3-manifold groups and nonpositive curvature

Michael Kapovich and Bernhard Leeb*

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Abstract. We prove that the fundamental group of any compact Haken manifold of zero Euler characteristic, which is neither *Nil* nor *Sol*, is nonpositvely curved on the large scale.

1 Introduction

Thurston's Hyperbolization Theorem provides metrics of constant negative curvature on closed atoroidal Haken 3-manifolds. More generally, Thurston's Geometrization Conjecture asserts that all closed 3-manifolds admit a canonical minimal decomposition into geometric pieces. In this paper, we are interested in closed irreducible 3-manifolds M with infinite fundamental group. They topologically decompose into Seifert and atoroidal components and, if the decomposition is non-trivial, these components can be equipped with geometric structures locally modelled on the nonpositively curved geometries $\mathbb{H}^2 \times \mathbb{R}$, respectively $\mathbb{H}^3$. In a large number of cases such manifolds M admit metrics of nonpositive sectional curvature, for instance, if M is not a graph manifold [L]. Many graph manifolds admit metrics of nonpositive curvature, but not all of them do [L, KL2, BK]. Another indication for the link between Haken manifolds and nonpositive curvature is the existence of automatic structures on the fundamental groups of all Haken manifold which are not *Nil*- or *Sol*-manifolds [E]. In particular, they admit a bicombing satisfying a fellow traveller property which is a weak version of the convexity of distance function for nonpositively curved spaces. The aim of this paper is to establish a stronger connection between Haken manifolds and nonpositive curvature; we show that fundamental groups of non-geometric Haken manifolds are nonpositively curved on the large scale in the following sense:

Theorem 1.1 *Let M be a Haken manifold of zero Euler characteristic, equipped with a Riemannian metric, which is neither Nil nor Sol. Then there exists a compact nonpositely curved 3-manifold N with totally-geodesic flat boundary and a bilipschitz homeomorphism between the universal covers of M and N which preserves the canonical decomposition. In particular, the fundamental groups $\pi_1(M)$ and $\pi_1(N)$ are quasi-isometric.*

The theorem has several direct implications for the large-scale geometry of 3-manifold groups $\pi_1(M)$:

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1. The fundamental groups of geometric components are undistorted (quasi-isometrically embedded) in $\pi_1(M)$. Moreover there are coarse Lipschitz retractions from $\pi_1(M)$ to the fundamental groups of its geometric components.

2. If M is not Nil or Sol then all rank-two abelian subgroups are undistorted and quasi-minimize area.

3. The isoperimetric inequality for $\pi_1(M)$ is exponential if M is Sol, cubic if M is Nil, linear if M is closed hyperbolic and quadratic otherwise. This had been previously known [E].

We use Theorem 1.1 in [KL3] to prove that quasi-isometries preserve the canonical decomposition of Haken manifold groups. This yields new quasi-isometry invariants for fundamental groups of Haken manifolds. Another result in this direction is a special case of a theorem of Schwartz: he shows that the fundamental groups of cusped hyperbolic 3-manifolds are quasi-isometric if and only if they are commensurable [Sch]. Other than that the quasi-isometry classification of Haken 3-manifold groups remains open. For instance, the following question has not yet been answered:

Question 1.2 *Are the fundamental groups of all (closed) graph manifolds quasi-isometric?*

Commensurable groups are quasi-isometric by trivial reasons. Admitting a non-positively curved metric is a commensurability invariant for Haken 3-manifolds [KL2]. Thus our theorem provides quasi-isometries between noncommensurable groups.

Haken manifolds which have hyperbolic components admit nonpositively curved metrics [L]. Therefore we will only investigate graph manifolds. *Graph-manifolds* are compact Haken manifolds with boundary of zero Euler characteristic which are obtained by gluing Seifert manifolds along boundary surfaces. We exclude from the class of graph-manifolds Sol- and Seifert manifolds. *Flip-manifolds* are special graph-manifolds which are constructed as follows: Take a finite collection of products of S^1 with compact oriented hyperbolic surfaces with geodesic boundary. Glue them along boundary tori by maps which interchange the basis and fiber directions. It is easy to construct metrics of nonpositive curvature on flip manifolds. In section 2 of this paper we prove that the fundamental group of any graph-manifold is quasi-isometric to the fundamental group of a flip-manifold. We also show that instead of flip-manifolds one can use manifolds fibred over the circle. Our construction generalizes an earlier example: Epstein, Mess and Gersten discovered independently that $\mathbb{H}^2 \times \mathbb{R}$ is quasi-isometric to the universal cover $\widetilde{PSL}(2,\mathbb{R})$ of the unit tangent bundle of $\mathbb{H}^2$.

In section 3 we apply Theorem 1.1 to study another quasi-isometry invariant, namely divergence. We prove that the fundamental groups of all graph-manifolds have quadratic divergence. This extends earlier results of Gersten [G1, G2] who shows that certain Hadamard spaces have quadratic divergence. We extend these results to all graph-manifolds. In particular, we show that for periodic geodesics in flip-manifolds the divergence is quadratic unless the geodesic is contained in a single Seifert component. In the latter case, the divergence is linear.

2 Construction of quasi-isometries between graph manifold groups

To motivate our discussion below, we recall the construction of bilipschitz homeomorphisms between the universal cover of the unit tangent bundle $UT(\mathbb{H}^2)$ of the hyperbolic plane and $\mathbb{H}^2 \times \mathbb{R}$, for details see [R]. We pick a base point $p_0 \in \mathbb{H}^2$ and identify the unit tangent circle $UT_{p_0}(\mathbb{H}^2)$ with S^1. If $v \in UT_x(\mathbb{H}^2)$ is a unit vector, we denote by $\phi(v) \in UT_{p_0}(\mathbb{H}^2)$ the vector obtained by parallel transporting v along the geodesic segment $[xp_0]$. The map $UT(\mathbb{H}^2) \to \mathbb{H}^2 \times S^1$ given by $v \mapsto (x, \phi(v))$ is bilipschitz because the area of geodesic triangles in $\mathbb{H}^2$ is linearily bounded in terms of the shortest side length. This bilipschitz homeomorphism lifts to the universal covers. In section 2.2 below we will give a relative version of this construction.

2.1 Finite covers of graph manifolds

We recall that graph manifolds are obtained by gluing finitely many Seifert manifolds with hyperbolic base orbifolds. We exclude *Sol* and Seifert manifolds and require that the gluing maps between the Seifert components do not identify (unoriented) Seifert fibers up to homotopy. We shall refer to the tori and Klein bottles separating adjacent Seifert components as *splitting surfaces.* The universal cover $\tilde{M}$ of a graph-manifold M splits as the union of universal covers of Seifert components. We call them Seifert components of the universal cover. We call surfaces separating Seifert components of $\tilde{M}$ *splitting flats.*

In this section, we construct for any graph manifold a finite cover whose Seifert components and gluing maps are as simple as possible. The type of constructions we use are well-known, see [He, MM].

Lemma 2.1 *Any graph-manifold M_0 has an orientable finite cover M_2 where all Seifert components are trivial circle bundles over (orientable) surfaces of genus ≥ 2. Furthermore, we can arrange that the intersection numbers of the fibres of adjacent Seifert components are ± 1.*

Proof: Step 1. By passing to the orientable cover we may assume that M_0 is orientable.

Step 2. Next we make the splitting surfaces orientable. Let $K_1, \ldots, K_m \subset M_0$ be the splitting Klein bottles. Each K_i has a neighborhood $N(K_i)$ homeomorphic to a twisted interval bundle over K_i. The boundary $\partial N(K_i)$ is a 2-torus. Let $\phi : \pi_1(M_0) \to (\mathbb{Z}/2\mathbb{Z})^m$ be the homomorphism given by the $\mathbb{Z}/2\mathbb{Z}$-intersection number with the K_i and let M_1 be the covering of M_0 corresponding to the kernel of ϕ. The Klein bottles K_i lift to 2-sided tori in M_1 because M_0 is orientable.

Step 3. Now we remove singular fibers. The components Z_j of the canonical decomposition of M_1 are Seifert manifolds with base-orbifolds O_j. The orbifolds O_j have incompressible boundary, they are in fact hyperbolic. Fix some integer $p \geq 7$ and denote by $\hat{O}_j$ the orbifold obtained by attaching a disc with one singular cone point of order p to each boundary component of O_j. The orbifolds $\hat{O}_j$ are hyperbolic and therefore have finite nonsingular orientable covers $\hat{Q}_j$ of genus ≥ 2. We remove from $\hat{Q}_j$ the inverse images of the inserted singular discs. The resulting surface Q_j

covers O_j so that the restriction of the covering to each boundary component has degree p. We have exact sequences

$$1 \to \mathbb{Z} \to \pi_1(Z_j) \xrightarrow{\psi_j} \pi_1(O_j) \to 1$$

and

$$1 \to \mathbb{Z} \to \psi_j^{-1}\pi_1(Q_j) \xrightarrow{\psi_j} \pi_1(Q_j) \to 1.$$

The subgroup $\psi_j^{-1}\pi_1(Q_j)$ of $\pi_1(Z_j)$ has finite index and the corresponding Seifert manifold covering Z_j is homeomorphic to $S^1 \times Q_j$. Therefore we can find a p-fold covering $S^1 \times Q_j \to S^1 \times Q_j$ and thus obtain a finite covering $\bar{Z}_j$ over Z_j which satisfies the properties:

1. $\bar{Z}_j$ is the product of the circle with the orientable hyperbolic surface Q_j of genus ≥ 2.
2. On each boundary torus of $\bar{Z}_j$ the covering $\bar{Z}_j \to Z_j$ restricts to the characteristic $p \times p$-fold cover of a boundary torus of Z_j.

This implies that we can glue copies of $\bar{Z}_j$ to obtain a finite covering M_1 of the original graph manifold, see [He, MM].

Step 4. Consider a Seifert component $Z = S \times S^1$ of M_1. For each boundary torus T_k of Z, we denote by n_k the intersection number in T_k of the fiber f of Z and the fiber f_k of the other Seifert component adjacent to T_k. We attach a disk with cone point of order $|n_k|$ to each boundary circle $S \cap T_k$ of S. The resulting orbifold is hyperbolic, because the genus of S is at least 2, and admits a finite covering by a surface. We extend the corresponding finite covering $S' \to S$ to a covering $Z' := S' \times S^1 \to S \times S^1 = Z$ by taking the product with $id : S^1 \to S^1$. The restriction of this covering to each boundary torus $T'_k \to T_k$ is determined by the subgroup of $\pi_1(T_k)$ generated by f and f_k. We repeat this for all Seifert components of M_1 and then we glue copies of the covers Z' to obtain a finite cover M_2 of M_1. By construction, the intersection numbers of fibres of adjacent Seifert components of M_2 are ± 1. $\square$

2.2 Quasiisometric change of gluing maps

In this section we prove our main Theorem 1.1. Let M be a graph manifold. By the discussion in section 2.1, we can assume that M is oriented and all Seifert components Z_j are trivial circle bundles over orientable hyperbolic surfaces: $Z_j = S_j \times S^1$. We may further assume that for each splitting torus the intersection numbers between fibers of adjacent Seifert components are ± 1.

Any choice of Riemannian metrics on the Seifert components, which are not required to be compatible on the splitting tori, yields a path metric on M. All these path metrics are bilipschitz equivalent. For our purposes, the following choice is convenient: We put negatively curved metrics on the base surfaces S_j so that all boundary components are totally geodesic of unit length. Then we equip Z_j with the product metric so that the fibers have length one. This induces canonical affine structures on the boundary tori of the Seifert components and we may assume that the gluing maps are affine.

Let Z, Z' be adjacent Seifert components (which may coincide) and let $T \subset \partial Z$, $T' \subset \partial Z'$ be tori which are identified by an affine gluing map $A: T \to T'$. The first homology groups $H_1(T, \mathbb{Z})$ and $H_1(T', \mathbb{Z})$ contain distinguished elements f_T and $f_{T'}$ corresponding to the Seifert fibres of Z and Z'. We consider a change of the gluing map $A: T \to T'$ of the following type: Define $B_T \subset SL(H_1(T, \mathbb{Z}))$ to be the stabiliser of f_T, and analogously define $B_{T'}$. Given linear transformations $s_T \in B_T$, $s_{T'} \in B_{T'}$ we replace A by the new gluing map $s_{T'} \circ A \circ s_T$. We do not modify the other gluing maps.

Proposition 2.2 *Let N be the manifold obtained by performing the modified gluing map $s_{T'} \circ A \circ s_T$. Then there exists a bilipschitz homeomorphism between the universal covers $\tilde{M}$ and $\tilde{N}$ which preserves their canonical decompositions. In particular, $\pi_1(M)$ and $\pi_1(N)$ are quasi-isometric .*

Proof: We lift the canonical splitting of M to the universal cover and denote by Λ the tree dual to this decomposition of $\tilde{M}$. Every vertex v of Λ corresponds to a Seifert component $\tilde{Z}_v$ which universally covers a Seifert component of M, and every edge e adjacent to v corresponds to a boundary flat $\tilde{T}_{ve} \subset \partial \tilde{Z}_v$ covering a splitting torus of M. For each oriented edge $e = [vv']$ there is an affine gluing map $\tilde{A}_{vv'} : \tilde{T}_{ve} \to \tilde{T}_{v'e}$. The group of decktransformations stabilizing an edge e acts as an integer lattice on the splitting flat $\tilde{T}_{ve} \cong \tilde{T}_{v'e}$.

The Seifert foliation of each boundary torus of a Seifert component of M induces a vertical foliation of the correponding splitting flat in $\tilde{M}$ by straight lines. The matrices $s_T, s_{T'}$ yield a collection of linear transformations $\tilde{s}_{ve}$ of the splitting flats $\tilde{T}_{ve}$ which preserve vertical lines and are well-defined up to integer translations. Pick a Seifert component $\tilde{Z}_v \cong \tilde{S}_v \times \mathbb{R}$ of $\tilde{M}$ and let $\tilde{\ell}_{ve} \subset \tilde{S}_v$ be the boundary geodesic corresponding to $\tilde{T}_{ve}$. The product decomposition $\tilde{T}_{ve} \cong \tilde{\ell}_{ve} \times \mathbb{R}$ yields natural coordinates and we can write the affine map $\tilde{s}_{ve}$ as

$$\tilde{s}_{ve}(x, t) = (x, \theta_{ve}(x) + t)$$

where $\theta_{ve} : \tilde{\ell}_{ve} \to \mathbb{R}$ is an affine function. The collection of differential 1-forms $d\theta_{ve}$ defines closed 1-forms on the boundaries of $\tilde{S}_v$ which are periodic with respect to deck-transformations. Lemma 2.4 below allows to construct bilipschitz maps $H_v : \tilde{Z}_v \to \tilde{Z}_v$ whose restrictions to boundary flats $\tilde{T}_{ve}$ equal $\tilde{s}_{ve}$ up to translations: According to Lemma 2.4, there exists a smooth Lipschitz function $h_v : \tilde{S}_v \to \mathbb{R}$ such that $dh_v|_{\partial \tilde{S}_v} = \alpha_v$. Hence the restriction of h_v to each boundary component $\tilde{\ell}_{ve}$ has the same slope as the corresponding function θ_{ve}. We define a bilipschitz homeomorphism H_v of $\tilde{Z}_v$ by the formula

$$H_v(x, t) = (x, h_v(x) + t).$$

The homeomorphisms H_v have uniform bilipschitz constant C. The restriction of H_v to each boundary flat $\tilde{T}_{ve}$ differs from $\tilde{s}_{ve}$ by a vertical translation. After enlarging C we may assume that this vertical translation is integral. The mappings H_v can then be combined to a bilipschitz map between $\tilde{M}$ and $\tilde{N}$. □

The following theorem implies our main theorem 1.1 stated in the introduction.

Theorem 2.3 *Let M be a graph manifold equipped with a Riemannian metric. Then there exists a nonpositively curved flip-manifold N and a bilipschitz homeomorphism* *between the universal covers $\tilde{M}$ and $\tilde{N}$ which preserves their canonical decompositions. As a consequence, the fundamental group of any graph-manifold M is quasi-isometric to the fundamental group of a flip-manifold.*

Furthermore, the fundamental group of any graph-manifold M is quasi-isometric to the fundamental group of a manifold fibered over the circle.

Proof: Pick two boundary tori T and T' of Seifert components of M which are glued via the identification map $A : T \to T'$. Fixed product decompositions of the Seifert components give us bases $\{f, b\}$ of $H_1(T, \mathbb{Z})$ and $\{f', b'\}$ of $H_1(T', \mathbb{Z})$. The elements f and f' correspond to the Seifert fibres. Due to Lemma 2.1 we can assume that the intersection number between Af and f' is ± 1. Hence there exist elements $s_T \in B_T$ and $s_{T'} \in B_{T'}$ so that

$$s_T^{-1}(A^{-1}f') = \pm b \qquad \text{and} \qquad s_{T'}(Af) = \pm b'.$$

Then the modified gluing map $\ddot{A} = s_{T'} \circ A \circ s_T$ satisfies the flip condition

$$\ddot{A}f = \pm b' \qquad \text{and} \qquad \ddot{A}^{-1}f' = \pm b.$$

Let N be the flip manifold obtained from M by changing all gluings in this fashion. Proposition 2.2 implies the existence of a bilipschitz homeomorphism between the universal covers $\tilde{M}$ and $\tilde{N}$ which preserves the canonical decomposition. Since we choose the metrics on the Seifert components as explained in the beginning of this section, the gluing maps between the Seifert components of N are isometries and the induced path metric is nonpositively curved. However the metric we obtain will be singular along splitting tori. We smooth out the singularity using the same procedure as in [L] to get a Riemannian metric of nonpositive curvature on N. This proves the first assertion of the theorem.

To prove the second assertion, observe that the "affine lines" $B_T \cdot b = A^{-1}(\pm f)' + \mathbb{Z} \cdot f$ and $A^{-1}(B_{T'} \cdot b') = \pm f + \mathbb{Z} \cdot A^{-1} f'$ in $H_1(T, \mathbb{Z})$ intersect at the point $\pm f + A^{-1}(\pm f')$. Therefore there are transformations $s_T \in B_T$ and $s_{T'} \in B_{T'}$ so that $s_T(b) = A^{-1} s_{T'}^{-1} b'$. The modified gluing map $\ddot{A} = s_{T'} \circ A \circ s_T$ then satisfies $\ddot{A}b = b'$. The new gluings are compatible with the foliations of the Seifert components by surfaces (given by the product structure). Therefore the manifold obtained from the modified gluings fibers over the circle. □

2.3 Lipschitz functions on universal covers of negatively curved manifolds

Let S be a smooth compact manifold with strictly negative sectional curvature and totally-geodesic boundary (in our application it will be a hyperbolic surface). Denote by $\tilde{S}$ the universal cover of S. Let α be a closed smooth 1-form on ∂S. We denote the pull-back of α to $\partial \tilde{S}$ by α as well.

Lemma 2.4 *There exists a smooth Lipschitz function h on $\tilde{S}$ satisfying $dh|\partial \tilde{S} = \alpha$.*

Proof: We first extend α to a smooth 1-form β on $\tilde{S}$ which is $\pi_1(S)$-invariant. The forms β and $d\beta$ are bounded. For $x, y \in \tilde{S}$, we denote by γ_{yx} the geodesic from y to

x, and by γ_x the geodesic connecting a fixed base point $p \in \tilde{S}$ to x. Set

$$g(x) := \int_{\gamma_x} \beta.$$

Consider a 1-connected smooth ruled surface Δ_{xy} bounded by the geodesic triangle $\Delta(p, x, y)$ in $\tilde{S}$. For instance, connect the vertex p by geodesic segments to the points on the segment γ_{yx}. The sectional curvature of the induced Riemannian metric on Δ_{xy} is bounded from above by the sectional curvature of S. Since S has a negative upper curvature bound, we have the following estimates for the area of the ruled surface:

$$area(\Delta_{xy}) \leq constant_1$$

and

$$area(\Delta_{xy}) \leq constant_2 \cdot d(x, y) \tag{1}$$

Since

$$g(x) - g(y) - \int_{\gamma_{yx}} \beta = \int_{\partial \Delta_{xy}} \beta = \int_{\Delta_{xy}} d\beta,$$

we have

$$|g(x) - g(y) - \int_{\gamma_{yx}} \beta| \leq area(\Delta_{xy}) \cdot \|d\beta\| \leq constant_3. \tag{2}$$

Furthermore,

$$|\int_{\gamma_{yx}} \beta| \leq constant_4 \cdot d(x, y)$$

and

$$|\int_{\Delta_{xy}} d\beta| \leq constant_5 \cdot d(x, y)$$

because of (1). Therefore

$$|g(x) - g(y)| \leq constant_6 \cdot d(x, y)$$

so g is Lipschitz. The closed form $dg - \alpha$ on $\partial \tilde{S}$ is exact. Hence we can write

$$dg - \alpha = df$$

The smooth function $f : \partial \tilde{S} \to \mathbb{R}$ is Lipschitz because g is Lipschitz and α is a bounded 1-form. We choose f so that it has a zero on each component of $\partial \tilde{S}$; estimate (2) then implies that f is bounded. Fix $\epsilon > 0$ sufficiently small so that the nearest-point-projection $\pi_{\partial \tilde{S}}$ to $\partial \tilde{S}$ is well-defined and distance-nonincreasing on the tubular neighborhood $N_\epsilon(\partial \tilde{S})$. Take a bump function $\sigma : [0, \infty) \to \mathbb{R}$ so that $\sigma(0) = 1$ and $\sigma(t) = 0$ for $t \geq \epsilon$. We extend f to $\tilde{S}$ by

$$\hat{f}(x) := \sigma(d(x, \partial \tilde{S})) \cdot f(\pi_{\partial \tilde{S}}(x)).$$

Since the function f is Lipschitz and bounded, the extension $\hat{f}$ is also Lipschitz and bounded. The function $h := g - \hat{f}$ on $\tilde{S}$ has the desired properties. $\Box$

3 Divergence of geodesics

In this section we apply our results to discuss another quasi-isometry invariant of geodesic metric spaces, namely divergence. This notion was introduced in [G1] and we recall the definition for the convenience of the reader. Let X be a complete geodesic metric space with one end. Pick a point $x \in X$ and a positive real number r. Consider the path metric on the complement $X - B_r(x)$ of a metric ball centered at x. We define $f(r)$ as the diameter of the intersection of the metric sphere $S_r(x)$ with the unbounded component of $X - B_r(x)$. The property that $f(r)$ grows exponentially or polynomially of degree d does not depend on x and is a quasi-isometry invariant for X. Analoguously, one can define the divergence of a complete minimizing geodesic $\ell : \mathbb{R} \to X$. Consider the complement of the R-ball $B(R)$ centered at $\ell(0)$ equipped with the path metric. For each $R > 0$, we measure the distance $div(R)$ between the points $\ell(\pm R)$ in $X \setminus B_R(\ell(0))$ equipped with the path metric. The growth rate of the function div is called the *divergence* of ℓ. The divergence of geodesics provides an estimate from below for the divergence of the entire space.

The next two results give lower estimates for the divergence of geodesics in CAT(0)-spaces. We start with an estimate for flip-manifolds.

Proposition 3.1 *Let M be a flip manifold with a natural metric of nonpositive curvature and let $\ell : \mathbb{R} \to \tilde{M}$ be a geodesic. Then the divergence of ℓ is linear if ℓ is contained in the union of finitely many Seifert components and superlinear otherwise.*

Proof: The divergence of geodesics contained in one Seifert component is clearly linear. Therefore the same is true for geodesics contained in a finite union of Seifert components of $\tilde{M}$.

Suppose that ℓ successively intersects infinitely many splitting flats F_n, $n \in \mathbb{Z}$, which divide ℓ into subsegments. Consider a shortest curve α_R which connects the points $\ell(\pm R)$ outside the ball $B_R(\ell(0))$. We choose successive points x_n on α_R so that $x_n \in F_n$. Clearly, the length of the portion of α_R between the flats F_{n-1} and F_{n+1} is at least $d(x_{n-1}, x_n) + d(x_n, x_{n+1})$. For sufficiently large R, this portion of α_R lies at distance at least $R/2$ from the shortest geodesic segment σ connecting F_{n-1} and F_{n+1}.

Lemma 3.2 $d(x_{n-1}, x_n) + d(x_n, x_{n+1}) \geq \frac{1}{2\sqrt{2}} \cdot R - constant.$

Proof: We call $Y_\pm$ the Seifert component between F_n and $F_{n\pm1}$. Let $p = \sigma \cap F_n$ and denote by $l_\pm$ the Seifert fiber of $Y_\pm$ containing p. Since $d(x_n, \sigma) \geq R/2$ we may assume without loss of generality that $d(x_n, l_+) \geq \frac{1}{2\sqrt{2}} R$. Since the nearest point projection of F_n to F_{n+1} is a strip of uniformly bounded width $\leq w$ we obtain the inequality $d(x_n, x_{n+1}) \geq d(x_n, F_{n+1}) \geq d(x_n, l_+) - w - d(F_n, F_{n+1}) \geq \frac{1}{2\sqrt{2}} \cdot R - constant.$ □

The lemma implies that the length of the portion of α_R between F_{n-1} and F_{n+1} is at least $R/4$ for sufficiently big R. This finishes the proof. □

We now give a better lower estimate for the divergence of *periodic* rank-one geodesics in arbitrary locally-compact CAT(0)-spaces. Recall that a periodic geodesic has rank one if and only if it does not bound a flat half-plane.

Proposition 3.3 *Let X be a locally-compact CAT(0)-space and ℓ be a complete geodesic which is invariant under a cyclic group of hyperbolic isometries. If ℓ has subquadratic divergence, then it bounds a flat half-plane and hence has linear divergence.*

Proof: We use the same notation α_R as in the proof of the previous proposition. Subquadratic divergence means that the length of α_R equals $\epsilon_R \cdot R^2$ with $\lim_{R\to\infty} \epsilon_R = 0$. Fix $h > 0$. Denote by $\pi : X \to \ell$ the nearest-point-projection. For sufficiently large R, we can find a subsegment $[a_1 a_2] \subset \ell(-R/2, R/2)$ of length h so that the portion of α_R which projects on $[a_1 a_2]$ via π has length at most $\epsilon_R hR$. Pick points $b_i \in \alpha_R$ with $\pi(b_i) = a_i$. Let $\rho_i : [0, L_i] \to X$ be the unit speed geodesic joining $a_i = \rho_i(0)$ to b_i. We have $L_i \geq R/2$. The function $\psi(t) := d(\rho_1(t), \rho_2(t))$ is convex, monotonically increasing on $[0, R/2]$ and satisfies

$$\psi(0) = h, \qquad \psi(R/2) \leq \epsilon_R Rh.$$

Therefore

$$h \leq \psi(h) \leq (1 + 2\epsilon_R h) \cdot h.$$

The quadrilateral with vertices a_i and $\rho_i(h)$ has three sides of length h, one side of length $\leq (1+2\epsilon_R h)\cdot h$ and angles $\geq \pi/2$ at a_i. We have a family of such quadrilaterals Q_R parametrized by R. Using the translations along ℓ, we transport the quadrilaterals Q_R to a fixed compact subset of X. The Hausdorff limit as R tends to infinity of a convergent subsequence of the translated quadrilaterals is isometric to a square of side-length h in $\mathbb{R}^2$. Hence for each h, we obtain a flat square of side-length h in X adjacent to ℓ. The local compactness of X implies the existence of a flat half-plane bounded by ℓ. $\square$

We resume the discussion of graph manifolds M of nonpositive curvature. Notice that a geodesic in the universal cover $\tilde{M}$ bounds a flat half-plane if and only if it is contained in a single Seifert component. Thus any periodic geodesic in $\tilde{M}$ which intersects a splitting flat has rank one and hence at least quadratic divergence. An upper bound for the divergence is easier to obtain: The divergence of any geodesic in $\tilde{M}$ and the divergence of $\tilde{M}$ itself is at most quadratic, see also [G2] where it is proved that the divergence is at most quadratic for fundamental groups of all graph-manifolds fibred over the circle. Therefore according to Theorem 2.3 the divergence in fundamental groups of all graph-manifolds is at most quadratic.

Corollary 3.4 *The fundamental group of any graph-manifold has quadratic divergence.*

This extends the following earlier results in [G1], [G2]:

- The divergence of fundamental groups of graph-manifolds fibered over the circle is at most quadratic.

- Let Σ be the once punctured torus and M be the mapping torus of a Dehn twist on Σ. Then the divergence of $\pi_1(M)$ is precisely quadratic.

References

[BK] S. Buyalo and V. Kobelskii, *Geometrization of graphmanifolds: isometric states,* St. Petersburg Math. Journal, Vol. **7** (1996), N 3.

[E] D. Epstein, J. Cannon, D. Holt, S. Levy, M. Paterson, W. Thurston, *Word-Processing in Groups,* Bartlett and Jones, Boston, 1992.

[G1] S. Gersten, *Quadratic divergence of geodesics in CAT(0)-spaces,* Geometric Analysis and Functional Analysis, Vol. **4**, no. 1 (1994), 37– 51.

[G2] S. Gersten, *Divergence in 3-manifold groups*, Geometric Analysis and Functional Analysis, Vol. **6**, no. 4 (1994), 633– 647.

[He] J. Hempel, *Residual finiteness for 3-manifolds*, Ann. Math. Stud., vol. **111** (1987), 373–396.

[KL1] M. Kapovich and B. Leeb, *On asymptotic cones and quasi-isometry classes of fundamental groups of 3-manifolds*, Geometric Analysis and Functional Analysis, Vol. **5**, no. 3 (1995), 582–603.

[KL2] M. Kapovich and B. Leeb, *Actions of discrete groups on Hadamard spaces*, Math. Ann., Bd. **306** (1996) 341–352.

[KL3] M. Kapovich and B. Leeb, *Quasi-isometries preserve the geometric decomposition of Haken manifolds,* Vol. 128, F. 2 (1997), p. 393–416.

[L] B. Leeb, *3-manifolds with(out) metrics of nonpositive curvature*, Invent. Math. Vol. **122** (1995), 277–289.

[MM] D. McCullough and A. Miller, *Manifold covers of 3-orbifolds with geometric pieces,* Topology and its Applications, Vol. **31** (1989), 169–185.

[R] E. Rieffel, *Groups coarse quasi-isometric to* $\mathbb{H}^2 \times \mathbb{R}$, PhD thesis, UCLA 1993.

[Sch] R. Schwartz, *The quasi-isometry classification of rank one lattices,* Publ. of IHES, vol. **82** (1996), 133– 168.

Michael Kapovich, Department of Mathematics, University of Utah, Salt Lake City, UT 84112, USA; kapovich@math.utah.edu

Bernhard Leeb, Mathematisches Institut, Universität Bonn, Beringstr. 1, 53115 Bonn, Germany; leeb@rhein.iam.uni-bonn.de