. □

**5.3. Case 2:  $X$  contains a parallel set with the full boundary.** In this section we prove the main theorem under the assumption that  $X$  contains a parallel set  $P$  whose ideal boundary is the entire  $\partial_\infty X$ .

**Proposition 16.** *Suppose that there is a convex product subset  $P = \mathbb{R} \times Y$  such that  $\partial_\infty S = \partial_\infty X$ . Then  $G$  is commensurable to the fundamental group of a 3-dimensional Seifert manifold. In particular,  $G$  contains  $\mathbb{Z}^2$ .*

*Proof.* We will assume that  $P$  is a maximal convex product subset in  $X$ . Since  $Y$  is Gromov–hyperbolic, it follows that the Tits boundary of  $S$  is the suspension of a discrete metric space which is the ideal boundary of  $Y$ . Therefore, since  $\partial_\infty P = \partial_\infty X$ , the group  $G$  preserves the ideal boundary of the geodesic  $l = \mathbb{R} \times \{y\}$ . Hence for each  $g \in G$  the geodesic  $g(l)$  is parallel to  $l$ , which (by the maximality assumption) implies that  $g(P) = P$ .

We have an induced isometric action  $\rho : G \curvearrowright Y$ . Since the suspension of  $\partial_\infty Y$  is homeomorphic to the 2-sphere  $\partial_\infty X$ , the ideal boundary of  $Y$  is homeomorphic to  $S^1$ . Thus the cocompact isometric action  $\rho : G \curvearrowright Y$  extends to a uniform (topological) convergence action  $G \curvearrowright \partial_\infty Y = S^1$ . Therefore, according to [10, 12, 13, 19], the action  $G \curvearrowright S^1$  is topologically conjugate to a Moebius action  $\rho'$ .

Let  $K$  denote the kernel of  $\rho'$ .

**Lemma 17.**  *$K$  contains an infinite cyclic subgroup of finite index.*

*Proof.* Let  $D = D(Y)$  denote the constant given by Lemma 4. Pick a point  $y \in Y$ . Then for each  $g \in K$ ,

$$d(y, g(y)) \leq D.$$

Therefore the  $K$ -orbit of  $y$  is contained in the metric ball  $B(y, D)$ . Thus for every  $x \in X$ , the  $K$ -orbit of  $x$  is contained in a  $D$ -neighborhood of the geodesic  $l = \{y\} \times \mathbb{R}$  (passing through  $x$ ). Therefore  $K$  is quasi-isometric to  $\mathbb{Z}$  and hence is virtually  $\mathbb{Z}$ . □

**Lemma 18.** *The action  $G \curvearrowright S^1$  is topologically conjugate to an action of a uniform lattice in  $Isom(\mathbb{H}^2)$ .*

*Proof.* The action  $\rho'(G) \curvearrowright \mathbb{H}^2$  is cocompact, therefore we have the following possibilities:

- (a)  $\rho'(G)$  is a cocompact discrete subgroup in  $Isom(\mathbb{H}^2)$ .
- (b)  $\rho'(G)$  is a solvable subgroup in  $Isom(\mathbb{H}^2)$ , which fixes a point in  $S^1$ . Then  $\rho'(G)$  is not virtually abelian which contradicts the fact that  $G$  is a CAT(0) group.
- (c)  $\rho'(G)$  is dense in  $PSL(2, \mathbb{R})$ . Then, the group  $\rho'(G)$  contains a nontrivial elliptic element  $\hat{g}$  and it also contains a sequence of elements  $\hat{h}_i$  which converge

to  $1 \in PSL(2, \mathbb{R})$ . Let  $g, h_i \in G$  be elements which map (via  $\rho'$ ) to  $\hat{g}$  and  $\hat{h}_i$  respectively. Clearly,  $\rho(g) \in Isom(Y)$  is elliptic as well, let  $y \in Y$  be its fixed point. By taking conjugates  $g_i := h_i g h_i^{-1}$ , we get an infinite collection of distinct elements  $\{g_i : i \in \mathbb{N}\}$  of  $G$  such that for each  $n \in \mathbb{Z}$ ,  $g_i(y \times \mathbb{R})$  is contained in  $N_R(y \times \mathbb{R})$  where  $R \in \mathbb{R}_+$  is independent of  $i$ . We note that since all  $g_i$  are pairwise conjugate, there exists  $C < \infty$  such that  $d(x, g_i(x)) < C$  for each  $x \in y \times \mathbb{R}$  and  $i \in \mathbb{N}$ . This contradicts discreteness of the action of  $G$  on  $X$ .  $\square$

The above two lemmas imply that the kernel of  $\rho$  is commensurable to  $\mathbb{Z}$  and the quotient  $\rho(G)$  is commensurable to the fundamental group of a 2-dimensional hyperbolic surface. Thus, after passing to a finite index subgroup in  $G$  we obtain a short exact sequence

$$(19) \quad 1 \rightarrow K \rightarrow G \rightarrow Q \rightarrow 1$$

where  $Q$  is the fundamental group of a closed oriented surface.

**Lemma 20.** *Suppose that for a group  $H$  we have a short exact sequence*

$$1 \rightarrow \mathbb{Z}/n\mathbb{Z} \rightarrow H \rightarrow Q \rightarrow 1.$$

*Then  $H$  contains a finite index surface subgroup.*

*Proof.* Let  $t$  denote the generator of  $\mathbb{Z}/n\mathbb{Z}$ . Let  $a_i, b_i, i = 1, \dots, n$  denote the lifts to  $H$  of the standard generators of  $Q$ . It suffices to consider the case when

$$[a_1, b_1] \cdots [a_n, b_n] = t$$

and  $t$  belongs to the center of  $H$ . Consider the finite Heisenberg group

$$H_n := \langle a, b, t : [a, b] = t, a^n = b^n = t^n = 1, [a, t] = 1, [b, t] = 1 \rangle.$$

Define the homomorphism  $\phi : H \rightarrow H_n$  by

$$\phi(a_1) = a, \phi(b_1) = b, \phi(a_i) = \phi(b_i) = 1, \forall i \geq 2.$$

Then the kernel  $H'$  of  $\phi$  is a torsion-free subgroup of finite index in  $H$ . It follows that the map  $H \rightarrow Q$  sends  $H'$  injectively to a finite index subgroup in  $Q$ . Therefore  $H'$  is a surface group.  $\square$

We now return to the exact sequence (19). As in the above lemma we let  $a_i, b_i, i = 1, \dots, n$  denote the lifts to  $G$  of the standard generators of  $Q$ . Let  $H \subset G$  denote the subgroup generated by these elements. If

$$t := [a_1, b_1] \cdots [a_n, b_n]$$

is an infinite order element of  $K$  then  $H$  is isomorphic to the fundamental group of a Seifert manifold (whose base is a surface with the fundamental group  $Q$ ). It is clear that  $H$  has finite index in  $G$ .

If  $t$  has finite order then, according to Lemma 20, after passing to a finite index subgroup in  $Q$  we can assume that  $t = 1$ . Pick an infinite order element  $k \in K$  which belongs to the center of  $G$ . Then the subgroups  $H$  and  $\langle k \rangle$  generate the product

$$\mathbb{Z} \times Q \subset G.$$

Again, clearly, this subgroup has finite index in  $G$ . Thus, in the both cases,  $G$  is commensurable to the fundamental group of a 3-dimensional Seifert manifold.  $\square$

Thus, the conclusion of Theorem 2 holds provided that  $X$  contains a parallel set with the full boundary.

**Assumption 21.** From now on we will assume that the ideal boundary of each parallel set of  $X$  is a proper subset of  $\partial_\infty X$ .

**5.4. Case 3: The ideal boundary of every parallel set in  $X$  is a proper subset of  $\partial_\infty X$ .** In this section we show that the *peripheral circles* of the ideal boundaries of nontrivial parallel sets in  $X$  can be used to construct a *small stable nontrivial isometric action* of  $G$  on an  $\mathbb{R}$ -tree. Then, by Rips theory,  $G$  admits a nontrivial splitting as an amalgam with virtually abelian edge groups. This, in turn, implies that the edge groups are virtually  $\mathbb{Z}^2$ .

According to Eberlein's theorem (see [11] in the smooth case and [6, Theorem 9.33] in general), the  $CAT(0)$  space  $X$  is either a visibility space or it contains a 2-flat  $F$ . Since in the former case,  $G$  is Gromov-hyperbolic, we assume that  $X$  contains a 2-flat  $F$ . In particular,  $X$  contains *nontrivial parallel sets*.

**Lemma 22.** *Suppose that  $P = Y \times \mathbb{R}$  is a nontrivial parallel set in  $X$ . Then  $\partial_\infty P$  contains a topological circle  $S$  which is geodesic in the Tits metric so that  $S$  bounds a disk in  $\partial_\infty X \setminus \partial_\infty P$ .*

*Proof.* Let  $\xi, \eta \in \partial_\infty P$  be the ideal points of a geodesic  $y \times \mathbb{R} \subset Y \times \mathbb{R} = P$ . Then the Tits boundary  $\partial_{Tits} P$  is the metric join  $S^0 \star \partial_{Tits} Y$ , which is the union of geodesic segments  $L_\mu$  of length  $\pi$  connecting  $\eta$  and  $\xi$  and passing through  $\mu \in \partial_{Tits} Y \subset \partial_{Tits} X$ . Clearly, if  $\mu \neq \mu'$  then  $S := L_\mu \cup L_{\mu'}$  is a topological circle which is geodesic in the Tits metric.

Let  $D$  be a component of  $\partial_\infty X \setminus \partial_\infty P$ . Then there is a point  $\zeta \in \partial D$  which belongs to  $L_\mu \setminus \{\xi, \eta\}$  for some  $\mu \in \partial_{Tits} Y$ . Clearly,  $\partial D$  is not contained in  $L_\mu$ , therefore there exists a point  $\zeta' \in \partial D$  which belongs to  $L_{\mu'} \setminus \{\xi, \eta\}$  for some  $\mu' \in \partial_{Tits} Y \setminus \{\mu\}$ . The reader will verify that the circle  $S = L_\mu \cup L_{\mu'}$  bounds  $D$ .  $\square$

We will refer to these circles  $S$  as in Lemma 22, as *peripheral circles* of  $\partial_\infty P$ . A flat in  $X$  whose boundary is a peripheral circle will be called a *peripheral flat*.

It follows from the properties of the Tits metric (discussed in section 2) that if  $F, F' \subset X$  are 2-flats then the intersection  $\partial_{Tits} F \cap \partial_{Tits} F' \subset \partial_{Tits} X$  is convex and either consists of two antipodal points or is a circular arc in  $\partial_{Tits} F$  of the length  $\leq \pi$ .

**Definition 23.** We say that totally-geodesic circles  $S, S' \subset Z$  *cross* if  $S$  contains points from each component of  $Z \setminus S'$  (in the visual topology). Note that *crossing* is a symmetric relation. We will say that the ideal boundaries of two parallel sets  $P, P'$  *cross* if at least one circle in  $\partial_{Tits} P$  crosses a circle in  $\partial_{Tits} P'$ .

Observe that if  $S$  and  $S'$  cross, the intersection  $S \cap S'$  consists of a pair of antipodal points.

**Lemma 24.** *Suppose that  $P = l \times Y \subset X$  is a parallel set for which  $\partial_\infty Y$  consists of at least 3 points (i.e.,  $P$  is not within finite Hausdorff distance from a flat) and  $F \subset X$  is a 2-flat which is not contained in  $P$ . Then  $\partial_\infty P$  and  $S = \partial_\infty F$  do not cross.*

*Proof.* Suppose to the contrary that  $\partial_\infty P$  and  $S = \partial_\infty F$  do cross. Recall that  $\partial_\infty P$  is the metric join of  $\{\eta, -\eta\} = \partial_\infty l$  and  $\partial_\infty Y$ . If  $S$  were to pass through  $\eta$  then, by convexity,  $S$  passes through  $-\eta$  as well and hence  $F$  would be contained in

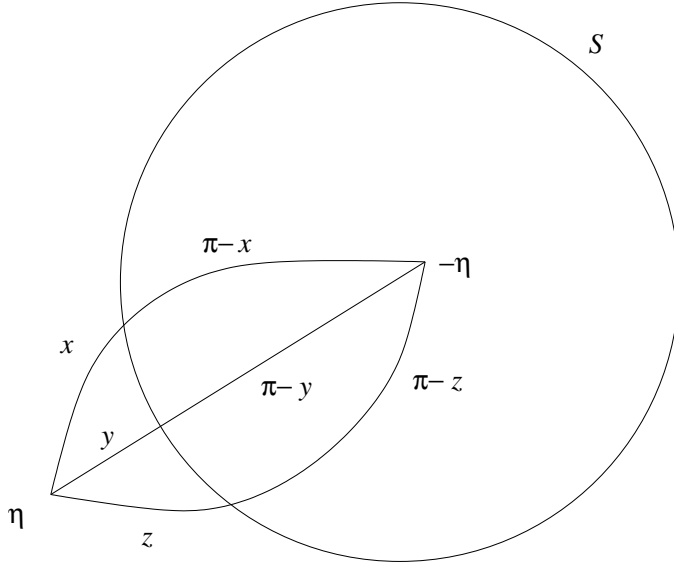


FIGURE 1.

the parallel set  $P$ . Therefore,  $S$  does not pass through  $\partial_\infty l$  and the configuration  $\{\partial_\infty P, S\}$  has to look like the one in Figure 1, where  $x, y, z$  denote the distances from  $\eta$  to the points of intersection between  $\partial_\infty P$  and  $S$ . It follows that  $x + y = \pi$ ,  $y + z = \pi$ ,  $x + z = \pi$  and thus

$$x = y = z = \pi/2.$$

This implies that the circle  $S$  is contained in  $\partial_\infty Y$ , thus  $Y$  cannot be Gromov-hyperbolic. This contradicts Lemma 15.  $\square$

We observe that, since  $G \curvearrowright X$  is properly discontinuous, the stabilizer of each flat  $F \subset X$  in the group  $G$  is virtually abelian. We assume that this stabilizer is virtually cyclic (possibly finite)—otherwise  $G$  contains  $\mathbb{Z}^2$ .

Suppose that we have three flats  $F, F', F'' \subset X$  with pairwise distinct ideal boundaries. We will say that  $F'$  separates  $F$  from  $F''$  if the following holds:

$$\partial_\infty F \subset \overline{D}, \partial_\infty F'' \subset \overline{D}'',$$

where  $D \sqcup D'' = Z \setminus \partial_\infty F'$ . We set the ternary relation  $\beta$  by:  $\beta(F, F', F'')$  if  $F'$  separates  $F$  from  $F''$ .

We leave it to the reader to verify that with this ternary relation the set  $\mathcal{P}$  of all peripheral flats in  $X$  satisfies the axioms of a pretree.

**Lemma 25.** *If  $U_0$  is a horoball in  $X$  then  $W := \partial_\infty U_0$  does not separate  $\partial_\infty X$ .*

*Proof.* Let  $\xi \in \partial_\infty X$  and consider the horoballs  $U_t = \{b_\xi(x) \leq t\}$ ,  $t \in \mathbb{R}$ , where  $b_\xi$  is the appropriately normalized Busemann function at  $\xi$ . Clearly  $\partial_\infty U_t = W$  for each  $t$ . Property (2) of the  $\mathcal{Z}$ -set compactification applied to the pairs  $(U_t, W)$  means that we have natural isomorphisms

$$(26) \quad H_c^i(U_t) \rightarrow \tilde{H}^{i-1}(W).$$

Suppose that  $[\zeta] \in H_c^i(U_t)$ . Then there exists  $s < t$  such that  $U_s$  is disjoint from the support set of the cocycle  $\zeta$ . Therefore  $[\zeta]$  maps trivially to  $H_c^i(U_s)$  and hence, by naturality of (26), it maps trivially to  $\tilde{H}^{i-1}(W)$ . We conclude that  $\tilde{H}^*(W) = 0$ . Therefore, by the Alexander duality on  $\partial_\infty X$ , the subset  $W = \partial_\infty U_0$  cannot separate  $\partial_\infty X$ .  $\square$

**Proposition 27.** *Let  $F, F''$  be flats in  $X$ . Then the set  $S(F, F'')$  of flats  $F'$  separating  $F$  from  $F''$  is compact with respect to the Gromov–Hausdorff topology.*

*Proof.* If  $\partial_\infty F = \partial_\infty F''$  then for each flat  $F'$  separating  $F$  and  $F''$  we have:  $\partial_\infty F' = \partial_\infty F$ . Therefore,  $S(F, F'')$  is compact by Corollary 14.

Whence we can assume that  $\partial_\infty F' \neq \partial_\infty F''$ . Suppose that  $F_i$  is a sequence of 2-flats in  $X$  which diverge to infinity, i.e.

$$\lim_i d(o, F_i) = \infty$$

where  $o \in X$  is a base-point. Then, as in the proof of Lemma 10, the limit of the distance functions to  $F_i$  (normalized at  $o$ ) subconverge to a Busemann function  $b_\xi$  in  $X$ . Let  $U$  be the horoball  $\{x : b_\xi(x) \leq 0\}$ .

If, say,  $\partial_\infty F \subset \partial_\infty U$  then the flat  $F$  is contained in the sublevel set of the Busemann function  $b_\xi$  and therefore  $X$  would contain a flat half-space  $\mathbb{R}_+^3$ , which contradicts Lemma 13. Thus both complements

$$\partial_\infty F \setminus \partial_\infty U, \quad \partial_\infty F'' \setminus \partial_\infty U$$

are nonempty.

**Lemma 28.** *1. In the Hausdorff topology on the set of closed subsets of  $X \cup \partial_\infty X$ , the sets  $F_i \cup \partial_\infty F_i$  subconverge into  $\partial_\infty U$ .*

*2.  $\partial_\infty F \cap \partial_\infty F'' \subset \partial_\infty U$ .*

*Proof.* 1. Suppose that the assertion is false. Then there exists a sequence of points  $x_i \in \partial_\infty F_i$  such that

$$\eta = \lim_i x_i \notin \partial_\infty U.$$

Clearly,  $\eta \in \partial_\infty X$ . Consider a parametrization  $\rho(t), t \in \mathbb{R}_+$  of the geodesic ray  $\overline{o\eta}$ . Then, since  $\eta \notin \partial_\infty U$ , there exists  $T \geq 0$  such that

$$(29) \quad b_\xi(\rho(t)) \geq 1, \quad \forall t \geq T.$$

The Busemann function  $b_\xi$  is the limit of the normalized distance functions

$$d_i(x) = d(x, F_i) - d(o, F_i).$$

Then  $d_i(o) = 0, d_i(x_i) \leq 0$  for all  $i$  and hence, by convexity,

$$d_i(y_i) \leq 0, \quad \forall y_i \in \overline{ox_i}.$$

This, together with the inequality (29), contradicts the assumption that the geodesics  $\overline{ox_i}$  converge to the geodesic ray  $\overline{o\eta}$ .

2. Observe that  $\partial_\infty F \cap \partial_\infty F'' \subset \partial_\infty F_i$  for each  $i$ . Thus (2) follows from (1).  $\square$

We continue the proof of Proposition 27. Pick points

$$\eta \in \partial_\infty F \setminus \partial_\infty U, \quad \eta'' \in \partial_\infty F'' \setminus \partial_\infty U.$$

Previous lemma implies that

$$\eta, \eta'' \notin \partial_\infty F \cap \partial_\infty F''$$

and that (since  $\partial_\infty U$  does not separate  $\partial_\infty X$ ) for large  $i$  the points  $\eta, \eta''$  belong to the same connected component of  $\partial_\infty X \setminus \partial_\infty F_i$ . This contradicts the assumption that  $F_i$  is between  $F, F''$  for all  $i$ .  $\square$

Now, let's pick a peripheral 2-flat  $F_0 \in \mathcal{P}$ , consider the set  $\{gF_0, g \in G\}$  and its closure  $\mathcal{F}$  in the Gromov–Hausdorff topology. The elements of  $\mathcal{F}$  are peripheral 2-flats in  $X$  and the group  $G$  acts naturally on  $\mathcal{F}$ . We note that since no flat in  $\mathcal{F}$  has cocompact stabilizer,  $\mathcal{F}$  contains no isolated points. After passing to a smaller  $G$ -invariant subset in  $\mathcal{F}$  we may assume that the action  $G \curvearrowright \mathcal{F}$  is minimal. The union

$$\tilde{\mathcal{L}} := \cup_{F \in \mathcal{F}} F$$

equipped with the Gromov–Hausdorff topology becomes a locally compact 2-dimensional lamination, the topological action  $G \curvearrowright \tilde{\mathcal{L}}$  is properly discontinuous and cocompact. The lamination  $\tilde{\mathcal{L}}$  has a continuous  $G$ -invariant leafwise flat metric. Therefore, since each leaf of  $\tilde{\mathcal{L}}$  is amenable, Plante's construction (see [17]) implies existence of a transversal  $G$ -invariant measure  $\mu$  on  $\tilde{\mathcal{L}}$ ; minimality of  $G \curvearrowright \mathcal{F}$  implies that this measure has full support.

**Lemma 30.** *Suppose that  $F \in \mathcal{F}$ ,  $g_n \in G$  is a sequence such that  $\lim_{n \rightarrow \infty} g_n F = F_\infty \in \mathcal{F}$ . Then there exist  $x_-, x_+ \in \mathcal{F}$  such that for all sufficiently large  $n$ ,  $g_n F \in [x_-, x_+]$  and  $F_\infty \in [x_-, x_+]$ .*

*Proof.* Since  $\lim_{n \rightarrow \infty} g_n F = F_\infty$ , the circles  $\partial_{Tits}(g_n F)$  converge to the circle  $\partial_{Tits} F_\infty$  in the Chabauty topology (we again are using here the visual topology on  $Z$ ). The circles in the collection

$$\{\partial_{Tits}(g_n F), \partial_{Tits} F_\infty, n \in \mathbb{N}\}$$

are all peripheral and hence do not cross each other (by Lemma 24). This implies that for all large  $n, m$  either  $\partial_{Tits}(g_n F)$  separates  $\partial_{Tits}(g_m F)$  from  $\partial_{Tits} F_\infty$  or  $\partial_{Tits} F_\infty$  separates  $\partial_{Tits}(g_n F)$  from  $\partial_{Tits}(g_m F)$ .  $\square$

The above lemma implies that the natural projection  $p : \tilde{\mathcal{L}} \rightarrow \mathcal{F}$  is continuous, where we give  $\mathcal{F}$  the order topology, whose basis consists of the open intervals  $(a, b)$ . It is also clear that  $p$  is a proper map in the sense that for each interval  $[a, b]$  the inverse image  $p^{-1}([a, b])$  consists of leaves of  $\tilde{\mathcal{L}}$  which intersect a certain compact subset in  $X$ : If a sequence of flats  $F_j$  leaves every compact subset in  $X$  then this sequence subconverges to a point in  $\partial_\infty X$ , but a point cannot separate one circle in  $\partial_{Tits} X$  from another.

The measure  $\mu$  on the pretree  $\mathcal{F}$  has no atoms and (since the measure  $\mu$  transversal to  $\tilde{\mathcal{L}}$  has full support) for each pair of distinct points  $x, x' \in \mathcal{F}$ ,  $\mu([x, x']) = 0$  iff the corresponding flats  $F, F'$  in  $X$  are not separated by any flat in  $\mathcal{F}$ . We let  $T$  be the quotient of  $\mathcal{F}$  by the equivalence relation: Points  $x, x' \in \mathcal{F}$  are equivalent iff  $\mu([x, x']) = 0$ . The  $G$ -action, the pretree structure, and the measure  $\mu$  project to  $T$  (we retain the notation  $\mu$  for the projection of the measure). As it was explained in section 3, the measure  $\mu$  yields a metric  $d$  on  $T$ . Local compactness of  $\tilde{\mathcal{L}}$  implies that the restriction of  $d$  to each interval in  $T$  is a complete metric. It is clear that the group  $G$  acts isometrically on  $T$ .

*Remark 31.* The map  $\mathcal{F} \rightarrow T$  has at most countable multiplicity. Moreover, all but countably many points in  $T$  have a unique preimage in  $\mathcal{F}$ .

**Lemma 32.** 1.  $T$  is an uncountable metric tree.

2. Stabilizers of nondegenerate arcs in  $T$  are virtually cyclic and the action  $G \curvearrowright T$  is stable.

3.  $G$  does not have a global fixed point in  $T$ .

*Proof.* 1. Follows from Lemma 8.

2. By our hypothesis, for each point  $F \in \mathcal{F}$  its  $G$ -stabilizer is virtually cyclic. Since  $\mathcal{F}$  is perfect, it is uncountable; hence, by Remark 31, uncountably many points in each nondegenerate arc  $[x, y] \subset T$  have a virtually cyclic stabilizer. Thus the action  $G \curvearrowright T$  is *small*. Since  $G$  is a  $CAT(0)$  group, each virtually cyclic subgroup of  $G$  is contained in a maximal virtually cyclic subgroup. Therefore, if  $I_1 \supset I_2 \supset \dots$  is a descending chain of arcs in  $T$ , then the sequence of their stabilizers in the group  $G$

$$G_{I_1} \subset G_{I_2} \subset \dots$$

is eventually constant. Thus the action  $G \curvearrowright T$  is *stable*.

3. The action  $G \curvearrowright \mathcal{F}$  is minimal, hence the action  $G \curvearrowright T$  is minimal as well. Since  $T$  is not a point it follows that  $G$  cannot fix a point in  $T$ .  $\square$

Since  $G$  acts properly discontinuously and cocompactly on the contractible space  $X$ , this group is finitely-presented. Therefore, by Lemma 32, we can apply [3] to conclude that the group  $G$  splits as an amalgam with a virtually solvable edge subgroup  $A$ . Since  $G$  is a  $CAT(0)$  group, the subgroup  $A$  is virtually abelian and finitely generated; let  $A' \subset A$  be a finite index free abelian subgroup. Since  $G$  splits over  $A$ , the pair  $(G, A)$  has at least two ends, and hence the same is true for the pair  $(G, A')$ . Since  $G$  is a 3-dimensional Poincaré duality group over  $\mathcal{R}$  this implies that  $A'$  has rank at least 2. This proves the main theorem.  $\square$

#### REFERENCES

- [1] W. BALLMANN, *Lectures on Spaces of Nonpositive Curvature*, DMV Seminar, Band 25, Birkhäuser, 1995.
- [2] M. BESTVINA, *Local homology properties of boundaries of groups*, Mich. Math. J., 43 (1996), pp. 123–139.
- [3] M. BESTVINA AND M. FEIGN, *Stable actions of groups on real trees*, Inventiones Math., 121 (1995), pp. 287–321.
- [4] R. BIERI, *Homological dimension of discrete groups*, Mathematics Department, Queen Mary College, London, 1976. Queen Mary College Mathematics Notes.
- [5] B. H. BOWDITCH AND J. CRISP, *Archimedean actions on median pretrees*, Math. Proc. Cambridge Philos. Soc., 130 (2001), pp. 383–400.
- [6] M. BRIDSON AND A. HAEFLIGER, *Metric spaces of non-positive curvature*, vol. 319 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], Springer-Verlag, Berlin, 1999.
- [7] K. BROWN, *Cohomology of Groups*, vol. 87 of Grad. Texts in Math., Springer, 1982.
- [8] S. BUYALO, *Euclidean planes in 3-dimensional manifolds of nonpositive curvature*, Mat. Zametki, 43 (1988), pp. 103–114.
- [9] H. CARTAN AND S. EILENBERG, *Homological algebra*, Princeton University Press, 1999. With an appendix by David A. Buchsbaum, Reprint of the 1956 original.
- [10] A. CASSON AND D. JUNGREIS, *Convergence groups and Seifert fibered 3-manifolds*, Invent. Math., 118 (1994), pp. 441–456.
- [11] P. EBERLEIN, *Geodesic flow on certain manifolds without conjugate points*, Transaction of AMS, 167 (1972), pp. 151–170.
- [12] D. GABAI, *Convergence groups are Fuchsian groups*, Ann. of Math. (2), 136 (1992), pp. 447–510.

- [13] A. HINKKANEN, *Abelian and nondiscrete convergence groups on the circle*, Trans. Amer. Math. Soc., 318 (1990), pp. 87–121.
- [14] M. KAPOVICH AND B. KLEINER, *Geometry of quasi-planes*. Preprint, 2004.
- [15] M. KAPOVICH AND B. LEEB, *Quasi-isometries preserve the geometric decomposition of Haken manifolds*, Invent. Math., 128 (1997), pp. 393–416.
- [16] L. MOSHER, *Geometry of cubulated 3-manifolds*, Topology, 34 (1995), pp. 789–814.
- [17] J. PLANTE, *Foliations with measure preserving holonomy*, Annals of Math., 102 (1975), pp. 327–361.
- [18] V. SCHROEDER, *Codimension one tori in manifolds of nonpositive curvature*, Geom. Dedicata, 33 (1990), pp. 251– 265.
- [19] P. TUKIA, *Homeomorphic conjugates of Fuchsian groups*, J. Reine Angew. Math., 391 (1988), pp. 1–54.

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