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Ideal triangles in Euclidean buildings and branching to Levi subgroups

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ABSTRACT

Let $\underline{G}$ denote a connected reductive group, defined and split over $\mathbb{Z}$, and let $\underline{M} \subset \underline{G}$ denote a Levi subgroup. In this paper we study varieties of geodesic triangles with fixed vector-valued side-lengths α, β, γ in the Bruhat–Tits buildings associated to $\underline{G}$, along with varieties of ideal triangles associated to the pair $\underline{M} \subset \underline{G}$. The ideal triangles have a fixed side containing a fixed base vertex and a fixed infinite vertex ξ such that other infinite side containing ξ has fixed "ideal length" λ and the remaining finite side has fixed length μ. We establish an isomorphism between varieties in the second family and certain varieties in the first family (the pair (μ, λ) and the triple (α, β, γ) satisfy a certain relation). We apply these results to the study of the Hecke ring of $\underline{G}$ and the restriction homomorphism $\mathcal{R}(\widehat{\underline{G}}) \to \mathcal{R}(\widehat{\underline{M}})$ between representation rings. We deduce some new saturation theorems for constant term coefficients and for the structure constants of the restriction homomorphism.

1. Introduction

Let $\underline{G}$ be a connected reductive group, defined and split over $\mathbb{Z}$, and fix a split maximal torus $\underline{T}$ also defined over $\mathbb{Z}$. Let $\widehat{G} = \widehat{G}(\mathbb{C})$ denote the Langlands dual group of $\underline{G}$, and let $\mathcal{R}(\widehat{G})$ denote its representation ring. Let $\mathcal{H}_G$ denote the (spherical) Hecke ring associated to $\underline{G}(\mathbb{F}_q((t)))$, as described in Section 2. The goal of this paper is to understand various connections between the rings $\mathcal{H}_G$ and $\mathcal{R}(\widehat{G})$. Both come with bases and associated structure constants $m_{\alpha,\beta}(\gamma)$, $n_{\alpha,\beta}(\gamma)$ parameterized by

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Table 1
Constants associated to $\mathcal{H}$ and $\mathcal{R}$.

$c_\mu(\lambda)$	$r_\mu(\lambda)$
$m_{\alpha,\beta}(\gamma)$	$n_{\alpha,\beta}(\gamma)$

the same set, namely triples α, β, γ of $\underline{G}$-dominant elements of the cocharacter lattice of $\underline{T}$. Moreover, given any Levi subgroup $\underline{M} \subset \underline{G}$, we have the constant term homomorphism

$$c_M^G : \mathcal{H}_G \to \mathcal{H}_M$$

and the restriction homomorphism

$$r_M^G : \mathcal{R}(\widehat{G}) \to \mathcal{R}(\widehat{M});$$

cf. Section 2. Assuming $\underline{M}$ contains $\underline{T}$, both maps can be described by collections of constants $c_\mu(\lambda)$ and $r_\mu(\lambda)$, where μ respectively λ ranges over the $\underline{G}$-dominant respectively $\underline{M}$-dominant cocharacters of $\underline{T}$; cf. Section 2. In this paper we are studying connections between entries appearing in Table 1.

The connection between the entries in the bottom row was studied in [KLM3] and [KM2]. In this paper we will establish connections between the entries in the top row, the entries in the first column and the entries in the second column. As a corollary we will establish saturation results for the entries in the top row.

It was established in [KLM3] that $m_{\alpha,\beta}(\gamma)$ "counts" the number of $\mathbb{F}_q$-rational points in the variety of triangles $\mathcal{T}(\alpha, \beta; \gamma)$ in the Bruhat–Tits building of $\underline{G}(\overline{\mathbb{F}}_p((t)))$. Similarly, fixing a parabolic subgroup $\underline{P} = \underline{M} \cdot \underline{N}$ with Levi factor $\underline{M}$, we will see that $c_\mu(\lambda)$ counts (up to a certain factor depending only on $\underline{P}$, q, and λ) the number of $\mathbb{F}_q$-points in the variety of ideal triangles $\mathcal{IT}(\lambda, \mu; \xi)$ with the ideal vertex ξ fixed by $\underline{P}(\overline{\mathbb{F}}_p((t)))$ (see Section 2 for the definition). Given λ, μ, we will find a certain range of α, β, γ depending on λ, μ, so that the varieties $\mathcal{IT}(\lambda, \mu; \xi)$ and $\mathcal{T}(\alpha, \beta; \gamma)$ are naturally isomorphic over $\mathbb{F}_p$, thereby providing a geometric explanation for the numerical equalities

$$c_\mu(\lambda) q^{\langle \rho_N, \lambda \rangle} |K_{M,q} \cdot x_\lambda| = m_{\alpha,\beta}(\gamma) \tag{1.1}$$

and

$$r_\mu(\lambda) = n_{\alpha,\beta}(\gamma). \tag{1.2}$$

Here $K_{M,q} := \underline{M}(\mathbb{F}_q[[t]])$.

Let us state our main results a little more precisely. The equality (1.2) has a short proof using the Littelmann path models for each side (see Section 4), and this proof gave rise to the definition of the inequality $\nu \geqslant^P \mu$ (see Section 3 for the definition). Now fix any coweight ν that satisfies this inequality, so that in particular $\nu + \lambda$ will be $\underline{G}$-dominant for any $\underline{M}$-dominant cocharacter λ appearing as a weight in $V_\mu^{\widehat{G}}$. We can now state our first main theorem (Theorem 3.2).

Theorem 1.1. *Suppose μ, λ are as above ν is any auxiliary cocharacter satisfying $\nu \geqslant^P \mu$. Then there is an isomorphism of $\mathbb{F}_p$-varieties*

$$\mathcal{T}\big(\nu + \lambda, \mu^*; \nu\big) \cong \mathcal{IT}(\lambda, \mu; \xi).$$

As detailed in Section 9, the number of top-dimensional irreducible components of $\mathcal{T}(\nu + \lambda, \mu^*; \nu)$ (resp. $\mathcal{IT}(\lambda, \mu; \xi)$) is simply the multiplicity $n_{\nu+\lambda,\mu^*}(\nu)$ (resp. $r_\mu(\lambda)$). Similarly, in Section 10 we show that the number of $\mathbb{F}_q$-points on $\mathcal{T}(\nu + \lambda, \mu^*; \nu)$ (resp. $\mathcal{IT}(\lambda, \mu; \xi)$) is given by $m_{\nu+\lambda,\mu^*}(\nu)$

(resp. $c_\mu(\lambda)$, up to a factor). Thus, Theorem 1.1 implies the numerical equalities (1.1) and (1.2). This is stated more completely in Theorem 3.3.

Because of the homogeneity properties of the inequality $\nu \geqslant^P \mu$, these equalities mean that saturation theorems for $n_{\alpha,\beta}(\gamma)$ respectively $m_{\alpha,\beta}(\gamma)$ imply saturation theorems for $r_\mu(\lambda)$ respectively $c_\mu(\lambda)$. The following summarizes part of Corollary 3.4.

Corollary 1.2. *The quantities $c_\mu(\lambda)$ satisfy a saturation theorem with saturation factor k_Φ, and the quantities $r_\mu(\lambda)$ satisfy a saturation theorem with saturation factor k_Φ^2.*

For the precise formulation of these results we refer the reader to Section 3. Since there are two groups of mathematicians interested in the results of this paper, we will present both algebraic and geometric interpretations of the concepts and results.

Here are a few words on the relation of this paper to the prior work. In the earlier works [KLM1, KLM2,KLM3], Leeb and the second and third named authors studied geometric and representation-theoretic problems Q1, Q2, Q3, Q4 (see page 1 of [KLM3] for the precise formulations). In the present paper we study the analogues of Q3, Q4 for group pairs (G, M). The problems analogous to Q1, Q2 for group pairs (G, M) were studied in [BeSj] and [F] respectively. The paper [BeSj] actually studies the problem for general group pairs G, M, where G is a reductive group and M is any reductive subgroup.

Let us give an outline of the contents of this article. In Section 2 we recall some standard definitions and notation and we also define the notions of based triangles and based ideal triangles in the building. In Section 3 we state our main results. In Section 4 we give a simple proof of one of main results using Littelmann paths, and thereby explain the origin of the inequality $\nu \geqslant^P \mu$. In Section 5 we give a detailed study of based ideal triangles and the corresponding Busemann functions. We translate Theorem 3.2 into a statement about retractions and study those retractions in Sections 6 and 7; the proof of Theorem 3.2 is given in Section 8. The rest of the paper until Section 12 is directed toward the proof of Theorem 3.3. We prove some a priori bounds on dimensions of the varieties of (ideal) triangles in Section 9; these give geometric interpretations for the numbers $n_{\alpha,\beta}(\gamma)$ and $r_\mu(\lambda)$ appearing in Theorem 3.3. Section 10 likewise gives necessary geometric interpretations for the quantities $m_{\alpha,\beta}(\gamma)$ and $c_\mu(\lambda)$. In Section 11 we put the pieces together and prove Theorem 3.3 and Corollary 3.4. In Section 12 we provide some equidimensionality statements which are related to those given in [Ha2] for fibers of convolution morphisms. Finally, in Appendix A we give an alternative, more geometric, proof of the main ingredient in the proof of Theorem 1.1, namely, the equality of the retractions $\rho_{-\nu,\Delta_G-\nu}$ and $\rho_{K_P,\Delta_M} = b_{\xi,\Delta_M}$ on each geodesic $\overline{oz}$ of Δ_G-length μ, when $\nu \geqslant^P \mu$.

2. Notation and definitions

2.1. Algebra

In what follows, all the algebraic groups will be over $\mathbb{Z}$. Let $\underline{G}$ be a split connected reductive group, and let $\underline{T} \subset \underline{G}$ be a split maximal torus. Fix a Levi subgroup $\underline{M} \subset \underline{G}$ which contains $\underline{T}$.

Choose a parabolic subgroup $\underline{P} \subset \underline{G}$ which has $\underline{M}$ as a Levi factor. Let $\underline{P} = \underline{M} \cdot \underline{N}$ be a Levi splitting. Then choose a Borel subgroup $\underline{B}$ of $\underline{G}$ which contains $\underline{T}$ and is contained in $\underline{P}$. Let $\underline{U} \subset \underline{B}$ be the unipotent radical of $\underline{B}$. We then have $\underline{N} \subset \underline{U}$.

Let Φ denote the set of roots for $(\underline{G}, \underline{T})$, let Φ_N denote the set of roots for $\underline{T}$ appearing in Lie($\underline{N}$) and let Φ_M denote all roots in Φ which belong to $\underline{M}$. We let $Q(\Phi^\vee)$ denote the coroot lattice and $P(\Phi^\vee)$ the coweight lattice.

The choice of $\underline{B}$ (resp. $\underline{B}_M := \underline{B} \cap \underline{M}$) gives a notion of positive (co)root, and $\underline{G}$-dominant (resp. $\underline{M}$-dominant) element of $\mathcal{A} := X_*(\underline{T}) \otimes \mathbb{R}$. Let ρ denote[1] the half-sum of the $\underline{B}$-positive roots Φ^+. Similarly, we define ρ_N respectively ρ_M to be the half-sums of all roots in Φ_N respectively positive roots in Φ_M. Recall that W, the Weyl group of $\underline{G}$, acts by reflections on $\mathcal{A}$ with fundamental domain

[1] We also use the symbol ρ in the context of retractions of buildings (see Section 6) but no confusion should result from this.

Δ_G which is the convex hull of the $\underline{G}$-dominant coweights. Also, we define Δ_M as the convex hull of the $\underline{M}$-dominant coweights, so that $\Delta_G \subset \Delta_M$ and Δ_M is the fundamental domain of $W_{\underline{M}}$, the Weyl group $N_{\underline{M}}(\underline{T})/\underline{T}$ for $\underline{M}$. We let $\widetilde{W}$ denote the extended affine Weyl group of $\underline{G}$, i.e., $\widetilde{W} = \Lambda \rtimes W$, where $\Lambda := X_*(\underline{T})$.

Given $\lambda \in X^*(\underline{T})$ or $X_*(\underline{T})$, define $\lambda^* := -w_0\lambda$, where $w_0 \in W$ is the longest element. Note that $\rho^* = \rho$. Set $k_\Phi = \mathrm{lcm}(a_1, \dots, a_l)$, where $\sum_{i=1}^l a_i\alpha_i = \theta$ is the highest root and α_i are the simple roots of Φ. Let $\langle \cdot, \cdot \rangle : X^*(\underline{T}) \times X_*(\underline{T}) \to \mathbb{Z}$ denote the canonical pairing.

We define $\widehat{G} := \widehat{\underline{G}}(\mathbb{C})$ and, similarly, define $\widehat{M}$ and $\widehat{T}$. Having fixed the inclusions $\underline{G} \supset \underline{M} \supset \underline{T}$, we can arrange that we also have $\widehat{G} \supset \widehat{M} \supset \widehat{T}$. We will identify $X^*(\widehat{T})$ with $X_*(T)$ and roots of $(\widehat{G}, \widehat{T})$ with coroots of (G, T).

Let $V_\mu^{\widehat{G}}$ denote the irreducible representation of $\widehat{G}$ having highest weight μ. Let $\Omega(\mu)$ denote the set of $\widehat{T}$-weights in $V_\mu^{\widehat{G}}$, i.e., the intersection of the convex hull of $W \cdot \mu$ with the character lattice of $\widehat{T}$. We shall also think of $\Omega(\mu)$ as consisting of certain cocharacters of $\underline{T}$.

For $\mu, \lambda, \alpha, \beta, \gamma \in X^*(\widehat{T})$, define

$$r_\mu(\lambda) = \dim \mathrm{Hom}_{\widehat{M}}\big(V_\lambda^{\widehat{M}}, V_\mu^{\widehat{G}}\big), \tag{2.1}$$

$$n_{\alpha,\beta}(\gamma) = \dim \mathrm{Hom}_{\widehat{G}}\big(V_\alpha^{\widehat{G}} \otimes V_\beta^{\widehat{G}}, V_\gamma^{\widehat{G}}\big). \tag{2.2}$$

Let $\mathcal{R}(\widehat{G})$ denote the representation ring of $\widehat{G}$. The numbers $n_{\alpha,\beta}(\gamma)$ are the structure constants for $\mathcal{R}(\widehat{G})$, relative to the basis of highest weight representations $\{V_\alpha^{\widehat{G}}\}$. Similarly, the $r_\mu(\lambda)$ are the structure constants for the restriction homomorphism $\mathcal{R}(\widehat{G}) \to \mathcal{R}(\widehat{M})$.

Let $\mathbb{F}_q$ denote the finite field with $q = p^n$ elements (for a prime p), let k denote the algebraic closure $\overline{\mathbb{F}}_p = \overline{\mathbb{F}}_q$. Define the local function fields $L = k((t))$ and $L_q = \mathbb{F}_q((t))$ and their rings of integers $\mathcal{O} = k[[t]]$ and $\mathcal{O}_q = \mathbb{F}_q[[t]]$.

Let $G := \underline{G}(L)$ and $G_q := \underline{G}(L_q)$, and similarly, we define B, M, N, P, T, U and B_q, M_q, etc. (Note that in what follows, we will often abuse notation and write G, M, B, etc., instead of G_q, M_q, B_q, etc. (resp. $\underline{G}, \underline{M}, \underline{B}$, etc.), letting context dictate what is meant.)

Set $K := \underline{G}(\mathcal{O})$ and $K_q := \underline{G}(\mathcal{O}_q)$. These are maximal bounded subgroups of $G = \underline{G}(L)$ respectively $G_q := \underline{G}(L_q)$. Set $K_M := K \cap M$, $K_{M,q} = K_M \cap M_q$, and $K_P := N \cdot K_M$.

Let $\mathcal{H}_G = C_c(K_q \backslash G_q / K_q)$ and $\mathcal{H}_M = C_c(K_{M,q} \backslash M_q / K_{M,q})$ denote the spherical Hecke algebras of G_q and M_q respectively (they depend on q, but we will suppress this in our notation $\mathcal{H}_G$). Convolution is defined using the Haar measures giving K_q respectively $K_{M,q}$ volume 1. For the parabolic subgroup $P = MN$ of G, the constant term homomorphism $c_M^G : \mathcal{H}_G \to \mathcal{H}_M$ is defined by the formula

$$c_M^G(f)(m) = \delta_P(m)^{-1/2} \int_{N_q} f(nm)\, dn,$$

for $m \in M_q$. Here, the Haar measure on N_q is such that $N_q \cap K$ has volume 1. Further, letting $|\cdot|$ denote the normalized absolute value on L_q, we have $\delta_P(m) := |\det(\mathrm{Ad}(m); \mathrm{Lie}(N))|$. We define in a similar way δ_B, δ_{B_M}, c_T^G, and c_T^M. If $U_M := U \cap M$, then we have $U = U_M N$, and so

$$\delta_B(t) = \delta_P(t)\delta_{B_M}(t)$$

for $t \in T_q$, and

$$c_T^G(f)(t) = \big(c_T^M \circ c_M^G\big)(f)(t).$$

The map c_T^G (resp. c_T^M) is the Satake isomorphism S^G for G (resp. S^M for M). Thus, the following diagram commutes:

$$\begin{tikzcd} \mathcal{R}(\widehat{G}) \arrow[r, "\cong"] \arrow[d, "rest."'] & \mathbb{C}[X_*(T)]^W \arrow[d, "incl."'] & \mathcal{H}_G \arrow[l, "S^G"'] \arrow[d, "c^G_M"'] \\ \mathcal{R}(\widehat{M}) \arrow[r, "\cong"] & \mathbb{C}[X_*(T)]^{W_M} & \mathcal{H}_M. \arrow[l, "S^M"'] \end{tikzcd} \tag{2.3}$$

Given a cocharacter $\lambda \in X_*(\underline{T})$, we set $t^\lambda := \lambda(t)$, where $t \in L$ is the variable. For a G-dominant coweight μ, let $f^G_\mu = \mathrm{char}(K_q t^\mu K_q)$, the characteristic function of the coset $K_q t^\mu K_q$. Let f^M_λ have the analogous meaning. When convenient, we will omit the symbols G and M in the notation for f^G_μ, f^M_μ. For G-dominant coweights α, β, γ define the *structure constants* for the algebra $\mathcal{H}_G$ by

$$f_\alpha * f_\beta = \sum_\gamma m_{\alpha,\beta}(\gamma) f_\gamma.$$

Note that the $m_{\alpha,\beta}(\gamma)$ are functions of the parameter q, however we will suppress this dependence.

For G-dominant μ and M-dominant λ, we define $c_\mu(\lambda)$ by

$$c^G_M\left(f^G_\mu\right) = \sum_\lambda c_\mu(\lambda) f^M_\lambda.$$

Like the $m_{\alpha,\beta}(\gamma)$, the numbers $c_\mu(\lambda)$ depend on q, but we will suppress this.

2.2. *Definition of based (ideal) triangles in buildings*

Let $\mathcal{B} = \mathcal{B}_G$ denote the Bruhat–Tits building of G. This is a Euclidean building. It is not locally finite, because L has infinite residue field; however this will cause us no problems. This building has a distinguished special point o fixed by K. We consider it as the "origin" in the base apartment $\mathcal{A}$ corresponding to $\underline{T}$. Later on, we shall need to consider also the base alcove $\mathbf{a}$ in $\mathcal{A}$: it is the unique alcove of $\mathcal{A}$ whose closure contains o and which is contained in the dominant Weyl chamber Δ_G.

In what follows, we will sometimes write Δ in place of Δ_G. Recall that the Δ-distance $d_\Delta(x, y)$ in $\mathcal{B}$ is defined as follows. Given $x, y \in \mathcal{B}$, find an apartment $\mathcal{A}' \subset \mathcal{B}$ containing x, y. Identify $\mathcal{A}'$ with the model apartment $\mathcal{A}$ using an isomorphism $\mathcal{A}' \to \mathcal{A}$. Then project the vector $\overrightarrow{xy}$ in $\mathcal{A}$ to a vector $\overrightarrow{\lambda}$ the positive chamber $\Delta_G \subset \mathcal{A}$, so that x corresponds to the origin o, the tip of Δ_G. Then

$$d_\Delta(x, y) := \lambda.$$

Thus, $d_\Delta(x, y) = d_\Delta(y, x)^*$. Given a coweight $\lambda \in \Delta \cap \Lambda$ and $t^\lambda \in T$, we let $x_\lambda := t^\lambda \cdot o$, a point in $\mathcal{B}$. Then $d_\Delta(o, x_\lambda) = W \cdot \lambda \cap \Delta$. For $x \in \mathcal{B}$ and $\lambda \in \Delta$ we define the λ-sphere $S_\lambda(x) = \{y \in \mathcal{B}\colon\ d_\Delta(x, y) = \lambda\}$. In the case when $x = o$ and $\lambda \in \Delta \cap \Lambda$, we have

$$S_\lambda(o) = K \cdot x_\lambda.$$

Definition 2.1. Given $\alpha, \beta, \gamma \in \Delta \cap \Lambda$ define the space of *based "disoriented" triangles* $\mathcal{T}(\alpha, \beta; \gamma)$ to be the set of triangles $[o, y; x_\gamma]$ with vertices o, y, x_γ, so that

$$d_\Delta(o, y) = \alpha, \qquad d_\Delta(y, x_\gamma) = \beta.$$

Note that only the point y is varying.

Observe that $\mathcal{T}(\alpha,\beta;\gamma)$ can be identified with the subset of the usual set of oriented triangles $\mathcal{T}(\alpha,\beta,\gamma^*)$ whose final edge is $\overrightarrow{x_\gamma o}$. Also, it is easy to see that

$$\mathcal{T}(\alpha,\beta;\gamma) = Kx_\alpha \cap t^\gamma Kx_{\beta^*}$$

under the identification given by the map $[o, y; x_\gamma] \to y$.

We need to define a variant of the distance function $d_\Delta(-,-)$, where one of the points is "at infinity" in a particular sense we will presently describe. We let $\overline{xy}$ denote the unique geodesic segment in $\mathcal{B}$ connecting x to y. We will always assume that such segments (and all geodesic rays in $\mathcal{B}$) are parameterized by arc-length. We let $\partial_{Tits}\mathcal{B}$ denote the Tits boundary of $\mathcal{B}$, which is a spherical building. The points of $\partial_{Tits}\mathcal{B}$ could be defined as equivalence classes of geodesic rays in $\mathcal{B}$: two rays are equivalent if they are *asymptotic*, i.e., are within bounded distance from each other. A ray in $\mathcal{B}$ is denoted $\overline{x\xi}$ where x is its initial point and $\xi \in \partial_{Tits}\mathcal{B}$ represents the corresponding point in $\partial_{Tits}\mathcal{B}$.

One says that two rays $\gamma_1(t), \gamma_2(t)$ in $\mathcal{B}$ are *strongly asymptotic* if $\gamma_1(t) = \gamma_2(t)$ for all sufficiently large t.

Each parabolic subgroup P of G fixes a certain cell in $\partial_{Tits}\mathcal{B}$. In what follows, we will pick a generic point ξ in that cell. Then P is the stabilizer of ξ in G.

Now assume that M is a Levi factor of the parabolic P corresponding to ξ. By analogy with the definition of Busemann functions in metric geometry, we will define vector-valued Busemann functions (normalized at o)

$$b_{\xi,\Delta_M} : \mathcal{B}_G \to \Delta_M.$$

We refer the reader to Section 5 for the precise definition. Intuitively, $b_{\xi,\Delta_M}(y)$ measures the Δ_M-distance from ξ to y relative to the Δ_M-distance from ξ to o. A fundamental property (to be proved in Lemma 5.3) is that

$$b_{\xi,\Delta_M}(y) = \lambda \quad \Leftrightarrow \quad y \in K_P x_\lambda. \tag{2.4}$$

This gives an algebraic characterization of the function b_{ξ,Δ_M}, and also shows that it agrees with the retraction ρ_{K_P,Δ_M} which we define and study in Subsection 6.3.

We can now define the space of based ideal triangles.

Definition 2.2. Fix coweights $\lambda \in \Delta_M$, $\mu \in \Delta_G$ and a generic point ξ in the face of $\partial_{Tits}\mathcal{B}$ fixed by P. Then we define the set of *based ideal triangles* $\mathcal{IT}(\lambda,\mu;\xi)$ to consist of the triples o, y, ξ, where

$$d_\Delta(o, y) = \mu, \qquad b_{\xi,\Delta_M}(y) = \lambda.$$

Note that once again, only y is varying.

In other words, in view of (2.4), we have the purely algebraic characterization (proved in Corollary 5.4)

$$\mathcal{IT}(\lambda,\mu;\xi) = S_\mu(o) \cap K_P \cdot x_\lambda.$$

2.3. Affine Grassmannians and algebraic structure of (ideal) triangle spaces

We need to endow $\mathcal{T}(\alpha,\beta;\gamma)$ and $\mathcal{IT}(\lambda,\mu;\xi)$ with the structure of algebraic varieties defined over $\mathbb{F}_p$. To do so we will realize them as subsets of the affine Grassmannian.

The *affine Grassmannian* $\mathrm{Gr}^G := G/K$ will be considered as the k-points of an ind-scheme defined over $\mathbb{F}_p$. We can identify this with the orbit $G \cdot o \subset \mathcal{B}_G$. (If $\underline{G}$ is semi-simple then $G \cdot o$ is contained in the vertex set of $\mathcal{B}_G$, in general it is a subset of the skeleton of the smallest dimension in the polysimplicial complex $\mathcal{B}_G$.)

For any G-dominant cocharacter μ, let $x_\mu = t^\mu K/K$, a point in Gr^G. It is well known that the closure $\overline{Kx_\mu}$ of the K-orbit $Kx_\mu = S_\mu(o)$ in the affine Grassmannian is the union

$$\overline{S_\mu(o)} = \coprod_{\mu_0 \preccurlyeq \mu} S_{\mu_0}(o).$$

Here μ_0 ranges over G-dominant cocharacters in $X_*(T)$, and the relation $\mu_0 \preccurlyeq \mu$ means, by definition, that $\mu - \mu_0$ is a sum of positive coroots.

Each $\overline{S_\mu(o)}$ (resp. $S_\mu(o)$) is a projective (resp. quasi-projective) variety of dimension $\langle 2\rho, \mu\rangle$, defined over $\mathbb{F}_p$. Therefore Gr^G, the union of the projective varieties $\overline{S_\mu(o)}$, is an ind-scheme defined over $\mathbb{F}_p$.

Now, K is the set of k-points in a group scheme defined over $\mathbb{F}_p$ (namely, the positive loop group $L^{\geqslant 0}(\underline{G})$) which acts (on the left) on Gr^G in an obvious way. The orbits Kx_μ are automatically locally-closed in the (Zariski) topology on Gr^G, and are defined over $\mathbb{F}_p$.

Moreover, the group $K_P = NK_M$ we defined earlier is the k-points of an ind-group-scheme defined over $\mathbb{F}_p$ which also acts on Gr^G. The orbit spaces $K_P x_\lambda$ are neither finite-dimensional nor finite-codimensional in general, however, since they are orbits under an ind-group, they are still automatically locally closed in Gr^G.

By the above discussion, our spaces of triangles can be viewed as intersections of orbits inside Gr^G

$$\mathcal{T}(\alpha, \beta; \gamma) = Kx_\alpha \cap t^\gamma Kx_{\beta^*},$$

$$\mathcal{IT}(\lambda, \mu; \xi) = K_P x_\lambda \cap Kx_\mu$$

and as such each inherits the structure of a finite-dimensional, locally-closed subvariety defined over $\mathbb{F}_p$. Thus, it makes sense to count $\mathbb{F}_q$-points on these varieties.

Remark 2.3. The Bruhat–Tits building $\mathcal{B}_{G_q}$ for the group G_q isometrically embeds in $\mathcal{B}_G$ as a sub-building. It is the fixed-point set for the natural action of the Galois group $\mathrm{Gal}(k/\mathbb{F}_q)$ on $\mathcal{B}_G$. The orbit $G_q \cdot o \subset \mathcal{B}_{G_q}$ can be identified with G_q/K_q and thus with the set of $\mathbb{F}_q$-points in Gr^G. Accordingly, the sets of $\mathbb{F}_q$-points in $T(\alpha, \beta; \gamma)$ and $\mathcal{IT}(\lambda, \mu; \xi)$ then become spaces of based triangles and based ideal triangles in $\mathcal{B}_{G_q}$. Then "counting" the numbers of triangles in $\mathcal{B}_{G_q}$ computes structure constants for $\mathcal{H}_G$ and (up to a factor) the constant term map c_M^G. On the other hand, algebro-geometric considerations are more suitable for the varieties of triangles in $\mathrm{Gr}^G \subset \mathcal{B}_G$, since the field k is algebraically closed. Therefore, in this paper (unlike [KLM3]), we almost exclusively work with the building $\mathcal{B}_G$ rather than $\mathcal{B}_{G_q}$.

3. Statements of results

We fix cocharacters $\mu \in \Delta_G$ and $\lambda \in \Delta_M$. In order to state our results we need the following definition. Recall that $\langle \cdot, \cdot \rangle : X^*(\underline{T}) \times X_*(\underline{T}) \to \mathbb{Z}$ denotes the canonical pairing.

Definition 3.1. Suppose $\mu, \nu \in X_*(\underline{T})$. We write $\nu \geqslant^P \mu$ if

- $\langle \alpha, \nu \rangle = 0$ for all roots α appearing in $\mathrm{Lie}(\underline{M})$;
- $\langle \alpha, \nu + \lambda \rangle \geqslant 0$ for all $\lambda \in \Omega(\mu)$ and $\alpha \in \Phi_N$.

Note that this relation satisfies a semigroup property: if $\nu_1 \geqslant^P \mu_1$ and $\nu_2 \geqslant^P \mu_2$, then $\nu_1 + \nu_2 \geqslant^P \mu_1 + \mu_2$. It is also homogeneous: for every integer $z \geqslant 1$, we have $\nu \geqslant^P \mu \Leftrightarrow z\nu \geqslant^P z\mu$.

Theorem 3.2. *Let μ, λ be as above. Then for any cocharacter ν with $\nu \geqslant^P \mu$, we have an equality of subvarieties in* Gr^G,

$$\mathcal{T}\big(\nu+\lambda, \mu^*; \nu\big) = t^\nu\big(\mathcal{IT}(\lambda, \mu; \xi)\big).$$

In particular, the varieties $\mathcal{T}(\nu+\lambda, \mu^*; \nu)$ and $\mathcal{IT}(\lambda, \mu; \xi)$ are naturally isomorphic as $\mathbb{F}_p$-varieties.

For the next results, recall that $k_\Phi = \mathrm{lcm}(a_1, \dots, a_l)$, where $\sum_{i=1}^l a_i\alpha_i = \theta$ is the highest root and α_i are the simple roots of Φ.

Theorem 3.3. *For $\lambda \in \Delta_M$, $\mu \in \Delta_G$ as above and for any ν with $\nu \geqslant^P \mu$, set $\alpha := \nu + \lambda$, $\beta := \mu^*$, $\gamma := \nu$. Then*:

(i) (*First column of Table* 1)

$$c_\mu(\lambda) q^{\langle \rho_N, \lambda\rangle} |K_{M,q} \cdot x_\lambda| = m_{\alpha,\beta}(\gamma).$$

(ii) (*Second column*) $r_\mu(\lambda) = n_{\alpha,\beta}(\gamma) = n_{\nu,\mu}(\nu+\lambda)$.
(iii) (*First row*)

$$r_\mu(\lambda) \neq 0 \quad \Rightarrow \quad c_\mu(\lambda) \neq 0 \quad \Rightarrow \quad r_{k_\Phi \mu}(k_\Phi \lambda) \neq 0.$$

Assume now that $\mu - \lambda$ (or, equivalently, $\lambda + \mu^*$) belongs to the coroot lattice of $\underline{G}$.

Corollary 3.4. i. (*Semigroup property for r.*) *The set of (μ, λ) for which $r_\mu(\lambda) \neq 0$ is a semigroup.*
ii. (*Uniform saturation for c.*)

$$c_{N\mu}(N\lambda) \neq 0 \quad \textit{for some } N \neq 0 \quad \Rightarrow \quad c_{k_\Phi \mu}(k_\Phi \lambda) \neq 0.$$

iii. (*Uniform saturation for r.*)

$$r_{N\mu}(N\lambda) \neq 0 \quad \textit{for some } N \neq 0 \quad \Rightarrow \quad r_{k_\Phi^2 \mu}\big(k_\Phi^2 \lambda\big) \neq 0.$$

In particular, for $\underline{G}$ of type A, the set of (μ, λ) such that $r_\mu(\lambda) \neq 0$, is saturated.

Remark 3.5. One can improve (using results of [KM1,KKM,BK,S] on saturation for the structure constants for the representation rings $\mathcal{R}(\widehat{G})$) the constants k_Φ as follows:

One can replace k_Φ (in ii.) and k_Φ^2 (in iii.) by:

(a) $k = 2$ for Φ of type B, C, G_2,
(b) $k = 1$ for Φ of type D_4,
(c) $k = 2$ for Φ of type D_n, $n \geqslant 6$.

Conjecturally (see [KM1]), one can use $k = 1$ for all simply-laced root systems and $k = 2$ for all non-simply-laced.

Remark 3.6. The paper [Ha2] states saturation results and conjectures for the numbers $m_{\alpha,\beta}(\gamma)$ and $n_{\alpha,\beta}(\gamma)$, when α and β are sums of $\underline{G}$-dominant minuscule cocharacters. Suppose that μ is a sum of $\underline{G}$-dominant minuscule cocharacters. Then we conjecture that the implications in ii. and iii. above hold, where k_Φ and k_Φ^2 are replaced by 1.

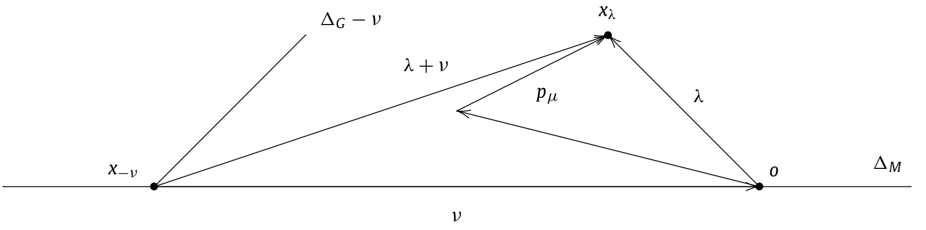

Fig. 1. The broken path p_μ from o to x_λ is an LS path of type μ. Then the Littelmann bigon with the sides p_μ and $\overline{ox_\lambda}$ yields a Littelmann triangle with the geodesic sides $\overline{x_{-\nu}o}$, $\overline{x_{-\nu}x_\lambda}$ and the broken side p_μ.

4. Relation to Littelmann's path model

There is a very short proof of Theorem 3.3 (ii) using Littelmann's path models, and our discovery of this proof was one of the starting points for this project. It gave rise to the notion of the inequality $\nu \geqslant^P \mu$ which plays a key role for us. For this reason, we present this proof here. A nice reference for the Littelmann path models used here is [Li].

We will prove the result in the following form: If $\nu \geqslant^P \mu$, then $r_\mu(\lambda) = n_{\nu,\mu}(\nu+\lambda)$.

Fix $\mu \in \Delta_G \cap \Lambda$. Write $\mathbb{B}_\mu$ for the set of all type μ LS-paths, that is, the set of all paths in $\mathcal{A}$ which result by applying a finite sequence of "raising" respectively "lowering" operators e_i respectively f_i to the path $\overrightarrow{o\mu}$ (the straight-line path from the origin to μ). Note that each path in $\mathbb{B}_\mu$ starts at the origin o and lies inside $\mathrm{Conv}(W_\mu)$, the convex hull of the Weyl group orbit of μ.

For any $x \in \mathcal{A}$, we can consider $x + \mathbb{B}_\mu$, the set of all type μ paths originating at x. Let $\mathbb{B}_\mu(x, y)$ denote the set of type μ paths which originate at x and terminate at y.

Now consider $\lambda \in \Delta_M \cap \Lambda$. The Littelmann path model for $r_\mu(\lambda)$ states that

$$r_\mu(\lambda) = \left|\mathbb{B}_\mu(o,\lambda) \cap \Delta_M\right|, \tag{4.1}$$

the cardinality of the set of type μ paths originating at o, terminating at λ, and contained entirely in Δ_M.

Now consider any $\nu \in \Delta_G \cap \Lambda$ such that $\nu + \lambda \in \Delta_G$. The Littelmann path model for $n_{\nu,\mu}(\nu+\lambda)$ states that

$$n_{\nu,\mu}(\nu+\lambda) = \left|\mathbb{B}_\mu(\nu,\nu+\lambda) \cap \Delta_G\right| \tag{4.2}$$

the cardinality of the set of type μ paths originating at ν, and terminating at $\nu+\lambda$, and contained entirely inside Δ_G.

Now assume $\nu \geqslant^P \mu$. This is defined precisely so that we have

$$\nu + \left(\mathbb{B}_\mu(o,\lambda) \cap \Delta_M\right) = \mathbb{B}_\mu(\nu,\nu+\lambda) \cap \Delta_G.$$

The equality of the quantities in (4.1) and (4.2) is now obvious. See Fig. 1.

5. Ideal triangles and vector-valued Busemann functions

5.1. *Based ideal triangles with ideal side-length λ and side-length μ*

As promised earlier in this paper we now make precise the definition of vector-valued Busemann function. Recall that a *flat* in a building is an isometrically embedded Euclidean subspace. Recall also that we fixed a parabolic subgroup $P \subset G$, with a Levi subgroup M and a point $\xi \in \partial_{Tits}\mathcal{B}$ fixed by P, where $\mathcal{B} = \mathcal{B}_G$. Then we also have an embedding $\mathcal{B}_M \subset \mathcal{B}_G$, where $\mathcal{B}_M$ splits as the product

$$\mathcal{B}_M = F \times \mathcal{Y}_M$$

where F is a flat in $\mathcal{B}$ and $\mathcal{Y}_M$ is a Euclidean building containing no flat factors. Pick a geodesic $\gamma \subset F$ asymptotic to ξ, and oriented so that the subray $\gamma(\mathbb{R}_+)$ is asymptotic to ξ. Then $\mathcal{B}_M = \mathcal{P}(\gamma)$, the subbuilding which is the *parallel set* of γ, i.e., the union of all geodesics which are at bounded distance from γ.

We fix an apartment $\mathcal{A}$ in $\mathcal{B}_M$ containing o; then $\mathcal{A}$ necessarily contains F (and, hence, γ) as well. Let Δ_M be a chamber of $\mathcal{B}_M$ in $\mathcal{A}$ with tip o; by our convention set in Section 2, Δ_M contains Δ, the chamber of $\mathcal{B}$ with the tip o. We note that Δ_M splits as $F \times \Delta'_M$.

In the case when $\mathcal{B}$ has rank one (i.e., is a tree) it is well known that the correct notion of the "distance" from a point $x \in \mathcal{B}$ to a point $\xi \in \partial_{Tits}\mathcal{B}$ is the value of the Busemann function $b_\xi(x)$, normalized to be zero at the base-point o. The usual Busemann function $b_\xi(x)$ is defined as follows:

$$b_\xi(x) := \lim_{t\to\infty}\bigl[d\bigl(\gamma(t), x\bigr) - d\bigl(\gamma(t), o\bigr)\bigr].$$

We note that

$$b_\xi(o) = 0.$$

In what follows we will define a Δ_M-valued Busemann function $b_{\xi,\Delta_M}(x)$ (which should be considered as the Δ_M-valued "distance" from ξ to x). Again, we note that b_{ξ,Δ_M} will be defined so that

$$b_{\xi,\Delta_M}(o) = 0.$$

5.2. The Δ_M-valued distance function $\widetilde{d}_{\Delta_M}$ on $\mathcal{B}$

Before defining vector-valued Busemann function, we first introduce a (partially defined) distance function $\widetilde{d}_{\Delta_M}$ on $\mathcal{B}$ which extends the function d_{Δ_M} defined on $\mathcal{B}_M$. We first note that for every $x \in \mathcal{B}$ there exists an apartment $\mathcal{A}_\xi \subset \mathcal{B}$ containing x, so that $\xi \in \partial_{Tits}\mathcal{A}_\xi$. For every two apartments $\mathcal{A}_\xi, \mathcal{A}'_\xi$ asymptotic to ξ there exists a (typically non-unique) isomorphism $\phi_{\mathcal{A}_\xi,\mathcal{A}'_\xi} : \mathcal{A}_\xi \to \mathcal{A}'_\xi$ which fixes the intersection $\mathcal{A}_\xi \cap \mathcal{A}'_\xi$ pointwise. Since this intersection contains a ray asymptotic to ξ, it follows that the extension of $\phi_{\mathcal{A}_\xi,\mathcal{A}'_\xi}$ to the ideal boundary sphere of $\mathcal{A}_\xi$ fixes the point ξ. Thus, the apartments $\{\mathcal{A}_\xi\}$ form an *atlas* on $\mathcal{B}$ with transition maps in $\widetilde{W}_M$, the stabilizer of ξ in $\widetilde{W}$. (Here we are abusing the notation and identify an apartment $\mathcal{A}_\xi$ in $\mathcal{B}$ and its isometric parameterization $\mathcal{A} \to \mathcal{A}_\xi$.) The only axiom of a building lacking in this definition is that not every two points in $\mathcal{B}$ belong to a common apartment $\mathcal{A}_\xi$. Nevertheless, for points $x, y \in \mathcal{B}$ which belong to some $\mathcal{A}_\xi$ we can repeat the definition of a chamber-valued distance function [KLM1]:

$$\widetilde{d}_{\Delta_M}(x, y) = d_{\Delta_M}\bigl(\phi_{\mathcal{A}_\xi,\mathcal{A}}(x), \phi_{\mathcal{A}_\xi,\mathcal{A}}(y)\bigr).$$

In particular, $\widetilde{d}_{\Delta_M}$ restricted to $\mathcal{B}_M$ coincides with d_{Δ_M}. Then $\widetilde{d}_{\Delta_M}$ is a partially-defined Δ_M-valued distance function on $\mathcal{B}$. As in the case of the definition of Δ-distance on $\mathcal{B}$, one verifies that $\widetilde{d}_{\Delta_M}$ is independent of the choices of apartments and their isomorphisms. It is clear from the definition that $\widetilde{d}_{\Delta_M}$ is invariant under the subgroup $P \subset G$ (but not under G itself).

5.3. The Busemann function $b_{\xi,\Delta_M}(x)$

We are now in position to repeat the definition of the usual Busemann function.

Pick $x \in \mathcal{B}$ and let γ' be a complete geodesic in $\mathcal{B}$ containing x and asymptotic to ξ. Let $\mathcal{P}(\gamma')$ be the parallel set of γ'. The following simple lemma is critical in what follows.

Lemma 5.1. *There exists t_0 such that for all $t \geqslant t_0$ we have $\gamma(t) \in \mathcal{P}(\gamma')$ and, furthermore, the subray $\gamma([t_0,\infty))$ and the point x belong to a common apartment $\mathcal{A}' \subset \mathcal{P}(\gamma')$. In particular, $\mathcal{A}'$ contains ξ in its ideal boundary and $\widetilde{d}_{\Delta_M}(\gamma(t), x)$ is well defined for $t \geqslant t_0$.*

Proof. We will give a proof only in the case when $\mathcal{B} = \mathcal{B}_G$ is the Bruhat–Tits building of a reductive group G over a nonarchimedean valued field, although the statement holds for general Euclidean buildings as well.

There exists a unipotent element $u \in G$ that carries $\mathcal{P}(\gamma)$ to $\mathcal{P}(\gamma')$ and fixes ξ. Since u is unipotent and fixes ξ, it also fixes an infinite subray of $\overline{o\xi}$. So there exists t_0 such that $\gamma([t_0,\infty)) \in \mathcal{P}(\gamma')$. Now choose an apartment $\mathcal{A}'$ in $\mathcal{P}(\gamma')$ which contains x and $\gamma(t_0)$. Since every apartment in $\mathcal{P}(\gamma')$ contains γ', we have $\xi \in \partial_{Tits}\mathcal{A}'$ and, consequently, $\gamma(t) \in \mathcal{A}'$ for $t \geqslant t_0$. $\square$

We now define $b_{\xi,\Delta_M}(x)$ by

$$b_{\xi,\Delta_M}(x) := \lim_{t\to\infty}\bigl[\widetilde{d}_{\Delta_M}\bigl(\gamma(t),x\bigr) - \widetilde{d}_{\Delta_M}\bigl(\gamma(t),o\bigr)\bigr].$$

We need to show that the limit on the right-hand side exists. We first claim that for each $t \geqslant t_0$, the difference vector

$$f(t,x) := \widetilde{d}_{\Delta_M}\bigl(\gamma(t),x\bigr) - \widetilde{d}_{\Delta_M}\bigl(\gamma(t),o\bigr)$$

is a well-defined element of Δ_M, where t_0 is as in Lemma 5.1 above. Indeed, by that lemma, $\widetilde{d}_{\Delta_M}(\gamma(t),x) \in \Delta_M$ is well defined for $t \geqslant t_0$. Next, $\gamma \subset F$ and $\Delta_M = F \times \Delta'_M$. Therefore,

$$-\widetilde{d}_{\Delta_M}\bigl(\gamma(t),o\bigr) = -\overrightarrow{\gamma(t)o} \in \Delta_M.$$

Hence, by convexity of Δ_M, $f(t,x) \in \Delta_M$.

Lemma 5.2. *$f(t,x)$ is constant in t for $t \geqslant t_0$.*

Proof. Let $\phi_{\mathcal{A}',\mathcal{A}} : \mathcal{A}' \to \mathcal{A}$ be an isomorphism of apartments as above fixing ξ. Hence $\phi_{\mathcal{A}',\mathcal{A}}(\gamma(t)) = \gamma(t)$ for $t \geqslant t_0$ and *by definition*

$$\widetilde{d}_{\Delta_M}\bigl(\gamma(t),x\bigr) = d_{\Delta_M}\bigl(\gamma(t),x'\bigr), \quad t \geqslant t_0,$$

where $x' := \phi_{\mathcal{A}',\mathcal{A}}(x)$. Then

$$f(t,x) = f\bigl(t,x'\bigr) = d_{\Delta_M}\bigl(\gamma(t),x'\bigr) - d_{\Delta_M}\bigl(\gamma(t),o\bigr), \quad t \geqslant t_0.$$

Accordingly, replacing x by x' we have reduced to the case where x', o and $\gamma(t), t \geqslant t_0$ are contained in the apartment $\mathcal{A}$. Now apply an element of the Weyl group of M (which fixes γ and, hence, $\gamma(t), \forall t$) to x' to obtain $x'' \in \Delta_M$, so that

$$\overrightarrow{\gamma(t)x''} = \widetilde{d}_{\Delta_M}\bigl(\gamma(t),x\bigr), \quad t \geqslant t_0.$$

The lemma now follows from

$$f\bigl(t,x'\bigr) = \overrightarrow{\gamma(t)x''} - \overrightarrow{\gamma(t)o} = \overrightarrow{ox''},$$

see Fig. 2. $\square$

Lemma 5.3. 1. *b_{ξ,Δ_M} is invariant under $K_P = NK_M$.*

2. *If $b_{\xi,\Delta_M}(x) = b_{\xi,\Delta_M}(y)$ then $K_P x = K_P y$.*

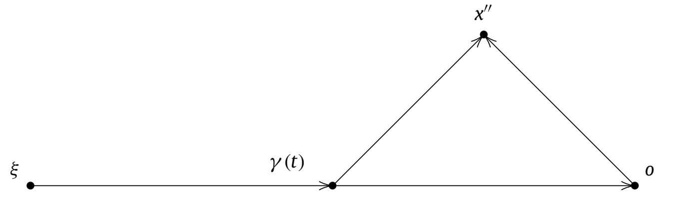

Fig. 2. $f(t, x')$ is constant for $t \geqslant t_0$.

Proof. 1. Let $u \in P$ be unipotent. Then u not only fixes ξ but, moreover, for every geodesic ray $\beta : \mathbb{R}_+ \to \mathcal{B}$ asymptotic to ξ, there exists t_u so that $u(\beta(t)) = \beta(t)$ for all $t \geqslant t_u$. Therefore, by the invariance property of $\widetilde{d}_{\Delta_M}$,

$$\widetilde{d}_{\Delta_M}\big(x, \gamma(t)\big) = \widetilde{d}_{\Delta_M}\big(u(x), u\big(\gamma(t)\big)\big) = \widetilde{d}_{\Delta_M}\big(u(x), \gamma(t)\big).$$

It then follows from the definition of b_{ξ,Δ_M} that it is invariant under u. The same argument works for u replaced with $k \in K_M$ since it suffices to know that (the entire ray) ρ is fixed by k.

2. For every $z \in \mathcal{B}$ there exists $n \in N$ so that $n(z) \in \mathcal{B}_M$. Therefore, applying elements of N to x, y, we reduce the problem to the case when $x, y \in \mathcal{B}_M$. For such x, y,

$$d_{\Delta_M}(o, y) = b_{\xi,\Delta_M}(y) = b_{\xi,\Delta_M}(x) = d_{\Delta_M}(o, x).$$

Therefore, x, y belong to the same K_M-orbit. $\square$

This lemma allows us to give a purely algebraic characterization of the space of based ideal triangles $\mathcal{IT}(\lambda, \mu; \xi)$:

Corollary 5.4. $\mathcal{IT}(\lambda, \mu; \xi) = K_P x_\lambda \cap K x_\mu = b^{-1}_{\xi,\Delta_M}(\lambda) \cap S_\mu(o)$.

6. Retractions

6.1. The retractions $\rho_{\mathbf{b},\mathcal{A}}$

Attached to any alcove $\mathbf{b}$ in an apartment $\mathcal{A}$ is a retraction $\rho_{\mathbf{b},\mathcal{A}} : \mathcal{B}_G \to \mathcal{A}$. It is distance-preserving and simplicial. Let us abbreviate it here by ρ. Recall that this retraction is defined as follows: Pick an apartment $\mathcal{A}' \subset \mathcal{B}$ containing alcoves $\mathbf{b}$ and $\mathbf{x}$. Then there exists a unique isomorphism of apartments $\phi : \mathcal{A}' \to A$ fixing $\mathbf{b}$. Then $\rho(\mathbf{x}) := \phi(\mathbf{x})$.

We need to review some of the basic properties of ρ. Let $\mathbf{x}$ denote any alcove in the building. Take a minimal gallery joining the base alcove $\mathbf{a} \subset \mathcal{A}$ to $\mathbf{x}$. Let $\mathbf{x}_0$ denote the next-to-last alcove in this gallery. Let F_0 denote the codimension one facet separating $\mathbf{x}_0$ from $\mathbf{x}$.

Let $\mathbf{c}' := \rho(\mathbf{x}_0)$, and let H denote the hyperplane in $\mathcal{A}$ which contains $\rho(F_0)$ and let s_H denote the corresponding reflection in $\mathcal{A}$. Let $\mathbf{c} := s_H(\mathbf{c}')$. Assuming we know $\mathbf{c}'$ by induction, what are the possibilities for $\rho(\mathbf{x})$? To visualize this, we will imagine $\mathbf{x}_0$ and F_0 as being fixed, and $\mathbf{x}$ as ranging over the affine line $\mathbb{A}^1$ consisting of the set of all alcoves $\mathbf{x} \neq \mathbf{x}_0$ containing F_0 as a face. Then one of the following holds:

Position 1. If $\mathbf{b}$ and $\mathbf{c}'$ are on the same side of H, then all $\mathbf{x} \in \mathbb{A}^1$ retract under ρ onto $\mathbf{c}$.

Position 2. If $\mathbf{b}$ and $\mathbf{c}'$ are on opposite sides of H, then one point in $\mathbb{A}^1$ retracts onto $\mathbf{c}$, and all remaining points of $\mathbb{A}^1$ retract onto $\mathbf{c}'$.

Suppose $\mathbf{x} \in \mathcal{B}_G$ is joined to the base alcove $\mathbf{a}$ by a gallery corresponding to a reduced word expression for $w \in \widetilde{W}$. It follows from the discussion above that $\rho_{\mathbf{b},\mathcal{A}}(\mathbf{x}) \leqslant w\mathbf{a}$ with respect to the Bruhat order $\leqslant$ on the set of alcoves in $\mathcal{A}$. (This Bruhat order is determined by the set of simple affine reflections corresponding to the walls of $\mathbf{a}$.)

Using this, it is not difficult to prove the following statement.

Lemma 6.1. *Let $x \in \overline{Kx_\mu}$. The $\rho_{\mathbf{b},\mathcal{A}}(x) \in \Omega(\mu)$.*

Finally, the following lemma is obvious.

Lemma 6.2. *For any $w \in \widetilde{W}$, we have*

$$\dot{w} \circ \rho_{\mathbf{b},\mathcal{A}} \circ \dot{w}^{-1} = \rho_{w\mathbf{b},\mathcal{A}}. \tag{6.1}$$

Recall that $\widetilde{W} = N_{\underline{G}}(\underline{T})(\mathbb{F}_p((t)))/\underline{T}(\mathbb{F}_p[[t]])$. On the left-hand side of (6.1), $\dot{w}$ denotes any lift of w in $N_{\underline{G}}(\underline{T})(\mathbb{F}_p((t)))$.

6.2. The retractions $\rho_{-\nu,\Delta_G-\nu}$

For any $\nu \in X_*(T)$, let $W_{-\nu}$ denote the finite Weyl group at $x_{-\nu} \in \mathcal{A}$, namely, the group generated by the affine reflections in $\mathcal{A}$ which fix the point $x_{-\nu}$. Regard $\Delta_G - \nu$ as the G-dominant Weyl chamber with apex $x_{-\nu}$. Then consider the retraction

$$\begin{aligned} \rho_{\Delta_G-\nu} : \mathcal{A} &\to \Delta_G - \nu, \\ v &\mapsto w(v) \end{aligned} \tag{6.2}$$

where $w \in W_{-\nu}$ is chosen so that $w(v) \in \Delta_G - \nu$.

Next choose any alcove $\mathbf{b} \subset \mathcal{A}$ with vertex $x_{-\nu}$. Then the composition

$$\rho_{\Delta_G-\nu} \circ \rho_{\mathbf{b},\mathcal{A}} : \mathcal{B}_G \to \Delta_G - \nu$$

is a retraction of the building onto the chamber $\Delta_G - \nu$. Because of (6.1), it is independent of the choice of $\mathbf{b}$. Hence we may set

$$\rho_{-\nu,\Delta_G-\nu} := \rho_{\Delta_G-\nu} \circ \rho_{\mathbf{b},\mathcal{A}}. \tag{6.3}$$

The following lemma will be useful later.

Lemma 6.3. *For any G-dominant cocharacter μ such that $\nu + \mu$ is also G-dominant, we have*

$$\rho^{-1}_{-\nu,\Delta_G-\nu}(x_\mu) = \left(t^{-\nu}Kt^{\nu}\right)x_\mu.$$

Proof. The special case

$$\rho^{-1}_{0,\Delta_G}(x_{\nu+\mu}) = Kx_{\nu+\mu} \tag{6.4}$$

is obvious. We have the identities (cf. Lemma 6.2)

$$\begin{aligned} t^{-\nu} \circ \rho_{\mathbf{b},\mathcal{A}} \circ t^{\nu} &= \rho_{\mathbf{b}-\nu,\mathcal{A}}, \\ t^{-\nu} \circ \rho_{\Delta_G} \circ t^{\nu} &= \rho_{\Delta_G-\nu}, \end{aligned}$$

and these together with the definition (6.3) yield the formula

$$t^{-\nu} \circ \rho_{0,\Delta_G} \circ t^{\nu} = \rho_{-\nu,\Delta_G-\nu}. \tag{6.5}$$

Now the desired formula follows from the special case (6.4) above. □

6.3. The retraction ρ_{K_P,Δ_M}

Recall that for a Borel subgroup $B = TU$, there is a corresponding retraction

$$\rho_{U,\mathcal{A}} : \mathcal{B}_G \to \mathcal{A}$$

which can be realized as $\rho_{\mathbf{b},\mathcal{A}}$ for $\mathbf{b}$ "sufficiently anti-dominant" with respect to the roots in $\mathrm{Lie}(U)$. We also have the retraction

$$\rho_{K,\Delta_G} := \rho_{0,\Delta_G} : \mathcal{B}_G \to \Delta_G$$

discussed above. We want to define a retraction

$$\rho_{K_P,\Delta_M} : \mathcal{B}_G \to \Delta_M$$

which interpolates between these two extremes, $\rho_{U,\mathcal{A}}$ and ρ_{K,Δ_G}.

Before defining ρ_{K_P,Δ_M} we shall review the construction of a similar retraction $\rho_{I_P,\mathcal{A}}$ which was introduced in [GHKR2].

Consider the following two properties of an element $\nu \in X_*(T)$:

(i) $\langle \alpha, \nu \rangle = 0$ for all roots $\alpha \in \Phi_M$;
(ii) $\langle \alpha, \nu \rangle >> 0$ for all roots $\alpha \in \Phi_N$.

We say ν is *M-central* if (i) holds and *very N-dominant* if (ii) holds.

Let $\mathbf{a}_M$ denote the base alcove in $\mathcal{A}$ determined by some set of positive roots Ψ_M^+ in M. (For example, we could take $\Psi_M^+ = \Phi_M^+ := \Phi^+ \cap \Phi_M$.) This means that $\mathbf{a}_M$ is the region of $\mathcal{A}$ which lies between the hyperplanes H_α and $H_{\alpha-1}$ for every $\alpha \in \Psi_M^+$. Let $I_M := I \cap M$, the Iwahori subgroup of M which corresponds to the alcove $\mathbf{a}_M$. Set $I_P := N \cdot I_M$.

Let $\mathcal{S}$ be any bounded subset of the building $\mathcal{B}_G$.

Lemma 6.4. *(See [GHKR2].) Consider the following properties of an alcove $\mathbf{b} \subset \mathcal{A}$:*

(a) $\mathbf{b} \subset \mathbf{a}_M$;
(b) $\mathbf{b}$ *has a vertex $x_{-\nu}$, where ν is M-central and very N-dominant* (*depending on $\mathcal{S}$*).

Then the corresponding retractions $\rho_{\mathbf{b},\mathcal{A}}$, for $\mathbf{b}$ satisfying (a) *and* (b), *all agree on the set $\mathcal{S}$.*

Proof. By [GHKR2], Lemma 11.2.1, there is a decomposition

$$G = \coprod_{w \in \widetilde{W}} I_P w I.$$

It follows that $\mathcal{B}_G$ is the union of all translates $g^{-1}\mathcal{A}$, as g ranges over I_P. Moreover, for $g \in I_P$, the map $g : g^{-1}\mathcal{A} \to \mathcal{A}$ is a simplicial map fixing the alcove $\mathbf{b}$ as long as $\mathbf{b} \subset \mathbf{a}_M$ and $\mathbf{b}$ is sufficiently anti-dominant with respect to N; it follows that, on $g^{-1}\mathcal{A}$, the map $g : g^{-1}\mathcal{A} \to A$ coincides with $\rho_{\mathbf{b},\mathcal{A}}$ for such $\mathbf{b}$.

To be more precise, for a bounded subset $\mathcal{S} \subset \mathcal{B}_G$ let us give a condition $(*_{\mathcal{S}})$ on alcoves $\mathbf{b}$ such that all retractions $\rho_{\mathbf{b},\mathcal{A}}$ for $\mathbf{b}$ satisfying $(*_{\mathcal{S}})$ will coincide on $\mathcal{S}$. There exists a bounded subgroup $I_{\mathcal{S}} \subseteq I_P$ such that $\mathcal{S} \subset \bigcup_{g \in I_{\mathcal{S}}} g^{-1}\mathcal{A}$. Let $I_{\mathbf{b}}$ denote the Iwahori subgroup fixing $\mathbf{b}$. Then condition $(*_{\mathcal{S}})$ can be taken to be

$$I_{\mathcal{S}} \subset I_{\mathbf{b}}. \tag{$*_{\mathcal{S}}$}$$

This condition suffices, because if $\mathbf{b}$ satisfies $(*_{\mathcal{S}})$, then $\rho_{\mathbf{b},\mathcal{A}}$ and $g \in I_{\mathcal{S}}$ will coincide as maps $g^{-1}\mathcal{A} \to \mathcal{A}$, since both are simplicial and fix $\mathbf{b}$. This proves the lemma. □

Denote by $\rho_{I_P,\mathcal{A}}(v)$ the common value of all retractions $\rho_{\mathbf{b},\mathcal{A}}(v)$ where $\mathbf{b}$ ranges over a set (depending on v) of alcoves as in the lemma. This defines a retraction

$$\rho_{I_P,\mathcal{A}} : \mathcal{B}_G \to \mathcal{A}.$$

As the notation indicates, it depends on the choice of the alcove $\mathbf{a}_M$ and the parabolic $P = MN$. The following result appeared in [GHKR2]. We give a proof for the benefit of the reader.

Lemma 6.5. *(See [GHKR2].) Suppose I_P is defined using P and $\mathbf{a}_M$ as above. Then*:

(i) *For any $g \in I_P$, we have $\rho_{I_P,\mathcal{A}}|_{g^{-1}\mathcal{A}} = g$.*
(ii) *For any alcove $\mathbf{x}$ in $\mathcal{A}$, we have $\rho_{I_P,\mathcal{A}}^{-1}(\mathbf{x}) = I_P\mathbf{x}$.*

Proof. Part (i) follows from the proof of the lemma above. It implies that $\rho_{I_P,\mathcal{A}}$ is I_P-invariant, in the following sense: thinking of $\rho_{I_P,\mathcal{A}}$ as a map $G/I \to \widetilde{W}I/I$ sending alcoves in $\mathcal{B}_G$ to those in $\mathcal{A}$, for each $w \in \widetilde{W}$, we have

$$\rho_{I_P,\mathcal{A}}^{-1}(w\mathbf{a}) \supset I_P w\mathbf{a}.$$

But since we have disjoint decompositions

$$G/I = \coprod_{w \in \widetilde{W}} I_P wI/I = \coprod_{w \in \widetilde{W}} \rho_{I_P,\mathcal{A}}^{-1}(w\mathbf{a}),$$

this containment must actually be an equality. This proves part (ii). □

We now turn to the variant of interest to us here, namely the retraction ρ_{K_P,Δ_M} onto the M-dominant Weyl chamber $\Delta_M \subset \mathcal{A}$. Here we recall $K_M = K \cap M$, and $K_P = N \cdot K_M$;

Definition 6.6. We define the retraction $\rho_{K_P,\Delta_M} : \mathcal{B}_G \to \Delta_M$ by setting

$$\rho_{K_P,\Delta_M} = \rho_{\Delta_M} \circ \rho_{I_P,\mathcal{A}}.$$

Here I_P is determined by $P = MN$ and the M-alcove $\mathbf{a}_M$ defined in terms of some set of positive roots Ψ_M^+ for M.

Of course, we need to show that this is indeed independent of the choice of $\mathbf{a}_M$.

Lemma 6.7. *The retraction $\rho_{\Delta_M} \circ \rho_{I_P,\mathcal{A}}$ is independent of the choice of M-alcove $\mathbf{a}_M$.*

Proof. Fix any bounded set $\mathcal{S} \subset \mathcal{B}_G$. Recall that on $\mathcal{S}$, $\rho_{I_P,\mathcal{A}}$ can be realized as the retraction $\rho_{\mathbf{b}-\nu,\mathcal{A}}$, for any M-central and very N-dominant cocharacter ν (depending on $\mathcal{S}$) and any alcove $\mathbf{b}$ whose closure contains o and which is contained in $\mathbf{a}_M$. Suppose $\mathbf{a}_M^1$ and $\mathbf{a}_M^2$ are two M-alcoves, and $\mathbf{b}_i \subset \mathbf{a}_M^i$ are two such alcoves. We need to show that $\rho_{\Delta_M} \circ \rho_{\mathbf{b_1}-\nu,\mathcal{A}} = \rho_{\Delta_M} \circ \rho_{\mathbf{b_2}-\nu,\mathcal{A}}$ on $\mathcal{S}$.

Let $w \in W_M$ be such that $w\mathbf{a}_M^1 = \mathbf{a}_M^2$. Then by Lemma 6.4, we know already that $\rho_{w\mathbf{b_1}-\nu,\mathcal{A}} = \rho_{\mathbf{b_2}-\nu,\mathcal{A}}$ on $\mathcal{S}$. So, it remains to see that $\rho_{\Delta_M} \circ \rho_{\mathbf{b_1}-\nu,\mathcal{A}} = \rho_{\Delta_M} \circ \rho_{w\mathbf{b_1}-\nu,\mathcal{A}}$ on $\mathcal{S}$.

Without loss of generality, we may assume $w = s_\alpha$, the reflection corresponding to a simple root α in the positive system Φ_M^+. We claim that $\rho_{w\mathbf{b_1}-\nu,\mathcal{A}}(\mathbf{x}) \in \{\rho_{\mathbf{b_1}-\nu,\mathcal{A}}(\mathbf{x}), w(\rho_{\mathbf{b_1}-\nu,\mathcal{A}}(\mathbf{x}))\}$. This will imply the lemma.

Denote by F the unique codimension 1 facet in $\mathcal{A}$ which separates $\mathbf{c}_1 := \mathbf{b}_1 - \nu$ from $\mathbf{c}_2 := w\mathbf{b}_1 - \nu$. Abbreviate $\rho_{\mathbf{c_i},\mathcal{A}}$ by ρ_i for $i = 1, 2$.

If $\mathbf{x} \subset \mathcal{A}$, then both retractions just give $\mathbf{x}$ and there is nothing to do. So, assume $\mathbf{x}$ is not contained in $\mathcal{A}$. Choose a minimal gallery $\mathbf{x} = \mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_d, F$ joining $\mathbf{x}$ to F. The notation means F is a face of $\mathbf{x}_d$ and $d \geqslant 0$ is the distance $d(\mathbf{x}, F)$ (by definition, for a facet $\mathcal{F}$, the distance $d(\mathbf{x}, \mathcal{F})$ is the minimal number of wall-crossings needed to form a gallery from $\mathbf{x}$ to an alcove $\mathbf{y}$ with $\mathcal{F} \subset \bar{\mathbf{y}}$).

There are two cases to consider. Assume first that $\mathbf{x}_d$ is one of the $\mathbf{c}_i$. Let us assume $\mathbf{x}_d = \mathbf{c}_1$ (the other case is similar). Then $d(\mathbf{x}, \mathbf{c}_1) = d$ and $d(\mathbf{x}, \mathbf{c}_2) = d + 1$. In this case $\mathbf{c}_2, \mathbf{c}_1, \mathbf{x}_{d-1}, \dots, \mathbf{x}$ is a minimal gallery joining $\mathbf{c}_2$ (and also $\mathbf{c}_1$) to $\mathbf{x}$, and it follows that $\rho_1(\mathbf{x}) = \rho_2(\mathbf{x})$.

Now assume that $\mathbf{x}_d$ is neither $\mathbf{c}_1$ nor $\mathbf{c}_2$. Then there are two minimal galleries

$$\mathbf{c}_1, \mathbf{x}_d, \dots, \mathbf{x}, \qquad \mathbf{c}_2, \mathbf{x}_d, \dots, \mathbf{x}$$

and $\mathbf{x}_d \not\subseteq \mathcal{A}$ (and hence no $\mathbf{x}_i$ lies in $\mathcal{A}$; see e.g. [BT], (2.3.6)). Thus these galleries leave $\mathcal{A}$ at F, and by looking at how they fold down onto $\mathcal{A}$ under ρ_1 and ρ_2, we see that $\rho_1(\mathbf{x}) = w(\rho_2(\mathbf{x}))$.

This proves the claim, and thus the lemma. □

Lemma 6.5(ii) above has the following counterpart. Using Lemma 5.3, we see from it that $\rho_{K_P,\Delta_M} = b_{\xi,\Delta_M}$.

Lemma 6.8. *For any M-dominant $\lambda \in X_*(T)$, we have*

$$\rho_{K_P,\Delta_M}^{-1}(x_\lambda) = K_P x_\lambda.$$

Proof. First we show that ρ_{K_P,Δ_M} is K_P-invariant; this will show $\rho_{K_P,\Delta_M}^{-1}(x_\lambda) \supseteq K_P x_\lambda$. Choose any M-alcove $\mathbf{a}_M$ and corresponding Iwahori I_M as in the definition of ρ_{K_P,Δ_M} as $\rho_{\Delta_M} \circ \rho_{I_P,\mathcal{A}}$. We can write $K_P = K_M N = I_M W_M I_M N$. By Lemma 6.5(ii), $\rho_{I_P,\mathcal{A}}$ is I_P-invariant, and so it suffices to show that ρ_{K_P,Δ_M} is W_M-invariant. On a bounded set $\mathcal{S}$, we realize this retraction as $\rho_{\Delta_M} \circ \rho_{\mathbf{b},\mathcal{A}}$ for some alcove $\mathbf{b}$ contained in $\mathbf{a}_M$ and sufficiently anti-dominant with respect to the roots in Φ_N. Now for $w \in W_M$ and $v \in \mathcal{S}$, we have

$$\rho_{\Delta_M} \circ \rho_{\mathbf{b},\mathcal{A}}(wv) = \rho_{\Delta_M}\bigl(w\bigl(\rho_{w^{-1}\mathbf{b},\mathcal{A}}(v)\bigr)\bigr) = \rho_{\Delta_M} \circ \rho_{w^{-1}\mathbf{b},\mathcal{A}}(v)$$

using Lemma 6.2 for the first equality. But by Lemma 6.7, the right-hand side is $\rho_{K_P,\Delta_M}(v)$, and the invariance is proved.

Next we show the opposite inclusion. We have

$$\rho_{K_P,\Delta_M}^{-1}(x_\lambda) = \bigcup_{w \in W_M} \rho_{I_P,\mathcal{A}}^{-1}(wx_\lambda).$$

By Lemma 6.5(ii), the right-hand side is contained in $I_P W_M x_\lambda$, which certainly belongs to $K_P x_\lambda$. This completes the proof. □

We deduce the following interpolation between the Cartan and Iwasawa decompositions of G:

Corollary 6.9. *The map $\widetilde{W} \to G$ induces a bijection*

$$W_M\backslash\Lambda = W_M\backslash\widetilde{W}/W \xrightarrow{\sim} K_P\backslash G/K.$$

The next lemma is a rough comparison between the retractions ρ_{K_P,Δ_M} and $\rho_{-\nu,\Delta_G-\nu}$. (Actually it concerns their restrictions to a given bounded subset $\mathcal{S} \subset \mathcal{B}_G$.) It will be made much more precise in the next section.

Lemma 6.10. *For any bounded set $\mathcal{S} \subset \mathcal{B}_G$, there are elements ν which are M-central and very N-dominant (depending on $\mathcal{S}$) such that*

$$\rho_{K_P,\Delta_M}|\mathcal{S} = \rho_{-\nu,\Delta_G-\nu}|\mathcal{S}.$$

Proof. First we remark that for ν which is M-central and very N-dominant, and for any alcove $\mathbf{b}$ having $x_{-\nu}$ as a vertex, every point $x \in \rho_{\mathbf{b},\mathcal{A}}(\mathcal{S})$ satisfies

$$\langle \alpha, \nu + x\rangle > 0$$

for $\alpha \in \Phi_N$. Thus the retraction of such an element x into $\Delta_G - \nu$ coincides with its retraction into Δ_M (and both are achieved by applying a suitable element of W_M). The lemma follows from this remark and the definitions. $\square$

7. Sharp comparison of ρ_{K_P,Δ_M} and $\rho_{-\nu,\Delta_G-\nu}$

7.1. Statement of key proposition

The following is the key technical device of this paper. It is a much sharper version of Lemma 6.10.

Proposition 7.1. *Let μ be a G-dominant element of $X_*(T)$. Suppose $\nu \geqslant^P \mu$. Then*

$$\rho_{-\nu,\Delta_G-\nu}|\overline{Kx_\mu} = \rho_{K_P,\Delta_M}|\overline{Kx_\mu}. \tag{7.1}$$

The first lemma deals with the images of the two retractions appearing in Proposition 7.1.

Lemma 7.2. *Assume $\nu \geqslant^P \mu$. Then the following statements hold.*

(a) *We have $\Omega(\mu) \cap \Delta_M = \Omega(\mu) \cap (\Delta_G - \nu)$.*
(b) *For $\lambda \in \Delta_M$, the intersection $K_P x_\lambda \cap \overline{Kx_\mu}$ is nonempty only if $\lambda \in \Omega(\mu)$.*
(c) *For $\lambda \in \Delta_G - \nu$, the intersection $(t^{-\nu}Kt^{\nu})x_\lambda \cap \overline{Kx_\mu}$ is nonempty only if $\lambda \in \Omega(\mu)$.*

Proof. Part (a) follows easily from the definitions. Part (b) is well known (cf. [GHKR1], Lemma 5.4.1). Let us prove (c). Assume $\lambda \in \Delta_G - \nu$ makes the intersection in (c) nonempty. By Lemma 6.3, we see that x_λ lies in $\rho_{-\nu,\Delta_G-\nu}(\overline{K\mu})$. By Lemma 6.1 and the definition of $\rho_{-\nu,\Delta_G-\nu}$, there exists $w_{-\nu} \in W_{-\nu}$ such that $w_{-\nu}(\lambda) \in \Omega(\mu)$. Write $w_{-\nu} = t^{-\nu}wt^{\nu}$ for some $w \in W$. Then we see that

$$w(\lambda + \nu) = w(\widetilde{\lambda}) + \nu \tag{7.2}$$

for some $\widetilde{\lambda} \in \Omega(\mu)$. Eq. (7.2) has the same form if we apply any element $w' \in W_M$ to it. By replacing w with a suitable element of the form $w'w$ ($w' \in W_M$), we may assume the right-hand side of (7.2) is

M-dominant. But then since $\nu \geqslant^P \mu$, the right-hand side is G-dominant. Then $w(\lambda+\nu)$ and $\lambda+\nu$ are both G-dominant, hence they coincide. This yields $\lambda = w(\widetilde{\lambda})$; thus λ belongs to $\Omega(\mu)$, as desired. □

We will rephrase Proposition 7.1 using the following lemma.

Lemma 7.3. *Fix a G-dominant cocharacter μ. Then the following are equivalent conditions on an element ν satisfying $\nu \geqslant^P \mu$:*

(i) $(t^{-\nu}Kt^{\nu})x_\lambda \cap \overline{Kx_\mu} = K_P x_\lambda \cap \overline{Kx_\mu}$, *for all* $\lambda \in \Omega(\mu) \cap \Delta_M$;
(ii) $\rho_{-\nu,\Delta_G-\nu}|\overline{Kx_\mu} = \rho_{K_P,\Delta_M}|\overline{Kx_\mu}$.

Proof. This is immediate in view of Lemmas 6.3, 6.8, and 7.2. □

In a similar way, we have the following result.

Lemma 7.4. *The following are equivalent conditions on cocharacters ν_1 and ν_2 which satisfy $\nu_i \geqslant^P \mu$ for $i = 1, 2$:*

(a) $(t^{-\nu_1}Kt^{\nu_1})x_\lambda \cap \overline{Kx_\mu} = (t^{-\nu_2}Kt^{\nu_2})x_\lambda \cap \overline{Kx_\mu}$ *for all* $\lambda \in \Omega(\mu) \cap \Delta_M$;
(b) $\rho_{-\nu_1,\Delta_G-\nu_1}|\overline{Kx_\mu} = \rho_{-\nu_2,\Delta_G-\nu_2}|\overline{Kx_\mu}$;
(c) $\rho_{\mathbf{b}_1,\mathcal{A}}(x) \in W_M(\rho_{\mathbf{b}_2,\mathcal{A}}(x))$, *for* $x \in \overline{Kx_\mu}$.

Here for $i = 1, 2$, $\mathbf{b}_i$ is any alcove having $x_{-\nu_i}$ as a vertex.

As discussed above in the context of $\rho_{-\nu,\Delta_G-\nu}$, in proving (c) we are free to use any alcove having $x_{-\nu_i}$ as a vertex we wish.

Proof. The equivalence of (a) and (b) is clear. The equivalence of (b) and (c) follows by the same argument which proved Lemma 6.10. □

If (a), (b), or (c) hold, then so does (ii) in Lemma 7.3, by virtue of Lemma 6.10. To see this, in (b) above, take $\nu = \nu_1$ and take $\nu_2 \geqslant^P \mu$ sufficiently dominant with respect to N so that, for $\mathcal{S} = \overline{Kx_\mu}$, the conclusion of Lemma 6.10 holds for it.

Thus, the following proposition will imply Proposition 7.1 (and in fact it is equivalent to Proposition 7.1).

Proposition 7.5. *Suppose $\nu_i \geqslant^P \mu$ for $i = 1, 2$. Then*

$$\rho_{-\nu_1,\Delta_G-\nu_1}|\overline{Kx_\mu} = \rho_{-\nu_2,\Delta_G-\nu_2}|\overline{Kx_\mu}.$$

7.2. Proof of Proposition 7.5

We will prove (c) in Lemma 7.4 holds. We make particular choices for the $\mathbf{b}_i$, namely, we set

$$\mathbf{b}_i := w_0\mathbf{a} - \nu_i$$

for $i = 1, 2$, where w_0 denotes the longest element in the Weyl group W (so that $w_0\mathbf{a}$ is the alcove at the origin which is in the *anti-dominant* Weyl chamber). Set $\rho_i = \rho_{\mathbf{b}_i,\mathcal{A}}$ for $i = 1, 2$. For these choices of $\mathbf{b}_i$, we will prove the following more precise fact:

$$\text{for } x \in \overline{Kx_\mu}, \text{ we have } \rho_1(x) = \rho_2(x).$$

Let $x \in \overline{Kx_\mu}$ and let $\mathbf{a} = \mathbf{a}_0, \mathbf{a}_1, \dots, \mathbf{a}_{r-1}, \mathbf{a}_r$ be any minimal gallery in $\mathcal{B}_G$ joining $\mathbf{a}$ to x (this means that x belongs to the closure of $\mathbf{a}_r$ and r is minimal with this property). It is obviously enough for us to prove that

$$\rho_1(\mathbf{a}_r) = \rho_2(\mathbf{a}_r).$$

We will prove this by induction on r. But first, we must formulate an alcove-theoretic proposition (Proposition 7.6 below). This involves the notion of μ-admissible alcove (cf. [HN,KR]). By definition, an alcove in $\mathcal{A}$ is *μ-admissible* provided it can be written in the form $w\mathbf{a}$ for $w \in \widetilde{W}$ such that $w \leqslant t_\lambda$ for some $\lambda \in W\mu$. Here $\leqslant$ is the Bruhat order on $\widetilde{W}$ determined by the alcove $\mathbf{a}$.

The set of μ-admissible alcoves is closed under the Bruhat order on any given apartment: if $\mathbf{a}_r$ is μ-admissible, and $\mathbf{a}_{r-1}$ precedes $\mathbf{a}_r$, then $\mathbf{a}_{r-1}$ is also μ-admissible. Moreover, if $x \in \Omega(\mu)$, then the minimal length alcove containing x in its closure is always μ-admissible. These remarks imply that the next proposition suffices to prove Proposition 7.5.

Proposition 7.6. *Suppose $\mathbf{a} = \mathbf{a_0}, \mathbf{a}_1, \dots, \mathbf{a}_r$ is any minimal gallery in $\mathcal{B}_G$ such that, in an apartment $\mathcal{A}'$ containing this gallery, the terminal alcove $\mathbf{a}_r$ (and thus every other alcove $\mathbf{a}_i$) is a μ-admissible alcove in $\mathcal{A}'$. Then we have*

$$\rho_1(\mathbf{a}_r) = \rho_2(\mathbf{a}_r).$$

Proof. We proceed by induction on r. There is nothing to prove for $r = 0$. Assume $r \geqslant 1$ and that $\rho_1(\mathbf{a}_{r-1}) = \rho_2(\mathbf{a}_{r-1})$. In particular, if F_r is the face separating $\mathbf{a}_{r-1}$ from $\mathbf{a}_r$, we have $\rho_1(F_r) = \rho_2(F_r)$. Let $H \subset \mathcal{A}$ denote the hyperplane containing $\rho_1(F_r)$; write s_H for the corresponding reflection in $\mathcal{A}$. Set $\mathbf{c}' := \rho_1(\mathbf{a}_{r-1})$ and $\mathbf{c} = s_H(\mathbf{c}')$.

We now recall the discussion of Subsection 6.1. For $i = 1, 2$, we have

$$\rho_i(\mathbf{a}_r) \in \{\mathbf{c}', \mathbf{c}\}.$$

Following Subsection 6.1, we need to show that the alcoves $\mathbf{b}_1$ and $\mathbf{b}_2$ are simultaneously in Position 1 (or 2) with respect to $\mathbf{c}'$ and H.

We claim that $\mathbf{b}_1$ and $\mathbf{b}_2$ are both on the same side of H. To see this, we may assume

$$H = H_{\alpha-k} = \{p \in X_*(T)_{\mathbb{R}} \mid \langle \alpha, p \rangle = k\}$$

for a positive root α. If $\alpha \in \Phi_M$, then our assertion follows from the fact that $\mathbf{b}_1$ and $\mathbf{b}_2$ belong to the same "M-alcove" (the anti-dominant one). If $\alpha \in \Phi_N$, it follows because $\nu_i \geqslant^P \mu$ and because H intersects the set $\mathrm{Conv}(W\mu)$. Indeed, let p be a point in $\mathrm{Conv}(W\mu) \cap H$. Then the condition

$$\langle \alpha, \nu_i + p \rangle \geqslant 0$$

implies that

$$\langle \alpha, -\nu_i \rangle \leqslant k,$$

which shows that $x_{-\nu_1}$ and $x_{-\nu_2}$ are on the same side of H, and this suffices to prove the claim.

This shows that the $\mathbf{b}_i$ are simultaneously in Position 1 (or 2) with respect to H and $\mathbf{c}'$. This is almost, but not quite enough by itself, to show that $\rho_1(\mathbf{a}_r) = \rho_2(\mathbf{a}_r)$. Position 1 is quite easy to handle (see below). As we shall see, in Position 2, $\mathbf{a}_r$ could retract under ρ_i to either $\mathbf{c}$ or $\mathbf{c}'$; we require an additional argument, given below, to show the two retractions must coincide.

First assume we are in Position 1, that is, $\mathbf{b}_i$ and $\mathbf{c}'$ are on the same side of H. Then as in Subsection 6.1, we see that $\rho_i(\mathbf{a}_r) = \mathbf{c}$ for $i = 1, 2$.

Now assume we are in Position 2, that is, $\mathbf{b}_i$ and $\mathbf{c}'$ are on opposite sides of H. In this case it is a priori possible that (say) $\rho_1(\mathbf{a}_r) = \mathbf{c}$ while $\rho_2(\mathbf{a}_r) = \mathbf{c}'$. Our analysis below will show that this cannot happen.

Choose any minimal gallery joining $\mathbf{b}_1 = w_0\mathbf{a} - \nu_1$ to F_r:

$$\mathbf{b}_1 = \mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n$$

where F_r is contained in the closure of $\mathbf{x}_n$ (and n is minimal with this property). Similarly, choose a minimal gallery joining $\mathbf{b}_2$ to F_r,

$$\mathbf{b}_2 = \mathbf{y}_0, \mathbf{y}_1, \ldots, \mathbf{y}_m.$$

Further, set $\mathbf{b} := w_0\mathbf{a} - \nu_1 - \nu_2$ and fix two minimal galleries in $\mathcal{A}$,

$$\mathbf{b} = \mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_p = \mathbf{b}_1,$$

$$\mathbf{b} = \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q = \mathbf{b}_2.$$

It is clear that ρ_1 (resp. ρ_2) fixes the former (resp. latter).

Claim. *The concatenation* $\mathbf{b}, \ldots, \mathbf{b}_1, \ldots, \mathbf{x}_n$ *is a* minimal *gallery.* (*The same proof will show that* $\mathbf{b}, \ldots, \mathbf{b}_2, \ldots, \mathbf{y}_m$ *is minimal.*) *This may be checked after applying the retraction* ρ_1. *But* $\rho_1(\mathbf{b}_1), \ldots, \rho_1(\mathbf{x}_n)$ *is a minimal gallery joining* $\mathbf{b}_1$ *to an alcove of* $\mathcal{A}$ *contained in the convex hull* $\mathrm{Conv}(W\mu)$, *and consequently this gallery is in the* J_P-positive *direction* (*see Subsection* 7.3 *below*). *The gallery* $\mathbf{b}, \ldots, \mathbf{b}_1$ *is also in the* J_P*-positive direction* (*and is fixed by* ρ_1).

It would be natural to hope that the concatenation of two galleries in the J_P-positive direction is always in the J_P-positive direction. Then we could invoke Lemma 7.8 below to prove that $\mathbf{b}, \ldots, \mathbf{b}_1 = \rho_1(\mathbf{b}_1), \ldots, \rho_1(\mathbf{x}_n)$ is minimal. But in general it is *not* true that the concatenation of two galleries in the J_P-direction is also in the J_P-positive direction (think of the extreme case $M = G$). However, in our situation, because $\mathbf{b}, \ldots, \mathbf{b}_1$ lies entirely in the same M-alcove (the anti-dominant one), the concatenation $\mathbf{b}, \ldots, \mathbf{b}_1 = \rho_1(\mathbf{b}_1), \ldots, \rho_1(\mathbf{x}_n)$ is nevertheless in the J_P-positive direction. Hence the concatenation $\mathbf{b}, \ldots, \mathbf{b}_1, \ldots, \mathbf{x}_n$ is minimal. The claim is proved.

It follows that the concatenations

$$\mathbf{u}_0, \ldots, \mathbf{u}_p = \mathbf{x}_0, \ldots, \mathbf{x}_n,$$

$$\mathbf{v}_0, \ldots, \mathbf{v}_q = \mathbf{y}_0, \ldots, \mathbf{y}_m$$

are two minimal galleries joining $\mathbf{b}$ to F_r. By a standard result (cf. [BT], (2.3.6)), any two such galleries belong to any apartment that contains both $\mathbf{b}$ and F_r. In particular, $\mathbf{x}_n = \mathbf{y}_m$. Now recall we are assuming that $\mathbf{c}'$, the common value of $\rho_1(\mathbf{a}_{r-1})$ and $\rho_2(\mathbf{a}_{r-1})$, is on the opposite side of H from the alcoves $\mathbf{b}_i$, and so $\mathbf{c}$ is on the same side of H as the $\mathbf{b}_i$. On the other hand, $\rho_1(\mathbf{x}_n)$ and $\rho_2(\mathbf{y}_m)$ are both equal to $\mathbf{c}$, since that is the alcove sharing the facet $\rho_i(F_r)$ with $\mathbf{c}'$ but on the same side of H as $\mathbf{b}_i$. In particular, we have $\mathbf{a}_{r-1} \neq \mathbf{x}_n$.

If $\mathbf{a}_r = \mathbf{x}_n = \mathbf{y}_m$, then we have $\rho_1(\mathbf{a}_r) = \mathbf{c} = \rho_2(\mathbf{a}_r)$. If $\mathbf{a}_r \neq \mathbf{x}_n$, then we have $\rho_1(\mathbf{a}_r) = \mathbf{c}' = \rho_2(\mathbf{a}_r)$. This completes the proof of Proposition 7.6. □

7.3. Galleries in the J_P-positive direction

We define $J \subset G$ to be the Iwahori subgroup corresponding to the "anti-dominant" alcove $w_0\mathbf{a}$. Set $J_M := J \cap M$ and $J_P := NJ_M$. One can think of J_P as the subset of G fixing every alcove in $\mathcal{A}$ which is contained in the M*-anti-dominant* alcove $\mathbf{a}_M^*$ and which is *sufficiently anti-dominant* with respect to the roots in Φ_N.

Now consider any hyperplane $H_{\beta-k}$ (where we assume $\beta \in \Phi^+$), and let s_H denote the corresponding reflection in $\mathcal{A}$. We can define its *J_P-positive side* $H^+_{\beta-k}$ and its *J_P-negative side* $H^-_{\beta-k}$, as follows. If $\beta \in \Phi_M$, we define $H^-_{\beta-k}$ to be the side of $H_{\beta-k}$ containing $\mathbf{a}^*_M$. If $\beta \in \Phi_N$, we define $H^+_{\beta-k}$ by

$$x \in H^+_{\beta-k} \quad \text{iff} \quad x - s_H(x) \in \mathbb{R}_{>0}\beta^\vee.$$

Suppose $\mathbf{z}'$ and $\mathbf{z}$ are adjacent alcoves, separated by the wall H. We say the wall-crossing $\mathbf{z}', \mathbf{z}$ is in the *J_P-positive direction* provided that $\mathbf{z}' \subset H^-$ and $\mathbf{z} \subset H^+$.

Definition 7.7. A gallery $\mathbf{z}_0, \mathbf{z}_1, \dots, \mathbf{z}_r$ is in the *J_P-positive direction* if every wall-crossing $\mathbf{z}_{i-1}, \mathbf{z}_i$ is in the J_P-positive direction.

The following lemma is a mild generalization of Lemma 5.3 of [HN], and its proof is along the same lines.

Lemma 7.8. *Any gallery $\mathbf{z}_0, \dots, \mathbf{z}_r$ in the J_P-positive direction is minimal.*

Proof. Let H_i denote the wall shared by $\mathbf{z}_{i-1}$ and $\mathbf{z}_i$. If the gallery is not minimal, there exists $i < j$ such that $H_i = H_j$. We may assume the $H_k \neq H_i$ for every k with $i < k < j$. By assumption $\mathbf{z}_{i-1} \subset H_i^-$, and $\mathbf{z}_i \subset H_i^+$. Since we do not cross H_i again before H_j, we have $\mathbf{z}_{j-1} \subset H_j^+$. This is contrary to the assumption that the wall-crossing $\mathbf{z}_{j-1}, \mathbf{z}_j$ goes from H_j^- to H_j^+. $\square$

7.4. Triangles in terms of retractions

Set $\rho_0 := \rho_{0,\Delta_G}$. Then the space of based triangles $\mathcal{T}(\alpha, \beta; \gamma)$ can be identified with

$$\rho_0^{-1}(x_\alpha) \cap t^\gamma \rho_0^{-1}(x_{\beta^*})$$

since $\rho_0^{-1}(x_\alpha) = S_\alpha(o)$ and $t^\gamma \rho_0^{-1}(x_{\beta^*}) = S_{\beta^*}(x_\gamma)$.

Now we fix a G-dominant cocharacter μ and an M-dominant cocharacter λ contained in $\Omega(\mu)$. Fix a parabolic subgroup $P = MN$ having M as Levi factor. Let $\xi \in \partial_{Tits}\mathcal{B}$ be a generic point fixed by P. Then

$$\mathcal{IT}(\lambda, \mu; \xi) := \rho^{-1}_{K_P,\Delta_M}(x_\lambda) \cap \rho_0^{-1}(x_\mu).$$

Note that this space depends only on P but not on ξ. Fix any auxiliary cocharacter ν which satisfies $\nu \geqslant^P \mu$. Then Proposition 7.1 shows that we can also describe this space as

$$\mathcal{IT}(\lambda, \mu; \xi) = \rho^{-1}_{-\nu,\Delta_G-\nu}(x_\lambda) \cap \rho_0^{-1}(x_\mu). \tag{7.3}$$

8. Proof of Theorem 3.2

Proof of Theorem 3.2. The desired equality

$$\mathcal{T}\big(\nu + \lambda, \mu^*; \nu\big) = t^\nu\big(\mathcal{IT}(\lambda, \mu; \xi)\big)$$

is just the equality

$$Kx_{\nu+\lambda} \cap t^\nu Kx_\mu = t^\nu(K_P x_\lambda \cap Kx_\mu)$$

which can obviously be rewritten as

$$\big(t^{-\nu} K t^{\nu}\big) x_\lambda \cap K x_\mu = K_P x_\lambda \cap K x_\mu.$$

But this follows from Proposition 7.1, or more precisely from its equivalent version, the equality stated in Lemma 7.3(i). □

9. On dimensions of varieties of triangles

9.1. Based triangles

Let $\alpha, \beta, \gamma \in \Delta_G \cap \Lambda$. Assume that $\alpha + \beta - \gamma \in Q(\Phi^\vee)$, which is a necessary condition for $\mathcal{T}(\alpha, \beta; \gamma)$ to be nonempty. It is clear that

$$\mathcal{T}(\alpha, \beta; \gamma) = K x_\alpha \cap t^{\gamma} K x_{\beta^*},$$

and, as we explained earlier, this shows that $\mathcal{T}(\alpha, \beta; \gamma)$ has the structure of a finite-dimensional quasi-projective variety defined over $\mathbb{F}_p$.

It also shows that we can identify $\mathcal{T}(\alpha, \beta; \gamma)$ with a fiber of the convolution morphism

$$\mathbf{m}_{\alpha,\beta} : S_\alpha(o) \mathbin{\tilde{\times}} S_\beta(o) \to \overline{S_{\alpha+\beta}(o)}.$$

Here

$$S_\alpha(o) \mathbin{\tilde{\times}} S_\beta(o) := \big\{(y, z) \in \mathcal{B}^2\colon\ y \in S_\alpha(o),\ z \in S_\beta(y)\big\}.$$

By definition, $\mathbf{m}_{\alpha,\beta}(y, z) = z$. It follows that

$$\mathcal{T}(\alpha, \beta; \gamma) = \mathbf{m}_{\alpha,\beta}^{-1}(x_\gamma).$$

We have the following a priori bound on the dimension of the variety of triangles.

Proposition 9.1. *Recall that ρ denotes the half-sum of the positive roots Φ^+. Then*

$$\dim \mathcal{T}(\alpha, \beta; \gamma) \leqslant \langle \rho, \alpha + \beta - \gamma \rangle.$$

Moreover, the number of irreducible components of $\mathcal{T}(\alpha, \beta; \gamma)$ of dimension $\langle \rho, \alpha + \beta - \gamma \rangle$ equals the multiplicity $n_{\alpha,\beta}(\gamma)$.

Proof. This can be proved using the Satake isomorphism and the Lusztig–Kato formula, following the method of [KLM3], §8.4. Alternatively, one can invoke the semi-smallness of $\mathbf{m}_{\alpha,\beta}$ and the geometric Satake isomorphism, see [Ha1,Ha2]. Compare [Ha1], Theorems 1.1 and Proposition 1.3. □

9.2. Based ideal triangles

Fix $\mu \in \Delta_G \cap \Lambda$ and $\lambda \in \Delta_M \cap \Lambda$, and assume $\mu - \lambda \in Q(\Phi^\vee)$, which is a necessary condition for $\mathcal{IT}(\mu, \lambda; \xi)$ to be nonempty.

As for the variety of based triangles $\mathcal{T}(\alpha, \beta; \gamma)$, we want to give an a priori bound on the dimension of $\mathcal{IT}(\lambda, \mu; \xi)$ and also give a relation between its irreducible components and the multiplicity $r_\mu(\lambda)$.

The key input is the following analogue of Proposition 9.1 for the intersections of N- and K-orbits in Gr^G. It is proved by considering (2.3) and manipulating the Satake transforms and Lusztig–Kato formulas for G and M, in a manner similar to [KLM3], §8.4.

We set $\mathcal{F} := N x_\lambda \cap K x_\mu$, a finite-type, locally-closed subvariety of Gr^G, defined over $\mathbb{F}_p$.

Proposition 9.2. *(See [GHKR1].) Let ρ_M denote the half-sum of the roots in Φ_M^+. Then*

$$\dim \mathcal{F} \leqslant \langle \rho, \mu + \lambda \rangle - 2\langle \rho_M, \lambda \rangle,$$

and the multiplicity $r_\mu(\lambda)$ equals the number of irreducible components of $\mathcal{F}$ having dimension equal to $\langle \rho, \mu + \lambda \rangle - 2\langle \rho_M, \lambda \rangle$.

The link between the dimensions of $\mathcal{F} = Nx_\lambda \cap Kx_\mu$ and $\mathcal{IT} = K_P x_\lambda \cap Kx_\mu$ comes from the following relation between these varieties. To state it, we first recall that the Iwasawa decomposition

$$G = NMK$$

determines a well-defined *set-theoretic* map

$$\begin{aligned} G/K &\to M/K_M, \\ nmkK &\mapsto mK_M. \end{aligned}$$

We warn the reader that this is *not* a morphism of ind-schemes $\mathrm{Gr}^G \to \mathrm{Gr}^M$; however, when restricted to the inverse image of a connected component of Gr^M, it does induce such a morphism. (Our reference for these facts is [BD], especially Sections 5.3.28–5.3.30.) Since any K_P-orbit belongs to such an inverse image, the following lemma makes sense.

Lemma 9.3. *The map $nmk \mapsto mK_M$ induces a surjective, Zariski locally-trivial fibration*

$$\pi : K_P x_\lambda \cap Kx_\mu \to K_M x_\lambda$$

whose fibers are all isomorphic to $Nx_\lambda \cap Kx_\mu$.

Proof. Let $K_M^\lambda = (t^\lambda K_M t^{-\lambda}) \cap K_M$ denote the stabilizer of x_λ in K_M. The essential point is that the morphism $f : K_M \to K_M/K_M^\lambda$ is a locally trivial principal fibration according to [Ha2, Lemma 2.1]. Next note that since π is K_M-equivariant, it suffices to trivialize π over a neighborhood V of x_λ. According to the result of [Ha2] there exists a Zariski open neighborhood $V \subset K_M/K_M^\lambda$ of x_λ and a section $s : V \to K_M$ of $f | f^{-1}(V)$. Hence for $x' \in V$ we have

$$s(x')x_\lambda = x'.$$

We will now prove that the induced map $\pi : \pi^{-1}(V) \to V$ is equivariantly equivalent to the product bundle $V \times \mathcal{F} \to V$ where $\mathcal{F} = Nx_\lambda \cap Kx_\mu$. Indeed, define $\Phi : \pi^{-1}(V) \to V \times \mathcal{F}$ by

$$\Phi(x) = \left(\pi(x), s\left(\pi(x)\right)^{-1} x\right).$$

Put $\pi(x) := x'$. Note that Φ does indeed map to $V \times \mathcal{F}$ because (from the equivariance of π) we have

$$\pi\left(s\left(\pi(x)\right)^{-1} x\right) = s\left(\pi(x)\right)^{-1} \pi(x) = s\left(x'\right)^{-1} x' = x_\lambda.$$

Now define $\Psi : V \times \mathcal{F} \to \pi^{-1}(V)$ by

$$\Psi\left(x', u\right) = s\left(x'\right)u.$$

Clearly Φ and Ψ are algebraic because s and π are. The reader will verify that Φ and Ψ are mutually inverse. □

Since $K_M x_\lambda$ is irreducible of dimension $2\langle\rho_M, \lambda\rangle$, we immediately deduce the following proposition from Proposition 9.2 and Lemma 9.3. It is an analogue of Proposition 9.1 for the ideal triangles.

Proposition 9.4. *We have the inequality*

$$\dim \mathcal{IT}(\lambda, \mu; \xi) \leqslant \langle\rho, \mu + \lambda\rangle,$$

and the number of irreducible components of $\mathcal{IT}(\lambda, \mu; \xi)$ having dimension equal to $\langle\rho, \mu + \lambda\rangle$ is the multiplicity $r_\mu(\lambda)$.

Remark 9.5. The structure group of the fiber bundle $\pi : K_P x_\lambda \cap K x_\mu \to K_M/K_M^\lambda$ is K_M^λ in the following sense. Choose trivializations $\Psi_1 : V \times \mathcal{F} \to \pi^{-1}(V)$ and $\Psi_2 : V \times \mathcal{F} \to \pi^{-1}(V)$ determined by sections s_1 and s_2 as above. Since s_1 and s_2 are K_M-valued functions on V we may define a new K_M-valued function $k(x')$ on V by $k(x') = s_1(x')^{-1} s_2(x')$. Hence, there exists a morphism $k : V \to K_M^\lambda$ with

$$s_2(x') = s_1(x')k(x').$$

It is then immediate that

$$\Psi_1^{-1} \circ \Psi_2(x', u) = (x', k(x')u).$$

10. Geometric interpretations of $m_{\alpha,\beta}(\gamma)$ and $c_\mu(\lambda)$

The above section gave the geometric interpretations of the numbers $n_{\alpha,\beta}(\gamma)$ and $r_\mu(\lambda)$, by describing them in terms of certain irreducible components of the varieties $\mathcal{T}(\alpha, \beta; \gamma)$ and $\mathcal{II}(\lambda, \mu; \xi)$. The purpose of this section is to give similar geometric interpretations for $m_{\alpha,\beta}(\gamma)$ and $c_\mu(\lambda)$. This will be used to deduce Theorem 3.3 from Theorem 3.2.

10.1. Hecke algebra structure constants

By applying Theorem 8.1 and Lemma 8.5 from [KLM3], and taking into account that $|S_\lambda(o)(\mathbb{F}_q)| = |S_{\lambda^*}(o)(\mathbb{F}_q)|$, we see that

Lemma 10.1. $m_{\alpha,\beta}(\gamma) = |\mathcal{T}(\alpha, \beta; \gamma)(\mathbb{F}_q)|$.

10.2. The constant term

The goal of this subsection is to give a geometric interpretation of the constants $c_\mu(\lambda)$. Fix the weights $\lambda \in \Delta_M$, $\mu \in \Delta_G$. In what follows we temporarily abuse notation and write G, K, M etc., in place of G_q, K_q, M_q, etc. The following lemma comes from integration in stages according to the Iwasawa decomposition $G = KNM$.

Lemma 10.2. *We have*

$$|Nx_\lambda \cap Kx_\mu| = q^{\langle\rho_N,\lambda\rangle} c_M^G(f_\mu)(t^\lambda) = q^{\langle\rho_N,\lambda\rangle} c_\mu(\lambda).$$

Proof. The Iwasawa decomposition $G = KNM$ gives rise to an integration formula, relating integration over G to an iterated integral over the subgroups K, N, and M, where if Γ is any of these unimodular groups, we equip Γ with the Haar measure which gives $\Gamma \cap K$ volume 1. For a subset $S \subset G$, write 1_S for the characteristic function of S. Using the substitution $y = knm$ in forming the iterated integral, the left-hand side above can be written as

$$\begin{aligned}
\int_G 1_{NK}\big(t^{-\lambda}y\big)1_{Kt^\mu K}(y)\,dy &= \int_G 1_{NK}\big(t^{-\lambda}y^{-1}\big)1_{Kt^\mu K}\big(y^{-1}\big)\,dy \\
&= \int_M\int_N\int_K 1_{NK}\big(t^{-\lambda}m^{-1}n^{-1}k^{-1}\big)1_{Kt^\mu K}\big(m^{-1}n^{-1}k^{-1}\big)\,dk\,dn\,dm \\
&= \int_M\int_N 1_{NK}\big(m^{-1}n^{-1}\big)1_{Kt^\mu K}\big(t^\lambda m^{-1}n^{-1}\big)\,dn\,dm \\
&= \int_M\int_N 1_{NK}(m)1_{Kt^\mu K}\big(t^\lambda mn\big)\,dn\,dm \\
&= \int_N 1_{Kt^\mu K}\big(t^\lambda n\big)\,dn \\
&= \delta_P^{-1/2}\big(t^\lambda\big)c_M^G(f_\mu)\big(t^\lambda\big),
\end{aligned}$$

which implies the lemma since $\delta_P^{1/2}(t^\lambda) = q^{-\langle\rho_N,\lambda\rangle}$. $\square$

Now we return to the previous notational conventions, where we distinguish between G, K, M etc. and G_q, K_q, M_q, etc.

Set

$$\begin{aligned}
\mathcal{I} &:= \mathcal{IT}(\lambda,\mu;\xi), \\
\tau_{\lambda\mu} &= \big|\mathcal{IT}(\lambda,\mu;\xi)(\mathbb{F}_q)\big|, \\
i_{\lambda,q} &:= \big|K_{M,q} : K_{M,q}^\lambda\big|,
\end{aligned}$$

where K_M^λ is the stabilizer of x_λ in K_M and $K_{M,q}^\lambda := K_{M,q} \cap K_M^\lambda$. In other words, $i_{\lambda,q}$ is the cardinality of the orbit $K_{M,q}(x_\lambda)$, and in particular, it is finite. Recall also that

$$\mathcal{F} = \mathcal{F}_{\lambda\mu} := Nx_\lambda \cap Kx_\mu.$$

By Lemma 9.3 the variety $\mathcal{I}$ fibers over $K_M/K_M^\lambda = K_M(x_\lambda)$ with fibers isomorphic to $\mathcal{F}$ via the map f which is the restriction of the N-orbit map on NK_Mx_λ to $\mathcal{I}$. In particular,

$$\big|\mathcal{I}(\mathbb{F}_q)\big| = i_{\lambda,q}\big|\mathcal{F}(\mathbb{F}_q)\big|.$$

It is proved in Lemma 10.2 that

$$\big|\mathcal{F}(\mathbb{F}_q)\big| = q^{\langle\rho_N,\lambda\rangle}c_M^G(f_\mu)\big(t^\lambda\big) = q^{\langle\rho_N,\lambda\rangle}c_\mu(\lambda).$$

Set

$$\varphi(\lambda,q) := \frac{1}{q^{\langle\rho_N,\lambda\rangle}i_{\lambda,q}}.$$

By combining Lemma 10.2 with Lemma 9.3, we obtain the following geometric interpretation of $c_\mu(\lambda)$:

Corollary 10.3.

$$c_\mu(\lambda) = q^{-\langle\rho_N,\lambda\rangle}\big|\mathcal{F}(\mathbb{F}_q)\big| = \varphi(\lambda, q)\tau_{\lambda,\mu} = \varphi(\lambda, q)\big|\mathcal{IT}(\lambda, \mu; \xi)(\mathbb{F}_q)\big|.$$

11. Proofs of Theorem 3.3 and Corollary 3.4

Proof of Theorem 3.3. (i) Since $\nu \geqslant^P \mu$, by Theorem 3.2, the $\mathbb{F}_p$-varieties $\mathcal{T}(\alpha, \beta; \gamma) = \mathcal{T}(\nu+\lambda, \mu^*; \nu)$ and $\mathcal{IT}(\lambda, \mu; \xi)$ are isomorphic. Since the cardinalities of their sets of $\mathbb{F}_q$-rational points are $m_{\alpha,\beta}(\gamma)$ and

$$c_\mu(\lambda) q^{\langle\rho_N,\lambda\rangle}|K_{M,q} \cdot x_\lambda|$$

respectively (see Lemma 10.1 and Corollary 10.3), we conclude that

$$m_{\alpha,\beta}(\gamma) = c_\mu(\lambda) q^{\langle\rho_N,\lambda\rangle}|K_{M,q} \cdot x_\lambda|.$$

(ii) We now prove the equality $r_\mu(\lambda) = n_{\nu,\mu}(\nu + \lambda)$. First, it is well known that $n_{\nu,\mu}(\nu + \lambda) = n_{\nu+\lambda,\mu^*}(\nu)$. Next, by Proposition 9.1, $n_{\lambda+\nu,\mu^*}(\nu)$ is the number of irreducible components of $\mathcal{T}(\lambda + \nu, \mu^*; \nu)$ of dimension

$$\big\langle\rho, \lambda + \nu + \mu^* - \nu\big\rangle = \langle\rho, \lambda + \mu\rangle.$$

On the other hand, by Proposition 9.4, $r_\mu(\lambda)$ is the number of irreducible components of $\mathcal{IT}(\lambda, \mu; \xi)$ of dimension $\langle\rho, \lambda + \mu\rangle$. Since $\mathcal{T}(\lambda + \nu, \mu^*; \nu) \cong \mathcal{IT}(\lambda, \mu; \xi)$, the equality follows.

(iii) Consider the implication

$$r_\mu(\lambda) \neq 0 \quad \Rightarrow \quad c_\mu(\lambda) \neq 0.$$

Set $\alpha = \nu + \lambda$, $\beta = \mu^*$, and $\gamma = \nu$. Using parts (i) and (ii), the implication is equivalent to the implication

$$n_{\alpha,\beta}(\gamma) \neq 0 \quad \Rightarrow \quad m_{\alpha,\beta}(\gamma) \neq 0.$$

Now if $n_{\alpha,\beta}(\gamma) \neq 0$, then by Proposition 9.1 the variety $\mathcal{T}(\alpha, \beta; \gamma)$ is nonempty and so $|\mathcal{T}(\alpha, \beta; \gamma)(\mathbb{F}_q)| \neq 0$ for all $q >> 0$. But the Hecke path model for $\mathcal{T}(\alpha, \beta; \gamma)$ then shows that $|\mathcal{T}(\alpha, \beta; \gamma)(\mathbb{F}_q)| \neq 0$ *for all* q (cf. [KLM3], Theorem 8.18). Thus $m_{\lambda,\beta}(\gamma) \neq 0$.

Consider next the implication

$$c_\mu(\lambda) \neq 0 \quad \Rightarrow \quad r_{k\mu}(k\lambda) \neq 0$$

for $k := k_\Phi$. If $c_\mu(\lambda) \neq 0$ then $m_{\lambda+\nu,\mu^*}(\nu) \neq 0$ for $\nu \geqslant^P \mu$ as above. By [KM2],

$$m_{\lambda+\nu,\mu^*}(\nu) \neq 0 \quad \Rightarrow \quad n_{k\cdot(\lambda+\nu),k\cdot\mu^*}(k \cdot \nu) \neq 0.$$

Since the inequality $\geqslant^P$ is homogeneous with respect to multiplication by positive integers, we get

$$k\nu \geqslant^P k\mu.$$

Therefore,

$$n_{k\cdot(\lambda+\nu),k\cdot\mu^*}(k \cdot \nu) = r_{k\mu}(k\lambda).$$

This concludes the proof of Theorem 3.3. $\square$

Proof of Corollary 3.4. Assume that $\mu - \nu \in Q(\Phi_G^\vee)$.

i. (Semigroup property for r.) Suppose $(\lambda_i, \mu_i) \in (\Delta_M \cap \Lambda) \times (\Delta_G \cap \Lambda)$ are such that $r_{\mu_i}(\lambda_i) \neq 0$ for $i = 1, 2$. For each i, choose ν_i with $\nu_i \geqslant^P \mu_i$. If we set $\alpha_i := \nu_i + \lambda_i$, $\beta_i := \mu_i^*$, and $\gamma_i = \nu_i$, for $i = 1, 2$, then Theorem 3.3(ii) gives us

$$n_{\alpha_i,\beta_i}(\gamma_i) = r_{\mu_i}(\lambda_i) \neq 0.$$

It is well known that the triples (α, β, γ) with $n_{\alpha,\beta}(\gamma) \neq 0$ form a semigroup, so we have $n_{\alpha_1+\alpha_2,\beta_1+\beta_2}(\gamma_1 + \gamma_2) \neq 0$. By the semigroup property of $\geqslant^P$, we have $\nu_1 + \nu_2 \geqslant^P \mu_1 + \mu_2$, so Theorem 3.3(ii) applies again and implies

$$n_{\alpha_1+\alpha_2,\beta_1+\beta_2}(\gamma_1 + \gamma_2) = r_{\mu_1+\mu_2}(\lambda_1 + \lambda_2).$$

The result is now clear.

ii. (Uniform saturation for c.) Consider the implication

$$c_{N\mu}(N\lambda) \neq 0 \quad \text{for some } N \neq 0 \quad \Rightarrow \quad c_{k_\Phi \mu}(k_\Phi \lambda) \neq 0.$$

As above, take $\nu \geqslant^P \mu$ and set $\alpha := \nu + \lambda$, $\beta := \mu^*$, $\gamma := \nu$. Then (with some positive factors $Const_1$, $Const_2$)

$$c_\mu(\lambda) = Const_1 \cdot m_{\alpha,\beta}(\gamma), \qquad c_{N\mu}(N\lambda) = Const_2 \cdot m_{N\alpha,N\beta}(N\gamma).$$

Note that our assumption $\mu - \lambda \in Q(\Phi^\vee)$ is equivalent to $\lambda + \mu^* \in Q(\Phi^\vee)$ and thus to $\alpha + \beta - \gamma \in Q(\Phi^\vee)$. Now the implication follows from the uniform saturation for the structure constants m for the Hecke ring $\mathcal{H}_G$ proved in [KLM3].

iii. (Uniform saturation for r.) Consider the implication

$$r_{N\mu}(N\lambda) \neq 0 \quad \text{for some } N \neq 0 \quad \Rightarrow \quad r_{k^2\mu}\bigl(k^2\lambda\bigr) \neq 0$$

for $k := k_\Phi$. Similarly to (ii), this implication follows from the implication

$$n_{N\alpha,N\beta}(N\gamma) \neq 0 \quad \text{for some } N \neq 0 \quad \Rightarrow \quad n_{k^2\alpha,k^2\beta}\bigl(k^2\gamma\bigr) \neq 0$$

proved in [KM2]. Since for type A root systems Φ, $k_\Phi = 1$, it follows that the semigroup $\{r_\mu(\lambda) \neq 0\}$, is saturated. $\square$

12. Remarks on equidimensionality

When computing the dimensions of various varieties, we may just as well work over the algebraic closure $k = \overline{\mathbb{F}}_p$ of $\mathbb{F}_p$. We will also consider all the following schemes and ind-schemes with reduced structure, as this has no effect on dimension questions.

By [Ha2], we know that when α and β are sums of minuscule cocharacters, then the variety $\mathcal{T}(\alpha, \beta; \gamma)$ is either empty, or it is equidimensional of dimension $\langle \rho, \alpha + \beta - \gamma \rangle$. Here we present some analogous results for $\mathcal{IT}(\lambda, \mu; \xi)$.

Corollary 12.1. *Let $\underline{G} = \mathrm{GL}_n$. Then $\mathcal{IT}(\lambda, \mu; \xi) = NK_M x_\lambda \cap K x_\mu$ (resp. $N x_\lambda \cap K x_\mu$) is either empty or it is equidimensional of dimension $\langle \rho, \mu + \lambda \rangle$ (resp. $\langle \rho, \mu + \lambda \rangle - 2\langle \rho_M, \lambda \rangle$).*

Proof. All coweights for GL_n are sums of minuscules. So for any choice of ν with $\nu \geqslant^P \mu$, the cocharacters $\alpha = \nu + \lambda$ and $\beta = \mu^*$ are sums of $\underline{G}$-dominant minuscule cocharacters. Hence the result for $\mathcal{IT}(\lambda, \mu; \xi)$ follows using Theorem 3.2 and the equidimensionality of $\mathcal{T}(\nu + \lambda, \mu^*; \nu)$ proved in [Ha2]. The statement on $N x_\lambda \cap K x_\mu$ follows from the statement on $\mathcal{IT}(\lambda, \mu; \xi)$ using Lemma 9.3. $\square$

With more work, one can show the following stronger result.

Proposition 12.2. *Let $\underline{G}$ be arbitrary and suppose μ is a sum of minuscule $\underline{G}$-dominant cocharacters. Then the conclusion of Corollary* 12.1 *holds for the pair (μ, λ). Consequently, we have $r_\mu(\lambda) \neq 0 \Leftrightarrow c_\mu(\lambda) \neq 0$.*

Unlike Corollary 12.1, this is not a direct application of [Ha2]. It would be in the situation where we can find ν such that $\nu \geqslant^P \mu$ and such that $\nu + \lambda$ is a sum of minuscules. However, there is no guarantee we can find ν with such properties in general.

This proposition is a special case of a more general result. For $\lambda \in \Delta_M$ and $\mu \in \Delta_G$ we set

$$\mathcal{Q}_\mu = K x_\mu,$$
$$S_\lambda^N = N x_\lambda.$$

These are finite-type $\mathbb{F}_p$-varieties which are locally closed in the affine Grassmannian Gr^G.

Theorem 12.3. *Let μ be a sum of minuscule $\underline{G}$-dominant cocharacters. Suppose the variety $S_\lambda^N \cap \overline{\mathcal{Q}}_\mu$ is nonempty, and let $C \subset S_\lambda^N \cap \overline{\mathcal{Q}}_\mu$ be an irreducible component whose generic point belongs to the stratum $\mathcal{Q}_\nu \subset \overline{\mathcal{Q}}_\mu$, for a $\underline{G}$-dominant coweight $\nu \leqslant \mu$. Then*

$$\dim(C) = \langle \rho, \nu + \lambda \rangle - 2\langle \rho_M, \lambda \rangle.$$

In particular, $S_\lambda^N \cap \mathcal{Q}_\mu$ is either empty, or it is equidimensional with dimension equal to $\langle \rho, \mu + \lambda \rangle - 2\langle \rho_M, \lambda \rangle$.

We will prove this theorem in the following section.

13. Proof of Theorem 12.3

13.1. Proof of Theorem 12.3 for $S_{w\mu}^N \cap \overline{\mathcal{Q}}_\mu$

In this subsection μ will denote an *arbitrary* $\underline{G}$-dominant coweight, but now we assume $\lambda = w\mu$ for some $w \in W$. We continue to assume that $\lambda = w\mu$ is $\underline{M}$-dominant (this puts some restrictions on w which we shall not need to use).

It is easy to prove the following generalization of a result of Ngô and Polo [NP], Lemma 5.2.

Proposition 13.1. *(See [NP].) The map $n \mapsto n x_{w\mu}$ gives an isomorphism of varieties*

$$\prod_{\substack{\alpha \in \Phi^+ \\ w\alpha \in \Phi_N}} \prod_{i=0}^{\langle \alpha, \mu \rangle - 1} U_{w\alpha, i} \xrightarrow{\sim} S_{w\mu}^N \cap \overline{\mathcal{Q}}_\mu,$$

where Φ_N is the set of $\underline{B}$-positive roots in $\mathrm{Lie}(N)$, *and $U_{\beta,i}$ is affine root group consisting of the set of elements of form $u_\beta(xt^i)$ $(x \in \overline{\mathbb{F}}_p)$, where $u_\beta : \mathbb{G}_a \to \underline{G}$ is the homomorphism determined by the root β, and where $i \in \mathbb{Z}$.*

In particular, we see that $S_{w\mu}^N \cap \overline{\mathcal{Q}}_\mu$ is just an affine space of dimension

$$\dim\big(S_{w\mu}^N \cap \overline{\mathcal{Q}}_\mu\big) = \sum_{\substack{\alpha \in \Phi^+ \\ w\alpha \in \Phi_N}} \langle \alpha, \mu \rangle$$

$$= \sum_{\alpha \in \Phi^+ \cap w^{-1}\Phi^+} \langle \alpha, \mu \rangle - \sum_{\alpha \in \Phi^+ \cap w^{-1}\Phi_M^+} \langle \alpha, \mu \rangle$$
$$= \langle \rho, \mu + w\mu \rangle - \sum_{\beta \in w\Phi^+ \cap \Phi_M^+} \langle \beta, w\mu \rangle,$$

where Φ_M^+ is the $\underline{B}_M$-positive roots in $\underline{M}$. We have used the identity

$$\sum_{\alpha \in \Phi^+ \cap w^{-1}\Phi^+} \alpha = \rho + w^{-1}\rho$$

in simplifying the first sum in the second line.

Now note that the inclusion $w\Phi^+ \cap \Phi_M^+ \subseteq \Phi_M^+$ and the $\underline{M}$-dominance of $w\mu$ show that in any event

$$\sum_{\beta \in w\Phi^+ \cap \Phi_M^+} \langle \beta, w\mu \rangle \leqslant 2\langle \rho_M, w\mu \rangle,$$

and thus

$$\dim\bigl(S_{w\mu}^N \cap \overline{\mathcal{Q}}_\mu\bigr) \geqslant \langle \rho, \mu + w\mu \rangle - 2\langle \rho_M, w\mu \rangle.$$

But by the dimension estimate in Proposition 9.2 this inequality is an equality, and we get the following result.

Proposition 13.2. *Assume $w\mu$ is $\underline{M}$-dominant. Then $\sum_{\beta \in w\Phi^+ \cap \Phi_M^+} \langle \beta, w\mu \rangle = 2\langle \rho_M, w\mu \rangle$, and*

$$S_{w\mu}^N \cap \overline{\mathcal{Q}}_\mu \cong \mathbb{A}^{\langle \rho, \mu + w\mu \rangle - 2\langle \rho_M, w\mu \rangle}.$$

Note that $S_{w\mu}^N \cap \overline{\mathcal{Q}}_\mu$ is irreducible in this case. If μ is minuscule, then any element $\lambda \in \Omega(\mu)$ is of the form $\lambda = w\mu$ for some $w \in W$. Thus Proposition 13.2 proves Theorem 12.3 for μ minuscule.

13.2. Some morphisms between affine Grassmannians

In the following we find it convenient to change some of our earlier notations. Let $*$ (resp. $*_M$) denote the obvious base-point in the affine Grassmannian $\mathcal{Q} = G/K$ (resp. $\mathcal{Q}^M = M/K_M$). Recall (Subsection 9.2), the Iwasawa decomposition $G = NMK$ gives rise to a well-defined set-theoretic map

$$\pi : G/K \to M/K_M,$$
$$nm* \to m *_M .$$

Recall that if c denotes a connected component of the ind-scheme $\mathcal{Q}^M$, then $(G/K)_c := \pi^{-1}(c) \subset G/K$ is a locally-closed sub-ind-scheme, and the restricted map

$$\pi : (G/K)_c \to M/K_M$$

is a morphism of ind-schemes. The upshot is that in the discussion below, all morphisms we will define set-theoretically using the Iwasawa decomposition are actually morphisms of (ind-)schemes.

For an r-tuple of $\underline{G}$-dominant coweights μ_i, we have the twisted product

$$\widetilde{\mathcal{Q}}_{\mu_\bullet} := \overline{\mathcal{Q}}_{\mu_1} \tilde{\times} \cdots \tilde{\times} \overline{\mathcal{Q}}_{\mu_r} \subset \mathcal{Q}^r$$

a finite-dimensional projective variety whose points are r-tuples $(g_1*, \dots, g_r*)$ such that for each $i = 1, \dots, r$,

$$g_{i-1}^{-1} g_i * \in \overline{\mathcal{Q}}_{\mu_i},$$

where by convention $g_0 = 1$.

Projection onto the last factor $(g_1*, \dots, g_r*) \mapsto g_r*$, gives a projective birational morphism

$$m_{\mu_\bullet} : \widetilde{\mathcal{Q}}_{\mu_\bullet} \to \overline{\mathcal{Q}}_{|\mu_\bullet|},$$

where by definition $|\mu_\bullet| = \sum_i \mu_i$.

It is a well-known fact due to Mirković–Vilonen [MV1,MV2] that $m_{\mu_\bullet}$ is semi-small and locally trivial in the stratified sense (cf. also [Ha2]). We shall make essential use of the local triviality, which means that around each $y \in \mathcal{Q}_\nu \subset \overline{\mathcal{Q}}_{|\mu_\bullet|}$, there is an open subset $V \subset \mathcal{Q}_\nu$ and an isomorphism

$$m_{\mu_\bullet}^{-1}(V) \cong V \times m_{\mu_\bullet}^{-1}(y)$$

which commutes with the projections onto V.

13.3. Preliminaries for the proof of Theorem 12.3 in general

We will denote by $\pi_{c_\bullet}$ any morphism of the form

$$\pi^r : (G/K)_{c_1} \times \cdots \times (G/K)_{c_r} \to M/K_M \times \cdots \times M/K_M,$$

for any suitable family of connected components c_i of M/K_M. We let $\pi_{c_\bullet,r}$ denote the composition of such a morphism with the projection onto the last factor of $(M/K_M)^r$.

Recall we are assuming μ is a sum of $\underline{G}$-dominant minuscule coweights. Let us fix an r-tuple of $\underline{G}$-dominant minuscule coweights μ_i, $i = 1, \dots, r$, such that $\mu = |\mu_\bullet| := \sum_i \mu_i$.

Definition 13.3. For $\nu_\bullet$ an r-tuple of $\underline{M}$-dominant coweights ν_i with $\nu_i \in W\mu_i$, we define $\sigma_i = (\nu_1, \dots, \nu_i)$ and an ind-scheme

$$\mathfrak{S}_{\sigma_i}^N = N\big(K_M t^{\nu_1} K_M \cdots K_M t^{\nu_i} K_M\big)K/K,$$

and set $\mathfrak{S}_{\nu_\bullet}^N := \mathfrak{S}_{\sigma_1}^N \times \cdots \times \mathfrak{S}_{\sigma_r}^N$. Further, define a scheme

$$\mathfrak{S}_{\nu_\bullet}^N \cap \widetilde{\mathcal{Q}}_{\mu_\bullet} = \big(\mathfrak{S}_{\sigma_1}^N \times \cdots \times \mathfrak{S}_{\sigma_r}^N\big) \cap \widetilde{\mathcal{Q}}_{\mu_\bullet},$$

the intersection being taken in $\mathcal{Q} \times \cdots \times \mathcal{Q}$. Finally, if λ is $\underline{M}$-dominant and $\lambda \leqslant_M |\nu_\bullet|$, then let us define the following scheme

$$\big(\mathfrak{S}_{\nu_\bullet}^N \cap \widetilde{\mathcal{Q}}_{\mu_\bullet}\big)_\lambda = \pi_{c_\bullet,r}^{-1}\big(t^\lambda *_M\big) \cap \big(\mathfrak{S}_{\nu_\bullet}^N \cap \widetilde{\mathcal{Q}}_{\mu_\bullet}\big),$$

where the sequence of connected components $c_\bullet$ is the unique one such that

$$\mathfrak{S}_{\nu_\bullet}^N \subset (G/K)_{c_1} \times \cdots \times (G/K)_{c_r}.$$

With this notation, we have the following crucial lemma.

Lemma 13.4. *We have an equality of locally-closed subvarieties*

$$m_{\mu_\bullet}^{-1}\left(S_\lambda^N \cap \overline{\mathcal{Q}}_\mu\right) = \bigcup_{\nu_\bullet}\left(\mathfrak{S}_{\nu_\bullet}^N \cap \widetilde{\mathcal{Q}}_{\mu_\bullet}\right)_\lambda.$$

Here $\nu_\bullet = (\nu_1, \dots, \nu_r)$ ranges over all ordered r-tuples of $\underline{M}$-dominant coweights with $\nu_i \in W\mu_i$ for all i, whose sum satisfies $\lambda \leqslant_M |\nu_\bullet|$.

Proof. Since $S_\lambda^N \cap \overline{\mathcal{Q}}_{|\mu_\bullet|}$ is the fiber over $t^\lambda *_M$ for the restriction of the morphism

$$\pi : G/K \to M/K_M$$

to $\overline{\mathcal{Q}}_{|\mu_\bullet|}$, it is clear that the right-hand side is contained in the left. Let $(g_1*, \dots, g_r*)$ belong to the left-hand side. By the Iwasawa decomposition, we may represent $g_1* \in \mathcal{Q}_{\mu_1}$ as $g_1* = x_1 m_1 t^{\nu_1}*$, where $x_1 \in N$, $\nu_1 \in W\mu_1$ is $\underline{M}$-dominant, and $m_1 \in K_M$. Further, $g_1^{-1} g_2 \in \mathcal{Q}_{\mu_2}$, so by the same reasoning we can represent g_2* by

$$g_2* = x_1 m_1 t^{\nu_1} x_2 m_2 t^{\nu_2} *$$

for some $x_2 \in N$, $m_2 \in K_M$, and $\nu_2 \in W\mu_2$ an $\underline{M}$-dominant coweight. Continuing we finally represent g_r* in the form

$$g_r* = x_1 m_1 t^{\nu_1} \cdots x_r m_r t^{\nu_r} *,$$

where $x_i \in N$, $m_i \in K_M$, and $\nu_i \in W\mu_i$ is $\underline{M}$-dominant, for all i. It is immediate that $(g_1*, \dots, g_r*) \in (\mathfrak{S}_{\nu_\bullet}^N \cap \widetilde{\mathcal{Q}}_{\mu_\bullet})_\lambda$. □

In preparation for the next lemma, let us recall some of the standard notation relating to loop groups over a function field $k((t))$. Let $LG = G(k((t)))$. Let $L^{\geqslant 0}G = G(k[[t]])$ $(= K)$, and let $L^{<0}G$ denote the kernel of the homomorphism

$$G\left(k\left[t^{-1}\right]\right) \to G(k),$$

determined by $t^{-1} \mapsto 0$. A fundamental fact about the topology of loop groups is that the multiplication map $L^{<0}G \times L^{\geqslant 0}G \to LG$ defines open immersions

$$gL^{<0}G \xrightarrow{\sim} gL^{<0}G * \subset \mathcal{Q};$$

see [BL].

Now let $\nu_\bullet$ be as in Lemma 13.4. Then using the above remarks applied to the loop group LM, one can show that Zariski-locally the twisted product $\widetilde{\mathcal{Q}}_{\nu_\bullet}^M = \mathcal{Q}_{\nu_1}^M \tilde{\times} \cdots \tilde{\times} \mathcal{Q}_{\nu_r}^M$ is isomorphic to the product $\mathcal{Q}_{\nu_1}^M \times \cdots \times \mathcal{Q}_{\nu_r}^M$; see e.g. [NP]. Further, since the map $L^{\geqslant 0}M \to \mathcal{Q}_{\nu_i}$ given by $m \mapsto mt^{\nu_i} *_M$ has a section Zariski-locally on the base (Lemma 9.3), we see that we may cover $\widetilde{\mathcal{Q}}_{\nu_\bullet}^M$ by sufficiently small Zariski-open subsets on which an element in the twisted product can be expressed in the form

$$\left(m_1 t^{\nu_1} *_M, m_1 t^{\nu_1} m_2 t^{\nu_2} *_M, \dots, m_1 t^{\nu_1} \cdots m_r t^{\nu_r} *_M\right),$$

for *unique* elements $m_i \in K_M$. Let V be such an open set.

The following is the key result allowing us to reduce the general case of Theorem 12.3 to the special case already treated in Proposition 13.2.

Lemma 13.5. *For any $\nu_\bullet$ as in Lemma 13.4 and any open subset $V \subset \widetilde{\mathcal{Q}}^M_{\nu_\bullet}$ as above, there is an isomorphism*

$$\left(\mathfrak{S}^N_{\nu_\bullet} \cap \widetilde{\mathcal{Q}}_{\mu_\bullet}\right) \cap \pi_{c_\bullet}^{-1}(V) \cong \left(S^N_{\nu_1} \cap \mathcal{Q}_{\mu_1} \times \cdots \times S^N_{\nu_r} \cap \mathcal{Q}_{\mu_r}\right) \times V.$$

Further, this induces an isomorphism

$$\left(\mathfrak{S}^N_{\nu_\bullet} \cap \widetilde{\mathcal{Q}}_{\mu_\bullet}\right)_\lambda \cap \pi_{c_\bullet}^{-1}(V) \cong \left(S^N_{\nu_1} \cap \mathcal{Q}_{\mu_1} \times \cdots \times S^N_{\nu_r} \cap \mathcal{Q}_{\mu_r}\right) \times \left(V \cap m_{\nu_\bullet}^{-1}\left(t^\lambda *_M\right)\right).$$

Here $m_{\nu_\bullet} : \widetilde{\mathcal{Q}}^M_{\nu_\bullet} \to \overline{\mathcal{Q}}^M_{|\nu_\bullet|}$ is the convolution morphism on the affine Grassmannian for $\underline{M}$.

Proof. It is easy to construct the desired isomorphism. By the construction of V and Proposition 13.1, we may express any element in $(S^N_{\nu_1} \cap \mathcal{Q}_{\mu_1} \times \cdots \times S^N_{\nu_r} \cap \mathcal{Q}_{\mu_r}) \times V$ as

$$\left(x_1 t^{\nu_1} *, \dots, x_r t^{\nu_r} *\right) \times \left(y_1 t^{\nu_1} *_M, y_1 t^{\nu_1} y_2 t^{\nu_2} *_M, \dots, y_1 t^{\nu_1} \cdots y_r t^{\nu_r} *_M\right)$$

for some uniquely determined $x_i \in N$ and $y_i \in K_M$. We send this to the element

$$\left(y_1 x_1 t^{\nu_1} *, \dots, y_1 x_1 t^{\nu_1} \cdots y_r x_r t^{\nu_r} *\right)$$

in $(\mathfrak{S}^N_{\nu_\bullet} \cap \widetilde{\mathcal{Q}}_{\mu_\bullet}) \cap \pi_{c_\bullet}^{-1}(V)$. It is easy to construct the inverse of this morphism and show that inverse is algebraic. □

Lemma 13.6. *Each space $(S^N_{\nu_1} \cap \mathcal{Q}_{\mu_1} \times \cdots \times S^N_{\nu_n} \cap \mathcal{Q}_{\mu_n}) \times (V \cap m_{\nu_\bullet}^{-1}(t^\lambda *_M))$ is equidimensional of dimension $\langle \rho, \mu + \lambda \rangle - 2\langle \rho_M, \lambda \rangle$.*

Proof. By Proposition 13.2 the first term in the product is an affine space of dimension

$$\sum_{i=1}^n \langle \rho, \mu_i + \nu_i \rangle - 2\langle \rho_M, \nu_i \rangle = \left\langle \rho, \mu + |\nu_\bullet| \right\rangle - 2\left\langle \rho_M, |\nu_\bullet| \right\rangle.$$

By the main result of [Ha2], the second term in the product is a union of irreducible components of dimension

$$\left\langle \rho_M, |\nu_\bullet| - \lambda \right\rangle.$$

Adding these, we see that we need to show that

$$\langle \rho, \mu + \lambda \rangle - 2\langle \rho_M, \lambda \rangle = \left\langle \rho, \mu + |\nu_\bullet| \right\rangle - 2\left\langle \rho_M, |\nu_\bullet| \right\rangle + \left\langle \rho_M, |\nu_\bullet| - \lambda \right\rangle.$$

The difference between the two sides is simply

$$\left\langle \rho - \rho_M, |\nu_\bullet| - \lambda \right\rangle = 0.$$

This is zero since $|\nu_\bullet| - \lambda$ is a sum of $\underline{B}_M$-positive simple coroots of M: each such $\alpha^\vee$ is also a simple $\underline{B}$-positive coroot of $\underline{G}$, and so $\langle \rho, \alpha^\vee \rangle = \langle \rho_M, \alpha^\vee \rangle = 1$. □

Corollary 13.7. *The locally closed varieties $(\mathfrak{S}^N_{\nu_\bullet} \cap \widetilde{\mathcal{Q}}_{\mu_\bullet})_\lambda$ are equidimensional of dimension $\langle \rho, \mu + \lambda \rangle - 2\langle \rho_M, \lambda \rangle$, a number independent of $\nu_\bullet$. Consequently by Lemma 13.4, $m_{\mu_\bullet}^{-1}(S^N_\lambda \cap \widetilde{\mathcal{Q}}_{\mu_\bullet})$ is equidimensional of the same dimension.*

13.4. Proof of Theorem 12.3 in general

Proof. Let $\mathfrak{U} \subset C$ be a dense open subset of C which misses all the other irreducible components of $S_\lambda^N \cap \overline{\mathcal{Q}}_\mu$. By shrinking $\mathfrak{U}$ if necessary, we may assume $\mathfrak{U} \subset \mathcal{Q}_\nu$ and even that $m_{\mu_\bullet}$ is trivial over $\mathfrak{U}$ (recall $m_{\mu_\bullet}$ is locally trivial in the stratified sense; see [Ha2], Lemma 2.1). Then for any fixed point $y \in \mathfrak{U}$, we have an isomorphism

$$m_{\mu_\bullet}^{-1}(\mathfrak{U}) \cong \mathfrak{U} \times m_{\mu_\bullet}^{-1}(y).$$

By the main theorem of [Ha2], the fiber is equidimensional of dimension

$$\dim\bigl(m_{\mu_\bullet}^{-1}(y)\bigr) = \langle \rho, \mu - \nu \rangle,$$

recalling that $|\mu_\bullet| = \mu$. Also, by Corollary 13.7, we see that $m_{\mu_\bullet}^{-1}(\mathfrak{U})$ is equidimensional of dimension

$$\dim\bigl(m_{\mu_\bullet}^{-1}(\mathfrak{U})\bigr) = \langle \rho, \mu + \lambda \rangle - 2\langle \rho_M, \lambda \rangle.$$

These facts imply that $\mathfrak{U}$ has dimension $\langle \rho, \nu + \lambda \rangle - 2\langle \rho_M, \lambda \rangle$, as desired. □

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Appendix A

To simplify the notation we set $\rho_\xi := b_{\xi,\Delta_M}$ and $\rho_{-\nu} := \rho_{-\nu,\Delta_G - \nu}$. Our goal is to give a geometric proof of Theorem 3.2, which can be restated as follows.

Theorem A.1. *If $\nu \geqslant^P \mu$ then for every geodesic $\gamma = \overline{oz} \subset \mathcal{B}_G$ of Δ-length μ, we have*

$$\rho_{-\nu}|\gamma = \rho_\xi|\gamma.$$

Proof. Throughout the proof we will be using the concept of a *generic* geodesic in a building introduced in [KM2]. A geodesic (finite or infinite) γ in $\mathcal{B}_G$ is *generic* if it is disjoint from the codimension 2 skeleton of the polysimplicial complex $\mathcal{B}_G$, except for, possibly, the end-points of γ. It is easy to see that generic segments are dense: Every geodesic contained in the apartment $\mathcal{A}$ is the limit of generic geodesics in $\mathcal{A}$.

We next review basic properties of the retractions ρ_ξ and ρ_ν. Both maps are isometric when restricted to each alcove in $\mathcal{B}_G$; hence, both maps are 1-Lipschitz, in particular, continuous.

Observe that for every $x \in \mathcal{B}_G$, there exists an apartment $\mathcal{A}_x \subset \mathcal{B}_G$ so that $\xi \in \partial_{Tits}\mathcal{A}_x$, $\mathcal{A}_x \cap \mathcal{A}$ has nonempty interior and contains an infinite subray in $\overline{o\xi}$. This follows by applying Lemma 5.1 to a generic geodesic $\gamma \subset \mathcal{A}$ asymptotic to ξ and passing through an alcove $\mathbf{a} \subset \mathcal{A}$ containing o. For such an apartment $\mathcal{A}_x$, there exists a unique isomorphism $\phi_x : \mathcal{A}_x \to \mathcal{A}$ fixing $\mathcal{A}_x \cap \mathcal{A}$. Then $y = \phi_x(x) \in \mathcal{A}$ is independent of the choice of $\mathcal{A}_x$ (although, ϕ_x does). By the definition of b_{ξ,Δ_M}, we see that $b_{\xi,\Delta_M}(x) = b_{\xi,\Delta_M}(y)$. Hence, ρ_x factors as the composition $\rho_{\Delta_M} \circ \rho_{\xi,\mathcal{A}}$, where $\rho_{\xi,\mathcal{A}}(x) = y$. (The map $\rho_{\xi,\mathcal{A}}$ equals the map $\rho_{I_P,\mathcal{A}}$ defined in Subsection 6.3.)

We next make observations about the geometric meaning of the partial order $\geqslant^P$. In what follows it will be convenient to extend the partial order $\nu \geqslant^P \mu$ to arbitrary vectors μ in $\mathcal{A}$ (not only cocharacters) (we will be still assuming however that $\nu \in X_*(\underline{T})$). We will also extend the definition of x_λ from $\lambda \in \Lambda$ to general vectors λ in the affine space $\mathcal{A}$: Given a vector λ in the apartment $\mathcal{A}$, we let $x_\lambda \in A$ denote the point so that $\overrightarrow{ox_\lambda} = \lambda$.

Given $\nu \in \Lambda$ we let $\mathbf{a}_\nu$ denote the alcove of $(\mathcal{A}, \widetilde{W})$ with the vertex x_ν and contained in the negative chamber $-\Delta_G + \nu$.

Lemma A.2. *Suppose that ν is annihilated by all roots of Φ_M. Then the following are equivalent*:

1. $$\nu \geqslant^P \mu.$$
2. $C_\mu := \mathrm{Conv}(W \cdot x_\mu) \cap \Delta_M$ *is contained in* $\Delta - \nu$.
3. *For every positive root* $\alpha \in \Phi \setminus \Phi_M$,
$$\alpha|_{C_\mu} \geqslant \alpha(x_{-\nu}).$$
4. *If a wall H of $(\mathcal{A}, \widetilde{W})$ intersects the set C_μ, then it does not separate $x_{-\nu}$ from ξ in the sense that the ray $\overline{x_{-\nu}\xi}$ does not cross H.*
5. *If a wall H of $(\mathcal{A}, \widetilde{W})$ has nonempty intersection with C_μ, then it does not separate $\mathbf{a}_{-\nu}$ from ξ in the sense that it does not separate any point of $\mathbf{a}_{-\nu}$ from ξ.*

Proof. The proof is straightforward and is left to the reader. We observe only that for every positive root α,

$$\max(\alpha|\mathbf{a}_{-\nu}) = \alpha(x_{-\nu}) = -\langle\alpha, \nu\rangle.$$

Thus, if $\nu \geqslant^P \mu$, then any wall H of $(\mathcal{A}, \widetilde{W})$ intersecting C_μ does not separate $\mathbf{a}_{-\nu}$ from ξ. $\square$

The next lemma establishes equality of the retractions $\rho_\xi, \rho_{-\nu}$ on certain subsets of $\mathcal{B}_M$.

Lemma A.3. 1. *If $\nu \geqslant^P \mu$ then the retractions $\rho_\xi, \rho_{-\nu}$ agree on* $\mathrm{Conv}(Wx_\mu) \subset \mathcal{A}$.
2. *Suppose that $x \in \mathcal{B}_M$ is such that $\rho_\xi(x) \in \Delta_G - \nu$. Then, again $\rho_\xi(x) = \rho_{-\nu}(x)$.*

Proof. 1. Let $x \in \mathrm{Conv}(Wx_\mu)$. There exists $w \in W_M$ such that $x' := w(x) \in \Delta_M$; then

$$x' \in C_\mu = \mathrm{Conv}(Wx_\mu) \cap \Delta_M.$$

Clearly, $x' = \rho_\xi(x)$. On the other hand, since $C_\mu \subset \Delta_G - \nu$, it follows that $\rho_{-\nu}(x) = \rho_{-\nu}(x') = x'$.

2. The proof is similar to (1). First, find $k \in K_M$ such that $k(x) \in \mathcal{A}$ and $w \in W_M$ such that $x' = wk(x) \in \Delta_M$. Then, by the definition of ρ_ξ, $wk(x) = \rho_\xi(x)$. On the other hand, $d_\Delta(x_{-\nu}, x) = d_\Delta(x_{-\nu}, x')$. Then $\rho_{-\nu}(x) = w'(x')$, for some $w' \in W_{-\nu}$. However, by our assumption, $x' \in \Delta_G - \nu$, hence $w'(x') = x'$. $\square$

Note that, given $\mu \in \Delta_G$, the segment $\overline{ox_\mu}$ is the limit of generic segments $\overline{ox_{\mu_i}} \subset \mathrm{Conv}(W \cdot x_\mu)$. In particular, $\mathrm{Conv}(W \cdot x_{\mu_i}) \subset \mathrm{Conv}(W \cdot x_\mu)$ and, therefore,

$$\nu \geqslant^P \mu \quad \Rightarrow \quad \nu \geqslant^P \mu_i, \quad \forall i.$$

Thus, since both the retraction $\rho_\xi, \rho_{-\nu}$ are continuous, it suffices to prove Theorem A.1 for μ such that the segment γ is generic.

We will assume from now on that μ is generic and $\nu \geqslant^P \mu$. Moreover, we will assume that μ is *rational*, i.e., $\mu \in \Lambda \otimes \mathbb{Q}$.

For convenience of the reader we recall the definition of a *Hecke path* in the sense of [KM2]. Let $\pi = \pi(t), t \in [0, r]$ be a piecewise-linear path in $\mathcal{A}$ parameterized by its arc-length. At each break-point t, the path π has two derivatives $\pi'_-(t), \pi'_+(t)$, which are unit vectors in $\mathcal{A}$. Then π is a *Hecke path* if the following holds for each break-point ([KM2], Definitions 3.1 and 3.26):

1. $\pi'_+(t) = dw(\pi'_-(t))$ for some $w \in \widetilde{W}_{\pi(t)}$, the stabilizer of $\pi(t)$ in $\widetilde{W}$.
2. Moreover, w is a composition of affine reflections

$$w = \sigma_m \circ \cdots \circ \sigma_1, \quad \sigma_i \in \widetilde{W}_{\pi(t)},$$

each σ_i is the reflection in an affine hyperplane $\{\alpha_i(x) = t_i\}$ through the point $\pi(t)$, $\alpha_i \in \Phi^+$, so that for each $i = 1, \ldots, m$,

$$\langle \alpha_i, \eta_i \rangle < 0, \quad i = 0, \ldots, m-1,$$

where $\eta_0 = \pi'_-(t)$, $\eta_i := d\tau_i(\eta_{i-1}), i = 1, \ldots, m$ and $\eta_m = \pi'_+(t)$. Thus, for $\pi'_-(t) \in \Delta_G$, we have $m = 0$ and, hence, $\pi'_-(t) = \pi'_+(t)$; this means that the corresponding Hecke path π is geodesic as it does not have break-points.

We will need another property of Hecke paths: Suppose that π is a rational Hecke path, i.e., it starts at o and ends at a *rational point*, i.e, a point in $\Lambda \otimes \mathbb{Q}$. Then there exists $N \in \mathbb{N}$ so that the path $N \cdot \pi$ is an LS path in the sense of Littelmann [Li]. The proof consists in unraveling the definition of an LS path as it was done in [KM2] and observing that all break-points of a rational LS path occur at rational points. We will also need the fact that for every geodesic segment $\sigma \subset \mathcal{B}_G$ and any $\nu \in \Lambda$, the image of σ under the retraction $\rho_{x_\nu, \Delta+\nu}$ is a Hecke path (see [KM2]). The next proposition generalizes Lemma 7.2(b) from $\mu \in \Delta_G \cap \Lambda$ to $\mu \in \Delta_G$.

Proposition A.4. *$\rho_\xi(\gamma) \subset C_\mu$. In particular, $\rho_{\xi,\mathcal{A}}(\gamma) \subset \mathrm{Conv}(Wx_\mu)$.*

Proof. We first prove an auxiliary lemma which is a weak version of Theorem A.1. (Cf. Lemma 6.10.)

Lemma A.5. *Given γ, if ν is M-central and very N-dominant (more precisely, $\langle \alpha, \nu \rangle \geqslant const(\gamma)$ for all roots $\alpha \in \Phi_N$), then $\rho_\xi|_\gamma = \rho_{-\nu}|_\gamma$.*

Proof. Consider geodesic rays $\overline{x\xi}$ from the points $x \in \mathcal{B}_G$ asymptotic to ξ. For every such ray there exists a unique point $x' = f_\xi(x) \in \overline{x\xi}$ so that the subray $\overline{x'\xi}$ is the maximal subray in $\overline{x\xi}$ contained in $\mathcal{B}_M$. Explicitly, the map f_ξ can be described as follows. First, recall that for $x \in \mathcal{B}_G$, there is a unique point $\bar{x} \in \mathcal{B}_M$ so that $\bar{x} = n(x)$ for some $n \in N$ (even though, the element $n \in N$ is non-unique). Moreover, for every alcove $\mathbf{a} \subset \mathcal{B}_G$, the element $n \in N$ can be chosen the same for all $x \in \mathbf{a}$. The function $x \mapsto \bar{x}$ is isometric on each alcove in $\mathcal{B}_G$, hence, it is continuous. Now, given $x \in \gamma$, find an element $n \in N$ so that $\bar{x} := n(x) \in \mathcal{B}_M$. Then, by convexity of $\mathcal{B}_M$ in $\mathcal{B}_G$, $n(\overline{x\xi}) \subset \mathcal{B}_M$. By the above observation, the image $n(\overline{x\xi})$ is independent of the choice of n. By convexity of $Fix(n)$, the intersection $Fix(n) \cap n(\overline{x\xi})$ is an infinite ray.

Claim A.6. *For every $n \in N$ such that $\bar{x} = n(x) \in \mathcal{B}_M$, we have*

$$\overline{x'\xi} = Fix(n) \cap n(\overline{x\xi}),$$

where $x' = f_\xi(x)$.

Proof. Since $n(x') \in n(\overline{x\xi}) \subset \mathcal{B}_M$, we have $n(x') = x'$. For the same reason, n fixes the entire sub-ray $\overline{x'\xi}$ pointwise. Thus, $\overline{x'\xi} \subset Fix(n) \cap n(\overline{x\xi})$. Let $y \in \overline{n(x)x'} \setminus \{x'\}$. Then $n^{-1}(y) \in \overline{xx'} \setminus \{x'\}$ and, hence, does not belong to the subbuilding B_M. Therefore, $y \notin Fix(n)$ and $Fix(n) \cap n(\overline{x\xi}) \subset \overline{x'\xi}$. $\square$

We next claim that the function f_ξ is continuous. Indeed, it suffices to verify its continuity on each alcove $\mathbf{a} \subset \mathcal{B}_G$. As observed above, n can be (and will be) taken the same for all points of $\mathbf{a}$. Then (by using the action of K_M) continuity of f_ξ reduces to the following

Claim A.7. *Let $n \in N$. Then the function $p \mapsto q$, $p \in \mathcal{A}$ defined by*

$$\overline{q\xi} = Fix(n) \cap \overline{p\xi}$$

is continuous.

Proof. The statement follows easily from the fact that the fixed-point set of n intersected with $\mathcal{A}$ is a convex polyhedron. $\square$

We now apply the continuous function f_ξ to the compact γ. Its image is a compact subset C' of $\mathcal{B}_M$. Thus $C'' := \rho_\xi(C') \subset \Delta_M$ is also compact. Then, for all M-central $\nu \in \Delta_G$ which are sufficiently N-dominant (depending on the diameter of C''), the set C'' is contained in the relative interior of $\Delta_G - \nu$ in Δ_M. We then claim that for such choice of ν, $\rho_\xi|_\gamma = \rho_{-\nu}|_\gamma$.

For every $x \in \gamma$, the segment $\overline{x'x''} := \overline{x'\xi} \cap \rho_\xi^{-1}(\Delta_G - \nu) \subset \mathcal{B}_M$ has positive length. According to part 2 of Lemma A.3, $\rho_\xi|_{\overline{x'x''}} = \rho_{-\nu}|_{\overline{x'x''}}$. Moreover, $\rho_\xi(\overline{xx''})$ is the unique geodesic segment in $\mathcal{A}$ containing the subsegment $\rho_\xi(\overline{x'x''})$ and having the same metric length as $\overline{xx''}$. (This follows from the fact that ρ_ξ restricts to an isometry on the ray $\overline{x\xi}$.)

We now claim that the projection $\rho_{-\nu}$ also sends $\overline{xx''}$ to a geodesic segment in $\Delta_G - \nu$. Indeed, the path $\pi := \rho_{-\nu}(\overline{x''x})$ is a Hecke path in $\mathcal{A}$. The unit tangent vector τ to π at $\rho_{-\xi}(x'')$ is contained in Δ_G since its opposite (pointing to ξ) is contained in $-\Delta_G$. Then the definition of a Hecke path above implies that π is geodesic.

The retraction $\rho_{-\nu}$ preserves metric lengths of curves [KM2], therefore, π is a geodesic of the same length as $\overline{xx''}$. Hence, $\rho_{-\nu}(x) = \rho_\xi(x)$. Lemma follows. $\square$

The only corollary of this lemma that we will use is

Corollary A.8. *$\rho_\xi(\gamma)$ is a Hecke path in $(\mathcal{A}, \widetilde{W})$ of the Δ-length μ in the sense of* [KM2].

Proof. By [KM2], the retractions $\rho_{-\nu} : \mathcal{B}_G \to \Delta_G - \nu$ send geodesics in $\mathcal{B}_G$ to Hecke paths preserving Δ-lengths. Now, the assertion follows from the above lemma. $\square$

We are now ready to prove Proposition A.4. According to Corollary A.8, the image $\rho_\xi(\gamma)$ is a Hecke path in Δ_M with the initial point o. Let π be a subpath of $\rho_\xi(\gamma)$ starting at the origin o. Assume for a moment that $\pi = \pi : [0, 1] \to \mathcal{A}$ is an LS path in the sense of Littelmann of the Δ-length β. Then the terminal point $\pi(1)$ of π is a weight of a representation $V_\beta^{\widehat{G}}$, see [Li]. Therefore, $\pi(1)$ is contained in $\mathrm{Conv}(Wx_\beta) \subset \mathrm{Conv}(Wx_\mu)$. Since π is contained in Δ_M, it then follows that $\pi \subset C_\mu$.

More generally, suppose that π is a subpath of $\rho_\xi(\gamma)$ which terminates at a rational point $\pi(1) \in \Lambda \otimes \mathbb{Q}$ of the apartment A. Then there exists $N \in \mathbb{N}$ so that $N \cdot \pi$ is an LS path of the Δ-length $N\beta$, where β is the Δ-length of π. Then, by the above argument,

$$N \cdot \pi(1) \in \mathrm{Conv}(W \cdot x_{N\beta})$$

and, hence, $\pi(1) \in \mathrm{Conv}(W \cdot x_\beta) \subset \mathrm{Conv}(W \cdot x_\mu)$. The general case follows by density of rational points in $\rho_\xi(\gamma)$. Thus, $\rho_\xi(\gamma) \subset C_\mu$. The second assertion of Proposition A.4 immediately follows from the first. $\square$

Proposition A.9. *For every point $x \in \gamma$ there exists an apartment $\mathcal{A}_x \subset \mathcal{B}_G$ connecting $\mathbf{a}_{-\nu}$, x and ξ, i.e., $\mathbf{a}_{-\nu} \cup \{x\} \subset \mathcal{A}_x$ and $\xi \in \partial_{Tits}\mathcal{A}_x$.*

Proof. Clearly, the assertion holds for $x = o$ since $o, \mathbf{a}_{-\nu}, \xi$ belong to the common model apartment $\mathcal{A}_o := \mathcal{A} \subset X$. We cover γ by alcoves $\mathbf{a}_i \subset \mathcal{B}_G$, $i = 0, \ldots, m$, where $\mathbf{a}_0 \subset \mathcal{A}$ is an alcove intersecting γ only at the point o. Recall that, by the genericity assumption, $\gamma_i = \mathbf{a}_i \cap \gamma$ is contained in the interior of $\mathbf{a}_i$ except for the end-points of this arc. Then, if the assertion holds for some point in the interior of γ_i, it holds for all points of γ_i. We suppose therefore that the assertion holds for points in the alcoves $\mathbf{a}_0, \ldots, \mathbf{a}_k$ and will prove it for the points of γ_{k+1}. We will mostly deal with the case $k \geqslant 1$ and explain how to modify the argument for $k = 0$. Let $x := \gamma_k \cap \gamma_{k+1}$ and y be such that $\overline{xy} = \gamma_{k+1}$.

Let $\mathcal{A}_x$ be an apartment as above. We assume that x belongs to a wall H in $\mathcal{B}_G$ and $\mathbf{a} = \mathbf{a}_{k+1}$ is not contained in A_x (for otherwise we again would be done). If $k = 0$, we take $H \subset \mathcal{A}$. In this case, $\nu \geqslant^P \mu$ implies that H does not separate ξ from $\mathbf{a}_{-\nu}$. Assume now that $k > 0$. Since γ is generic, the germ $H \cap \mathbf{a}$ of H at x is contained in $\mathcal{A}_x$. Therefore, without loss of generality, we can assume that $H \subset \mathcal{A}_x$.

Claim A.10. *H does not separate $\mathbf{a}_{-\nu}$ from ξ in $\mathcal{A}_x$.*

Proof. In view of Lemma A.2, it suffices to show that H does not separate $x_{-\nu}$ from ξ.

Let $H' \subset \mathcal{A}$ be the (unique) wall containing $\rho_\xi(H \cap \mathbf{a})$. Since $\nu \geqslant^P \mu$, $\rho_\xi(\gamma) \subset C_\mu = \mathrm{Conv}(Wx_\mu) \cap \Delta_M$ (Proposition A.4), it follows that $H' \cap C_\mu \neq \emptyset$. Hence, H' does not separate $\mathbf{a}_{-\nu}$ from ξ.

We recall that the map $\rho_\xi | \mathcal{A}_x$ is obtained in two steps: First, an isomorphism $\phi : \mathcal{A}_x \to \mathcal{A}$ fixing $\mathbf{a}_{-\nu}$, and then applying the projection $\rho_{\Delta_G - \nu} : \mathcal{A} \to \Delta_G - \nu$ (obtained by acting on $\phi(p)$, $p \in \mathcal{A}_x$, by an appropriate element $w \in W_M$). Let $w \in W_M$ be the element which sends $\phi(\mathbf{a}_k)$ (and, hence, $\phi(H \cap \mathbf{a})$) to Δ_M. Note that w fixes ξ and $x_{-\nu}$. Since H' did not separate ξ from $x_{-\nu}$ in $\mathcal{A}$, it then follows that $w^{-1}(H')$ does not separate either. Since $\phi : \mathcal{A}_x \to \mathcal{A}$ is an isomorphism fixing ξ and $\mathbf{a}_{-\nu}$, it then follows that $H = \phi^{-1} w^{-1}(H')$ also does not separate ξ and $x_{-\nu}$. The claim follows. □

Since $\mathcal{B}_G$ is a thick building, there exists a half-apartment $\mathcal{A}_y^+ \subset \mathcal{B}_G$ containing the alcove $\mathbf{a}$, so that $\mathcal{A}_y^+ \cap \mathcal{A}_x = H$. Let $\mathcal{A}_x^-$ denote the half-space in $\mathcal{A}_x$ bounded by H and containing $\mathbf{a}_{-\nu}$; hence, $\xi \in \partial_\infty \mathcal{A}_x^-$. Then $\mathcal{A}_y := \mathcal{A}_x^- \cup \mathcal{A}_y^+$ is an apartment, $\mathbf{a}_{-\nu} \subset \mathcal{A}_y$, $y \in \mathcal{A}_y$ and $\xi \in \partial_\infty \mathcal{A}_y$. Proposition follows. □

We now can finish the proof of the main theorem. Pick $x \in \gamma$. We will show that $\rho_{-\nu}(x) = \rho_\xi(x)$.

The map $\rho_{-\nu} : \mathcal{B}_G \mapsto \Delta_G - \nu$ is the composition of two maps: First the canonical isomorphism of the apartments $\psi_x : \mathcal{A}_x \to \mathcal{A}$ which fixes the intersection $\mathcal{A}_x \cap \mathcal{A}$, and then the quotient map $\rho_{\Delta_G - \nu} : \mathcal{A} \to \Delta_G - \nu$. The intersection $V := \mathcal{A}_x \cap \mathcal{A}$ has nonempty interior in $\mathcal{A}$ (since $\mathbf{a}_{-\nu}$ does) Similarly, the projection $\rho_\xi : \mathcal{B}_G \mapsto \Delta_M$ is obtained by first taking the isomorphism of apartments

$$\rho_{\xi,\mathcal{A}} | \mathcal{A}_x : \mathcal{A}_x \to \mathcal{A}$$

(again, fixing V) and then applying ρ_{Δ_M}. Since V has nonempty interior, it follows that the isomorphisms of apartments ψ_x and $\rho_{\xi,\mathcal{A}_x} | \mathcal{A}_x$ agree on the entire apartment $\mathcal{A}_x$. Hence,

$$\rho_{\xi,\mathcal{A}}(x) = \psi_x(x).$$

By Proposition A.4, $\rho_{\xi,\mathcal{A}}(x) \in \mathrm{Conv}(Wx_\mu)$. By Lemma A.3, part 1,

$$\rho_{\Delta_M} | \mathrm{Conv}(Wx_\mu) = \rho_{\Delta_G - \nu} | \mathrm{Conv}(Wx_\mu).$$

Therefore,

$$\rho_\xi(x) = \rho_{\Delta_M} \circ \rho_{\xi,\mathcal{A}}(x) = \rho_{\Delta_G - \nu} \circ \psi_x = \rho_{-\nu}(x). \qquad \square$$

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