Rigidity of groups and group actions and some (un)related questions

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1 Definitions and background

While considering groups as geometric objects one is inevitably confronted with the fact that Cayley graphs of groups (as metric spaces) are not uniquely determined by a group, but are unique up to quasi-isometry.

Definition 1.1. A map $f : X \to Y$ between two metric spaces is called an (L, A) quasi-isometric embedding if for every $x, y \in X$ we have

$$\frac{1}{L}d(x, y) - A \leq d(f(x), f(y)) \leq Ld(x, y) + A.$$

A map f is an (L, A) *quasi-isometry* if every point in Y is within distance $\leq A$ from some $f(x)$, $x \in X$.

Therefore, the key meta-questions are:

To which extent quasi-isometries preserve algebraic properties of groups?

In particular,

To which extent quasi-isometry between groups implies an isomorphism?

Recall that commensurable groups are quasi-isometric and an extension with finite kernel implies quasi-isometry. Therefore, one has to be content with comparing quasi-isometry and the *weak commensurability* equivalence relation on groups induced by passing to finite index subgroups and extensions with finite kernel.

Given a group G one defines the *abstract commensurator* $Comm(G)$ as follows. The elements of $Comm(G)$ are equivalence classes of isomorphisms between finite index subgroups of G. Two such isomorphisms $\psi : G_1 \to G_2, \phi : G'_1 \to G'_2$ are equivalent if their restrictions to further finite index subgroups $G''_1 \to G''_2$ are equal. The composition and the inverse are defined in the obvious way, making $Comm(G)$ a group.

Let X be a metric space or a group G. Call X *strongly QI rigid* if each (L, A)-quasi-isometry $f : X \to X$ is within finite distance from an isometry $\phi : X \to X$ or an element ϕ of $Comm(G)$ and moreover $d(f, \phi) \leq C(L, A)$.

Call a group G *QI rigid* if any group G' which is quasi-isometric to G is weakly commensurable to G.

Call a class of groups $\mathcal{G}$ *QI rigid* if each group G which is quasi-isometric to a member of $\mathcal{G}$ is actually weakly commensurable to a member of $\mathcal{G}$.

Among quasi-isometric invariants of groups are:

1. Rate of growth.

2. Dehn function, and, hence, solvability of the word problem.

3. Amenability (since Følner property is manifestly geometric).

4. Finite presentability and, more generally, various "higher" finiteness properties, like F_n, F_∞, FP_n, FP_∞, see [17].

5. Virtual nilpotence [16], virtual nilpotence degree [25] and Hirsch rank for solvable groups [32]. Among nilpotent groups, real cohomology rings and associated graded Lie algebras.

6. Hyperbolicity.

7. (Co)homological dimension among groups of finite (co)homological dimension [32].

8. Being weakly commensurable to a lattice in a semisimple algebraic group (over Archimedean or non-Archimedean local field), this is a combination of [19, 10, 37].

9. Being a virtually free group.

10. Admitting a splitting over a finite subgroup [33]. Admitting (virtually) a splitting over a (virtually) cyclic subgroup [26].

11. Being weakly commensurable to the fundamental group of a closed surface (follows, for instance, from the characterization of PD(2) groups in [8, 7]), or a closed 3-manifold (combination of the geometrization conjecture, [18] for generic Haken manifolds, [16] for Euclidean and Nil-manifolds and [11] for Sol-manifolds).

12. Being (virtually) the mapping class group of a surface [2].

Among strongly QI rigid metric spaces are:

1. Nonpositively curved symmetric spaces and Euclidean buildings except for those which have factors isometric to the real-hyperbolic space, complex-hyperbolic space or a tree.

2. Curve complex for closed surfaces of genus ≥ 4.

3. Mapping class groups of closed surfaces of genus ≥ 3.

4. Locally compact thick 2-dimensional hyperbolic buildings [4, 38].

On the other hand, various properties of groups are not quasi-isometry invariant:

1. Virtual solvability (A. Erschler [6]).

2. Property (T) (S. Gersten, M. Ramachandran).

3. Linearity and residual finiteness (they is not even weak commensurability invariant), see e.g. [24].

2 Quasi-isometric rigidity and other questions on solvable groups

Problem 2.1. *Let G_1, G_2 be quasi-isometric nilpotent groups. Then G_1, G_2 are lattices in the same connected nilpotent Lie group.*

Note that such groups need not be weakly commensurable. It was proven by R. Sauer [32] that quasi-isometric nilpotent groups have isomorphic *real* cohomology rings. To give positive answer to the above question, one has to push this result to the level of *real minimal models* (real Malcev completions), see e.g. [15, 1] for the background.

Problem 2.2. *Given a group Γ quasi-isometric to a lattice Λ in a Lie group G is Γ virtually a lattice in some Lie group G'?*

Note that G and G' are necessarily quasi-isometric. It is expected that the answer is yes. For Λ non-uniform, the result should be stronger and under mild irreducibility assumptions, in the non-uniform setting, one should not need to switch G for a quasi-isometric G'. The non-uniform case is probably somewhat harder than the uniform one.

A special case of this is the following conjecture that appears in [11].

Conjecture 2.1. *Any group quasi-isometric to a polycyclic group is virtually polycyclic.*

This is the same as asking that any group quasi-isometric to a lattice in a solvable Lie group is virtually a lattice in a solvable Lie group. Various cases of this can be done by work of Eskin, Fischer and Whyte, more are done by Peng [11, 12, 13, 27, 28]. Work of Dymarz is also an important tool in many cases [5]. The general case is probably accessible by similar methods, though a better understanding of the geometry of a general solvable Lie group is needed.

Problem 2.3 (D. Fischer)**.** *Given a group* Γ *quasi-isometric to a Lie group* G*, is* Γ *virtually a lattice in another Lie group* G'*? In particular, if no Lie group quasi-isometric to* G *admits a lattice, are there no finitely generated groups quasi-isometric to* G*?*

It is easy to see that the odd phrasing is necessary. There are groups G with no lattices where some obvious G' has a lattice. E.g. take G to be the Borel subgroup of a simple Lie group. The only cases of this question where the answer is known involve G which is reductive or does not admit lattices because it is non-unimodular.

Problem 2.4 (D. Fischer)**.** *Find a subclass of amenable groups that contains solvable groups and is closed under quasi-isometries? Identify the QI closure of solvable groups?*

At the moment, the only known class of amenable groups that cannot be quasi-isometric to solvable groups consists the groups of intermediate growth. Probably there are other examples that can be shown to be non-solvable by some invariant, but this is a long way from answering the question. The question is motivated in part by results on quasi-isometric rigidity of lamplighter groups. This is the “easiest” quasi-isometry class of amenable groups that contains both solvable and non-solvable groups and we now know that it is (essentially) closed under the relation of quasi-isometries [11, 12, 13].

Another family of questions involves the quasi-isometry classification of lattices in solvable analytic Lie groups defined over various fields. Actually, it is currently seems that it is better to ask this question about arithmetic subgroups than about lattices. For example the Lamplighter group is arithmetic but not a lattice. For Archimedean fields, we at least have a good idea what all the solvable analytic Lie groups are. This is not true in general. Some first problems are:

Problem 2.5 (D. Fischer)**.** *Describe all solvable, analytic Lie groups over non-Archimedean local fields. Determine which of these groups contain arithmetic groups. Determine which contain finitely generated arithmetic groups. (Finite generation is not automatic in this context.) Also classify arithmetic groups in products of solvable, analytic groups over different fields, some Archimedean, some not.*

One motivation for this question is Wortman's observation that many lamplighter groups are virtually arithmetic subgroups in algebraic groups defined over fields of positive characteristic. Another is that, at least in those examples, techniques for quasi-isometric rigidity theorems seem to transfer to this setting easily.

More profoundly, one knows that the class of all polycyclic groups is much better behaved than the class of all solvable groups. The question is looking for a larger class of solvable groups that would remain conducive to study, but be much larger. The guess is that the right class to study is the answer to the question above.

The problem probably has a reasonable answer if one assumes the group is algebraic. For non-algebraic groups, one needs to see if there is an analogue of Auslander's structure theory. This is likely true in characteristic zero, but it seems unclear in positive characteristic. The class of groups that are arithmetic in products of solvable groups over (possibly) different local fields might be called *S-polycyclic groups*. The analogous theory is quite well developed for semisimple groups, but appears almost non-existent for solvable groups.

The following is motivated by Mostow's characterization of polycyclic groups as (virtually) lattices in solvable Lie groups.

Problem 2.6 (D. Fischer)**.** *Find an algebraic characterization of S-polycyclic groups.*

This of course depends on the answer to the earlier question. Another set of questions is:

Problem 2.7 (D. Fischer)**.** *Formulate and prove your favorite rigidity theorem (QI rigidity, strong rigidity, superrigidity) for S-polycyclic groups.*

Some of this is much harder than it looks, see the paper of Lifschitz and Witte Morris for the difficulties in proving even strong rigidity for S-arithmetic nilpotent groups [20]. If one assumes the existence of an infinite place in S, then Witte has already proven some strong results about super and strong rigidity [36]. As pointed out in [20], it is well known that unipotent p-adic algebraic groups have no lattices. I believe this is also true in the solvable case, but do not know a proof. There are S-arithmetic examples in products of algebraic p-adic and real groups groups.

Another natural question is:

Problem 2.8 (D. Fischer)**.** *Is there a unimodular, homogeneous graph that is not quasi-isometric to a finitely generated group?*

Definition 2.1. A graph is called *unimodular* if its automorphism group is unimodular, i.e. admits a bi-invariant measure?

If the word unimodular in the above problem is dropped, then this question was asked by Woess in the early 90's and answered by the Eskin, Fischer and Whyte in 2005. Even with the word unimodular added, the answer should be yes. I believe something stronger should be true:

Problem 2.9 (D. Fischer)**.** *Is there a reasonable, and provable, version of the statement "a random homogeneous graph is not quasi-isometric to a Cayley graph"? Same question with unimodular added.*

Though one knows many definitions of random graph and random group, it seems a bit tricky to find a good working definition of random homogeneous graph. One might try to define random totally disconnected groups and then define random homogeneous graphs as their Schreier graphs.

3 Measure equivalence and von Neumann rigidity

Groups Γ_1, Γ_2 are quasi-isometric if and only if there exists a locally compact topological space X and commuting cocompact properly discontinuous actions $\Gamma_i \curvearrowright X$, $i = 1, 2$. This leads to a natural generalization of quasi-isometry:

Definition 3.1. Groups Γ_1, Γ_2 are called *measure-equivalent* (ME) if there exists a measure space X and commuting, free, measure-preserving actions $\Gamma_i \curvearrowright X$ $(i = 1, 2)$ each of which admits a measurable fundamental domain of finite measure.

Examples of measure equivalent properties are:

1. A group is amenable iff it is ME to $\mathbb{Z}$.
2. Property (T) and Haagerup property are ME invariant.
3. Being a lattice in a higher rank irreducible Lie group is a ME invariant.

On the other hand, admitting a decomposition over a finite group is not a ME invariant. Hyperbolicity is not ME invariant.

Problem 3.1. *Let Γ be a group with property (T) and Λ a group measure equivalent to Γ. Is there a locally compact group G which (virtually) contains both Γ and Λ as lattices?*

The last question is the ME equivalent of the following older question of Connes. In this question all groups are assumed ICC and $L(G)$ is the von Neumann algebra of G.

Problem 3.2. *Let Γ be a group with property (T) and Λ another group with $L(\Gamma) \cong L(\Lambda)$, is $\Gamma \cong \Lambda$?*

Question 3.1 is also an ME variant of a QI question where "property (T)" is replaced by "one-ended hyperbolic" and ME by QI. It may be that one needs additional conditions on Γ and Λ to make the question interesting. The following seems more likely to be true, though equally inaccessible by current technology.

Problem 3.3. *Given a random group Γ with property (T), is any group Λ which is ME to Γ virtually Γ.*

Again, this is something of a "back-translation" of the analogous QI question about random (hyperbolic) groups. The most concrete case of question 3.1 that is open is:

Problem 3.4. *Let Γ be a lattice in G where G is either $Sp(1,n)$ or F_4^{-20}. Is any group ME to Γ virtually a lattice in G?*

The answer to this question should be yes. An closely related question about orbit equivalence of actions of G is due to Zimmer.

To return to the ME question, an inconceivable way to answer it is to answer the following question positively and then proceed by cases:

Problem 3.5. *Is every measure preserving ergodic action of a group with property (T) built out of the following types of pieces*

1. *discrete spectrum*
2. *affine algebraic*
3. *generalized Bernouilli shift, or*
4. *Gaussian.*

By built from is meant to include constructions involving both products and skew products or any more general procedure that would lead to a tower of extensions with some sort of relative version of the above properties. One might consider the question to be "generalized, measurable Zimmer program" which seems difficult as the "smooth Zimmer program" is already quite hard and involves a lot more structure. It is important to note that here we seek a weaker notion of "equivalence" asking only for measurable rather than smooth conjugacy. Most experts seem to believe the answer to this question should be no.

4 Groups acting on CAT(0) spaces, property (T) and variants

By a $CAT(0)$ Hilbert manifold we mean a complete $CAT(0)$ space all of whose tangent cones are Hilbert spaces. This includes the case of Riemannian manifolds of non-positive curvature, the case of Hilbert spaces and much more.

Problem 4.1. *If Γ is a group with property (T) for which all finite dimensional linear representations over a field of characteristic zero are unitary, is it true that every action of Γ on a $CAT(0)$ Hilbert manifold has a fixed point?*

This is known for groups which arise as fundamental groups of certain simplicial complexes by work of Izeki and Nayatani. I believe this class includes all uniform lattices in isometry groups of higher rank buildings, but do not know if there is a proof in the literature. This should follow from combining Garland-type inequalities with the work of Izeki and Nayatani. A much harder question is whether this holds for non-uniform lattices in the isometry groups of higher rank buildings. Also random groups with property (T) in Zuk's "triangular" model are of this type, though this is not said explicitly in Zuk's paper, see e.g. Ollivier's book. Gromov has apparently indicated that the answer to the question should be no for proper hyperbolic quotients of lattices in $SP(1,n)$ (I learned this from a conversation with Pete Storm). One is supposed to be able to build actions of these groups on finite dimensional negatively curved manifolds that are reminiscent of actions of Kleinian groups on $\mathbb{H}^n$.

The following two questions present an interesting contrast, particularly as they both seem to be attributed to Gromov.

Problem 4.2. *Does every group with a finite $K(\pi,1)$ admit a proper action on a $CAT(0)$ space?*

Problem 4.3. *Is there a finitely generated group all of whose action on complete $CAT(0)$ spaces have fixed points?*

A good candidate for a positive answer to the second question is an infinite torsion group with uniformly bounded torsion, e.g. Grigorchuk's "monster". These groups are conjectured by Shalom to have property (T) and one might consider this question a strengthening of that conjecture. One can weaken/strengthen the questions by replaced $CAT(0)$ by some weaker notion of non-positive curvature or uniform convexity.

A recent result of Shalom allows one, for the first time, to construct linear groups with property (T) which are not lattices. Shalom shows $SL(n,\mathbb{Z}[x])$ has property (T). By evaluating x at a transcendental, one gets a non-lattice with (T). The groups considered by Shalom are "generalized lattices" in some obvious sense and Shalom calls them *universal lattices*. So there remains a question as to whether there is a linear group with property (T) that is not anything like a lattice, one variant of this is:

Problem 4.4. *Is there a linear group with property (T) which does not contain a lattice with property (T)?*

This is a vaguer version of an older question variant due to Zimmer:

Problem 4.5. *Is there an infinite index subgroup of $SL(3,\mathbb{Z})$ with property (T)?*

5 Groups acting on manifolds

The following two questions are due to Zimmer.

Problem 5.1. *Let Γ be a group with property (T). Is any smooth volume preserving action of Γ on a compact surface virtually trivial?*

Problem 5.2. *Let Γ be a lattice in a semi-simple Lie group G with property (T) and no compact factors. Let $d = d(G)$ be the minimal dimension of a non-trivial representation of G. Is every smooth volume preserving action of Γ on a compact manifold of dimension less than d virtually trivial?*

Zimmer has conjecture that the answer to both questions is yes. The second question is sometimes generalized to non-volume preserving actions by decreasing the dimension by one.

Problem 5.3. *Let Γ be a group with the property that any Γ action on a $CAT(0)$ Hilbert manifold has a fixed point. Is every volume preserving Γ action on any compact manifold isometric? virtually trivial?*

The last question is the most reasonable approach I know to a claim by Gromov that random groups do not act on compact manifolds, though it is only an approach in the volume preserving case. In the context of the last three questions, one knows how to produce an invariant Riemannian metric of very low regularity, i.e. measurable with a fairly weak growth condition. So one can view the question as one about regularity of invariant metric. For the first two question, producing an invariant metric of almost any better regularity allows one to prove the theorem. However all recent progress has been in the context of actions on surfaces and has made no use of this information. See work of Franks-Handel and Polterovitch.

Here are three other questions related to group actions that I learned recently from Etienne Ghys. He conjectures that the answer to all three questions is yes. All three questions are based on analogies between linear groups and groups of diffeomorphisms of compact manifolds.

Problem 5.4. *Let M be a compact manifold. Does there exist $k(M)$ such that any finite group $F < Diff(M)$ has an abelian normal subgroup of index k?*

This is an analogue of Jordan's theorem for finite subgroups of $GL(n, \mathbb{R})$. The answer is yes for closed manifolds of dimension ≤ 3: In dimension 2 it follows, e.g. from the solution of Nielsen realization problem, in dimension 3 it follows from the solution of the Geometrization Conjecture. In higher dimensions not much is known. Note that if we project the finite groups to the outer automorphism group of the fundamental group, the problem becomes group-theoretic and, then, the answer is again positive for hyperbolic fundamental groups [30].

An old result of L.N. Mann and J.C. Su [23] on p-groups in diffeomorphism groups taken with the classification of finite simple groups, implies that for any compact M, at most finitely many simple groups embed in $Diff(M)$.

Problem 5.5. *Let M be a compact manifold and Γ a finitely generated subgroup of $Diff(M)$. If Γ does not preserve a measure on M, does Γ contains a free non-abelian subgroup?*

This is an analogue of the Tits' alternative. The only case that is known is when $M = S^1$ where it is due to Margulis. Another variant of Tits' alternative is:

Problem 5.6. *Does any finitely generated group G of analytic diffeomorphisms of a compact manifold satisfy Tits' alternative? I.e. if G is not virtually solvable then G contains a free non-abelian group on at least two generators.*

The reason not to ask this last question in the smooth category is the realization of Thompson's group as a group of smooth diffeomorphisms of the circle [14].

6 Some other questions about groups

The following question is motivated by recent discussion on non-commutative versions of Freiman's theorem on Terry Tao's blog [34]. One wants to understand all subsets of all groups that have a property called "small-tripling".

Problem 6.1. *Given a group Γ, is it true that any subset of $A \subset \Gamma$ with $|A{\cdot}A{\cdot}A| = O(|A|)$ has a large subset A' which is a ball in some nilpotent subgroup of Γ. Here "large subset" means that A' contains some fixed proportion of the elements in A.*

For abelian groups this is a theorem, see e.g. the book by Tao and Vu [35]. The book also contains some background for this result, e.g. why we consider $A{\cdot}A{\cdot}A$ and not just $A{\cdot}A$ or quadruples in A. Detailed discussion of the question can be found at [35]. It has been suggested by Lindenstrauss that this question might be related to a quantative version of Gromov's theorem on groups of polynomial growth. It seems interesting to look for negative answers in groups of intermediate growth. Known results are mostly for linear groups.

A particularly interesting group is $\mathbb{Z}^2 \wr \mathbb{Z}_2$. This is a two dimensional analogue of the lamplighter group. Computing the geodesics in this group is equivalent to solving the Travelling Salesman Problem. So in particular a polynomial time computation (in reasonable initial data) would imply that $P = NP$ and so seems unlikely. This does make the following question seem important:

Problem 6.2. *Understand the geometry of* $\mathbb{Z}^2 \wr \mathbb{Z}_2$*?*

One can ask more precise variants of this question. The following is probably due to de la Harpe:

Problem 6.3. *Find all groups quasi-isometric to* $\mathbb{Z}^2 \wr \mathbb{Z}_2$*? Do this for* $\mathbb{Z}^n \wr \mathbb{Z}_m$*?*

This is not as implausible as one might first think. It seems conceivable that one can answer the question only knowing about quasi-geodesics and not geodesics. Having an algorithm to produce reasonable quasi-geodesics says very little about producing geodesics since the group has exponential growth.

The following question is from Lyons, Pemantle and Peres.

Problem 6.4. *Is there an inward biased random walk on* $\mathbb{Z}^2 \wr \mathbb{Z}_2$ *have positive rate of escape?*

In [21], the notion of inward bias is quite precise and means biasing the walk radially back towards the identity. Lyons, Pemantle and Peres ask the same question for inward biased random walks on any finitely generated amenable group of exponential growth and prove that the answer is yes for $\mathbb{Z} \wr \mathbb{Z}_2$. Their proof generalizes to the case of $\mathbb{Z} \wr \mathbb{Z}_m$. For other groups, like $\mathbb{Z}^2 \wr \mathbb{Z}_2$, the fact that one can't compute geodesics makes it difficult to work with radial bias. The question is appealing for solvable Lie groups and their lattices, though even there, one probably does not know enough about geodesics to follow the proof in [21].

One might vary the problem by considering other notions of inward bias that avoid the problems concerning computability of geodesics.

7 Questions collected during the workshop

7.1 QI invariants

Problem 7.1 (Mosher)**.** *Find QI invariants of free-by-cyclic groups (other than those found by N. Macura). In particular find QI invariants of free-by-cyclic groups with exponentially growing monodromy.*

A candidate for a specific invariant:

Definition 7.1. Fix $\phi : F_n \to F_n$. Given $\gamma \in F_n$, define $\lambda_\gamma = \lim_{k\to\infty} |\phi^k(\gamma)|^{1/k}$. Only finitely many numbers occur, and they can be obtained (in a non-obvious way) from the train track representation of ϕ. Let $\Lambda_\phi = \lambda_1 \leq \cdots \leq \lambda_m$ be the list of these numbers, counted with an appropriate multiplicity. Let $\log \Lambda_\phi$ be the projective class $[\log \lambda_1 : \cdots : \log \lambda_m]$.

Problem 7.2 (Mosher)**.** *Is* $\{\log \Lambda_\gamma, \log \Lambda_{\gamma^{-1}}\}$ *a QI invariant? (Is it even an isomorphism invariant?)*

Problem 7.3 (Kapovich)**.** *Can the Pansu conformal dimension of the boundary be related to* $\{\log \Lambda_\gamma, \log \Lambda_{\gamma^{-1}}\}$*? Note that Pansu conformal dimension* is *a QI invariant.*

In case ϕ is fully irreducible, there is no information contained in the above proposed invariant.

Problem 7.4 (Mosher)**.** *Find free-by-cyclic groups with fully irreducible monodromy which are not QI to one another.*

Problem 7.5 (Behrstock)**.** *Are there groups with Menger curve boundary which are QI rigid?*

Remark 7.1. This question is a special case of the well known question: is a random group QI rigid. Since a random group is hyperbolic with Menger curve boundary, it would be nice to have such an example. Misha remarked that this might follow from work of Bourdon and Pajot.

7.2 Decidability and quasi-isometry

It was noted that solvability of the word problem is a QI invariant of finitely presented groups (since solvability is equivalent to having a recursive upper bound for the Dehn function).

Problem 7.6 (Osin). *Is the class of recursively presented groups QI rigid? Within this class, is solvability of the word problem a QI invariant?*

Problem 7.7 (Osin). *Is decidability of the elementary theory an invariant of quasi-isometry?*

8 Solvable groups

Eskin, Fisher and Whyte have recently made enormous progress on understanding groups quasi-isometric to particular solvable groups, especially lattices in Sol, and wreath products (lamplighter groups) $F \wr \mathbb{Z}$ for F finite.

Problem 8.1 (Fisher). *What groups are QI to*

1. $\mathbb{Z}_2 \wr \mathbb{Z}^2$ *(more generally* $F \wr \mathbb{Z}^n$ *for* F *finite; all* F *with the same order yield QI groups)?*

2. $\mathbb{Z} \wr \mathbb{Z}$*?*

Remark 8.1. Understanding either of these questions is expected to be hard, but the second (about $\mathbb{Z} \wr \mathbb{Z}$) is expected to be much harder.

8.1 Questions of an analytic flavor

Let X be a metric graph or a Riemannian manifold. Then X is said to have *(strong) Liuouville property* if every bounded (resp. positive) harmonic function on X is constant, see [31, 9] for more detailed discussion of these properties. It is known that both properties are not QI unvariant for both graphs and Riemannian manifolds [22, 3]. However, none of these examples admits a geometric (cocompact isometric properly discontinuous) group action.

Problem 8.2 (Saloff-Coste). *1. Is there a pair of quasi-isometric groups* G_1, G_2 *and geometric actions* $G_i \curvearrowright X_i$*, where* X_i *are either graphs of Riemannian manifolds, so that* X_1 *has Liuouville property, while* X_2 *does not?*

2. Is there such an example with $G = G_1 = G_2$ *and* X_1, X_2 *Cayley graphs of* G*?*

Problem 8.3 (M. Kapovich). *Is Property (T) a QI invariant of two-dimensional groups?*

A related question:

Problem 8.4 (M. Kapovich)**.** *1. Find some QI invariant which distinguishes between CAT*(0) *square complexes and CAT*(0) *triangle complexes. (One has to assume that the complexes are thick to avoid obvious counter-examples.)*

2. Find a QI invariant which distinguishes between CAT(0) *triangular complexes X_1, X_2, where all vertex links in X_1 have $\lambda_1 > 1/2$ and all vertex links in X_2 have $\lambda_1 < 1/2$.*

Remark 8.2. The point here is that groups acting geometrically on CAT(0) cube complexes have the Haagerup property, whereas groups acting geometrically on (generic) two-dimensional triangle complexes have (T).

(See also Questions 9.7 and 9.8.)

9 JSJ

Problem 9.1 (Kapovich)**.** *It is known that the JSJ decomposition over $\mathbb{Z}$ is QI invariant (in the sense that the collection of QI types of simple pieces is a QI invariant?). What about higher dimensional JSJs?*

Remark 9.1 (Mosher)*.* A lot of results can be found in Mosher-Sageev-Whyte and in Papasoglu, but there is much room for improvement.

This led to Misha proposing a specific question which couldn't be handled by known techniques:

Problem 9.2 (Kapovich)**.** *Leeb and Scott show that there is a JSJ for closed non-positively curved manifolds of dimension ≥ 4. Is this decomposition a QI invariant of the fundamental groups of such manifolds?*

Remark 9.2. Misha remarked that a student of his (J. Lee) has shown that any codimension 1 flat lies in one of the pieces of the Leeb-Scott decomposition.

9.1 QI embeddings

Problem 9.3 (Freedman/Shalom)**.** *If G has exponential growth, does it have a QI embedded (or uniformly embedded?) binary tree?*

Problem 9.4 (Dranishnikov)**.** *If G has asymptotic dimension n, does it embed in a product of n trees?*

Answer is "yes" for a product of $n+1$ trees.

Problem 9.5 (Fisher)**.** *Does a symmetric space of non-positive curvature always QI embed in some finite product of bounded valence trees?*

Remark 9.3. Dranishnikov remarked that if "bounded valence" is removed, then the answer is yes.

Bonk and Schramm showed that any hyperbolic group can be QI embedded into $\mathbb{H}^n$ for some n.

Problem 9.6 (Kapovich)**.** *When can the Bonk-Schramm embedding be made "equivariant"? That is, when does the QI embedding extend to an action by uniform quasi-isometries on $\mathbb{H}^n$? Can this ever happen for an infinite property (T) group? In the other direction, would such an extension imply the Haagerup property?*

Problem 9.7 (Fisher)**.** *Is $QC(\S^n)$ a-(T)-menable (i.e. has the Haagerup property)?*

Problem 9.8 (Valette?)**.** *Is the Haagerup property a QI invariant?*

Remark 9.4. Probably it would be better to consistently refer to a-(T)-menability or the Haagerup property and not alternate.

Problem 9.9 (Behrstock)**.** *Is there a finitely presented group with quasi-flats, but no $\mathbb{Z} \oplus \mathbb{Z}$? (Note that there exists finitely generated groups, which yield counterexamples if the finite presentability assumption is dropped.)*

9.2 Bestvina problem list

11.1 Still not understood for $n = 2$.

11.2 Closed, by recent work of Eskin, Fisher and Whyte.

11.3 Open, and the same question should be asked for the Mostow–Siu examples.

11.4 Open and important, see Problems 7.1 through 7.4 for more specific questions.

11.5 Closed.

11.6 Open.

9.3 Techmüller questions

Problem 9.10 (Kapovich)**.** *Is the quasi-isometry group of Teichmüller space abstractly commensurable to the mapping class group?*

Problem 9.11 (Manning)**.** *What about the QI group of the curve complex? In particular, are there parabolic QIs of the curve complex?*

This was recently solved by Rafi and Schleimer [29] in "almost all" cases.

Problem 9.12. *Is there a finitely generated group QI to Teichmüller space with the Teichmüller metric? With the WP metric?*

The expected answer is no.

9.4 Other questions

Problem 9.1. *Are there finitely generated (amenable) groups which are quasi-isometric but not bi-Lipschitz homeomorphic?*

Problem 9.2. *Construct an example of a hyperbolic group with Menger curve boundary, which is QI rigid.*

Problem 9.3. *Let G be a* random *k-generated group, $k \geq 2$. Is G QI rigid?*

Randomness can be defined for instance as follows. Consider the set $B(n)$ of presentations

$$\langle x_1, ..., x_k | R_1, ..., R_l \rangle$$

where the total length of the words $R_1, ..., R_l$ is $\leq n$. Then a class C of k-generated groups is said to consist of random groups if

$$\lim_{n \to \infty} \frac{|B(n) \cap C|}{|B(n)|} = 1.$$

Here is another notion of randomness: fix the number l of relators, assume that all relators have the same length n; this defines a class of presentations $S(k, l, n)$. Then require

$$\lim_{n \to \infty} \frac{|S(k, l, n) \cap C|}{|S(k, l, n)|} = 1.$$

Problem 9.13 (Schwartz)**.** *Let* $S = \{2^n \mid n \geq 0\}$*, and let* $T = \{3^n \mid n \geq 0\}$*. Are the Cayley graphs of* $\mathbb{Z}$ *with respect to* S *and* T *quasi-isometric?*

Problem 9.14 (Kapovich)**.** *Let* $H < G$ *so that the quotient of the Cayley graph of* G *by* H *is quasi-isometric to* $\mathbb{R}$ *or* $\mathbb{R}^+$*. Does it follow that* H *is "virtually normal" in* G *with infinite cyclic quotient?*

Problem 9.15 (Mosher)**.** *Study "patterned quasi-isometry", in which left cosets of some particular subgroups are required to be sent to sets which are finite Hausdorff distance from other left cosets. Does this help with understanding the Gromov-Thurston (or Mostow-Siu) examples?*

Problem 9.16 (Behrstock)**.** *QI classify* 3*-manifold groups.*

More specifically:

Problem 9.17 (Behrstock)**.** *For non-geometric* 3*-manifolds with only hyperbolic pieces does QI imply commensurable?*

Problem 9.18 (Behrstock)**.** *QI classify right angled Artin groups.*

Problem 9.19 (Mosher)**.** *Suppose a* 2*-dimensional group* G *has the following property: Any two proper planes have infinite diameter coarse intersection. Can* G *be other than a surface or Baumslag-Solitar group?*

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