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FLAT CONFORMAL STRUCTURES ON 3-MANIFOLDS

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Abstract

We prove an existance theorem for flat conformal structures on finite-sheeted coverings over a wide class of Haken manifolds.

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Introduction

Flat conformal structure on the manifold M (of dimension $n>2$) is a maximal atlas $K=\{(U_i,\phi_i),\ \phi_i:U_i\to V_i\subset \mathbb{S}^n,\ i\in I\}$ with conformal transition maps $\phi_i\circ\phi_j^{-1}$. There is another (more classical) definition of flat conformal structure (FCS) as a conformal class of conformally- euclidean riemannian metrics on M. This definition is equivalent to former one (see [Ku 1], [Kul] e.g.). The most well known way to construct FCS is the so called *uniformization*: if a Kleinian group Γ acts freely and discontinuously on a domain $D\subset \mathbb{S}^n$ then a flat conformal structure K_Γ naturally arises on the factor-manifold $M=D/\Gamma$. For this structure K_Γ the covering $p: D\to M$ is conformal map. Such structures are called *uniformizable* and Γ is called *uniformizating* group. It should be noticed also that among eight 3-dimensional geometries [Sc] there are five conformally-euclidean ones: $\mathbb{S}^3$, $\mathbb{E}^3$, $\mathbb{H}^2\times\mathbb{R}$, $\mathbb{S}^2\times\mathbb{R}$, $\mathbb{H}^3$.

The following result of W.Thurston is well known

THEOREM H [T1],[Mo]. Let M be a closed atoroidal Haken 3-manifold. Then M admits a hyperbolic structure.

Hence on a manifold of this wide class a FCS may be introduced. Also FCS exists on connected sum of conformally-flat manifolds [Kul]. On other hand W.Goldman [Go] has shown that any closed 3-manifold M, modelled on Sol or Nil geometry, does not admit a flat conformal structure.

The main aim of this paper is to prove the following theorem concerning existence of FCS on more wide class of 3-manifolds than provided by the theorems of Thurston and Kulkarni.

THEOREM 5.1. Let M be a closed Haken 3-manifold with unsolvable fundamental group such that the canonical composition of M from hyperbolic and Seifert components does not include gluing of hyperbolic manifolds with hyperbolic or Euclidean ones. Then some finite-sheeted covering of M admits an uniformizable flat conformal structure.

REMARK. Euclidean manifold (in sense of [Sc]) is a compact manifold N such that *int* N admits a complete euclidean structure. There are only three Euclidean 3-manifolds with boundary, all of them are covered by $\mathbb{S}^1\times\mathbb{S}^1\times[0,1]$. Therefore, if a closed 3-manifold M is glued of hyperbolic and euclidean components H and E then 2-sheeted covering of M is glued of two copies of the manifold H.

The first Russian version of the theorem 5.1 was published in [Ka 2], where the condition on hyperbolic-euclidean gluing was mistakenly dropped (the true russian exposition is in [Ka 5]).

The theorem 5.1 combined with the Kulkarni's result on conformal connected sum (see above) makes the following conjecture be probable.

CONJECTURE. Let M be a closed 3-manifold satisfying the Thurston's geometrization conjecture [T 1], i.e. M is the result of toroidal gluing and connected sum of manifolds possessing a geometric structure. Let us suppose also that the decomposition of M into connected sum of prime components does not include Sol- or Nil-manifolds. Then some finite-sheeted covering of M admits an uniformizable flat conformal structure.

The proof of the theorem 5.1 is organized in several stages. In the § 2 we shall prove the theorem 5.1 for the class of Seifert manifolds. More precisely

THEOREM 2.1. Let $S(g,e)$ be a total space of a circle bundle over a closed orientable surface S_g of genus g having euler number $e \in \mathbb{Z}$ such that $0<e\leq(g-1)/11$. Then $S(g,e)$ admits an uniformizable FCS.

An analogous result was independently obtained in joint work of M.Gromov, H.B.Lawson and W.Thurston [G L T] (see [Ku 3] for further discussion). It should be noticed that for $e=0$ flat conformal structure on $S(g, e)$ always exists, but for $e\neq 0$, $g=1$ the manifold $S(g, e)$ does not admit any FCS [Go].

Limit sets of groups $H(g, e)$ uniformizating $S(g, e)$ are tame unknotted topological circles in S^3 (Corollary 2.3). Such groups are called *pseudofuchsian*. Pseudofuchsian groups (probably with parabolic elements) provide one type of building blocks for proof of the theorem 5.1, they uniformizate finite-sheeted coverings of Seifert components in the canonical decomposition of M.

The other type of bilding blocks is a class of "hyperbolic" groups that uniformizate interiors of hyperbolic components of the canonical splitting of M. The main problem is to find small deformations of constructed pseudofuchsian

and "hyperbolic" groups such that conformal gluing of uniformizated hyperbolic and Seifert manifolds is possible. For this purpose we choose deformation of these groups such that : parabolic $\mathbb{Z}\oplus\mathbb{Z}$ become $\mathbb{Z}\oplus \mathbb{Z}_n$ (generated by loxodromic and elliptic transformations). At the same time cyclic parabolic subgroups of pseudofuchsian groups become loxodromic ones, which are conjugated to subgroups of corresponding $\mathbb{Z}\oplus \mathbb{Z}_n$. Arising elliptic elements disappear after transition to finite-index subgroups. Such deformations of pseudofuchsian groups are considered in § 4.

In the § 3 we state some auxiliary results concerning construction of some pseudofuchsian groups and above-mentioned deformation problems. In the § 5 we present the direct construction of a Kleinian group uniformizating finite-sheeted covering of M. This construction is preceded by two illustrating examples. The main tool here is Klein-Maskit Combination theorems and some results of Hempel, McCullough and Miller related to the residual finiteness property of 3-manifold groups.

These results together with some basic facts about Kleinian groups are collected in § 1. An example of closed orientable 3-manifold which does not admit any FCS but has conformally-flat finite-sheeted covering is presented in the § 6. This manifold is obtained by gluing of two boundary components of some Seifert manifold. This example shows that Thurston's conjecture about geometric realization of smooth actions of finite groups on geometric 3-manifolds (see [M S]) does not valid for conformal geometry.

In conclusion I express acknowledgements to my former advisors prof. S.L.Krushkal' and N.A.Gusevskii for help and general support and for participants of prof. S.L.Krushkal's seminar for fruitful discussions. I am grateful for all those who have sent to me their preprints and reprints. This long list includes prof. W.Goldman, R.Kulkarni, Y.Kamishima, H.Lawson, M.Gromov, N.Kuiper and many other mathematicians.

§ 1. Definitions and some basic facts of the theory of Kleinian groups and related topics

1.1. Let $\mathfrak{M}_n$ be the group of all orientation-preserving Mobius transformations of n-sphere $\mathbb{S}^n = \overline{\mathbb{R}}^n = \mathbb{R}^n \cup (\infty)$. The fixed-point set of $\gamma \in \mathfrak{M}_n$ is denoted by $Fix(\gamma) = \{x \in \mathbb{S}^n : \gamma(x) = x\}$. For the group $\Gamma \subset \mathfrak{M}_n$ the discontinuity set $R(\Gamma) = \{ x \in \mathbb{S}^n$: the point x possesses a neighbouorhood $U(x)$ such that the intersection $U(x) \cap \gamma(U(x)) = \emptyset$ for all but finite elements $\gamma \in \Gamma \}$.

Any Mobius transformation $\gamma \in \mathfrak{M}_n$ may be extended to the element $\tilde{\gamma} \in \mathfrak{M}_{n+1}$ which has a closed invariant ball $B \subset \mathbb{S}^{n+1}$ with boundary $\mathbb{S}^n$. The element γ is said to be *loxodromic* if $Fix(\tilde{\gamma}) \cap B = (p, q) \subset \mathbb{S}^n$, $p \neq q$. The element γ is said to be *parabolic* if $Fix(\tilde{\gamma}) \cap B = (p) \subset \mathbb{S}^n$; and γ is said to be *elliptic* in ether case ($Fix(\tilde{\gamma}) \cap int\ B \neq \emptyset$). If a loxodromic element γ is conjugate in $\mathfrak{M}_n$ to homothety $q : x \longrightarrow kx$, $x \in \mathbb{R}^n$, then γ is said to be *hyperbolic* element.

Fundamental set for the Kleinian group G is a subset Φ of $R(G)$ such that the orbit $G \cdot \Phi$ coincides with $R(G)$ and $g(\Phi) \cap \Phi = \emptyset$ for any $g \in G \setminus (\omega)$.

For a closed connected hypersurface S in $\mathbb{R}^n$ the compact component $int(S)$ of $\mathbb{R}^n \setminus S$ is called *interior* of this hypersurface. Analogously, $ext(S) = \mathbb{S}^n \setminus cl\ int(S)$ is called *exterior* of it.

Let h be a loxodromic transformation of $\mathbb{S}^3$, ℓ be any h-invariant proper arc of circle $\mathcal{L}$, that pass through $Fix(h)$.

DEFINITION 1. The pair $(h, \ell) = \overrightarrow{h}$ is called *directed loxodromic transformation*. Two directed transformations h_1, h_2 are called conjugated if there exists a transformation $f \in \mathfrak{M}_3$ such that (1) $f h_1 f^{-1} = h_2$ and (2) $f(\ell_2) = \mathcal{L}_1 \setminus cl(\ell_1)$.

Assume that the complex plane is included in $\mathbb{R}^3$ in the standard way: $\mathbb{C} = \{ (x_1, x_2, 0), x_1 + x_2 i \in \mathbb{C} \}$. Then a loxodromic transformation h is conjugated in $\mathfrak{M}_3$ to an element h^* preserving $\mathbb{C}$, $h^* : z \longmapsto k(h) \cdot z$, $z \in \mathbb{C}$, $k(h) \in \mathbb{C}^*$. The complex number $k(h)$ is independent of choice of h^* up to conjugation $k(h) \longmapsto \overline{k(h)}$, we shall suppose that $Im(k(h)) \geq 0$.

DEFINITION 2. The complex number $k(h)$ is the *complex coefficient* of the loxodromic transformation h.

Let (M, d) be a metric space, $X \subset M$, $Y \subset M$. Then we put:

$dist(X, Y) = sup \{ inf \{ d(x, y), y \in Y \}, x \in X \}$.

Let $g \in \mathfrak{M}_n$ be an element such that $g(\infty) \neq \infty$. Then the isometric sphere of the element g is the set $I(g) = \{ x \in \mathbb{R}^n, det(g'(x)) = 1 \}$, where $g'(x)$ is the Jacoby matrix for the map g. Let G be a Kleinian group such that $\infty \in R(G)$. Then the set $\bigcap_{g \in G} ext\ I(g)$ is called *isometrical fundamental polyhedron* of the group G.

1.2. Combination theorems.

DEFINITION 3. Let J be a subgroup of a group $G \subset \mathfrak{M}_n$, B be a subset of $\mathbb{S}^3$. Then B is called *precisely invariant under* J *in the group* G if (1) $J(B) = B$ and (2) for any $g \in G \setminus J$ we have $g(B) \cap B = \emptyset$.

DEFINITION 4. Let J be a cyclic loxodromic or trivial subgroup of $G \subset \mathfrak{M}_3$. Then compact manifold B, which is precisely invariant under $J \subset G$ is called (G, J)*- block* if $B \cap R(G) = B \cap R(J)$.

THEOREM 1.1 (FIRST MASKIT COMBINATION THEOREM). Let J be a cyclic loxodromic or trivial subgroup of discrete groups $G_1, G_2 \subset \mathfrak{M}_3$. Assume that $J \neq G_1$, $J \neq G_2$ and there is a closed embedded surface W dividing $\mathbb{S}^3$ into two compact submanifolds B_1, B_2, where B_m is a (J, G)- block, $m=1, 2$. Let D_m be a fundamental set for G_m such that :

(1) $D_m \cap B_m$ is a fundamental set for action of J in B_m, (2) $D_1 \cap W = D_2 \cap W$, (3) the set $D_m \cap B_{3-m}$ has non-empty interior, $m = 1, 2$.

Set $D = (D_1 \cap B_2) \cup (D_2 \cap B_1)$ and $G = \langle G_1, G_2 \rangle$.

Then the following statements hold.

(i) $G \cong G_1 *_J G_2$ -free product with amalgama J. (ii) The group G is Kleinian. (iii) D is a fundamental set for G. (iv) Let Q_m be the union of the G_m -translates of $int(B_m)$ and let R_m bee the complement of Q_m. Then $R(G)/G = (R_1 \cap R(G_1))/G_1 \cup (R_2 \cap R(G_2))/G_2$, where these manifolds are identified along their commong boundary $(W \cap R(G))/J$.

Now we consider the Second Combination Theorem. We shall assume that $f \in \mathfrak{M}_3$, J_1, J_2 are cyclic loxodromic (or trivial) subgroups of a discrete group $G_0 \subset \mathfrak{M}_3$. Two compact manifolds B_1, $B_2 \subset \mathbb{S}^3$ are *jointly f -blocked* if B_m is (J, G_0) -block $(m = 1, 2)$, f maps exterior of B_1 onto interior of B_2 and $f\cdot J\cdot f^{-1}=J$. If B_1 and B_2 are jointly f-blocked, then let A be equal to $ext(B_1 \cup B_2)$, $A_0= S^3 \setminus G_0(B_1 \cup B_2)$.

THEOREM 1.2 (SECOND MASKIT COMBINATION THEOREM). Let J_1, $J_2 \subset G_0$, $f \in \mathfrak{M}_3$ be as above. Assume that B_1 and B_2 are jointly f-blocked compact submanifolds of S^3, and that $A_0 \neq \emptyset$. Let D_0 be a fundamental set for G_0 such that
(1) $D_0 \cap B_m$ is a fundamental set for action of J_m on B_m,
(2) $f(D_0 \cap W_1) = D_0 \cap W_2$, where $W_m = \partial B_m$.
we set $G=\langle G_0, f\rangle$, $D=D_0 \cap (A \cup W_1)$. Then the following statements hold:
(i) $G \cong G_0 *_f$ is the HNN-extension of G_0 by f.
(ii) G is discrete. (iii) D is a fundamental set for G.
(iv) The set A_0 is precisely invariant under G_0 in G. Let $Q= clA_0 \cap R(G_0)$; then $R(G)/G$ is equal to Q/G_0, where the two boundary components $(W_1 \cap R(G_1))/J_1$ and $(W_2 \cap R(G_2))/J_2$ are identified, this identification is given by f.

REMARK 1. We don't formulate the Combination Theorems in greatest generality, but our formulations are sufficient for the purposes of this article. Some words on proofs of the theorems 1.1, 1.2.

These theorems are really due to Klein and Maskit. Our formulations are follow [Mk 1], however we drop all essentially 2-dimensional assertions of [Mk 1, Ch. VII, Th C.2, Th. E.5]. The various generalizations [Iv], [KAG], [Ap] of Combination Theorems to higher dimensions, repeat Maskit's original arguments [Mk 3]. So, the theorems 1.1, 1.2 may be proved in the same manner (rewriting proofs of [Mk 1]) or deduced from [Iv], [KAG, p.169- 170], [Ap, Th. 4.2, 4.5].

1.3. 3-manifolds.

We suppose that reader is familiar with basic concepts of 3-dimensional topology such as : *incompressible surfaces*, *canonical decomposition* of a *Haken* manifold into *hyperbolic* and *Seifert* manifolds (we shall consider them as total spaces of fiber bundles over 2-dimensional *orbifolds*)- see [He 1], [J S], [Sc] for references.

For construction of finite-sheeted coverings of 3-manifolds we shall frequently use the following results of J.Hempel [He

2] and D.MmCullough- A.Miller [M M].

THEOREM 1.3. Let Γ be a finitely generated subgroup of $PSL(2,\mathbb{C})$. Then for all but finite primes $p \in \mathbb{N}$ the group Γ contains a normal torsion-free subgroup Γ_0 of finite index such that intersection of Γ_0 with any maximal parabolic subgroup $P \subset \Gamma$ is by subgroup $\{\gamma^p : \gamma \in P\}$.

THEOREM 1.4. Let M be a Seifert fibered space over an orbifold O, $\rho: \tilde{O} \to O$ be a finite-sheeted covering (here $\tilde{O}$ is an orbifold and we consider ρ in sense of orbifold-theory), $n \in \mathbb{N}$ such that for any component $b \subset \partial\tilde{O}$ the restriction ρ to b is n-sheeted covering. Then there exist a Seifert fiber space $\tilde{M}$ with the base $\tilde{O}$ and a covering $\tilde{p}: \tilde{M} \to M$ such that the induced map of bases is $\rho: \tilde{O} \to O$ and the regular fiber of $\tilde{M}$ n times covers the regular fiber of M.

Proof of the theorem is not difficult [M M, Prop. 4.1]).

Let the manifold M be glued of finitely many components M_j by identification of incompressible boundary surfaces S_{kj}. Let us suppose that Γ_j are normal finite- index subgroups of $\Pi_j=(i_j)_*(\pi_1 M_j)$ (where $i_j: M_j \to M$ are natural inclusions) such that $\Gamma_j \cap (i_j)_*(\pi_1 S_{kj}) = \Gamma_m \cap (i_m)_*(\pi_1 S_{ml})$ where $i_k(S_{kj}) = i_m(S_{ml})$.

THEOREM 1.5. Under the above-stated conditions there exists a finite-index normal subgroup $\Gamma \subset \pi_1 M$ such as $\Gamma \cap \Pi_j = \Gamma_j$ for any j.

Proof of this theorem is easy also [M M, Prop. 1.1].

Let S be a closed surface, $D_1, \dots, D_{2r}$ -are pairwise disjoint closed discs in S, $\Sigma = S \setminus (intD_1 \cup \dots \cup intD_{2r})$. Then for any positive integer n there exists n-sheeted ramified cyclic covering $\rho: \tilde{S} \to S$ such as exactly one branch point of order n lies in every disc D_i (see e.g. [EEK]).

DEFINITION 5. The restriction of the covering ρ to the surface $\tilde{\Sigma} = \Sigma \setminus \rho^{-1}(intD_1 \cup \dots \cup intD_{2r})$ is *standard n-sheeted covering of the surface* Σ.

DEFINITION 6. Let M be the product $\Sigma \times S^1$, ρ be the standard n-sheeted covering of Σ. Then the covering $\tilde{p}: \tilde{M} \to M$ (that is constructed by the theorem 1.4) is *standard n^2-sheeted covering* of the manifold M.

§ 2. Uniformization of Seifert manifolds

2.1. Let M be a Seifert manifold with zero Euler number and hyperbolic base. Then there exists certain $\mathbb{H}^2\times\mathbb{R}$- structure on M (see [Sc]), hence $M=\mathbb{H}^2\times\mathbb{R}/\Gamma$, where Γ is a torsion-free discrete isometry group of $\mathbb{H}^2\times\mathbb{R}$. This group may be chosen so that it's cyclic normal subgroup is generated by the displacement $t:(z,\varphi)\longrightarrow(z,\varphi+2\pi)$, where $z\in\mathbb{H}^2$, $\varphi\in\mathbb{R}$. Let $q: \mathbb{H}^2\times\mathbb{R}\longrightarrow \mathbb{H}^2\times\mathbb{S}^1 = \{(x_1, x_2, x_3)\in\mathbb{R}^3: x_2^2+x_3^2>0\}$ be a cylindrical coordinates map, the deck-transformation group of this covering is $\langle t\rangle$. This map induces a homomorphism $q_*: \Gamma\longrightarrow\Gamma_0\subset \mathfrak{M}_3$. The group Γ_0 acts freely and discontinuously on $\mathbb{H}^2\times\mathbb{S}^1$ and the manifold $\mathbb{H}^2\times\mathbb{S}^1/\Gamma_0$ is homeomorphic to $\mathbb{H}^2\times\mathbb{R}/\Gamma=M$. So the manifold M admits a flat conformal structure which is uniformizated by a "fuchsian" group Γ. Since the geometries $\mathbb{E}^3$ and $\mathbb{S}^2\times\mathbb{R}$ can be realized in $\mathbb{R}^3$ as $(\mathbb{R}^3, |dx|^2)$ and $(\mathbb{R}^3\setminus(0), |dx|^2/|x|^2)$ we have that *any Seifert manifold with zero Euler number admits an uniformizable f.c.s.*

In contrast to that, any Seifert manifold with non-zero Euler number and euclidean base-orbifold admits no any FCS (see [Go]). The main purpose of this paragraph is to prove the following

THEOREM 2.1. Let $S(g, e)$ be the total space of the circle bundle over the closed orientable surface S_g of genus g which has the Euler number $e\in\mathbb{Z}$ such that $0<e\le(g-1)/11$. Then the manifold $S(g,e)$ admits an uniformizable flat conformal structure.

2.2. We shall need the following description of the manifold $S(g, e)$. Let $\Sigma_g = S_g\setminus int\, B^2$, where B^2 is a closed disc, $x\in\partial B^2$, $\mathfrak{N}=\Sigma_g\times\mathbb{S}^1$, $t=(x)\times\mathbb{S}^1\subset\partial\mathfrak{N}$, $\beta=\partial B^2\times(\varphi)$, where $\varphi\in\mathbb{S}^1$, $T=\partial B^2\times\mathbb{S}^1$ is the boundary of $\mathfrak{N}$. Let $\mathfrak{T}=B^2\times\mathbb{S}^1$ be a solid torus, $\tau=(x)\times\mathbb{S}^1\subset\partial\mathfrak{T}$, $\varkappa=\partial B^2\times(\varphi)\subset\partial\mathfrak{T}$. We shall denote the corresponding elements of $\pi_1(T)$ and $\pi_1(\mathfrak{T})$ by the same symbols : t , β , τ , $\varkappa$. The manifold $S(g,e)$ is glued of $\mathfrak{T}$ and $\mathfrak{N}$ so that the loop t is glued to τ and the loop β is glued to $\varkappa\cdot t^e$.

2.3. PROOF of the theorem 2.1. Our main purpose is to construct a Kleinian group $H=H(g, 1)$ such that $R(H)/H=M(H)$ is homeomorphic to $S(g, 1)$, where $g=12$. A fundamental polyhedron Φ for action of H on $R(H)$ is homeomorphic to a solid torus and satisfy the following properties:

(a) Faces of Φ which are Q_1 , R'_1 , Q'_1 , R_1 ,..., Q_g , R'_g , Q'_g , R'_g , Q_1 , lie on Euclidean spheres in $\mathbb{R}^3$ and they all are topological annuli. Two neighbouring faces (which are successively situated

in this chain of faces) intersect each other by Euclidean circle all other pairs of faces have empty intersection (see figure 1).

Faces of Φ are paired by Mobius transformations $A_1:Q_1\longrightarrow Q'_1$, $B_1:R_1\longrightarrow R'_1$,..., $A_g:Q_g\longrightarrow Q'_g$, $B_g:R_g\longrightarrow R'_g$ which generate the group H . Let x_0 be a point of the circle $Q_1\cap R_g$, $x_1=B_1^{-1}\circ A_1^{-1}\circ B_1\circ A_1(x_0)=[A_1,B_1](x_0)\in Q_2\cap R_1$ and so on, $x_g=[A_g, B_g]\circ\ldots\circ[A_1,B_1](x_0)\in R_g\cap Q_1$.

(b) We require x_g to be equal x_0 and the sum of dihedral angles of the polyhedron Φ to be equal 2π . Then Φ is a fundamental domain for the group $H=\langle A_1, B_1, \ldots, A_g, B_g: [A_g, B_g]\cdot\ldots\cdot[A_1, B_1]=1\rangle$. To see this it is sufficient to continue the polyhedron Φ to the hyperbolic space $\mathbb{H}^4=\mathbb{R}^4_+=\{(x_1, x_2, x_3, x_4): x_4>0\}$ (each sphere is continued to geodesic hyperplane) and then apply the Poincare-Maskit theorem on fundamental polyhedra [Mk 1].

Let α_1 be a simple closed curve on Q_1 which connects points x_0 and $A_1^{-1}\cdot B_1\cdot A_1(x_0)$, curve $\gamma_1\subset R_1$ connects the point $A_1(x_0)$ with x_1 , $\alpha'_1=A_1(\alpha_1)$, $\gamma'_1=B_1(\gamma_1)$ (see the figure 1). By analogy we construct curves α_2, α'_2, γ_2, γ'_2 , ..., α_g , α'_g , γ_g , γ'_g . Their union η is a simple closed curve on $\partial\Phi$.

(c) Let us suppose that the linking number of the curve η and the axis of the solid torus $\mathbb{S}^3\setminus\Phi$ is equal $|e|=1$. It is easy that this condition is equivalent to the following one:

the loop η is homotopic on $\partial\Phi$ to the loop $t+k$, where $t=Q_1\cap R_g$ and the class $[k]$ generates the kernel of $\pi_1(\partial\Phi)\longrightarrow\pi_1(\Phi)$ (under appropriate choice of orientations on the above mentioned loops).

2.4. Now we are to show that conditions (a)-(c) are sufficient for the group H uniformizate the manifold $S(g,1)$.

Let $T'\subset\Phi$ be a torus which is parallel to $\partial\Phi$, $\mathfrak{Y}$ be a component of $\Phi\setminus T'$ lying between $\partial\Phi$ and T'. The manifold $M(H)=R(H)/H$ is homeomorphic to the Φ/H. Let $q:\Phi\longrightarrow M(H)$ be a natural projection, $\mathfrak{N}=q(\mathfrak{Y})$, $\beta=q(\beta')$ where β' is a loop on T' parallel to η in $\Phi\setminus\mathfrak{Y}$. The manifold $\mathfrak{N}$ is homeomorphic to $\Sigma_g\times\mathbb{S}^1$ and the manifold $M(H)$ is glued of $\mathfrak{N}$ and the solid torus $\mathfrak{T}=q(\Phi\setminus\mathfrak{Y})$ essentially in the same way as in the item 2.2 (where we put $|e|=1$). Therefore we have $M(H)=S(g,1)$.

2.5. Construction of polyhedron Φ in the case $g=12$, $e=1$.

Let us notice that on the twisted strip L_1 (figure 2) the linking number of the boundary curve η and the "middle line" λ is equal to 1. On the same figure 2 the strip L_2 is drawn so that it is equivalent to L_1 and has no overlaps .

3

3 лицо

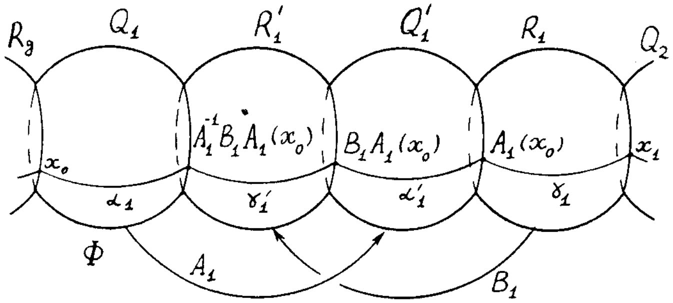

Figure 1

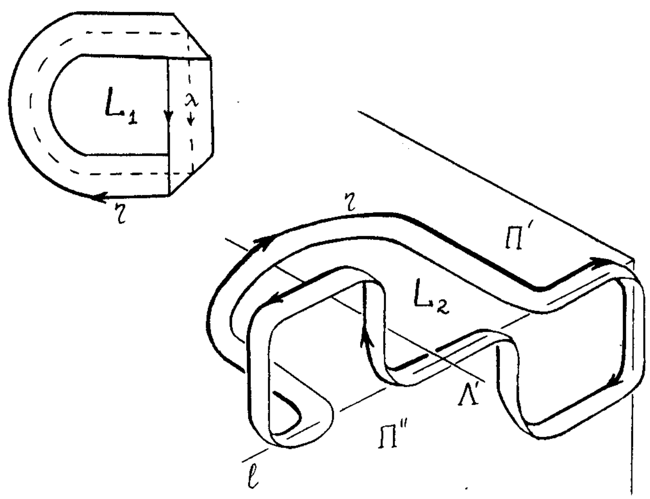

Figure 2

Our aim is to cover L_2 by spheres so that the conditions (a)-(c) of the item 2.3 are satisfied.

We single out two parts of the strip L_2 : the part L'_2 which is contained in the horizontal plane Π' and the part L''_2 on which the middle line λ lies in the vertical plane Π''. Let l be the intersection $\Pi' \cap \Pi''$ and $\Lambda' \subset \Pi'$ be the axis of symmetry of the substrip L''_2 , $O = l \cap \Lambda'$. We shall consider l and Λ' as a coordinate axises on the plane Π'.

Let O_1 and O_2 be points on the plane Π' with coordinates (0,1) and (2,1) respectively; $l_1 \subset \Pi'$ is a straight line which pass through points O_1 and O_2 .Next we put $\alpha=\pi/8$, $\varepsilon=\pi/24$ and point $C_1 \in \Pi'$ has the coordinates $(1, 1-tg(\alpha/2))$.

We choose the sphere Q_1 to be a sphere with the center C_1 and radius $r=tg(\alpha/2)/cos(\varepsilon/2)$ (the same letter Q_1 will denote those face of the polyhedra Φ that lies on this sphere). Spheres R'_1 , Q'_1 , R_1 and Q_2 arise as the result of rotation of the sphere Q_1 around the axis $O_2^{\perp}$ with the angles α , 2α , 3α, 4α . By analogy, spheres R_{12} , Q'_{12} , R'_{12} and Q_{12} arise as the result of the rotation of Q around the axis $O_1^{\perp}$ with the same angles (see the figure 3). It is easy to see that angles between the neighbouring spheres are equal to ε and centers of R_1 and Q_1 are lying on the axis l . So we have construct the necessary "covering" of the strip L'_2 .

Let J_1 be the inversion in the sphere Q_1 and σ_1 be a symmetry in the plane that passes through $O_2^{\perp}$ and the center of R'_1 , then we put $A_1=\sigma_1 \circ J_1$. Similarly, let I_1 be an inversion in the sphere R_1 , θ_1 be a symmetry in the plane that passes through $O_2^{\perp}$ and the center of Q'_1 , $B_1=\theta_1 \circ I_1$. It is easy to see that $A_1(Q_1)=Q'_1$, $B_1(R_1)=R'_1$, $A_1(Q_1 \cap R'_1)= R'_1 \cap Q'_1$ and so on.

Now we are going over to the consideration of the strip L'' . Let $\Lambda'' \subset \Pi''$ be a straight line orthogonal to l and passing through the point O. We shall consider l and Λ'' as coordinate axises on Π'' (see the figure 3). Let $O_3=(2, 1)$, $O_4=(1, 0)$ be points on the plane Π'', $Q_3^{\perp}$, $Q_4^{\perp}$ be straight lines passing through Q_3 , Q_4 orthogonally to Π''. Then the spheres R'_2 , Q'_2 , R_2 ,..., R_4 , Q_5 arise as the result of the rotation of Q_2 around $Q_3^{\perp}$ with angles α , 2α , 3α ,..., 11α , 12α . All these spheres are orthogonal to Π'' and have angles of intersection equal to ε . Finally, the spheres R'_5 , Q'_5 and R_5 arise as the result of the rotation of Q_5 around $O_4^{\perp}$ with angles α , 2α , 3α . The center of the sphere R_5 lies on the line l .

3 оборот

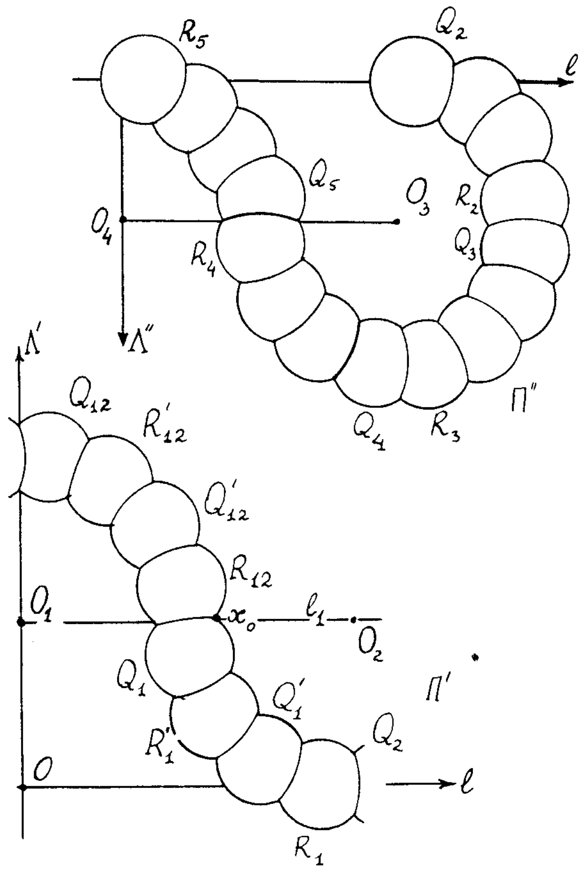

Figure 3

The system of spheres Q_6 , R'_6 ,..., Q'_{11} , R_{11} is obtained of the family Q_{12} , R'_{12} ... Q'_5 , R_5 due to the symmetry in the line Λ'. An angle between any two neighbouring spheres is equal to ε . The intersection $ext(Q_1) \cap ... \cap ext\ (R_{12})$ is precisely those polyhedra Φ that we were looking for.

Really, the sum of it's dihedral angles is equal to $48\varepsilon = 2\pi$. The generators A_2 , B_2 ,..., A_{12} , B_{12} may be chosen in the same way as A_1 and B_1 : $A_i = \sigma_i \circ J_i$, $B_i = \theta_i \circ I_i$ where I_i and J_i are inversions in Q_i and R_i , the transformations σ_i and θ_i are symmetries in the euclidean perpendicular bisectors of the lines joining centers of Q_i , Q'_i and R_i ,R'_i correspondingly.

Let $x_0 \in Q_1 \cap l_1$ be the points closest to O_2 . It is easy to see that $[A_{12}, B_{12}] \circ ... \circ [A_1, B_1](x_0) = x_0$ and the curve η on $\partial\Phi$ (which is constructed accordingly to the item 2.3) has a linking number 1 with respect to the axis λ of the solid torus $S^3 \setminus \Phi$. So the group $H=H(12, 1)$ have been constructed.

2.6. Here we shall demonstrate that for any g and e (such that $1 \le |e| \le (g-1)/11$) there exists a group $H(g,e)$ uniformizating the manifold $S(g,e)$. Let H be a subgroup in the group $H(12, 1)$ of index j . Then we have $H=H(11j+1, j)$ by the Lemma 3.5 of [Sc] and the Riemann-Hurwitz formula. Therefore, for any given $e=j > 0$ we have constructed a group $H(g, e)$ with $g=11e+1$ or equivalently $e=(g-1)/11$. So to complete the proof of the theorem 2.1 we only have to construct the group $H(g,e)$ for $g = 11e + k$ for any $k > 0$.

Let's denote by Π the euclidean plane that pass through the line l_1 orthogonally to Π' and let B be those component of $\mathbb{R}^3 \setminus \Pi$ which contains the sphere Q_{12} , next we put $\bar{\Pi} = \Pi \cup (\infty)$ and $\bar{B} = B \cup (\infty)$. The hyperbolic transformation $[A_{10}, B_{10}] \circ ... \circ [A_1, B_1]$ we denote by h . The fixed point set of h is the intersection of the straight line l_1 and the circle $C \subset \Pi'$ with center O_1 and radius $1-r^2 sin^2(\varepsilon/2)$. It is easy to see that the sphere $\bar{\Pi}$ is precisely invariant in the group $H(12, 1)$ with respect to $\langle h \rangle$.

We can choose a subgroup H of any prescribed index e in $H(12, 1)$ such that $H \supset \langle A_{11}, B_{11}, A_{12}, B_{12} \rangle$. So the group H is the result of Maskit combination of groups $\langle A_{11}, B_{11}, A_{12}, B_{12} \rangle$ and $G(11e-1, e)$.

To construct a group $H(11e+1+k, e)$ for any $k > 0$ it is sufficient to replace the subgroup $\langle A_{11}, B_{11}, A_{12}, B_{12} \rangle$ by a free fuchsian group $F_{2(2+k)}$ of rank $2(2+k)$ such that

(1) the circle C is invariant under the action of this group,

(2) $\langle h\rangle \subset [F_{2(2+k)}, F_{2(2+k)}]$,

(3) the ball $\mathbb{R}^3\setminus B$ is precisely invariant in $F_{2(2+k)}$ with respect to $\langle h\rangle$.

The groups $F_{(2+k)k}$ and $G(11e-1, e)$ satisfy to conditions of Maskit combination theorem (Th. 1.1) with amalgamated subgroup $\langle h\rangle$. It is not hard to see that the group $\langle F_{2(2+k)}, G(11e-1, e)\rangle$ uniformizate manifold $S(11e-1+2+k, e)= S(11e+1+k, e)$ which is glued of $\mathbb{S}^1\times\Sigma_{2+k}$ and $\mathbb{S}^1\times\Sigma_{11e-1}$. For more details see § 3, items 3.2 - 3.4 .

So the theorem 2.1 is proved. QED.

2.7. It should be noticed that, for the extension $\tilde{H}(g, e)$ of the group $H(g, e)$ into the space $\mathbb{H}^4$, the manifold $\mathbb{H}^4/\tilde{H}(g,e)$ is homeomorphic to the plane bundle over S_g with euler number e . This follows from the next considerations. Let's choose a fundamental polyhedron Φ for $H(g,e)$ such that $\partial\Phi$ consists of annuli lying on euclidean spheres (c.f. the item 2.3) . The convex hull $\tilde{\Phi}$ of Φ in $\mathbb{H}^4$ is a fundamental polyhedron for the action of $\tilde{H}(g,e)$ in $\mathbb{H}^4$. The polyhedron $\tilde{\Phi}$ admits a natural $\mathbb{R}^2$-fibration which is invariant under action of $\tilde{H}(g, e)$. This fibration projects into fibration of $M(H(g, e)) = \mathbb{H}^4\cup R(H(g,e))/\tilde{H}(g,e)$ which restriction to $\partial M(H(g,e))$ is a circle fibration over $S(g,e)$.

So for any g and e such that $0< e \le (g-1)/11$ the fibered space $\tilde{S}(g, e)$ with the base S_g and Euler number e admits a complete hyperbolic structure (c.f. [G L T], [Ku 3]), however we wouldn't go into details.

2.8. Corollary 2.1 . Any Seifert fibered space with hyperbolic base is virtually conformally-flat (i.e. it has a finite-sheeted covering space which admits a FCS).

Proof. It is sufficient to consider only orientable Seifert manifolds M with a non-zero Euler number. There take place the short exact sequence

$$1 \longrightarrow \mathbb{Z} \longrightarrow \pi_1(M) \xrightarrow{\varphi} F \longrightarrow 1$$

where F is a discrete subgroup of $Isom(\mathbb{H}^2)$. Hence the group F contains a finite-index subgroup F_o isomorphic to $\pi_1(S_g)$ where $g\ge 12$. The group $G_o=\varphi^{-1}(F_o)$ has the presentation $\langle a_1, b_1, \dots, a_g, b_g, t : [a_i, t]=[b_j, t]=[a_1, b_1]\cdot\ldots\cdot[a_g, b_g]\cdot t^{-e}=1\rangle$, where $e\ne 0$. If we put $\tau=t^e$ then the subgroup $G_o'=\langle a_1, b_1, \dots, a_g, b_g, \tau : [a_1, b_1]\cdot\ldots\cdot[a_g, b_g]=\tau\rangle$ has a

finite index in $\pi_1(M)$ and defines a covering $M_o\longrightarrow M$ such that M_o admits a flat conformal structure (due to the theorem 2.1). QED.

2.9. Application to quasiconformal groups.

We remind that the group Γ of homeomorphisms acting on $\mathbb{S}^n$ is said to be (*uniformly*) *quasiconformal* if there exists a number $K<\infty$ such that each element $\gamma\in\Gamma$ is K-quasiconformal map (see [Ma]). In papers [Tu], [F S], [Ma] examples were constructed which disproved the conjecture that any quasiconformal group is conjugate to conformal one (via a homeomorphism). Articles [F S] and [Ma] provide discrete examples of such groups.

Below we show how to construct an analogous example of action of the group $\mathbb{Z}_n\times\pi_1(S_g)$ on $\mathbb{S}^3$. Let $H=H(12, 1)$ be the group has been constructed in the theorem 2.1. Let $\varphi : M(H)\longrightarrow M(H)$ be an order n diffeomorphism isotopic to identity (it exists due to $\mathbb{S}^1$ -action on $M(H)= S(12, 1)$) . This diffeo- morphism admits a lift $\tilde{\varphi}$: $R(H)\longrightarrow R(H)$ of order n . The restriction of $\tilde{\varphi}$ to the compact fundamental domain Φ of the group H is smooth and, hence, is K-quasiconformal map. For any $h\in H$ we have $h\circ\tilde{\varphi}= \tilde{\varphi}\circ h$, therefore the map $\tilde{\varphi}$ is K-quasiconformal itself. It is sufficient to repeat considerations of B.Maskit [Mk 2] to prove that $\tilde{\varphi}$ admits a homeomorphic continuation f to the whole sphere $\mathbb{S}^3$. Furthermore, considerations of L.Bers [Bs, Lemma 2] imply that the map f is K-quasiconformal. The group $\Gamma=\langle H, f\rangle$ is isomorphic to $\mathbb{Z}_n\times\pi_1(S_g)$ and defines a K-quasiconformal action on $\mathbb{S}^3$. We may apply above reasoning to construct $\mathbb{S}^1$-action on $\mathbb{S}^3$ which is H-equivariant and $L(H)$ is fixed-point set for this $\mathbb{S}^1$-action. Hence $L(H)$ is a tame unknotted topological circle in $\mathbb{S}^3$ and the homeomorphism f is topologically conjugate to some euclidean rotation. Any element of $\Gamma\setminus\langle f\rangle$ is "hyperbolic" (in sense of [G M]) and hence is topologically conjugate to some mobius transformation. Consequently any element of Γ is conformal up to conjugation, however the following statement holds

Corollary 2.2. The group Γ is not topologically conjugate to any subgroup of $\mathfrak{M}_3$.

Proof. Suppose that such a conjugation g exists, then under the action of the group $G=g\cdot\Gamma\cdot g^{-1}\subset\mathfrak{M}_3$ the euclidean circle $Fix(g\cdot f\cdot g^{-1})$ is invariant. But the manifold $M(g\cdot H\cdot g^{-1})$ is homeomorphic to $M(\Gamma)$ and has a non-zero Euler number. However this contradicts with existence of $\mathbb{H}^2\times\mathbb{R}$ structure on the manifold $M(g\cdot H\cdot g^{-1})$. QED.

For another interesting example of quasiconformal group see the item 6.5 .

COROLLARY 2.3. Let M be a closed Seifert manifold with a hyperbolic base. Let Γ be a Kleinian group such that $M=\Omega/\Gamma$, where Ω is an invariant component of Γ, and Γ acts freely on Ω. Then $\Omega=R(\Gamma)$ and the limit set $L(\Gamma)=S^3\setminus\Omega$ is a tame unknotted topological circle.

PROOF. Proof follows from the proof of the Corollary 2.2 (see also [Ka 1] for the case of zero Euler number). QED.

This Corollary is answer to some question of Kuiper [Ku 3].

2.10. Flat conformal structures on manifolds $S(g, e)$, $e\neq 0$ provide us another interesting example of pathology - disconnectedness of the moduli space $C(M)$ of all FCS on the manifold $M=S(g, e)$. Definitions of topology on this space may be found in [L], [G C E]. Let $\nu(e,g)$ be equal to $[g-1/11e]$ - the greatest integer $\le (g-1)/11$.

THEOREM 2.2. Let M be a manifold $S(g, e)$. Then the space $C(M)$ consists of at least $\nu(e, g)$ connected components.

We drop here a detailed proof of this theorem since it would lead us far away from main subject of this paper. We only indicate below $\nu(g, e)$ structures on M which lie in different components of $C(M)$.

Consider the set of manifolds $\mathfrak{S}=\{S(n\cdot e, g) : 0 < n \le \nu(e, g)\}$. All manifolds of $\mathfrak{S}$ admit uniformizable FCS K_n, due to the theorem 2.1 . It is easy to see that there exists a covering $p : S(g, e) \to S(g, e\cdot n)$ and hence the structures K_n lift to structures $\tilde{K}_n$ on the manifold $S(g, e)$. Then the holonomy groups of the structures $\tilde{K}_n$ are groups $H(g, n\cdot e)$. The groups $H(g, m\cdot e)$ and $H(g, n\cdot e)$ can not be deformed one to other in the space of all pseudofuchsian groups (if $n\neq m$). Therefore, results of [Ka 1], [Ka 2] imply that the structures $\tilde{K}_n$ and $\tilde{K}_m$ lie in different components of $C(M)$.

§ 3. Some auxiliary results and constructions

In this paragraph we shall construct some Kleinian groups playing role of "building blocks" in proof of the theorem 5.1.

3.1. Hyperbolic Dehn-Thurston surgery.

Let N be a compact hyperbolic manifold with toroidal boundary $\partial N = T_1 \cup ... \cup T_k$, torsion-free discrete group $\Gamma \subset PSL(2, \mathbb{C})$ uniformizates $int N$ and $\rho_0 : \pi_1(N) \to \Gamma \subset PSL(2, \mathbb{C})$ is a natural representation. Let $Def(\Gamma) = Hom(\pi_1(N), PSL(2, \mathbb{C}))/ad\ PSL(2, \mathbb{C})$ denotes the deformation space of the group Γ . The following result is due to W.Thurston

[T, Ch. 5, Th. 5.6] (see also [C S] , [N Z],[Ka 4]).

THEOREM 3.1. The space $Def(\Gamma)$ near the point $[\rho_0]$ is a smooth complex manifold of complex dimension k . Furthermore, for any collection of prime elements $u_i \in \pi_1(T_i) \subset \pi_1(N)$, $i=1, ..., k$ one can find a number $\varepsilon > 0$ such that for any $\tau_i \in [2-\varepsilon, 2]$, $\tau=(\tau_1, ..., \tau_k)$ there exists a representation $\rho_\tau : \pi_1(N) \to PSL(2,\mathbb{C})$ with property: $|tr\ \rho_\tau(u_i)|=\tau_i$ and ρ_τ depends continuously on τ, $\rho_0=\rho_{(2,..,2)}$.

Let $\tau_i = t_n = 2\cdot cos\ |2\pi/n|$, where a sufficiently large positive integer n is one and the same for all i ; $\Gamma(n)=\rho_\tau(\Gamma)$. Then the group $\Gamma(n)$ is discrete (since some finite-index subgroup of it is a holonomy group of closed hyperbolic manifold). Let $\ell(i, n)$ be a common axis of the elliptic element $u_i(n)=\rho_\tau(u_i)$ and the loxodromic one $\upsilon_i(n)=\rho_\tau(\upsilon_i)$, where $\pi_1(T_i)=\langle u_i\rangle \oplus \langle \upsilon_i\rangle$. Also we denote by $\mathcal{K}(\ell(i, n), \theta)=\{x\in\mathbb{H}^3 : ch\ d(x,\ell(i, n)) \le 1/cos\ \theta\}$ the cone with the axis $\ell(i, n)$ and the vertex angle 2θ . More generally, let x , y be different points of S^3 and ℓ be an arc of a circle which connects x and y . There exists a mobius transformation γ such that $\gamma(x)=0$ and $\gamma(y)=\infty$. Let $\mathcal{K}_*(\ell, \theta)$ be the euclidean cone with the axis $\gamma(\ell)$. and the vertex angle 2θ . Then the set $\mathcal{K}(\ell, \theta)=\gamma^{-1}(\mathcal{K}_*(\ell, \theta))$ will be called a cone with the axis ℓ and angle θ . The boundary of the cone $\mathcal{K}(\ell, \theta)$ will be denoted by $K(\ell, \theta)$.

LEMMA 3.1. For any real $\theta\in(0, \pi/2)$ there exists a number n_0 with the property: if $\gamma_n\in\Gamma(n)$ is such that $\gamma_n(\mathcal{K}(\ell(n, j), \theta)) \cap \mathcal{K}(\ell(n, i)) \neq \emptyset$ then $i=j$ and $\gamma_n\in\langle u_i(n), \upsilon_i(n)\rangle$ = stabilizer of the cone $\mathcal{K}(\ell(n, i), \theta)$ in the group $\Gamma(n)$.

PROOF. Suppose that the statement of the Lemma is not true. We can conjugate the group $\Gamma(n)$ in $PSL(2, \mathbb{C})$ to obtain a group $\Gamma^*(n)$ where the element $\upsilon_i^*(n)$ has the fixed point set $\{0, \infty\}$ (an element of the group $\Gamma(n)$ conjugated to $\gamma\in\Gamma(n)$ will be denoted by γ^*). Evidently we have : $lim\ u_i^*(n) = lim\ \upsilon_i^*(n) = 1 \in PSL(2, \mathbb{C})$. Since we have $\gamma_n\mathcal{K}(\ell(n, j), \theta) \cap \mathcal{K}(\ell(n, i),\theta) \neq \emptyset$ for infinitely many n- s then (up to a subsequence) we obtain : $d(\ell(i, n), \gamma_n\ell(j, n)) < C$ for some $C<\infty$ which is independent of n . Hence there exists a sequence $c_n^*\in\langle u_i^*(n)\rangle \oplus \langle \upsilon_i^*(n)\rangle$ such that for some point $q\in \ell^*(i, n)$ we have $d(c_n^*\gamma_n^*(\ell^*(j, n)), q) < C+1$. Therefore the sequence $c_n^*\gamma_n^*\ell^*(j, n)$ converges to some limit line and $lim\ e_n=1$, where $e_n = c_n^*\gamma_n^*u_j^*(n)(c_n^*\gamma_n^*)^{-1}$. So for sufficiently large n the group $\langle\upsilon_n^*, e_n\rangle$ is not elementary and $|tr\ \upsilon_n^* - 4| + |tr([\upsilon_n^*, e_n^*]) - 2| < 1$ that contradicts to the Jorgensen inequality [Be, Ch 5] since the group

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$\Gamma^*(n)$ is discrete. QED.

3.2. Relative Euler class.

Let Σ be a compact orientable surface with boundary $\partial\Sigma = \beta_1 \cup ... \cup \beta_r$, $M = \Sigma \times S^1$ is a trivial fiber bundle over Σ, $\sigma : \partial\Sigma \to M$ is a partial section of this bundle, $\sigma(\beta_i) = \sigma_i \subset \partial M$.

DEFINITION. The Euler class of M relatively σ is equal to the $(-o_1(\sigma))$, where $o_1(\sigma)$ is the first obstruction for continuation of σ to the section $\Sigma \to M$, $o_1(\sigma) \in H^2(\Sigma, \partial\Sigma; \pi_1(S^1)) \simeq \mathbb{Z}$ (the last isomorphism is determined by the choice of orientation on Σ and S^1). The corresponding integer number $e(M, \sigma)$ is called the Euler number of M relatively to σ.

It is not hard to see that if σ, σ' are sections $\partial\Sigma \to M$ and $e(M, \sigma) = e(M, \sigma')$ then there exists an automorphism f of the fiber bundle $\Sigma \times S^1 \to \Sigma$ such that $f \circ \sigma = \sigma'$. In what follows recently we shall denote $\sigma(\partial\Sigma)$ by σ also. More geometri- cally $e(M, \sigma)$ may be described in the following way. It is easy that $[\sigma_1] + ... [\sigma_1] = e(M, \sigma) \cdot [t]$, where $[\gamma]$ is the homology class of the loop γ, t is the fiber of the fiber bundle $\Sigma \times S^1 \to \Sigma$ (orientations of Σ and t are supposed to be fixed).

Let $p : \tilde{M} \to M$ be a standard n-sheeted covering over M (see the item 1.3), $\sigma : \partial\tilde{\Sigma} \to \tilde{M}$ be a lift of $\sigma : \partial\Sigma \to M$. It is easy that $e(\tilde{M}, \tilde{\sigma}_1 \cup ... \cup \tilde{\sigma}_r) = e(M, \sigma_1 \cup ... \cup \sigma_r)$. Let M_1, M_2 be a trivial circle bundles over Σ_1, Σ_2, $\sigma_i : \partial\Sigma_i \to M_i$ be sections, the manifold M is glued of M_1 and M_2 along some boundary components as follows. The fiber is glued to fiber (preserving the orientation) and a section is glued to section with change of orientation, $\delta \subset \partial M$ is the set of loops remained after the gluing. Then $e(M, \delta) = e(M, \sigma_1) + e(M, \sigma_2)$.

3.3. Relative Euler class of Kleinian groups.

Let $e=1$ and $G(10, 1)$ be the group have been constructed in the item 2.6 (other definitions may be found there also), $H = H(12, 1)$. The cylinder $\Pi \setminus Fix(h)$ projects onto incompressible torus T under the covering $q : R(H) \to S(12, 1)$. Let $\tilde{\sigma}$ be an open segment of the straight line ℓ_1 (see the item 2.5) bounded by fixed points of the hyperbolic element h. Next we put $\sigma = q(\tilde{\sigma})$. The torus T divides $M = S(12, 1)$ in two components $M_1 = S^1 \times \Sigma_2$ and $\mathfrak{N}(10) = S^1 \times \Sigma_{10}$ and it is easy to see that σ can be obtained as an image of a section $\partial\Sigma_{10} \to \mathfrak{N}(10)$. Evidently we have $e(M_1, \sigma) = 0$ and hence $e(\mathfrak{N}(10), \sigma) = 1$, since $e(M, \emptyset) = 1$. The preimage of $\mathfrak{N}(10)$ under projection q:

$R(H) \to \mathfrak{N}(10)$ is the complement in $R(H)$ of the orbit $G(10, 1) \cdot B$.

In this situation it is natural to call 1 be a relative Euler number with respect to the pair $(\bar{\Pi}, \tilde{\sigma})$. More generally, let G be a Kleinian group, $h_1, ..., h_m$ -collection of non-conjugated loxodromic elements of G such that: (1) there exist cones $\mathcal{K}_i = \mathcal{K}(\ell(h_i), \theta_i)$ which are precisely invariant in G with respect to $\langle h_i \rangle$, $i = 1, ..., m$, (2) orbits of $\mathcal{K}_i$ are pairwise disjoint, (3) the manifold $M^*(G) = (R(G) \setminus G \cdot \bigcup_{i=1}^{m} \mathcal{K}_i)/G$ is homeomorphic to $\Sigma \times S^1$, where Σ is a compact surface, (4) $h_1 \cdot ... \cdot h_m \in [G, G]$. Let $\tilde{\sigma}_i \subset K_i = \partial\mathcal{K}_i$ be an infinite arc which is invariant under $\langle h_i \rangle$, $i=1, ..., m$; $\tilde{\sigma} = \bigcup_{i=1}^{m} \tilde{\sigma}_i$, σ is projection of $\tilde{\sigma}$ to $M^*(G)$. The orientation on σ_i is given by choice of h_i -generators of their fundamental groups.

DEFINITION. The *reletive Euler number* of G with respect to $\tilde{\sigma}$ is the number $e(G, \tilde{\sigma}) = e(M^*(G), \sigma)$.

REMARK 1. We suppose that the orientation in S^3 is fixed.

REMARK 2. For the group $G = G(10, 1)$ we have: $m=1$, $K_1 = \bar{\Pi} \setminus Fix(h)$, $h_1 = h$, $\theta_1 = \pi/2$, $\ell(h_1) = C \cap B$ (definitions see in the item 2.6).

3.4. Some properties of Riemannian surfaces with boundary.

Consider a compact surface S (with hyperbolic ametric) which is "pants" [Ab, Ch. II], i.e. ∂S consists of 3 geodesic curves and $H^1(S, \partial S; \mathbb{Z}) = 0$. It is known that S is uniquely determined (up to isometry) by ℓ_1, ℓ_2, ℓ_3 -lengths of it's boundary curves α_1, α_2, α_3. Furthermore, for any $(\ell_1, \ell_2, \ell_3) \in (\mathbb{R}_+)^3$ there exist corresponding "pants" $S = S(\ell_1, \ell_2, \ell_3)$ [Ab, Ch. II, § 3].

DEFINITION. If $\alpha_1 \subset \partial S$ is a boundary curve then the w-*collar* of α_1 is the set $\{x \in S : dist(x, \alpha) \le w\}$. If a collar is homeomorphic to the annulus, then it is called to be *regular*.

Under $\ell_1 \to \infty$ the component $\alpha_1 \subset \partial S$ is degenerated to the puncture (which is infine distant of any finite point). Hence for any fixed $0 < \ell_2, \ell_3, w < \infty$ there exists $\lambda > 0$ such that for all $0 < \ell_1 < \lambda$ the curve $\alpha_1 \subset \partial S(\ell_1, \ell_2, \ell_3)$ has regular w-collar.

LEMMA 3.2. (a) For any finite w, $\ell \ge 0$ there exists an integer $g_0 = g_0(w, \ell)$ such that : there exists a compact surface Σ_g of genus g and one (geodesic) boundary curve which has the length ℓ and the regular w-collar.

(b) If $w=0$ then $g_0(w, \ell) = 1$.

PROOF. Firstable we are proving the assertion of (b). Let $S = S(\ell$

, ℓ , ℓ) be "pants", $\alpha = \alpha_1 \subset \partial S$. The necessary surface of genus 1 is obtained of S via gluing of it's boundary curves α_2 , α_3 . A surface of arbitrary genus can be obtained of S by consecutive gluing along α_2 and α_3 of $2g$ pants of the kind $S(\ell , \ell , \ell)$ and after that - pairwise gluing of $2g$ boundary curves. So the assertion (b) is proved.

Consider the general case: $w \geq 0$. By the remark preceding Lemma 3.2 , for some $n \in \mathbb{N}$ there exist the pants $S(\ell/n , \ell , \ell)$ such that $\alpha_1 \subset \partial S(\ell/n , \ell , \ell)$ has a regular w-collar. For this surface we construct a covering $S \rightarrow S(\ell/n , \ell , \ell)$ such that the loop α_1 is n times covered by a component $\alpha \subset \partial S$. Then a length of α is equal to ℓ and the loop α has a regular w-collar in S . Denote the genus of S by h and the number of it's boundary components - by m . Due to the assertion (b) of this Lemma we can glue to each component of $\partial S \setminus \alpha$ a surface of genus 1 and the unique (geodesic) boundary component of which has the same length as β . The obtained surface Σ_{g_0} has the genus $g = h+m-1$ and precisely one boundary curve α which possesses a regular w-collar.

If we exchange some glued surface of genus 1 by a surface of genus $(g-g_0+1)$ (due to (b)) then the constructed surface Σ_g will satisfy the assertion (a) of the Lemma. QED.

3.5. Deformations of Schottky-type groups.

Here we introduce some notations: $\Delta_R = \{z \in \mathbb{C} : |z| < R\}$, $O_R = \partial \Delta_R$ is the circle of the radius R , this circle is provided with the counterclockwise orientation. Let us suppose that the positive integers $r \geq 2$, m , s are given. We put $R = 10m+6$. Next we construct a Schottky-type group $H \subset \mathfrak{M}_3$ which has r free hyperbolic generators and s parabolic ones such that:

(1) The disk Δ_R is invariant under H .

(2) The hyperbolic generators h_i are conjugate in $\mathfrak{M}_3$ to element $h \in G(10, 1)$ (see the item 3.2).

(3) The Euclidean radii of the isometric spheres of h_i and parabolic generators c_j $(i \leq r$, $j \leq s)$ are equal to 1/8 .

(4) This spheres are pairwise disjoint except of $I(c_j)$, $I(c_j^{-1})$ which are tangent $(j=1 ,..., s)$.

(5) There are no isometric spheres of this generators between $I(h_i)$ and $I(h_i^{-1})$ $(i=1 ,..., r)$ (see the figure 4).

(6) The element $h_{r+1}^{-1} = c_s \cdot ... \cdot c_1 \cdot h_r \cdot ... \cdot h_1$ belongs to $[H,H]$.

(7) The Euclidean radius of the axes $A(h_i) \subset \mathbb{H}^2 = \Delta_R$ of the element h_i is less than 1/8 , $i=1 ,..., r$.

(8) The Euclidean distance ($dist$) between any two neighbouring (on O_R) isometric spheres (except of $I(c_s^{-1})$ and $I(h_1)$) is less than 1 (figure 4) .

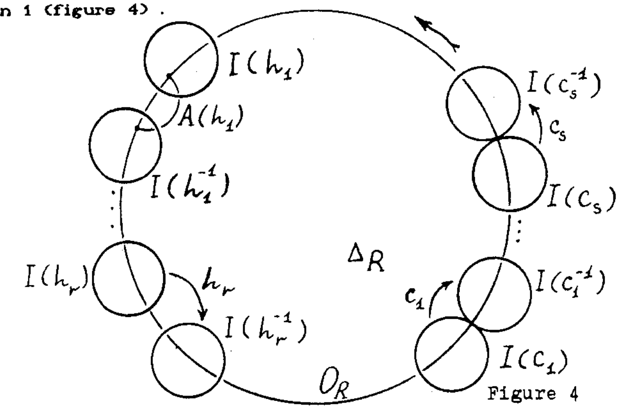

Figure 4

The group H may be easily constructed by means of the Klein's Combination theorem (see §1). So the domain

$$P = \bigcap_{i=1}^{r} \mathrm{Ext}(I(h_i) \cup I(h_i^{-1})) \cap \bigcap_{j=1}^{s} \mathrm{Ext}(I(c_j) \cup I(c_j^{-1}))$$

is a fundamental domain for action of H in S^3. It is easy to see that one of arcs $O_R \setminus Fix(h_{r+1})$ is precisely invariant under $\langle h_{r+1} \rangle$ in H , this arc will be denoted by $\ell(h_{r+1})$. More than, the axis $A(h_{r+1}) \subset \Delta_R$ of the element h_{r+1} and all strip in Δ_R between $A(h_{r+1})$ and $\ell(h_{r+1})$ are precisely invariant too.

Let $\mathcal{K}(h_{r+1}) = \mathcal{K}(\ell(h_{r+1}), \vartheta)$ be the cone with axis $\ell(h_{r+1})$ and so small vertex angle ϑ that (i) $\vartheta < \pi/2$, (ii) the euclidean distance $dist(\mathcal{K}(h_{r+1}), \ell(h_{r+1}))$ is less than 1/4 . The hyperbolic distance $dist(z, A(h_{r+1}))$ is equal to $w = arch(1/\sin(\vartheta))$ for any $z \in \partial\mathcal{K}(h_{r+1}) \cap \Delta_R$. We shall denote the hyperbolic length of $A(h_{r+1})/\langle h_{r+1} \rangle$ by ℓ . It is easy to see that the elements h_i have precisely invariant cones $\mathcal{K}(h_i)$ with the vertex angles $\pi/2$ and the axises lying in O_R .

We shall need the following result of A.Weil [W]. Let G be a Lie group and Γ is it's subgroup generated by elements γ_1 , γ_2 , ..., γ_n such that $\gamma_1 \cdot ... \cdot \gamma_n = 1$.

THEOREM 3.2. Let us suppose that $H^0(\Gamma, Ad) = 0$. Let $W: G^n \rightarrow G$ be the map $W: (g_1 ,..., g_n) \longmapsto g_1 \cdot ... \cdot g_n$. Then the restriction of W to $ad(G)(\gamma_1) \times ... \times ad(G)(\gamma_n)$ is a submersion near the point $(\gamma_1 ,..., \gamma_n)$.

Remark. For a semisimple Lie group G the condition $H^0(\Gamma, Ad)=0$ is equivalent to the next one: the centralizer of Γ in G is finite.

Corollary 3.2. Let G be the Lorentz group SO(N,1) with a metric $|\ ,\ |$, $\rho_0:H\to G$ be a representation such that $\rho_0(H)$ has the finite centralizer in G. Then for any sufficiently small ε, for any collection $(g_1, ..., g_s)\in G$ which is ε-distant of $(\rho_0(c_1), ..., \rho_0(c_s))$ there exists a representation $\rho_\varepsilon:H\to G$ such that :

(1) $\rho_\varepsilon(c_i)=g_i$, $i=1,..., s$;

(2) $\rho_\varepsilon(h_{r+1})=\rho_0(h_{r+1})$, $|\rho_\varepsilon(h_j), \rho_0(h_j)|<\delta(\varepsilon)$ and $\rho_\varepsilon(h_j)$ is conjugate with $\rho_0(h_j)$ in G, $j=1, ..., r$; (3) $\lim_{\varepsilon\to 0}\delta(\varepsilon)=0$.

This result will be used twice. Firstable, let G= PSL(2, ℂ), $\rho_0:H\to G$ be the natural inclusion. Then (due to the Corollary) there exists a representation $\rho:H\to G$ such that:

(1) $\rho(c_i)=c_i$, $\rho(h_{r+1})=h_{r+1}$, $i=1, ..., s$;

(2) the elements h_j and $\rho(h_j)$ are conjugate in G ;

(3) the group $\rho(H)$ has no an invariant euclidean circle;

(4) $dist(I(\rho h_j^{\pm 1}), O_R)<1/4$;

(5) the precisely invariant cones (listed before) after small angle-preserving perturbation remain to be precisely invariant in $\rho(H)$ and $dist(\mathcal{K}(\rho h_j), O_R)<1/8$, $j=1, ..., r$.

The group ρH will be denoted later by H (and elements $\rho(h_i)$- by h_i), we shall denote the isometric fundamental polyhedron of H by $\mathcal{P}$ since the initial group (that kept Δ_R invariant) is unnecessary in forthcoming considerations. The domain $\mathcal{P}$ is bounded by isometric spheres $I(h_j^{\pm 1})$ and $I(c_i^{\pm 1})$, $i\le s$, $j\le r$.

Evidently the centralizer of H in $\mathfrak{M}_3$=G is trivial and we can repeat the application of the Corollary 3.2. Let a positive ε_1 be so small that:

if $|h_j(\varepsilon), h_j|<\varepsilon_1$, $|c_i(\varepsilon), c_i|<\varepsilon_1$, then $dist(I(c_i^{\pm 1}(\varepsilon)), I(c_i^{\pm 1}))$ and $dist(I(h_j^{\pm 1}(\varepsilon)), I(h_j^{\pm 1}))$ are less than δ_1= $1/8\cdot min\{radius\ I(c_i), radius\ I(h_j), d(\mathcal{D}, \mathcal{D}')$ for all spheres $\mathcal{D}$, $\mathcal{D}'\subset\partial\mathcal{P}$ which are mutually disjoint, $j\le s$, $i\le r$ }. More than, we shall suppose that the distance between the points of $Fix(h_j(\varepsilon))$ is less than 1/4 and $dist(Fix(h_j(\varepsilon)), O_R)\le 1/4$, $i=1, ..., r$. Here $d(\ ,\ )$ and $dist(\ ,\ )$ are euclidean distances.

We shall consider only those elements $c_i(\varepsilon)$ for which $I(c_i(\varepsilon))\cap I(c_i^{\pm 1}(\varepsilon))=\emptyset$ ($c_i(\varepsilon)$ is loxodromic) and $c_i(\varepsilon)$ admits an invariant circle $L_i(\varepsilon)$ converging to O_R under $\varepsilon\to 0$. We choose a smallest arc $\ell_i(\varepsilon)$ among $O_R\setminus Fix(c_i(\varepsilon))$ and put $\vec{c}_i(\varepsilon)$ be the pair $(c_i(\varepsilon), \ell_i(\varepsilon))$.

Such elements $c_i(\varepsilon)$ are called admissible.

So due to the Corollary 3.2 there exists $\varepsilon_0<\varepsilon_1$ such that for all $\varepsilon<\varepsilon_0$ and admissible elements $c_i(\varepsilon)$ with the property $dist(\mathcal{K}(\ell_i(\varepsilon), 3\pi/4), O_R)<1/2$ a representation $\rho_\varepsilon:H\to\mathfrak{M}_3$ may be found such that:

(a) $\rho_\varepsilon(h_{r+1})=h_{r+1}$, (b) $\rho_\varepsilon(c_i)=c_i(\varepsilon)$, $i=1 ..., s$; (c) $|h_j, h_j(\varepsilon)=\rho_\varepsilon(h_j)|<\varepsilon_1$, $j=1 ..., r$.

Definition. Deformations ρ_ε satisfying all listed properties will be called *admissible*.

As the result of admissible deformation we obtain the family $H_\varepsilon=\rho_\varepsilon(H)$, $0<\varepsilon<\varepsilon_2$, of the rank $r+s$ Schottky groups that have fundamental polyhedrons $\mathcal{P}(\varepsilon)$ bounded by the isometric spheres of $c_i^{\pm 1}(\varepsilon)$, $h_j^{\pm 1}(\varepsilon)$, $1\le i\le s$, $1\le j\le r$. The domain $\mathcal{P}(\varepsilon)$ has the following properties:

(1) $dist(\partial\mathcal{P}(\varepsilon), O_R)<1/2$, (2) the cones $\mathcal{K}(\ell(h_j(\varepsilon), \pi/2)$ are precisely invariant under $\langle h_j(\varepsilon)\rangle\subset H(\varepsilon)$, where $\ell(h_j(\varepsilon))$ are euclidean segments joining $Fix(h_j(\varepsilon))$, (3) the same is true for cones $\mathcal{K}(h_{r+1})$ and $\mathcal{K}(c_i(\varepsilon))$, $i=1 ..., s$, (4) for all mentioned cones $\mathcal{K}$ we have $dist(\mathcal{K}, O_R)<1/2$ and $\mathcal{K}\cap\mathcal{P}(\varepsilon)$ is a fundamental domain for action in $\mathcal{K}$ of it s stabilizer. In what follows we shall denote the element h_{r+1} by $h_{r+1}(\varepsilon)$.

The manifold $H^*(H_\varepsilon)=(R(H_\varepsilon)\setminus H_\varepsilon\cdot(\bigsqcup_{i=1}^{s}\mathcal{K}(c_i(\varepsilon))\cup\bigsqcup_{j=1}^{r+1}\mathcal{K}(h_j(\varepsilon))))/H_\varepsilon$ is homeomorphic to $S^1\times\Sigma$, where Σ is a compact surface with $r+s+1$ boundary curves and zero genus.

3.6. "Shortest" arcs on boundaries of invariant cones.

In the complement to pair of distinct points p_1, $p_2\in S^3$ we introduce the standard conformally-euclidean metric, which is invariant under action of stabilizer of $\{p_1, p_2\}$ in S^3 and has a scalar curvature 1. This metric we restrict to any cone $K=\partial\mathcal{K}$ with vertexes $\{p_1, p_2\}$.

Let g be a mobius transformation preserving $\mathcal{K}$ such that the argument of the complex coefficient $A(g)$ is not a negative number (see § 1). Let $x\in K\setminus\{p_1, p_2\}$ be any point and μ be a shortest geodesic segment joining x and $g(x)$.

Definition. The infinite arc $\nu=\bigsqcup_{n\in\mathbb{Z}}g^n(\mu)$ is called *shortest directed arc corresponding to* (K, g). The orientation on ν is given by action of g.

The shortest directed arcs corresponding to $(K(h_j(\varepsilon)), h_j(\varepsilon))$

and $(K(c_i(\varepsilon))$, $c_i(\varepsilon))$ will be denoted by $\tilde{\gamma}_j(\varepsilon)$ and $\tilde{\beta}_i(\varepsilon)$ respectively. Because $A(c_i(\varepsilon))\to 1$ under $\varepsilon\to 0$ then $\tilde{\beta}_i(\varepsilon)$ is defined correctly and depends continuously on ε. The same is true for $\tilde{\gamma}_j(\varepsilon)$ since $A(h_j(\varepsilon))>0$.

3.7. Some torus constructions.

Let $O(P, r)$ be the circle with center P and radius r lying in the plane $\pi\subset\mathbb{R}^3$. Let $\ell\subset\pi$ be a straight line such that $dist(P, \ell)=r+R$, where $r\cdot R>0$. We shall denote by $T(R, r)$ the torus obtained from $O(P, r)$ by rotation around the axis ℓ (see the figure 5). Let O_R be the circle of the radius $R=10m+6$ (see item 3.5), line ℓ passes through the point O orthogonally to the disc Δ_R . Let Q be any point of O_R , $D(Q, 0.5)$ is the disc with center Q and radius 0.5, that is orthogonal to O_R .

The solid torus $\mathfrak{T}(m)$ arises of $D(Q, 0.5)$ via rotation around the axis ℓ . Then $S^3\setminus\mathcal{P}(\varepsilon)$ and orbits of cones $\mathcal{K}(h_i(\varepsilon))$, $\mathcal{K}(c_j(\varepsilon))$ under $H(\varepsilon)$ lay in $\mathfrak{T}(m)$. More than, the circle O_R is so large that it is possible to arrange mutually disjoint balls $B(P_i, 8)$ with centers $P_i\in O_{R+2}$ and radii 8 , $j=1,\dots,m$.

Let π_j be a plane that passes through ℓ and point P_j ; line $\ell_j\subset\pi_j$ is parallel to ℓ and $dist(P_j, \ell_j)=2$. Let us denote by $T'(1, 1)$ the torus with rotation axis ℓ_j , $\ell_j^{\perp}$ is the perpendicular from P_j to the line ℓ , $T(1, 1)$ is the torus, which is obtained of $T'(1, 1)$ by euclidean rotation around the axis $\ell_j^{\perp}$ to the angle $\pi/2$. It is easy to compute that $int T(1,1)\cap\mathfrak{T}(m)=\emptyset$ and these solid tori form a link in S^3 with index 1 (see the figure 6) and $T(1, 1)\subset int\, B(P_j, 8)$.

On other hand, consider the image of $T'(1, 1)$ under inversion in the unit sphere $S(P_j, 1)$ (that tangents the $T'(1, 1)$ along the large circle). Let $T_*(1, 1)$ be the image of resulting torus under homothety with center P_j and coefficient 7.5 . An easy calculation shows that $T_*(1, 1)\subset B(P_j, 8)$, $\mathfrak{T}(m)\cap int\, T_*(1, 1)=\emptyset$ and these solid torii form a link in S^3 with index 1 (see the figure 7)

Finally we introduce the following notations: the clockwise-directed loop $\Delta_R\cap\partial\mathfrak{T}(m)$ will be denoted by $\tilde{\delta}$. The directed loop $\tilde{\theta}=\partial D(Q, 0.5)$ is oriented so that the pair $(\tilde{\delta}, \tilde{\theta})$ provides $\partial\mathfrak{T}(m)$ with the orientation induced from $ext(\mathfrak{T}(m))$ (see the figure 8) .

3.8. Hyperbolic Dehn-Thurston surgery and shortest arcs.

Let N and Γ be hyperbolic manifold and discrete group of the item 3.1 . Let B_i be open horoballs in $\mathbb{H}^3$ which are precisely

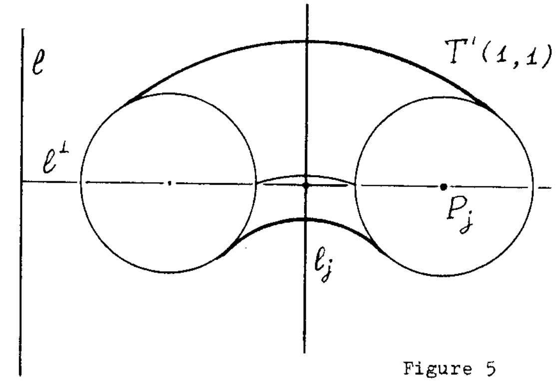

Figure 5

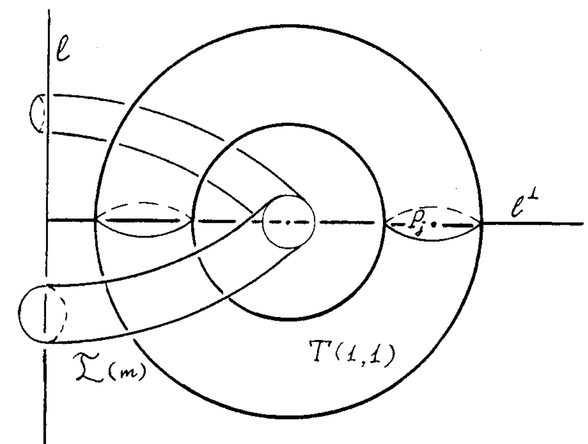

Figure 6

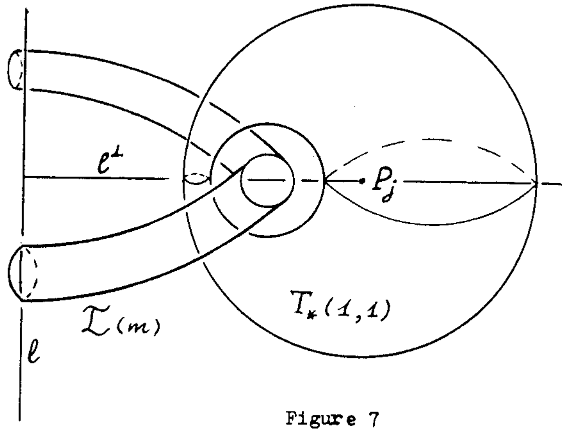

Figure 7

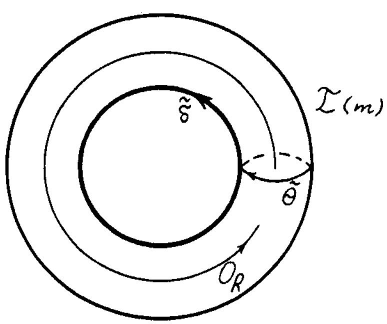

Figure 8

invariant under $\langle u_i\rangle \oplus \langle v_i\rangle \subset \Gamma$ (see the item 3.1), then we can assume that N is homeomorphic to $(\mathbb{H}^3_* \equiv \mathbb{H}^3 \setminus \Gamma(B_1 \cup \dots \cup B_k))/\Gamma$. We shall denote by π_* the projection from $\mathbb{H}^3_*$ to N.

Suppose that $B_1 = \{(x_1, x_2, x_3): x_3 > 1,\ x_1 + ix_2 \in \mathbb{C}\}$, $u_1: z \to z + 1$, $v_1: z \to z + w_1$, where $-1/2 < \mathrm{Re}(w_1) \le 1/2$. Let $\rho_T: \Gamma \to \Gamma(n)$ be a small deformation of Γ (see the item 3.1) such that $\rho_T(u_1) = u_1(n): z \to e^{2\pi i/n} z + \lambda_n$. There exist development maps $d_n: N \to \mathbb{H}^3$ such that $\lim_{n\to\infty} d_n = d_0 \equiv (\pi_*)^{-1}$, $(d_n)_* = \rho_{T(n)}$ are holonomy representations of corresponding incomplete hyperbolic structures. The development maps may be chosen so that some component of $d_n(T_i)$ is the cone $K(i, n) \equiv K(\ell(i, n), \theta_n)$, $\lim_{n\to\infty} \theta_n = \pi/2$, $i=1, \dots, n$. Let V be a loop on T_1 representing element v_1 of the group Γ, $\bar{V}(n)$ be the component of $d_n(V)$ which joints the points x and $v_1(n)(x) \in K(1, n)$.

LEMMA 3.3. Let $\tilde{v}(n)$ be the shortest arc on $K(1, n)$ which joint the points x, $v_1(n)(x) \in K(1, n)$ (see the item 3.6). Then for all but finite $n \in \mathbb{N}$ the arcs $\tilde{v}(n)$ and $\bar{V}(n)$ are homotopic on $K(1, n)$ (rel $\{x, v_1(n)(x)\}$).

PROOF. Without a loss of generality we may suppose that limit of $\bar{V}(n)$ is the segment $[(0,0,1), (\mathrm{Re}\ w_1, \mathrm{Im}\ w_1, 1)]$, $x = (0, 0, 1)$. Consider the line $\ell(1, n)$ as the axis of cylindrical coordinates in $\mathbb{H}^3 \setminus \ell(1, n)$. Then for all but finite n the variation of angle (of these coordinates) along arc $\bar{V}(n)$ is less than π. Now, the assertion of Lemma follows from direct calculations in cylindrical coordinates. QED

COROLLARY 3.3. Let $\mu(n)$ be a shortest infinite arc corresponding to $(K(\ell(1, n), \pi/4), v_1(n))$; let Γ_0 be a subgroup of Γ constructed for a prime n due to the theorem 1.3, $\Gamma_0(n) \equiv \rho_{T(n)}(\Gamma_0)$. Then (for all but finite primes $n \in \mathbb{N}$) the projection of $\tilde{\mu}(n)$ to the manifold $M(n) \equiv (\mathbb{H}^3 \setminus \Gamma_0(n) \cdot \ell(1, n))/\Gamma_0(n)$ is homotopic to a component of the lift of the loop $V(n)$ via a covering $M(n) \to int(N)$.

§4. Uniformization of Seifert manifolds, II.

4.1. In this paragraph we shall construct the groups $\mathfrak{G}$ which uniformizate Seifert components of Haken manifolds. The group $\mathfrak{G}$ arises as Maskit combination of two types of Kleinian groups: $G(10, 1)$ which has the Euler number 1 (see the item 3.3) and subgroups of $H(\varepsilon)$ which have zero Euler number (see items 3.5, 3.6).

4.2. THEOREM 4.1. Let $e \in \mathbb{Z}$, g, m , $s/2 \in \mathbb{N}$ be numbers such that $2g+m-|e|>0$, $(\vec{V}_j(p))$ be a sequence of directed loxodromic transformations indexed by the system of all primes $p \in \mathbb{N}$, $1 \le j \le s$. Let us suppose that $\lim_{p \to \infty} \vec{V}_j(p)=1$. Then for all but finite p there exists a Kleinian group $\mathfrak{G}=\mathfrak{G}(e, m, s, p) \subset \mathfrak{M}_3$ such as:

(1) The group $\mathfrak{G}$ contains s directed loxodromic elements $(\vec{Y}_j)^p$ possessing mutually disjoint precisely invariant cones $\mathcal{K}_j$ with vertex angles $3\pi/4$ and the same axises as $\vec{Y}_j$, $1 \le j \le s$.

(2) The elements $\vec{V}_j(p)$ and $\vec{Y}_j$ are conjugated in $\mathfrak{M}_3$, $1 \le j \le s$.

(3) The group $\mathfrak{G}$ possesses a fundamental set $\mathfrak{F}$ which contains the solid torus $S^3 \setminus int(\mathfrak{T}(m))$. Furthermore, $\mathfrak{F} \cap \mathcal{K}_j$ is a fundamental domain for action of $\langle \vec{Y}_j \rangle$ on $\mathcal{K}_j$, $\mathcal{K}_j \cap \mathfrak{T}(m) = \emptyset$.

(4) Let $R^*(\mathfrak{G})$ be the domain $R(\mathfrak{G}) \setminus \mathfrak{G}(S^3 \setminus \mathfrak{T}(m) \cup \dot{\bigcup}_{j=1}^{s} \mathcal{K}_j)$. Then the manifold $M(\mathfrak{G})= R(\mathfrak{G})/\mathfrak{G}$ is homeomorphic to $S^1 \times \mathfrak{R}$, where $\mathfrak{R}$ is a compact surface with $m+1$ boundary curves and genus

(*) $\tilde{g}= (p-1)g +(p-1)(m+ s)/2 -p +1$.

(5) Let $\tilde{\delta}$ be the directed loop from item 3.7, $\tilde{\beta}_j \subset K_j = \partial \mathcal{K}_j$ be the shortest directed arcs which correspond to (K_j , Y_j). Then the relative Euler number $e(M^*(\mathfrak{G}), \beta_1 \cup \ldots \cup \beta_s \cup \delta)$ is equal to e , where β_j , δ are projections of the corresponding loops $\tilde{\beta}_j$, $\tilde{\delta}$.

REMARK. The meaning of the condition (*) will be explained in the item 5.5.

4.3. PROOF **of the theorem 4.1.**

Denote the number $max(2, |e|)$ by r. Let's choose so large prime number p_o that:

(**) $\quad \xi(p_o)=p_o(g+m/2-r) -m/2 -9r +1 \ge g_o(w, I)$.

Here $g_o(\, , \,)$ is the function has been constructed in the Lemma 3.2, $I=\ell$ is the length of the hyperbolic displacement h_{r+1} (item 3.5) and w is the size of the collar computed due to the item 3.5, p. 22, accordingly to parameters (r, s, m) .

4.4. Case 1 : s is positive.

Let S be a riemannian surface of genus o with s punctures and $r+1$ geodesic geodesic boundary components (i.e. S is the Nielsen's kernel of Δ_R/H_o where H_o is a Schottky-type group of the item 3.5) . Because s is an even number there exists a regular p-sheeted covering $\tilde{S} \to S$ such that $\tilde{S}$ has s punctures and each boundary curve of $\tilde{S}$ maps injectively. It is easy to see that the genus of $\tilde{S}$ is equal to $g'=1-p+s(p-1)/2$. We shall suppose that $p \ge p_o$.

Let b_j be components of $\partial\tilde{S}$, $1 \le j \le (r+1)p$. Next we glue the following compact surfaces, with unique geodesic boundary component, to $\tilde{S}$:(a) to each component of $\partial S \setminus (b_1 \cup \ldots \cup b_{r+1})$ we glue a genus 1 compact surface; (b) to the components $b_1 , \ldots, b_r$ we glue genus 10 compact surfaces; (c) along the component b_{r+1} we glue a genus $\xi(p)$ surface whose boundary has the length I and possesses a regular w-collar (due to the condition (**)).

The resulting surface $\tilde{S}^*$ has genus $(9 +p)r -p +s(p -1)/2 + \xi(p)$ and s punctures, the area of $\tilde{S}^*$ is finite. The genus of $\tilde{S}^*$ is equal to $\tilde{g}$ (see (*)). If we remove $(m+1)$ disjoint closed discs from $\tilde{S}^*$ then we obtain a surface homeomorphic to $int\mathfrak{R}$.

4.5. Case 2: s is equal to zero.

We have $r \ge 2$, $p \ge 2$, therefore (**) implies the inequality $\xi'(p_o)= p_o(2g+m-2)/2 +1 -m/2 -10r \ge g_o(I, w)$. Then we put $\tilde{S}= S \to S$ be the trivial covering. Let $p \ge p_o$. In the same manner as in the Case 1 we glue surfaces of genus 10 to $b_1, \ldots, b_r \subset \partial S$ and a genus $\xi'(p)$ surface (possessing a regular w- collar along unique geodesic boundary component) is glued to b_{r+1} . The resulting surface $\tilde{S}^*$ has the same genus $\tilde{g}$ as $\mathfrak{R}$.

Next we put $\hat{p}=p$ in the Case 1 and $\hat{p}=1$ in the Case 2.

In the following items 4.6- 5.0 we shall construct via Maskit Combination the necessary group $\mathfrak{G}$. The combination process corresponds to the construction of the surface $\tilde{S}^*$ above.

4.6. Search of finite-index subgroup in $H(\varepsilon_p)$.

Remind that $\lim V_j(p)=1$. Then for all but finite p there exist small admissible deformations $c_j(\varepsilon_p)$ of $c_j \in H$ satisfying the condition (d) of (3.5) such that the elements $\vec{c}_j(\varepsilon_p)= (c_j(\varepsilon_p), l_j(\varepsilon_p))$ are conjugated to $\vec{V}_j(p)$, $j=1,\ldots,s$. We shall suppose that it is so for all $p \ge p_1 \ge p_o$.

The fundamental set $\mathcal{P}(\varepsilon_p)$ is not well enough for the Maskit Combination (along $\langle h_i \rangle$) even in the Case 2. The source is evident- $\mathcal{K}(h_{r+1}(\varepsilon_p)) \cap \mathcal{P}(\varepsilon_p)$ is not a fundamental set for action of $\langle h_{r+1} \rangle$ in this cone. Hence we are to change $\mathcal{P}(\varepsilon_p)$.

Since $\mathcal{P}(\varepsilon_p)$ is a fundamental domain for $H(\varepsilon_p)$ then $h_{r+1}(\varepsilon_p)(I(h_1(\varepsilon_p))\equiv S'(r+1)) \cap (S(r+1) \equiv I(h_1(\varepsilon_p))= \emptyset$ .Hence the exterior of $S(r+1) \cup S'(r+1)$ is a fundamental set for the group $\langle h_{r+1}(\varepsilon_p) \rangle$. The intersection $\mathcal{K}(l(h_{r+1}), \pi/2) \cap Int(I(c_s^{-1}(\varepsilon_p)) \setminus Int(S'(r+1))$ is a segment $\nabla=\nabla(r+1)$ which we glue to the set $\mathcal{P}(\varepsilon_p)$ (see figure 9).

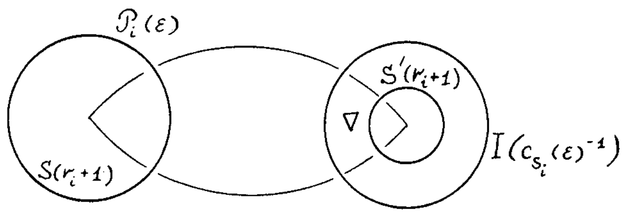

Figure 9

To save the fundamentality we have to cut out the orbit $(H(\varepsilon_p)\setminus 1)(\nabla)$ from the set $\mathcal{P}(\varepsilon_p)$. The resulting set we shall denote by $P^o(\varepsilon_p)$. Properties of the group $H(\varepsilon_p)$ provide that $\mathbb{S}^3\setminus P^o(\varepsilon_p)\subset \mathfrak{X}(m)$, where $\mathfrak{X}(m)$ is the solid torus of the item 3.7.

Now we are ready to construct the fundamental set for the $\hat{p}$-index subgroup in $H(\varepsilon_p)$.

The surface S is the Nielsen kernel of Δ_R/H, hence the regular covering $\tilde{S}\to S$ corresponds to a normal subgroup $H_p\subset H$ of index $\hat{p}$. The group $\rho_\varepsilon(H_p)$ will be denoted later by $H(p)$.

It is easy to see that the following decomposition holds

$$H(\varepsilon_p)= 1\cdot H(p)+ c_1(\varepsilon_p)\cdot H(p) +\ldots+ (c_1(\varepsilon_p))^{p-1}P^o(\varepsilon_p).$$

Let $\varphi_j\equiv\varphi_{j'+q(r+1)}\equiv(c_1(\varepsilon_p))^q\ \varphi_{j'}(c_1(\varepsilon_p))^{-q}$ $(0\le q<\hat{p})$ be representatives of conjugancy classes in $H(\varepsilon_p)$ corresponding to the components of $\partial\tilde{S}$. Then each element φ_j possesses a precisely invariant cone $\mathcal{K}(\varphi_j)=(c(\varepsilon_p))^q(\mathcal{K}(\varphi_{j'}))$, $1\le j'\le r$. For all values $j'=r+1\neq j$ we choose a precisely invariant cone $\mathcal{K}(\varphi_j)$ which has the vertex angle $\pi/2$ and the common axis with $(c_1(\varepsilon_p))^q(\mathcal{K}(\varphi_{j'}))$. All these cones lie in $\mathbb{S}^3\setminus P(\varepsilon_p)$ and, hence, the complement to the set $P^-(\varepsilon_p)=P^o(\varepsilon_p)\setminus(\bigsqcup_{j=1}^{\hat{p}(r+1)}\mathcal{K}(\varphi_j)\cup\bigsqcup_{l=1}^{s}{}^*\mathcal{K}(c_l(\varepsilon_p^{\hat{P}}))$ is situated inside of the solid torus $\mathfrak{X}(m)$ too.

4.7. Fundamental sets for groups $\langle\varphi_j\rangle$ and $H(p)$.

Let $j'<r+1$. Then we put $P\langle\varphi_{j'}\rangle\equiv ext(I(\varphi_{j'}))\cap ext(I(\varphi_{j'}^{-1}))$ be the isometric fundamental domain, $P\langle\varphi_j\rangle\equiv c_1(\varepsilon_p)^q(P\langle\varphi_{j'}\rangle)$. In the same manner, for $j'=r+1$ we put $P\langle\varphi_{j'}\rangle\equiv ext\,(S(r+1)\cup S'(r+1))$, and for $j=j'+q(r+1)$ let $P\langle\varphi_j\rangle\equiv c_1(\varepsilon_p)^q(P\langle\varphi_{j'}\rangle)$.

The fundamental set $P^o(\varepsilon_p)$ has some defect- the intersection $P^o(\varepsilon_p)\cap\mathcal{K}(c_l(\varepsilon_p))$ is not a fundamental set for action of $\langle c_l(\varepsilon_p)^{\hat{P}}\rangle$ in this cone, $2\le l\le s$. For this reason we need the following surgery on $P^o(\varepsilon_p)$. We put

$$P\langle c_l(\varepsilon_p)^{\hat{P}}\rangle\equiv\bigsqcup_{q=1}^{\hat{p}-1}c_1^{-q}(\varepsilon_p)(\mathcal{K}(c_l(\varepsilon_p)\cap P^o(\varepsilon_p)).$$

It is easy to see that this set is a fundamental domain for action of the group $\langle c_l(\varepsilon_p)^{\hat{P}}\rangle$ in $\mathcal{K}(c_l(\varepsilon_p))$. Furthermore the set

$$P(p)\equiv P^-(\varepsilon_p)\cup\bigsqcup_{l=1}^{s}P\langle c_l(\varepsilon_p)^{\hat{P}}\rangle\cup\bigsqcup_{j=1}^{\hat{p}(r+1)}P\langle\varphi_j\rangle$$

is a fundamental set for the group $H(p)$.

In what follows we shall denote the manifold

$$(R^*(H(p))\equiv R(H(p)\setminus H(p)\cdot\bigsqcup_j\mathcal{K}(\varphi_j))/H(p)$$

by $M^*(H(p))$. Also let us denote a shortest infinite arc corresponding to $(K(\varphi_j),\varphi_j)$ by $\tilde{\gamma}_j$.

4.8. Construction of Kleinian groups corresponding to surfaces were glued to $\tilde{S}$.

Let l_j be equal to $log(|k(\varphi_j)-1|)$, where $k(\ )$ is the complex coefficient. If $\varphi_j\in Isom(\mathbb{H}^2)$ then l_j is the "length" of this hyperbolic displacement.

Consider firstable the generic case : $|e|\ge 2$, i.e. $|e|=r$. For the elements φ_j ($j>r+1$) we choose a riemannian surface $S(j)$ of genus 1 with unique geodesic boundary curve of the length l_j (conf. the item 4.5). If $j=r+1$ then we choose a surface of genus $\zeta(p)$ or $\zeta'(p)$ (due to the Case 1 or Case 2 of the item 4.5). Let $F(j)$ be a Schottky subgroup of $Isom(\mathbb{H}^2\equiv\mathbb{R}^2_+)\subset\mathfrak{M}_3$ such that $S(j)$ isometric to the Nielsen's kernel of $\mathbb{H}^2/F(j)$. Let $\varphi'_j\in F(j)$ be an element corresponding to generator of $\pi_1(\partial S(j))$. Furthermore we put $A(\varphi_j)\subset R(F(j))$ be a complemental segment in $\partial\mathbb{H}^2$ to $Fix(\varphi'_j)$, $\mathcal{K}(\varphi_j)$ be the cone with the axis $A(\varphi'_j)$ and the vertex angle $\alpha=\pi/2$ (in the case $j\neq r+1$) and $\alpha=\pi-\theta$ (in the case $j=r+1$).

When $j\neq r+1$ the intersection $\mathcal{K}(\varphi'_j)\cap\mathbb{H}^2$ is contained in the complement to the Nielsen's convex hill $N(j)$ of $F(j)$; if $j=r+1$ then $\mathcal{K}(\varphi'_j)\cap N(j)$ lies in the w-collar of $\partial N(j)$. In any case the cone $\mathcal{K}(\varphi'_j)$ is precisely invariant under $\langle\varphi_j\rangle\subset F(j)$.

Surfaces of genus 10 (which have been glued to $\tilde{S}$) correspond to r copies of the group $G(10,1)=F(j)$, that have been constructed in the item 3.3, $1\le j\le r$. We shall use r copies of the ball $B\subset\mathbb{S}^3$ as precisely invariant cones $\mathcal{K}(\varphi'_j)$, where $\varphi'_j\equiv h\in G(10,1)$ (see item 3.3), $j=1,\ldots,r$.

In the exceptional cases $e=0$, $|e|=1$ we replace $r-|e|$ copies of the group G(10, 1) by a fuchsian Schottky groups $F(j)$. These groups uniformizate a genus 10 surface in the same manner as $F(i)$, $i\geq r+1$ (see the generic case). The vertex angles α in these cases are equal to $\pi/2$.

4.9. Construction of fundamental sets $P(j)$ for groups $F(j)$.

By the choice of vertex angles for the cones $\mathcal{K}(\varphi'_j)$ there exist mobius transformations ζ_j which map $ext\mathcal{K}(\varphi'_j)$ onto $int\mathcal{K}(\varphi_j)$. These transformations may be chosen with the additional property: ζ_j maps attractive fixed points of φ'_j to attractive fixed points of φ_j, $1\leq j\leq \hat{p}(r+1)$. Hence we have ($\zeta^*_j(\varphi'_j)\equiv \zeta_j\varphi'_j\zeta^{-1}_j$) $=\varphi_j$. The domain $\zeta^{-1}(P(\varphi_j))\equiv P(\varphi'_j)$ is fundamental for $\langle\varphi'_j\rangle$. Then we put $P(j)\cap \mathcal{K}(\varphi'_j)\equiv P(\varphi'_j)\cap \mathcal{K}(\varphi'_j)$. Furthermore, let us choose an arbitrary fundamental set $P(j)\cap (S^3\setminus\mathcal{K}(\varphi'_j))$ for action of the group $F(j)$ on $S^3\setminus F(j)(\mathcal{K}(\varphi'_j))$. So we obtain the fundamental set

$$P(j)\equiv (P(\varphi'_j)\cap \mathcal{K}(\varphi'_j)) \cup (P(j)\cap (S^3\setminus \mathcal{K}(\varphi'_j)))$$

for action of the group $F(j)$ on S^3. This fundamental set is well enough for Maskit Combination along $\langle\varphi'_j\rangle$.

Denote the manifold $(R(F(j))\setminus F(j)\cdot\mathcal{K}(\varphi'_j))/ F(j)$ by $M^*(F(j))$.

4.10. Construction of the group $\mathfrak{G}$.

Let p be a prime number greater than p_1. We shall combine the following list of groups :

$$H(p)\ ,\quad \zeta^*_j(F(j))\equiv F^*(j)\ ,\ j=1\ ,\dots,\ (r+1)\hat{p}\ .$$

The group $H(p)$ has the fundamental set $P(p)$ (see the item 4.7). The groups $F^*(j)$ have the fundamental sets $P^*(j)\equiv \zeta_j(P(j))$ such that: $(P^*(j)\setminus ext\ (\mathcal{K}(\varphi_j))\cap (P^0(p)\setminus int\ \mathcal{K}(\varphi_j))$ is a fundamental domain for action of the group $\langle\varphi_j\rangle$. Hence the conditions of the 1-st Combination Theorem are satisfied and the group $\mathfrak{G}= \langle H(p),\ F(j)^*,\ 1\leq j\leq (r+1)\hat{p} \rangle$ is a Kleinian group and the set Φ

$$\Phi= (P(p)\setminus \bigcup_{j=1}^{(r+1)\hat{p}} \mathcal{K}(\varphi_j)) \cup \bigcup_{j=1}^{(r+1)\hat{p}} (P^*(j)\setminus\ ext\ (\mathcal{K}(\varphi_j))$$

is a fundamental set for this group. Evidently, $S^3\setminus\Phi$ is contained inside of the solid torus $\mathfrak{T}(m)$.

Next we put $\vec{Y}_j\equiv \vec{c}_j(\varepsilon_p)$ and $\mathcal{K}_j\equiv \mathcal{K}(\ c_j(\varepsilon_p))$. From construction of the set Φ and item 4.7 it follows that for the set Φ the assertion (4) of the theorem 4.1 is valid. Next we are to verify the property (5). Due to the Combination Theorem we have: the manifold $M^*(\mathfrak{G})$ is glued of the manifolds $M^*(H(p))$ and $M^*(F(j))$, $1\leq j\leq \hat{p}(r+1)$. All these manifolds are trivial Seifert fibered spaces. The gluing homeomorphisms are lifted to the maps ζ_j of the cones $K(\varphi_j)$, $K(\varphi'_j)$. Hence, the fibers of these bundles are glued one to other (with preserving of orientation). Therefore, the manifold $M^*(\mathfrak{G})$ is a trivial circle bundle too. By the construction of $\mathfrak{G}$ and the items 4.4, 4.5, the base of this bundle is homeomorphic to the surface $\mathfrak{R}$ (see the item 4.1). Here $m+1$ boundary curves of $\mathfrak{R}$ correspond to the cones $K(Y_j)$, $1\leq j\leq s$ and the torus $\partial\mathfrak{T}(m)$.

Now we are to compute the relative Euler number for the group $\mathfrak{G}$. We remind that in the item 3.6 the "infinite shortest arcs" $\gamma_j\subset K(h_j(\varepsilon_p))$ and $\beta_i\subset K(Y_1)$ where introduced and we have

$$e(H(\varepsilon_p),\ \tilde{\beta}_1\cup\dots\cup \tilde{\beta}_s\cup \tilde{\gamma}_1\cup\dots\cup \tilde{\gamma}_{r+1})= 0.$$

When we pass to the p-index subgroup in $H(\varepsilon_p)$ the arcs $c_1(\varepsilon_p)^q(\tilde{\gamma}_j)\equiv\tilde{\gamma}_j$, $0\leq q\leq p-1$, become shortest arcs corresponding to φ_j. We put $\tilde{\beta}= \tilde{\beta}_1\cup\dots\cup \tilde{\beta}_s$. Therefore the Euler number $e(\ H(p),\ \tilde{\beta}\cup \tilde{\gamma}_1\cup\dots\cup \tilde{\gamma}_{\hat{p}(r+1)})$ is equal to zero. All $\tilde{\gamma}_j$ are arcs of euclidean circles in S^3, hence $e(F(j),\ \zeta^{-1}_j(\tilde{\gamma}_j))=0$ for any $j > |e|$ (since $F(j)$ is a fuchsian group for such j). If $j\leq |e|$ then $e(F(j),\ \zeta^{-1}_j(\tilde{\gamma}_j))= +1$ because $F(j)=$ G(10, 1) (see the item 3.6). So $e(\mathfrak{G},\ \tilde{\beta})= |e|$ due to the item 3.2.

REMARK. The orientation of fibers of the manifold $M^*(\mathfrak{G})$ is induced by the orientation of the loop $\tilde{\theta}\subset \partial\mathfrak{T}(m)$ (see the item 3.7). It is easy to see that this orientation is consistent with the orientation of fiber of the manifold $\mathfrak{N}(10)$ (see the items 3.2, 3.3).

However we need groups $\mathfrak{G}$ with negative Euler numbers too. For this reason, let us consider the reflection J in the plane Π' (see the item 2.5). Then $J\circ h= h\circ J$, $J(B)=B$ and the group $G(10,-1)= J\circ G(10,\ 1)\circ J$ has the Euler number $e(\ G(10,\ -1)\ ,\ \tilde{\sigma})$ equal to (-1), if the orientation of fiber is given by the map ζ_j: $K(\varphi_j)\to \Pi= K(\varphi'_j)$, $j=1\ ,\dots,\ |e|$. Then the group $\mathfrak{G}= \mathfrak{G}(-|e|)\equiv \langle H(p),\ F(j)^*,\ |e|\leq j\leq (r+1)\hat{p}\ ,\ \zeta^*_i(G\ (10,\ -1)),\ 1\leq i\leq |e|\rangle$ possesses all properties of the group $\mathfrak{G}(|e|)$ but it's relative Euler number $e(\mathfrak{G},\ \tilde{\beta})$ is equal to $(-|e|)$.

So, for all $e\in\mathbb{Z}$ we have obtained the group $\mathfrak{G}$ such that $e(\mathfrak{G},\ \tilde{\beta})=e$. The relative Euler number $e(S^3\setminus\mathfrak{T}(m),\ \tilde{\delta})$ is equal to zero (if we consider $S^3\setminus\mathfrak{T}(m)$ as a trivial circle bundle with an ordinary fiber $\tilde{\theta}$). Then we have: $e(M^*(\mathfrak{G}),\ \beta_1\cup\dots\cup \beta_s\cup \delta)=e$. All properties (1)-(6) are verified for the group $\mathfrak{G}$.

The Theorem 4.1 is proved. QED

§ 5. Conformal sewing of hyperbolic and Seifert manifolds

In this section we prove the main theorem of this article.

THEOREM 5.1. Let M be a closed Haken 3-manifold with unsolvable fundamental group such that in the canonical composition of M from Seifert and hyperbolic parts there are no gluings of hyperbolic components with hyperbolic or euclidean ones. Then some finite-sheeted covering of M admits an uniformizable flat conformal structure.

5.1. Two examples.

Before the proof of this theorem we produce two examples which explain forthcoming constructions and illustrate arising difficulties.

Example 1. Let $Z_i=\Sigma_i\times S^1$, $i=1, 2$, where Σ_i are surfaces of genus $g_i\neq 0$ and have connected boundary. The decomposition of Z_i into direct product introduces in $\pi_1(\partial Z_i)$ a "natural basis" (see the item 5.3). Let us suppose that the manifold M is glued of Z_i via a homeomorphism $f:\partial Z_1\to\partial Z_2$ which is defined (in natural bases) by a matrix $A\in GL(\mathbb{Z}, 2)$ with $a_{21}=1$. If the numbers g_i are sufficiently large with respect to $|a_{ij}|$ then there exist the groups $H(g_1, |a_{22}|)$, $H(g_2, |a_{11}|)$ (theorem 2.1). These groups uniformizate the manifolds $S(g_1, |a_{22}|)$, $S(g_2, |a_{11}|)$. Next we dispose the constructed groups in S^3 in such way that the complements of their fundamental domains (that look like twisted unknotted solid tori) define a link of index 1 in S^3. It is not hard to see that the group $G=H(g_1, |a_{22}|) * H(g_2, |a_{11}|)$ uniformizates the manifold M. However it is impossible to avoid the condition $|a_{21}|=1$ (for the circumscribed construction of the group G). Our aim is to find a finite-shoeted covering over M such that the corresponding coefficients a_{21} are equal to 1.

Example 2. Let G_1 be a torsion-free discrete subgroup of PSL(2, $\mathbb{C}$), $p: \mathbb{H}^3\to\mathbb{H}^3/G_1=\mathfrak{N}(G_1)$ is the universal covering, the manifold $\mathfrak{N}(G_1)$ is compact and contains a simple closed geodesic γ. Let us suppose that some component $\gamma\subset p^{-1}(\tilde{\gamma})$ has hyperbolic stabilizer $\langle g\rangle$ in G_1. Then γ has an open ε-neighbourhood $U_\varepsilon(\tilde{\gamma})$ which is precisely invariant under $\langle g\rangle$. It isn't hard to notice that the manifold $\mathfrak{N}^*=\mathfrak{N}(G_1)\setminus p(U_\varepsilon(\tilde{\gamma}))$ is hyperbolic [Koj]. We shall denote by C the euclidean circle that contains the arc $\tilde{\gamma}$.

Let G_2 be a rank $2r$ Schottky subgroup of $\mathfrak{M}_3$ such that: (1) the circle C is invariant under G_2, (2) $g\in[G_2, G_2]$ (3) the domain $S^3\setminus cl(U_\varepsilon(\tilde{\gamma}))$ is precisely invariant under $\langle g\rangle\subset G_2$. Then the group $G=\langle G_1, G_2\rangle$ uniformizates a manifold M which is glued of $\mathfrak{N}^*$ and $\Sigma\times S^1$ along the boundary tori. Here Σ is a compact genus r surface with connected boundary. However only few sewings may be realized in such way and the hyperbolicity of g is very restrictive condition. Both reasons force us to waive of utilizing Schottky groups with invariant circles. Instead of them we shall use groups $\mathfrak{G}$ that have been constructed in the theorem 4.1.

PROOF OF THE THEOREM 5.1.

5.2. Suppose that the manifold M is glued of the oriented Seifert components $M_1, \dots, M_\mathfrak{z}$ and hyperbolic ones $M_{\mathfrak{z}+1}, \dots, M_{\mathfrak{z}+\mathfrak{b}}$. Because $M\notin$ (Sol)∪ (Nil) and the theorem 2.1 is proved, now we may suppose that M isn't a Seifert manifold and there are no components of type $[0, 1]\times T^2$ among M_i-s. Let us agree to denote i-th component of ∂M_j by $\partial_i M_j$ and the sewing homeomorphism- by $f^{ij}_{kl}:\partial_i M_j\to\partial_k M_l$, where $(f^{ij}_{kl})^{-1}=(f^{kl}_{ij})$ and f^{ij}_{kl} changes the induced orientation of boundary (the manifold M is oriented). If $i\le\mathfrak{z}$, then those components of ∂M_i which are glued to hyperbolic manifolds will be called hyperbolic, and other components of ∂M_i will be called Seifert. Any manifold $int(M_i)$, $i>\mathfrak{z}$, is uniformizated by a discrete group $\Gamma_i\subset$ PSL(2, $\mathbb{C}$), $int(M_i)=\mathbb{H}^3/\Gamma_i$, where $\mathbb{H}^3$ is realized as $\mathbb{R}^3_+\equiv\{(x_1, x_2, x_3): x_3>0\}$.

5.3. Let O_i be the base-orbifold of M_i, $i\le\mathfrak{z}$. This orbifold has the boundary components $\mathfrak{b}_{1i}, \dots, \mathfrak{b}_{r_i i}$. Let us consider the orbifold O^+_i obtained of O_i via gluing with discs D_{ji} (which have one singular point of prime order $p>1$) along each $\mathfrak{b}_{ji}$. The orbifold O^+_i is finitely covered by an orientable surface Σ^+_i, the multiplicity of this covering is chosen to be even number. After removing preimages of D_{ji} from Σ^+ we obtain a surface Σ_i with even number of boundary curves, this surface finitely covers the orbifold O_i.

In this situation the theorem 1.4 is applicable and there exists the induced covering $\Sigma_i\times S^1\to M_i$. By virtue of the theorems 1.3, 1.5 we construct a finite-sheeted covering space of M, in canonical splitting of which all Seifert components have the type $\Sigma_i\times S^1$, where Σ_i is an orientable surface with an even number of boundary curves. If some M_i ($i\le\mathfrak{z}$) is lifted to $T^2\times[0, 1]$ via this covering, then this $T^2\times[0, 1]$ borders with some Seifert component- $\Sigma_j \times S^1$ and we shall unite them.

The constructed covering manifold also will be denoted by M and it's components - by M_i , we preserve the introduced notations for sewing homeomorphisms and numbers of hyperbolic and Seifert components too. We shall denote the number of boundary components of $\Sigma_i \times S^1$ by $\delta_i = \mathfrak{z}_i + \mathfrak{h}_i$, where $\mathfrak{z}_i$ is the number of Seifert boundary components and $\mathfrak{h}_i$ is the number of hyperbolic ones, all these numbers are even. The genus of Σ_i will be denoted by g_i . Without a loss of generality we may suppose that for regular fiber $u_{kl} \subset \partial_k M_l$ the loop $f^{ij}_{kl}(u_{kl})$ is not isotopic in M_j to a regular fiber $u_{ij} \subset \partial_i M_j$ ($k, i \le \mathfrak{z}$) . Else, the manifold $M_l \cup M_j$ may be exchanged by one Seifert manifold.

For a given number $i \le \mathfrak{z}$ we shall orient all fiber loops $u_{ij} \subset \partial M_j$ consistently. If $\partial_k M_l$ is glued with $\partial_i M_j$ ($l \le \mathfrak{z} < j$) then we put $u_{kl} \equiv (f^{kl}_{ij})(-u_{ij}) \subset \partial_k M_l$, where the directed loop $(-u_{ij})$ is obtained of u_{ij} via change of orientation. By the same symbol u_{ij} the corresponding element of $\pi_1(M_j)$ will be denoted too.

Let us consider a parabolic element $u_{kl} \in \Gamma_l$ which maps to $u_{kl} \in \pi_1(M_k)$ under the isomorphism $\Gamma_l \simeq \pi_1(M_l)$. By virtue of a conjugation of Γ_l in Isom $(\mathbb{H}^3)$ we can choose the element u_{kl} to be a translation $u_{kl}: z \longmapsto z+1$, $z \in \bar{\mathbb{C}} = \partial \mathbb{R}^3_+$. The second generator of the group $\mathfrak{U}_{kl} \simeq \pi_1(\partial_k M_l)$ is a parabolic element $\upsilon_{kl}: z \longmapsto z + w_{kl}$. Without a loss of generality we may suppose that $-1/2 < \mathrm{Re}\,(w_{kl}) \le 1/2$. By the same symbol υ_{kl} we shall denote the corresponding element of $\pi_1(M_l)$ as well as directed simple loop on $\partial_k M_l$. Analogously, $\upsilon_{ij} = f^{ij}_{kl}(\upsilon_{kl})$ denotes the directed loop on $\partial_i M_j$ and the corresponding element of $\pi_1(M_j)$.

5.4. Construction of the uniformizable covering, I.

In this item we construct a covering space over M for which any sewing map of Seifert components has the type of Example 1 .

Denote by σ_{ij} , $j=1$, ... , δ_j oriented components of the image of $\partial\Sigma_j$ under the section $\Sigma_j \to \Sigma_j \times S^1$. If a component $\partial_i M_j$ of ∂M_j is Seifert then the loop σ_{ij} will be also denoted by υ_{ij} .

DEFINITION. The introduced pairs of directed loops $(u_{ij}, \sigma_{ij}) \subset \partial M_j$ $(j \le \mathfrak{z})$ and $(u_{ij}, \upsilon_{ij}) \subset \partial M_j$ $(j > \mathfrak{z})$ define the *natural bases* for $\pi_1(\partial_i M_j) \cong \mathbb{Z} \oplus \mathbb{Z}$.

The sewing map $f = f^{ij}_{kl}$ in these bases is determined uniquely (up to isotopy) by the matrix $A(k, l) = (a_{pq}(k, l)) \in GL^-(2, \mathbb{Z})$, $f_*(u_{kl}, \upsilon_{kl}) = A(k, l)^* (u_{ij}, \sigma_{ij})$.

REMARK. For definiteness we put here $j \le l$. If $l > \mathfrak{z}$, then $a_{11}(k, l) = -1$, $a_{21}(k, l) = 0$ and the matrix $A(k, l)$ has the type

$$\begin{pmatrix} -1 & a_{12}(k, l) \\ 0 & 1 \end{pmatrix}$$

(the necessary sign of a_{22} may be obtained via exchange $w_{kl} \longmapsto -w_{kl}$). So, for any $i > \mathfrak{z}_j$ the loop υ_{ij} projects injectively to the base Σ_j as well as for $i \le \mathfrak{z}_j$ and a relative Euler number of M_j with respect to υ_{ij} -s can be calculated.

Now we are to find a covering space of M for which $a_{21}(k, l) = 1$ for any $k \le \mathfrak{z}$. Let us denote (for fixed i , j , k , l) the sewing map f^{ij}_{kl} by f . Then we put : $\sigma_{ij} \equiv a_{21}(k, l)\ \sigma_{ij}$, $\sigma_{kl} \equiv a_{21}(k, l)\ \sigma_{kl}$ - elements of $\pi_1(\partial_i M_j)$, $\pi_1(\partial_k M_l)$. Therefore, we have $f_*(u_{kl}, \tilde{\sigma}_{kl}) = \tilde{A}(k, l)^* (u_{ij}, \tilde{\sigma}_{ij})$, where

$$\tilde{A}(k,l) = \begin{pmatrix} a_{11}(k, l) & a_{21}(k, l)\, a_{12}(k, l) \\ 1 & a_{22}(k, l) \end{pmatrix}$$

If $\partial_i M_j$ is a hyperbolic component of ∂M_j then we put $\tilde{\sigma}_{ij} \equiv \sigma_{ij}$ ($1 \le j \le \mathfrak{z}$) . Further we construct a finite-sheeted covering $p_j : \tilde{\Sigma}_j \to \Sigma_j$ such that the defining subgroups for restriction of p_j to $\partial\tilde{\Sigma}_j$ are $\langle \tilde{\sigma}_{ij} \rangle \subset \pi_1(\partial_i \Sigma_j)$. The induced covering of Seifert manifolds will be $p_j \times id : \tilde{M}_j = \tilde{\Sigma}_j \times S^1 \to \Sigma_j \times S^1$. For hyperbolic manifolds M_l we put $\tilde{M}_l \equiv M_l \to M_l$ be the trivial coverings.

If u_{ij} , $\tilde{\sigma}_{ij} \subset \partial_i M_j$ then components of their preimages under this covering will be denoted by $\tilde{u}_{ij}$, $\tilde{\sigma}_{ij}$. Then by virtue of the theorem 1.5 we construct a finite-sheeted covering space over M which is glued of components of type $\tilde{M}_i$. The gluing homeomorphisms $(\tilde{f}^{kl}_{ij})$ of Seifert components of $\tilde{M}$ are defined by the matrices $\tilde{A}(i, j)$ in the indicated natural bases $(\tilde{u}_{ij}, \tilde{\sigma}_{ij})$. So, we have constructed the covering space with necessary matrices of gluing maps. For simplification of notations we shall drop all signs "~ " , preserving them only for matrices $\tilde{A}(\ ,\ )$. We shall preserve notations $\mathfrak{z}$, $\mathfrak{h}$ ... for numbers of components.

5.5. Construction of the uniformizable covering, II.

In this item we construct the necessary finite covering $M_0 \to M$ (such that the manifold M_0 admits an uniformizable FCS).

Let us consider some Seifert component M_i of M .

Case A. Let us suppose that the number $\mathfrak{z}_i$ of Seifert boundary components of ∂M_i is positive. Then, without a loss of generality, we can suppose that $\upsilon_{ji} = \sigma_{ji}$ $(j \ge \mathfrak{z}_i)$, since a section defined on hyperbolic components of $\partial\Sigma_i$ may be continued to all Σ_i. We put $e_i \equiv a_{22}(i, 1) + \ldots + a_{22}(i, \mathfrak{z})$.

Case B. If $\mathfrak{z}_i = 0$ then we put $e_i \equiv e(M_i, \upsilon_{1i} \cup \ldots \cup \upsilon_{\mathfrak{b}_i i})$-obstruction to the section of the Case A.

Notice that if for standard p^2-sheeted coverings $\tilde{M}_i \to M_i$, $i \le \mathfrak{z}$, we construct a covering $\tilde{M} \to M$ (due to the theorems 1.3, 1.5), then the numbers $\tilde{m}_i$, $\tilde{s}_i$, $\tilde{e}_i$, associated with canonical splitting of M, remains the same as m_i, s_i, e_i. However the genus of $\tilde{\Sigma}_i$ tends to infinity if $p \to \infty$. So, without a loss of generality we may suppose that $g_i + m_i/2 - max(\,|e_i|\,, 2\,) > 0$.

Next we choose a prime number p_0 such that

(a) For all $l > \mathfrak{z}$, $n \ge p_0$ and groups $\Gamma_l(n)$ the conclusion of the Lemma 3.1 holds.

(b) For all primes $p \ge p_0$, for numbers e_i, g_i, m_i, s_i (see above) and for the sequence $(\vec{V}_j(p))_i \equiv (\upsilon^*_{kl}(p), l(\upsilon^*_{kl}))$ (see the item 3.1) there exists a Kleinian group $\mathfrak{G}_i = \mathfrak{G}(e_i, m_i, s_i, p)$ from the Theorem 4.1. Here (k, l) are indexes of those components $\partial_k M_l \subset \partial M_l$ $(l > \mathfrak{z})$ which are glued to $\partial_j M_i$ $(i \le \mathfrak{z})$.

REMARK. The genus $\tilde{g}_i$ of the surface $\mathfrak{R}_i$ (see the Theorem 4.1) is equal to the genus of the standard p-sheeted covering $\tilde{\Sigma}_i$ over Σ_i.

5.6. Construction of groups $\mathfrak{G}_k$ uniformizating hyperbolic manifolds M_k.

By choice of the number p_0 (see above), for any $p \ge p_0$, $k \ge \mathfrak{z}$, there exists a representation $\rho_p : \Gamma_k \to \Gamma_k(p) \subset PSL(2, \mathbb{C})$ such that $\rho_p(u_{kl}) = u_{kl}(p)$ are elliptic elements of order p and $\upsilon_{kl}(p) = \rho_p(\upsilon_{kl})$ are loxodromic transformations (conjugated to X_{ij}). Then (for all $k \ge \mathfrak{z}$) we pass to a normal subgroups $\Gamma^0_k \subset \Gamma_k$ with the properties : $|\,\Gamma_k : \Gamma^0_k\,| < \infty$, and $\Gamma_k \cap \langle u_{kl}, \upsilon_{kl}\rangle = \langle u^p_{kl}, \upsilon^p_{kl}\rangle$ (see the theorem 1.3). The groups $\mathfrak{G}_k = \rho_p(\Gamma^0_k)$ are those that we are looking for.

5.7. Construction of the covering M_0 over M.

Let $\tilde{M}_i = \Sigma_i \times S^1 \to M_i = \Sigma_i \times S^1$ be the standard p-sheeted covering, where $p = p_0$ (see above), $i \le \mathfrak{z}$. If $k > \mathfrak{z}$, then we introduce the coverings $\tilde{M}_k \to M_k$, determined by the inclusions $\Gamma^0_k < \Gamma_k$, $int\ M_k = \mathbb{H}^3/\Gamma_k$. Further, by virtue of the Theorem 1.5, we construct a finite-sheeted covering $\hat{\pi} : M_0 \to M$. The manifold M_0 is those that we need. This manifold is glued of hyperbolic and Seifert components homeomorphic to $\tilde{M}_i$ and $\tilde{M}_k$, $1 \le i \le \mathfrak{z} < k \le \mathfrak{z} + \mathfrak{b}$; the restrictions of $\hat{\pi}$ to these components are equivalent to the described coverings $\tilde{M}_i \to M_i$ and $\tilde{M}_k \to M_k$.

Let us denote Seifert components of M_0 by $X_1, \ldots, X_{\mathfrak{z}}$ and hyperbolic - by $X_{\mathfrak{z}+1}, \ldots, X_{\mathfrak{z}+\mathfrak{b}}$. We shall denote the sewing maps by $\tilde{f}^{kl}_{ji} : \partial_j X_i \to \partial_k X_l$, and preserve notation δ_i for number of boundary components of the manifold X_i. Any manifold X_i covers some component M_i, of M. Next we associate to each manifold X_i a copy G_i of the group $\mathfrak{G}_i$, (see above). If $i > \mathfrak{z}$, then elements of G_i corresponding to $\vec{\upsilon}_{ji}(p)^p$ will be denoted by $\vec{b}_{ji}$ and we put $(b_{ji})^{1/p} \equiv \upsilon_{ji}(p)$. If $i \le \mathfrak{z}$, then we shall denote by b_{ji} the element corresponding to $(Y_{j'i'})^p \in \mathfrak{G}_{i'}$; also we put $(b_{ji})^{1/p} \equiv Y_{j'i'}$. Fundamental sets Φ_i for such groups were constructed in §4. Now we are to find "good" fundamental sets for the groups G_l $(l > \mathfrak{z})$.

Let $\mathcal{K}_{kl}$ be the cone with vertex angle $\pi/4$ and the same axis as $\vec{b}_{kl}$. These cones are precisely invariant under $\langle b_{kl}\rangle < G_l$ and their orbits under G_l are pairwize disjoint (due to the Lemma 3.1). Firstable we shall choose a fundamental domains for action of $\langle b_{kl}\rangle$ in $\mathcal{K}_{kl}$. The boundary torus $\partial_k X_l$ is glued to the boundary torus $\partial_j X_i$ of some Seifert component of M_0. Then, by construction of the group $G_i = \mathfrak{G}_{i'}$, there exists a mobius transformation μ^{ji}_{kl} conjugating $(\vec{b}_{kl})^{1/p}$ and $(\vec{b}_{ji})^{1/p}$. This transformation maps the cone $\mathcal{K}_{kl}$ to the cone $cl(\,ext\mathcal{K}_{ji})$. The intersection $\mathcal{K}_{ji} \cap \Phi_i$ is a fundamental domain for action of $\langle b_{ji}\rangle$ in $\mathcal{K}_{ji}$. Therefore we put: $\Phi_l \cap (K_{kl} \equiv \partial\mathcal{K}_{kl}) = \mu^{kl}_{ji}(K_{ji} \cap \Phi_i)$. Here and below we suppose that $\mu^{kl}_{ji} = (\mu^{ji}_{kl})^{-1}$. Next we can extend $\Phi_l \cap K_{kl}$ to a fundamental domain $\Phi_l \cap \mathcal{K}_{kl}$ for action of $\langle b_{kl}\rangle$ in $\mathcal{K}_{kl}$. Finally we extend $\Phi_l \cap (\,K_{1l} \cup \ldots \cup K_{\delta_l l}\,)$ to a fundamental set of G_l in $\mathbb{H}^3 \setminus G_l \cdot (\,\mathcal{K}_{1l} \cup \ldots \cup \mathcal{K}_{\delta_l l})$.

5.8. Realization of sewing maps via mobius transformations.

We have constructed above the mobius transformations $\mu^{ji}_{kl} : K_{kl} \to K_{ji}$ for $i \le \mathfrak{z} \le l$. These maps shall realize sewing homeomorphism f^{ji}_{kl} between hyperbolic and Seifert manifolds. Let $i \le \mathfrak{z}$, $R_i = 10m_i + 6$, $m_i = \mathfrak{z}_i$. Then we repeat the disposition of balls $B(P_{ji}, 8)$ on O_{R_i} from the item 3.7. These balls shall be filled by tori of kind $T(1, 1)$ or $T_*(1, 1)$ as follows. We provide $\mathbb{N}^2$ with the lexicographic order. If $i, l \le \mathfrak{z}$, $(l, k) > (i, j)$ and $\partial_j X_i$

is glued with $\partial_k X_l$ then the ball $B(P_{kl}, 8)$ is filled by $T(1, 1)$ and the ball $B(P_{ji}, 8)$ is filled by $T_*(1, 1)$ (details see in the item 3.7). These tori will be denoted by T_{kl} and T_{ji} correspondingly. Let us denote by Φ_i^- the domain

$$\Phi_i \setminus \Big[\bigcup_{j=1}^{m_i} int(T_{ji}) \cup \bigcup_{j=1}^{s_i} int(\mathfrak{X}_{ji}) \Big]$$

For any i the manifold $\Phi_i^- \setminus \mathfrak{X}(m_i)$ is homeomorphic to $S^1 \times E_i$, where E_i is a surface of genus 0 with $m_i + 1$ boundary curves; also we have $\partial(\Phi_i^- \setminus \mathfrak{X}(m_i)) = T(m_i) \cup T_{1i} \cup ... \cup T_{m_i i}$. We choose simple directed loops $\tau_{ji} \subset T_{ji}$ which are parallel to the directed loop $\tilde{\theta}_i \subset T(m_i)$ in the manifold $S^1 \times E_i$ (the loop θ_i was introduced in the item 3.7). Further we put $\varkappa_{ji} = T_{ji} \cap \Delta_{R_i}$ be simple loops with with the clockwise orientation. The corresponding elements of $\pi_1(T_{ji})$ will be denoted by the same symbols τ_{ji}, $\varkappa_{ji}$.

By construction of the tori $T_*(1, 1)$, $T(1, 1)$, if $\partial_j X_i$ is glued with $\partial_k X_l$ $(l, i \le \mathfrak{z})$, then there exists a mobius transformation $\mu = \mu_{kl}$: $int\, T_{kl} \to ext\, T_{ji}$. We choose μ such that $\mu_*(\tau_{kl}) = \varkappa_{ji}$, $\mu_*(\varkappa_{kl}) = \tau_{ji}$.

Next we introduce a shortest directed infinite arcs χ_{kl} corresponding to $(K_{kl}, b_{kl}^{1/p})$, $l > \mathfrak{z}$. These arcs may be chosen so that $\mu(\chi_{kl}) = \tilde{\beta}_{ji} \equiv \chi_{ji}$. For $i \le \mathfrak{z}$ let $\theta_{ji} \subset K_{ji}$ be a simple directed loop homotopic to θ_i in $S^3 \setminus O_{R_i}$. If $k > \mathfrak{z}$ and $\partial_j X_i$ is glued with $\partial_k X_l$, then we put $\theta_{kl} = \mu_{ji}^{kl}(-\theta_{ji})$, where the sign "minus" means change of orientation. Let us remind that for $i \le \mathfrak{z}$ we have $e(G_i, \chi_{1i} \cup ... \cup \chi_{s_i i}) = e_i$.

5.9. Computation of matrixes of mobius sewing maps.

Let $\lambda_{ji} \in \pi_1(T_{ji})$ be equal to $\tilde{a}_{22}(j, i)\tau_{ji} + \varkappa_{ji}$, where $\tilde{a}_{22}(j, i)$ is coefficient of the matrix $\tilde{A}$ for the gluing homeomorphism f_{ji}^{kl}: $\partial_j X_i \to \partial_k X_l$. Then direct calculations show that $(\mu_{ji}^{kl})_* (\tau_{ji}, \lambda_{ji}) = \begin{pmatrix} \tilde{a}_{11}(j, i) & 1 \\ \tilde{a}_{12}(j, i) & \tilde{a}_2(j, i) \end{pmatrix} \begin{pmatrix} \tau_{kl} \\ \lambda_{kl} \end{pmatrix}$, where $a_{11}(j, i) = -a_{22}(k, l)$, $a_{12}(j, i) = a_{12}(k, l) = a_{11}a_{22} + 1$. So the maps $(\mu_{kl}^{ji})_*$ and $(f_{kl}^{ji})_*$ are given by the same matrixes in the bases $(\tau_{ji}, \lambda_{ji})$ and $(\tilde{u}_{ji}, \tilde{v}_{ji})$. Here and below, $(\tilde{u}_{ji}, \tilde{v}_{ji})$ is lift of the natural base on ∂M_j, to $\partial \tilde{M}_j \equiv \partial X_j$. Next we prove an analogous fact about maps of cones K_{ji}. Let $i \le \mathfrak{z}$, X_i be a Seifert manifold; first we suppose that $0 = m_i = \mathfrak{z}_i$ - number of Seifert components of ∂X_i.

Then we drop arcs χ_{ji} to the loops $\bar{\chi}_{ji} \subset \partial X_i$ via the covering $G_i \cdot \Phi_i^- \to X_i$. As we have seen above, for these loops the equality $e(X_i, \bar{\chi}_{1i} \cup ... \cup \bar{\chi}_{b_i i}) = e_i$ holds. Let $\bar{\theta}_{ji}$ be projection of $\theta_{ji} \subset K_{ji}$ to ∂X_i. Since $e(X_i, \tilde{v}_{1i} \cup ... \cup \tilde{v}_{b_i i}) = e_i$ then (up to automorphism of the fiber bundle $X_i \to \tilde{\Sigma}_i$) we have $(\bar{\theta}_{ji}, \bar{\chi}_{ji}) = (\tilde{u}_{ji}, \tilde{v}_{ji})$.

Essentially the same holds in the case $m_i > 0$. Then the relative euler number $e(X_i, \tilde{v}_{1i}, ..., \tilde{v}_{\delta_i i})$ is equal to 0, since $e(M_{i'}, v_{1i'} \cup ... \cup v_{\delta_i i'}) = 0$ (see the item 5.4). On other hand, $e(X_i, \bar{\chi}_{1i} \cup ... \cup \bar{\chi}_{s_i i} \cup \bar{\lambda}_{s_i+1,i} \cup ... \cup \bar{\lambda}_{\delta_i i}) = 0$, due to construction of the loops λ_{ji} and since

$$e(G_i, \bar{\chi}_{1i} \cup ... \cup \bar{\chi}_{b_i i}) = e_i = \sum_{k=1}^{m_i} \tilde{a}_{22}(i, s_i + k).$$

So we can suppose that $(\bar{\theta}_{ji}, \bar{\chi}_{ji}) = (\tilde{u}_{ji}, \tilde{v}_{ji})$, $(\tilde{\theta}_{ki}, \bar{\lambda}_{ki}) = (\tilde{u}_{ki}, \tilde{v}_{ki})$, $i \le \mathfrak{z}$, $j \le \mathfrak{z}_i < k$.

Now consider hyperbolic components X_i of M_o. Then $\bar{\chi}_{ji} = \tilde{v}_{ji}$ (up to homotopy) due to the Corollary 3.3 and since $\bar{\chi}_{ji}$ is projection of a shortest arc with respect to $(K_{ji}, (b_{ji})^{1/p})$. Further, we have $\bar{\theta}_{ji} = \tilde{u}_{ji}$ (up to homotopy) because the Dehn surgery on X_i, which annihilates $[u_{ji}]$, $j \le \delta_i$, gives us the manifold $\mathbb{H}^3/G_i$.

Let K_{ji}, $i > \mathfrak{z}$, and K_{kl}, $l \le \mathfrak{z}$, be a cones are paired by transformation $\mu = \mu_{ji}^{kl}$. Then $\mu(\chi_{ji}) = \chi_{kl}$, $\mu(\theta_{ji}) = -\theta_{kl}$ (see the item 5.8). So, projection of $\bar{\mu}_{ji}^{kl}: \partial_j X_i \to \partial_k X_l$ of μ_{ji}^{kl} is isotopic to the gluing homeomorphism $(\tilde{f}_{ji}^{kl})$.

5.10. Combination of Kleinian groups G_i.

Now we are ready to combine (by induction) the Kleinian groups G_i to obtain the necessary group G uniformizating M_o. We start with the group $G_o^* \equiv G_1^* \equiv G_1$, restricted fundamental set $\Phi_o^* \equiv \Phi_1^-$, subset $\mathfrak{C} = \{ id \} \subset \mathfrak{M}_3$ and empty subset $\mathfrak{S} \subset \mathfrak{M}_3$.

Step of induction. Next let us suppose that the groups G_1^*, ..., G_n^* are combined to the group G_o^*, accordingly to subset $\mathfrak{S}$ of mobius gluing maps of the item 5.8, $\mathfrak{S} = \{\mu_{kl}^{ji} : i, l \le n, j \in$ subset of $\langle 1, \dots, \delta_i \rangle$, $k \in$ subset of $\langle 1, \dots, \delta_l \rangle \}$. Suppose also that we have constructed a set of mobius transformations ν_i, $\nu_i^*(G_i) = G_i^*$, $\nu_i^{**}: g \longmapsto \nu_i g \nu_i^{-1}$, and some restricted fundamental set Φ_o^*. Let $\mu_{kl}^{ji} \notin \mathfrak{S}$ be a mobius sewing map such that $l \le n < i$. Then we put $\nu_i \equiv \nu_l \circ \mu_{ji}^{kl}$, $G_i^* \equiv \nu_i^*(G_i)$. The torus $\nu_l(T_{kl}) = \nu_i(T_{ji})$ (if $i, l \le \delta$), or the cone $\nu_l(K_{kl}) = \nu_i(K_{ji})$ (if $l \le \delta < i$) will be denoted by W_{kl}^*. It is precisely invariant in G_o^* and G_i^* under subgroup $\langle 1 \rangle$ (or cyclic loxodromic group $\nu_l^*(\langle b_{kl} \rangle) \equiv \langle b_{kl}^* \rangle$ in the cone case). Morethan, we have $\Phi_l \equiv \nu_l(\Phi_l^-) \subset int\ W_{kl}^*$. On the other hand, $\Phi_o^* \equiv \bigcup_{m < n+1} \Phi_m^* \subset ext\ W_{kl}^*$. So, the domain $cl(int\ W_{kl}^*)$ is a $(G_o^*, \langle b_{kl} \rangle)$-block (in the sense of § 1) and the domain $cl(ext\ W_{kl}^*)$ is a $(G_i^*, \langle b_{kl} \rangle)$-block. The intersection $\Phi_o^* \cap \Phi_i^*$ is equal to $\Phi_o^* \cap W_{kl}^* = \Phi_i^* \cap W_{kl}^*$. Now it is easy to see that for the groups G_o^* and G_i^* the conditions of 1st Combination Theorem are fulfilled, where the joint subgroup is $\langle 1 \rangle$ or $\langle b_{kl} \rangle$. The amalgamated free product $\langle G_i^*, G_o^* \rangle$ will be denoted by G_o^* again, $\mathfrak{S} := \mathfrak{S} \cup \{ \mu_{kl}^{ji} \}$, $\mathfrak{C} := \mathfrak{C} \cup \{\nu_i\}$, $\Phi_o^* := \Phi_o^* \cup \Phi_i^*$.

Arguing analogously we consider the case $i, l \le n$, when there takes place an HNN- extension of the group G_o^* by means of $\nu_i \circ \nu_l^{-1}$ (the 2-nd Combination Theorem). The group $\langle G_o^*, \nu_i \circ \nu_l^{-1} \rangle$ will be denoted then by G_o^*.

We shall repeat the above combination process until all mobius gluing maps μ_{kl}^{ji} will be used. The arising group G_o^* is denoted by G. Applying the 1-st and 2-nd Maskit Combination Theorems we obtain that the set Φ_o^* is fundamental for action of the group G in the invariant component $R_o \subset R(G)$ which contains the infinity. The properties of transformations μ_{kl}^{ji} (see the item 5.9) imply that R_o/G is homeomorphic to M_o. The theorem 5.1 is proved. QED.

§ 6. An example of orientable 3-manifold which does not admit a flat conformal structure but has a conformally flat finite-sheeted covering

6.1. Construction of the manifold.

Let $\mathcal{O}$ be the orbifold supported by $\mathbb{S}^1 \times [0, 1]$ and possessing

one singular cone point of order 2. Then we choose a Seifert fibration $N \longrightarrow \mathcal{O}$ over $\mathcal{O}$, such that $H = \pi_1(M) \simeq \langle a, b, c, t : c^2 = t,\ abc = 1,\ [a, t] = [b, t] = 1 \rangle$. The manifold N has two boundary tori T_1 and T_2, let $i_l : T_l \longrightarrow N$ be inclusions, $l = 1, 2$. The fundamental groups of T_1 and T_2 are generated by $\{a_1, t_1\}$ and $\{b_2, t_2\}$, where $i_{1*}(a_1, t_1) = (a, t)$ and $i_{2*}(b_2, t_2) = (b, t)$. There exists an orientation reversing homeomorphism $f : T_1 \longrightarrow T_2$ such as $f_*(a_1) = t_2$, $f_*(t_1) = b_2$. Let M be the manifold $N/(x \equiv f(x))$. It is easy to see that M obeys the conditions of the theorem 5.1 (since there are no hyperbolic and $\mathbb{E}^3$- components in canonical splitting of M). Then a finite-sheeted covering $M_o \longrightarrow M$ exists, such that M_o possesses a FCS.

6.2. THEOREM 6.1. The manifold M does not admit a FCS.

PROOF. Let us suppose that a FCS $\mathcal{K}$ on the manifold M exists; $d_*: \pi_1(M) \longrightarrow \mathfrak{M}_3$ is its holonomy representation. If $g \in G = \pi_1(M)$, then we put $g^* = d_*(g)$. The group G is an HNN- extension of H, $G \simeq \langle H, \varphi : \varphi^{-1} a \varphi = t,\ \varphi^{-1} t \varphi = b \rangle$.

LEMMA 6.1. For the group $G^* = d_*(G)$ one of the following assertions hold:

(a) G^* is almost abelian, (b) G^* has a two-point invariant set in $\mathbb{S}^3$, (c) the group G^* is conjugate in $\mathfrak{M}_3$ to some subgroup of SO(4), (d) G^* has an invariant euclidean circle L and point in $\mathbb{S}^3$, (e) G^* has an invariant euclidean sphere Σ and point in $\mathbb{S}^3$.

6.3. PROOF of the Lemma 6.1.

(1) First let us suppose that $t^* = 1$. Then $a^* = b^* = c^* = 1$ and $G^* = \langle \varphi^* \rangle$ and the assertion (a) holds.

(2') Let $1 \ne t^*$ be an elliptic transformation. If t^* has no fixed points in $\mathbb{S}^3$, then the extension of t^* to $\mathbb{H}^4$ has unique fixed point q there. The condition $[a^*, t^*] = [b^*, t^*] = 1$ implies that $a^*(q) = b^*(q) = q$ and $\langle q \rangle$ is invariant under H^*. Since $\varphi^{*-1} a^* \varphi^* = t^*$, then $\varphi^*(q) = q$ and the assertion (c) is true.

(2'') So we have $Fix(t^*) = l_t$ is a circle in $\mathbb{S}^3$. This circle we shall identify with $L = \{(x, 0, 0) \in \mathbb{R}^3 : x \in \mathbb{R}\}$. The half-plane $\overline{\mathbb{R}}_+^2 = \{(x_1, x_2, 0) : x_1 \in \mathbb{R},\ x_2 \ge 0\} \cup \{\infty\}$ will be denoted by Π. Since a^*, b^* are conjugate with t^*, then $Fix(a^*) = l_a$, $Fix(b^*) = l_b$ are euclidean circles. The commutation $[a^*, t^*] = [b^*, t^*] = 1$ implies that the following alternative holds :

(i) one of the circles l_a, l_b coincides with l_t, or
(ii) l_a and l_b are orthogonal to Π and their centers lie on L.

Consider (i). If $l_a = l_t = L$ then $\varphi^*(L) = L$, $l_b = \varphi^*(l_t) = L$ and, hence, $G^*(L) = L$. So, G^* is $\mathbb{Z}_2$-extension of an abelian group and the assertion (a) holds.

Consider (ii). Then Π is invariant under the group $\langle a^*, b^* \rangle$. Hence, $(c^* = (a^*b^*)^{-1})(\Pi) = \Pi$ and $(t^* = c^{*2})(\Pi) = \Pi$. Therefore, we have $t^* = 1$ (this case has been considered above).

(3) Let t^* be a loxodromic transformation, $Fix(t^*) = \{0, \infty\} \subset \overline{\mathbb{R}}^3$. Then the elements a^*, b^* are loxodromic too and $Fix(a^*) = Fix(b^*) = \{0, \infty\}$ (due to commutativity of $\langle a, t \rangle$, $\langle b, t \rangle$). Therefore, $\varphi^*(\{0, \infty\}) = \{0, \infty\}$ and the assertion (b) holds.

(4) Now we have the last case- t^* is parabolic, $t^*(\infty) = \infty$.

(4') First we suppose that $t^*(\vec{x}) = U\vec{x} + \vec{e}_1$, where $U \in SO(3) \setminus (1)$ is a rotation around the axis L. Then there holds: $a^*(L) = b^*(L) = L$, $a^*(\infty) = b^*(\infty) = \infty$, a^* and b^* are translations with rotation around L. Hence, $\varphi^*(L) = L$ and $\varphi^*(\infty) = \infty$, so the assertion (d) holds.

(4'') It remains the possibility- $t^*(\vec{x}) = \vec{x} + \vec{e}_1$. Then a^* and b^* are translations on vectors $\vec{a}$ and $\vec{b}$ correspondingly. The condition $t^* = (a^*b^*)^{-2}$ implies that $\vec{e}_1 = -2\vec{a} - 2\vec{b}$. Hence, the element φ^* can not be elliptic or parabolic (else $|\vec{e}_1| = |\vec{a}| = |\vec{b}|$, which is impossible).

So, the transformation φ^* is loxodromic. The group H^* has an invariant straight line L (if $\vec{a}$ and $\vec{b}$ are linearly dependent) or a plane P (if these vectors are independent). This line (or plane) may be chosen invariant under φ^*. These two cases correspond to the assertions (d) and (e). The lemma is proved. ∎

6.4. <u>Now we return to the proof of the theorem 6.1.</u>

Consider (a). Then $\pi_1(M)$ is almost abelian [Ku 2], that is impossible, since $M \not\approx (\mathbb{E}^3)$.

Consider (b). Then the image of the development map $d\colon \widetilde{M} \to \mathbb{S}^3$ is equal to $\mathbb{R}^3 \setminus (0)$ and $M = (\mathbb{R}^3 \setminus (0))/G^*$ (see [Ka 1]). This contradicts to asphericity of M.

Consider (c). Then M is the elliptic manifold (cf. [G K]) and $\pi_1(M)$ is finite, which isn't true.

The case (d) is impossible due to [Ka 1, Th. 3] .

There remains the possiblity (e). Let us consider the set $d^{-1}(\Sigma) \subset \widetilde{M}$, where $p\colon \widetilde{M} \to M$ is the universal covering. Then $S = p \circ d^{-1}(\Sigma)$ consists of incompressible surfaces [Ka 3] . For any component M_1 of $M \setminus S$ we have the restriction of d to a component of $p^{-1}(clM_1)$ is an equivariant homeomorphism onto $cl(int\ \Sigma) \setminus (p)$ or onto $cl(int\ \Sigma) \setminus (p)$ [Ka 3] . The last contradicts with compactness of M_1. This contradiction completes the proof of theorem 6.1. QED.

6.5. REMARK. Recently N.Isachenko [Is] showed that any representation of the group G into $\mathfrak{M}_3$ has a solvable image. Due to this fact he has constructed an example of discrete quasiconformal group Γ acting on $\mathbb{S}^3$, which isn't isomorphic to any subgroup of $\mathfrak{M}_3$. The group Γ is a finite extension of the Kleinian group G_o uniformizating finite covering M_o over M.

REFERENCES

[Ab] W.Abikoff, The real analytic theory of Teichmuller space. New York- Heidelberg- Berlin: Springer, 1980.

[Ap] B.Apanasov, Discrete groups and structures of manifolds. Novosibirsk, 1983 (in russian).

[Be] A.F.Berdon, The geometry of discrete groups. N.Y.- Heidelberg- Berlin: Springer, 1983.

[Bs] L.Bers, On moduli of Kleinian groups, Uspehi Mat. Nauk, 29 (1974), 86- 102 (in russian)

[CS] M.Culler, P.Shalen, Varieties of representations and splittings of 3-manifolds, Ann. of Math., 117 (1983), 109- 146.

[ECG] D.B.A.Epstein, R.Canary, P.Green, Notes on Notes of Thurston. In: London Math. Soc Lect. Notes, 111 (1987) 3-92

[EEC] A.Edmonds, J.Ewing, R.Kulkarni, Regular tesselations of surfaces and (p, q, 2) -triangle groups, Ann. of Math., 116 (1982), 113- 132.

[FS] M.Freedman, R.Scora, Strange actions of groups on spheres, J. Diff. Geom. 25 (1987), 75- 98.

[Go] W.Goldman, Conformally flat manifolds with nilpotent holonomy, Trans. Amer. Math. Soc., 278 (1983), 573- 583.

[G K] N.A.Gusevskii, M.E.Kapovich, Conformal structures on 3- manifolds, Soviet Math. Dokl. 34 (1987) 314- 318.

[GLT] M.Gromov, H.B.Lawson, W.Thurston, Hyperbolic 4-manifolds and conformally flat 3-manifolds, Publ. IHES, Vol. 68, 1989.

[GM] F.Gehring, G.Martin, Convergence groups, MSRI preprint, Berkley, 1986.

[H 1] J.Hempel, 3- manifolds, Princeton Univ. Press, 1976.

[H 2] J.Hempel, Residual finiteness for 3-manifolds. In: Ann. of Math. Stud. N 111, 1987, P. 373- 396.

[Is] N.Isachenko, Dokl. Acad. Sci. USSR (to appear)

[Iv] D.Ivascu, On Klein- Maskit combination theorem, Lecture Notes in Math. 743 (1976), 115- 124.

[J S] W.Jaco, P.Shalen, Seifert Fibre Spaces in 3-manifolds.- Memoirs of AMS, 1979, N 2.

[Ka 1] M.E.Kapovich, Some properties of developments of conformal structures on 3-manifolds, Soviet. Math. Dokl. 35 (1987), 146- 149.

[Ka 2] M.E.Kapovich, Flat conformal structures on 3-manifolds, Siberian Branch of USSR Acad. of Sci., Institute of mathematics, 1987, preprint 17 (in russian).

[Ka 3] M.E.Kapovich, On conformal structures with fuchsian holonomy, Dokl. Acad. of Sciences of USSR, 301 (1988), 23- 26 (in russian).

[Ka 4] M.E.Kapovich, Deformations of representations of fundamental groups of 3- manifolds (to appear)

[Ka 5] M.E.Kapovich, Kleinian groups and flat conformal structures. Dissertation, Institute of Mathematics of Siberian Branch of the Soviet Academy of Sci., 1988.

[Ka 6] M.Kapovich, Deformation spaces of flat conformal structures, Answers & Questions in General Topology (to appear)

[Kam] Y.Kamishima, Conformally flat manifolds whose development maps are not surjective, Trans. Amer. Math. Soc., 294 (1986), 607- 621.

[Koj] S.Kojima, Isometry groups of hyperbolic 3-manifolds, Topology and its Appl., 29 (1988) 297- 307.

[Ku 1] N.Kuiper, On conformally- flat spaces in the large, Ann. of Math. 50 (1949), 916- 924.

[Ku 2] N.Kuiper, On compact conformally euclidean spaces of dimension > 2, Ann. of Math., 52 (1950), 478- 490.

[Ku 3] N.Kuiper, Hyperbolic 4- manifolds and tesselations,

2.оборот

Math. Publ. IHES , vol. 68, 1988.

[Kul] R.Kulkarni, On principle of uniformmization, J. Diff. Geom., 13 (1978), 109- 138.

[K A G] S.L.Krushkal, B.N.Apanasov, N.A.Gusevskii, Kleinian groups and uniformization in examples and problems, Transl. of AMS, Vol. 62, 1986.

[L] L.Lok, Deformations of locally homogenious spaces, Thesis, Columbia Univ., 1984.

[Mar] G.Martin, Discrete quasiconformmal groups, Ann. Acad Sci. Fenn., Ser.AI Math., 11 (1986), 179- 202.

[Mk 1] B.Maskit, Kleinian groups, New York- Heidelberg- Berlin: Springer, 1987.

[Mk 2] B.Maskit, Self- maps of Kleinian groups, Amer. Math. J. 93 (1971), 840- 856.

[Mk 3] B.Maskit, On Klein combination theorem, III. In: Advances in the theory of Riemann surfaces, Princ. Univ. 1971,

[Mo] J.Morgan, On Thurston's uniformization theorem for 3-dimensional manifolds. In: Smith Conjecture, Acad. Press, 1984

[MM] D.McCullough, A.Miller, Manifold covers of 3-orbifolds with geometric pieces, Preprint, 1987.

[MS] W.Meeks, P.Scott, Finite group actions on 3- manifolds, Inv. Math. 86 (1986), 287- 346.

[NZ] W.Newmann, D.Zagier, Volumes of hyperbolic manifolds, Topology, 24 (1985), 307- 332.

[R] Raymond F., Classification of actions of the circle, Trans. Amer. Math. Soc., 131 (1968), 51- 78.

[Sc] P.Scott, The geometries of 3- manifolds, Bull. London Math. Soc., 15 (1983), 401- 478.

[T1] W.Thurston, Hyperbolic structures on 3-manifolds. I: Ann. of Math. 124 (1986), 203- 246.

[T2] W.Thurston, Geometry and topology of 3- manifolds, Princeton Univ. Lecture Notes, 1978.

[Tu] P.Tukkia, A quasiconformal group not isomorphic to Mobius, Ann. Acad Sci. Fenn., Ser.AI Math., 6 (1986), 149- 160.

[W] A.Weil, Remarks on cohomology of groups, Ann. of Math., 80 (1964), 149- 157.