MÖBIUS STRUCTURES ON 3-MANIFOLDS

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CHAPTER 1

INTRODUCTION

A Möbius structure on an n-manifold is a maximal atlas with values in $\mathbb{S}^n$ such that the transition maps are restrictions of Möbius transformations in $M\ddot{o}b(\mathbb{S}^n)$, where $M\ddot{o}b(\mathbb{S}^n)$ is the group of all Möbius transformations of $\mathbb{S}^n$. Under the assumption $n \geq 3$, a Möbius structure is nothing but a flat conformal structure. A way to construct Möbius manifolds is the following : If a discrete group $G < M\ddot{o}b(\mathbb{S}^n)$ acts properly discontinuously and freely on a domain $\Omega \subset \mathbb{S}^n$, then the quotient manifold Ω/G admits a Möbius structure.

We restrict our attention to Möbius structures on 3-manifolds. Any manifold modeled on one of $\mathbb{E}^3, \mathbb{S}^3, \mathbb{H}^3, \mathbb{S}^2 \times \mathbb{R}$ or $\mathbb{H}^2 \times \mathbb{R}$ admits a Möbius structure ([10]). On the other hand, any closed manifold modeled on Nil or Sol does not admit a Möbius structure.

A Möbius structure exists on connected sum of two Möbius manifolds ([8]). The main theorem in this thesis is Theorem 5.3 :

Let M be a closed oriented 3-manifold. Then there exists a 3-manifold N so that the connected sum of M and N admits a Möbius structure.

An outline of each chapter is as follows.

Chapter 2 describes the background and machinery that we use in this thesis to glue two Möbius manifolds with toral boundary so that the resulting manifold admits a Möbius structure which extends these two given structures. We define Möbius structures on 3-manifolds without boundary and then have a discussion of Möbius structures on 3-manifolds with boundary in terms of Möbius thickenings.

Chapter 3 presents Alexander's theorem and its relative version, Corollary 3.9 : *For a given closed oriented 3-Möbius manifold M, there exists a simple branched* *covering of* $\mathbb{S}^3$ *whose singular locus is a link in a 3-ball* B *in* M. We consider the manifold M_1 gotten by removing the interior of a regular neighborhood of the singular locus from M. The manifold M_1 is a compact Möbius manifold whose boundary is a disjoint union of tori. We glue the Möbius structure on M_1 to appropriate Möbius manifolds with toral boundary. The resulting manifold is a connected sum since the boundary sphere of B is a separating 2-sphere.

Chapter 4 proves the main theorem in the special case that the branched locus is a round circle in $\mathbb{S}^3$. We use a Fuchsian group to construct an appropriate Möbius manifold with toral boundary. We present a specific construction of a Fuchsian group using Poincaré's fundamental polyhedron theorem and we get the Möbius manifold which is the product of the surface with connected boundary and $\mathbb{S}^1$.

Chapter 5 proves the main theorem in the general case that the branch locus is a link in $\mathbb{S}^3$. To deal with the general case, we construct quasi-Fuchsian groups with prescribed fundamental domains (the closure of their complements in $\mathbb{S}^3$ are isotopic to regular neighborhoods of the given polygonal knots in $\mathbb{S}^3$). We obtain Theorem 5.1 : *For a given polygonal knot* L_0 *in* $\mathbb{R}^3$*, there exist a quasi-Fuchsian group* G *and a compact fundamental domain* Φ *for* G *acting on* $\mathbb{S}^3$ *such that* $\overline{\mathbb{S}^3 - \Phi}$ *is isotopic to a regular neighborhood* $Nbd(L_0)$ *of* L_0. We take a regular neighborhood $Nbd(\partial\Phi)$ in Φ, denoted Φ'. The manifold Φ'/G is a Möbius manifold with toral boundary. It is homeomorphic to the product of the surface with connected boundary and $\mathbb{S}^1$. We obtain the total space of its 2-fold covering which is a Möbius manifold with toral boundary. We discuss the procedure of gluing such structures along the boundary to the Möbius manifold M_1.

CHAPTER 2

MÖBIUS STRUCTURES

Let X be a connected, simply connected, oriented n-dimensional manifold and let G be a group of diffeomorphisms of X onto itself. An n-dimensional manifold M admits an *(X, G)-structure*, if there exist an open cover $\{U_i\}$ of M and a set of diffeomorphisms $\{\varphi_i\}$ with $\varphi_i : U_i \to \varphi_i(U_i) \subset X$ such that if $U_i \cap U_j \neq \varnothing$ then the restriction of $\varphi_j \circ \varphi_i^{-1}$ to each connected component of $\varphi_i(U_i \cap U_j)$ is the restriction of an element of G. $\{(U_i, \varphi_i)\}$ is called an *atlas defining the (X, G)-structure* and there is a unique maximal atlas which contains $\{(U_i, \varphi_i)\}$. Note that any atlas defining an (X, G)-structure on M determines a unique maximal structure. In general, the extension to the maximal structure on M is done without further comment.

A diffeomorphism of $\mathbb{S}^n$ onto itself is called a *Möbius transformation* of $\mathbb{S}^n$ if it carries round $(n-1)$-spheres to themselves. Let $Möb(\mathbb{S}^n)$ denote the full group of Möbius transformations of the n-sphere $\mathbb{S}^n = \overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$.

Definition 2.1 *A* Möbius structure *is an ($\mathbb{S}^n$, $Möb(\mathbb{S}^n)$)-structure. A manifold with a given Möbius structure is called a* Möbius manifold.

Definition 2.2 *Let M and N be Möbius manifolds of dimension n. A map $f : M \to N$ is* locally Möbius *if for each $x \in M$ there exist $(x \in U, \varphi)$ and $(f(U), \psi)$, in the Möbius structures on M and N, such that $\psi \circ f \circ \varphi^{-1}$ is a restriction of a Möbius transformation in $Möb(\mathbb{S}^n)$. A locally Möbius map is called a* Möbius morphism. *If a Möbius morphism is bijective, it is called a* Möbius isomorphism.

Remark 2.3 *We have* Liouville's theorem *as follows: Let U, V be open connected subsets of $\mathbb{S}^n$, $n \geq 3$, and $f : U \to V$ be a conformal map. Then f is a restriction of a Möbius transformation g of $\mathbb{S}^n$ and g is uniquely determined by f.*

Under the assumption $n \geq 3$, a flat conformal structure on an n-dimensional manifold M is nothing but a Möbius structure on M by Liouville's theorem. So the notions of a conformally flat manifold and a Möbius manifold are equivalent for $n \geq 3$.

If M is a simply connected Möbius manifold, then there exists a Möbius morphism $dev : M \to \mathbb{S}^n$. It is called a *developing map* of the Möbius manifold M. Tautologically, the Möbius structure on M is the pull-back structure of the canonical Möbius structure on $\mathbb{S}^n$ by the developing map dev. This developing map is unique up to postcomposition with an element of $M\ddot{o}b(\mathbb{S}^n)$.

Let M be a Möbius manifold. Lifting the Möbius atlas to the universal cover $\widetilde{M}$ of M, we have a developing map $dev : \widetilde{M} \to \mathbb{S}^n$. We also call it a *developing map* of M. It is considered as a multi-valued map from M to $\mathbb{S}^n$. In this case, the fundamental group $\pi_1(M)$ of M acts on $\widetilde{M}$ as a group of Möbius automorphisms of $\widetilde{M}$. By the uniqueness of the developing map, there exists a unique $\rho(\gamma) \in M\ddot{o}b(\mathbb{S}^n)$ such that $dev \circ \gamma = \rho(\gamma) \circ dev$ where $\gamma \in \pi_1(M)$. This gives rise to a representation $\rho : \pi_1(M) \to M\ddot{o}b(\mathbb{S}^n)$ which is called the *holonomy representation.* It is determined uniquely up to a conjugacy by an element in $M\ddot{o}b(\mathbb{S}^n)$ by the uniqueness of the developing map. In particular, the pair of dev and ρ is an invariant of the Möbius structure.

In this thesis, we consider only Möbius structures on orientable 3-manifolds, that is, $(\mathbb{S}^3, M\ddot{o}b^+(\mathbb{S}^3))$-structure, where $M\ddot{o}b^+(\mathbb{S}^3)$ is the full group of orientation-preserving Möbius transformations of $\mathbb{S}^3$.

Definition 2.4 *Let M be a 3-manifold with boundary. Suppose that M_1 is a Möbius manifold containing M as a submanifold with the Möbius structure $\mathcal{C}_1$ (the flat conformal structure). The Möbius manifold $(M_1, \mathcal{C}_1)$ is called a* Möbius thickening *of M. Two Möbius thickenings $(M_1, \mathcal{C}_1)$ and $(M_2, \mathcal{C}_2)$ of M are* equivalent *if there exists a Möbius thickening $(M_3, \mathcal{C}_3)$ of M such that $(M_3, \mathcal{C}_3) \subset (M_i, \mathcal{C}_i)$ for* $i = 1, 2$.

Definition 2.5 *Let M be a 3-manifold with boundary. A* Möbius structure *on M is an equivalence class of Möbius thickenings of M.*

Suppose M and N are compact oriented Möbius manifolds with boundary. Let $Nbd(\partial M)$ (resp. $Nbd(\partial N)$) be a neighborhood of ∂M (resp. ∂N) in a thickening of M (resp. N). If there exists a Möbius isomorphism $g : Nbd(\partial M) \to Nbd(\partial N)$ such that $g(\partial M) = \partial N$ and $g|_{\partial M}$ is orientation-reversing, then the attaching manifold $M \cup_{g|_{\partial M}} N$ by the map $g|_{\partial M} : \partial M \to \partial N$ admits a Möbius structure which extends Möbius structures of M and N.

If f is a homeomorphism from ∂M to ∂N isotopic to such a map $g|_{\partial M} : \partial M \to \partial N$, then $M \cup_f N$ also admits a Möbius structure.

Theorem 2.6 *Let M_1 and M_2 be compact oriented Möbius manifolds with toral boundary T_1 and T_2 respectively. Let $dev_i : Nbd(T_i) \to \mathbb{S}^3$ be the restriction of a developing map to a neighborhood $Nbd(T_i)$ of T_i in a thickening of the Möbius structure on M_i, for $i = 1, 2$. Suppose that single-valued branches $f : Nbd(T_1) \to \mathbb{S}^3$ and $h : Nbd(T_2) \to \mathbb{S}^3$ of dev_i exist and that their restriction $f : T_1 \to T_1' \subset \mathbb{S}^3$ and $h : T_2 \to T_2' \subset \mathbb{S}^3$ are 2-fold coverings between tori. If there exists a Möbius transformation $g \in M\ddot{o}b^+(\mathbb{S}^3)$ such that $g(T_1') = T_2'$, $g : T_1' \to T_2'$ reverses orientations (induced from M_i , $i = 1, 2$) and $g_*(f_*(\pi_1(T_1))) = h_*(\pi_1(T_2))$, then $M_1 \cup_{\tilde{g}} M_2$ admits a Möbius structure which extends the Möbius structures on M_i, where $\tilde{g} : T_1 \to T_2$ is a lifting of g.*

Proof. Consider f just on a neighborhood $Nbd(T_1)$ in a thickening of M_1. Pull back the Riemannian metric from $\mathbb{S}^3$ by $f : Nbd(T_1) \to \mathbb{S}^3$. Put the path metric on $Nbd(T_1)$ as a distance function. Denote by $B_r(x)$ the open metric ball of radius r centered at x. By local injectivity of f, there exists $\delta > 0$ such that $f|_{B_\delta(x)}$ is injective for each $x \in T_1$. Choose $\epsilon > 0$ so that $\delta > 3\epsilon$. Define $N_\epsilon(T_1) = \cup_{x \in T_1} B_\epsilon(x)$. Then $f(N_\epsilon(T_1)) = \cup_{x \in T_1} f(B_\epsilon(x)) = \cup_{x \in T_1} B_\epsilon(f(x)) = \cup_{x' \in T_1'} B_\epsilon(x') = N_\epsilon(T_1')$.

We claim that $f : N_\epsilon(T_1) \to N_\epsilon(T_1')$ is a 2-fold covering. Note that it is a local isometry by the construction. Take $y \in N_\epsilon(T_1')$. Also, $y \in B_\epsilon(z)$ for some

$z \in T_1'$. $(f|_{T_1})^{-1}(z)$ consists of two points x_1 and x_2 in T_1, since $f|_{T_1}$ is a 2-fold covering. So $f^{-1}(B_\epsilon(z)) \supseteq B_\epsilon(x_1) \cup B_\epsilon(x_2)$. Since $\delta > 3\epsilon$ and $f|_{B_\delta(x_1)}$ is injective, we obtain that $x_2 \notin B_\delta(x_1)$, which implies $B_\epsilon(x_1) \cap B_\epsilon(x_2) = \varnothing$. Assume there exists $x \in N_\epsilon(T_1) - (B_\epsilon(x_1) \sqcup B_\epsilon(x_2))$ such that $f(x) \in B_\epsilon(z)$. Then $x \in B_\epsilon(x_3)$ for some $x_3 \in T_1$ with $f(x_3) \neq z$ and $f(x_3) \in B_\epsilon(f(x_3)) \cap B_\epsilon(z)$. Letting $B_\epsilon(x_1) \cap B_\epsilon(x_3) \neq \varnothing$, we have $\epsilon \leq d(x, x_1) < 3\epsilon < \delta$ and also $f(x) \in B_\epsilon(z)$. It contradicts that $f|_{B_\delta(x_1)}$ is injective and hence $f^{-1}(B_\epsilon(z)) = B_\epsilon(x_1) \sqcup B_\epsilon(x_2)$.

If ϵ is small enough, there exists two 2-fold coverings $f : N_\epsilon(T_1) \to N_\epsilon(T_1')$ and $h : N_\epsilon(T_2) \to N_\epsilon(T_2')$. Since $g \in M\ddot{o}b(\mathbb{S}^3)$ satisfies $g(T_1') = T_2'$, we obtain $g : N_\epsilon(T_1') \to N_\epsilon(T_2')$ which is an isometry by taking push-forward metric. Since $f_*(\pi_1(N_\epsilon(T_1))) \cong f_*(\pi_1(T_1)) \cong h_*(\pi_1(T_2)) \cong h_*(\pi_1(N_\epsilon(T_2)))$, there exists a lifting $\eta : N_\epsilon(T_1) \to N_\epsilon(T_2)$ such that the following diagram commutes :

$$\begin{array}{ccc} N_\epsilon(T_1) & \xrightarrow{\ \eta\ } & N_\epsilon(T_2) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle h} \\ N_\epsilon(T_1') & \xrightarrow[g]{} & N_\epsilon(T_2') \end{array}$$

By the construction, $\eta : N_\epsilon(T_1) \to N_\epsilon(T_2)$ is a Möbius isomorphism and $\eta|_{T_1} = \tilde{g} : T_1 \to T_2$ is a lifting of $g : T_1' \to T_2'$. Therefore the attaching manifold $M_1 \cup_{\tilde{g}} M_2$ is a Möbius manifold. □

Remark 2.7 *In case that two developing maps of M_i are single-valued on M_i in Theorem 2.6, we obtain a single-valued developing map dev on $M_1 \cup_{\tilde{g}} M_2$. It is also a local homemorphism. Since $M_1 \cup_{\tilde{g}} M_2$ is compact, dev is a covering projection. Indeed, it is a homeomorphism because the base space $\mathbb{S}^3$ is simply-connected. Hence dev : $M_1 \cup_{\tilde{g}} M_2 \to \mathbb{S}^3$ is a Möbius isomorphism.*

The above case is not interesting, because $M_1 \cup_{\tilde{g}} M_2$ is Möbius isomorphic to $\mathbb{S}^3$ which has the canonical Möbius structure. We want that at least one of two developing maps of M_i is multi-valued on M_i.

CHAPTER 3

BRANCHED COVERINGS OF $\mathbb{S}^3$

The concept of branched coverings came from the theory of Riemann surfaces. We denote by $p_k : B^2 \to B^2$ the restriction of the complex map $z \mapsto z^k$, $k \geq 1$, to the unit disk $B^2 \subset \mathbb{C}$. Branched coverings between surfaces are locally equivalent to the map p_k. If $k \geq 2$, we call the point $z = 0$ a *singular point of index* k and its image $w = 0$ a *branch point* where $p_k : z \mapsto w = z^k$.

Definition 3.1 *A map $p : X \to Y$ between two closed surfaces is called a* branched covering of degree d *if p is finite-to-one and there exists a minimal finite set $B \subset Y$ such that the restriction $p|_{p^{-1}(Y-B)}$ is a d-fold covering.*

We call B the *branch set* of p. The *singular set* of p is the set of points $x \in X$ where the branched covering p fails to be a local homeomorphism.

Example 3.2 *The typical example of branched coverings of degree d is the map $f_d : \mathbb{S}^2 \to \mathbb{S}^2$ defined by $z \mapsto z^d$ for some $d \geq 2$. It has two singular points 0, ∞ of index d and two branch points 0, ∞.*

Definition 3.3 *Two branched coverings p, $p' : X \to Y$ are said to be* equivalent *if there exist homeomorphisms $h_1 : X \to X$ and $h_2 : Y \to Y$ such that the following diagram commutes :*

$$\begin{array}{ccc} X & \xrightarrow{h_1} & X \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ Y & \xrightarrow[h_2]{} & Y \end{array}$$

Definition 3.4 *A branched covering $p : X \to Y$ of degree d is* simple *if $|p^{-1}(y)| \geq d-1$ for all $y \in Y$.*

If $y \in Y$ is a branch point of a simple branched covering $p : X \to Y$ of degree d, then $|p^{-1}(y)| = d-1$.

Example 3.5 *Consider the map $f_3 : \mathbb{S}^2 \to \mathbb{S}^2$ when $d = 3$ in Example 3.2. Then f_3 is not simple. See the local model $p_3 : \overline{B^2} \to \overline{B^2}$ around a singular point of f_3. We modify p_3 to $p_3' : \overline{B^2} \to \overline{B^2}$ as in Figure 3.1. Since $p_3|_{\partial\overline{B^2}} = p_3'|_{\partial\overline{B^2}}$, we can also modify $f_3 : \mathbb{S}^2 \to \mathbb{S}^2$ to $f_3' : \mathbb{S}^2 \to \mathbb{S}^2$ which is a simple branched covering of degree 3.*

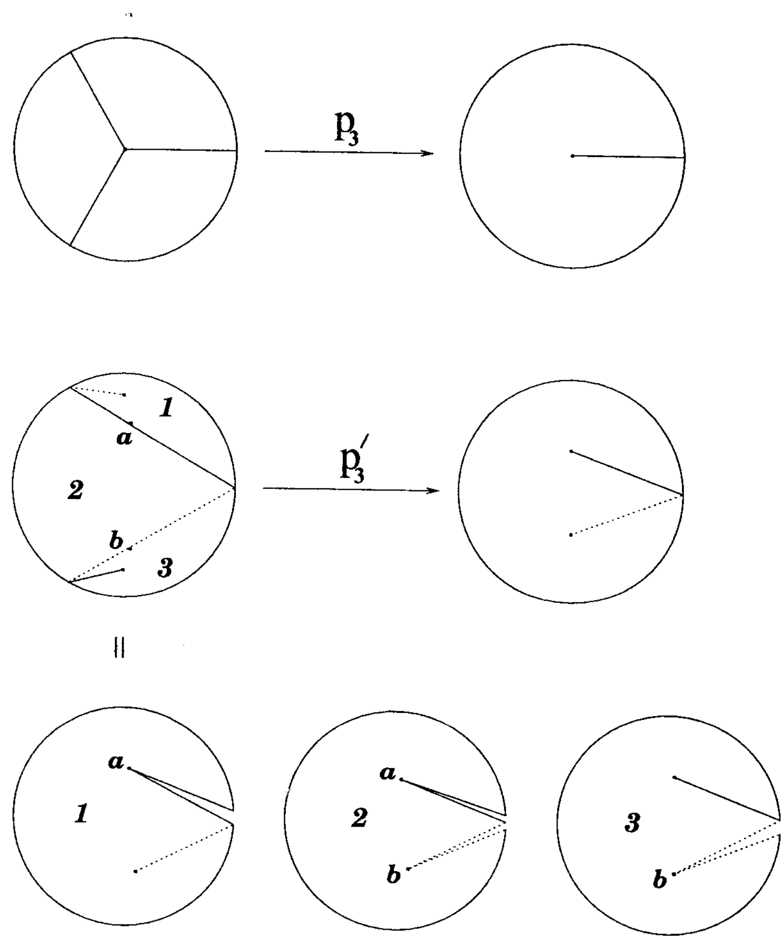

Figure 3.1. Modification of p_3 to p_3'

In general we can modify $f_d : \mathbb{S}^2 \to \mathbb{S}^2$ to $f'_d : \mathbb{S}^2 \to \mathbb{S}^2$ so that f'_d is a simple branched covering of degree d with $2d-2$ singular points of index 2 and $2d-2$ branch points.

Theorem 3.6 (Lüroth) ([4]) *If simple branched coverings $p : \mathbb{S}^2 \to \mathbb{S}^2$ and $q : \mathbb{S}^2 \to \mathbb{S}^2$ have the same degree d, then they are equivalent.*

Now branched coverings between 3-dimensional manifolds is defined similarly by replacing 2-dimensional local equivalence.

Definition 3.7 *A map $f : M \to N$ between 3-manifolds is called a* branched covering of begree d *if there exists a 1-submanifold K of M such that $f|_{M-K} : M - K \to N - f(K)$ is a d-fold covering and f is locally equivalent to the map $p_m \times id : B^2 \times I \to B^2 \times I$ with $(z,t) \mapsto (z^m, t)$, $m \geq 1$.*

The points $x \in M$ corresponding to $(0,t)$ with $m \geq 2$ belong to the *singular locus* of f, denoted Σ_f, and $f(\Sigma_f)$ is called the *branch locus* of the branched covering f, denoted B_f. Note that Σ_f is contained in $K = f^{-1}(B_f)$. Each connected component K_i of Σ_f corresponds to an integer m_i as in Definition 3.7, which is called the *index* of K_i. A branched covering $f : M \to N$ of degree d is *simple* if $|f^{-1}(y)| \geq d-1$ for all $y \in N$, that is, each point in the branch locus B_f of f has $d-1$ preimages. If f is simple, then for each $y \in B_f$ there is a unique $x \in \Sigma_f$ such that $y = f(x)$. Two branched coverings $f, f' : M \to N$ are said to be *equivalent* if there exist homeomorphisms $h_1 : M \to M$ and $h_2 : N \to N$ such that the following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{h_1} & M \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ N & \xrightarrow[h_2]{} & N \end{array}$$

Lemma 3.8 *Suppose two branched coverings p, $p' : \mathbb{S}^2 \to \mathbb{S}^2$ are equivalent. If p extends to a branced covering $\hat{p} : \overline{B^3} \to \overline{B^3}$, then p' extends to a branched covering which is equivalent to $\hat{p}$.*

Proof. Since p and p' are equivalent, there exist homeomorphisms $h_i : \mathbb{S}^2 \to \mathbb{S}^2$, $i = 1, 2$, such that $h_2 \circ p = p' \circ h_1$. Any homeomorphism $h : \mathbb{S}^2 \to \mathbb{S}^2$ extends to a homeomorphism $\hat{h} : \overline{B^3} \to \overline{B^3}$ by a radial extension, known as *Alexander's trick.* So h_i extends to a homeomorphism $\hat{h}_i : \overline{B^3} \to \overline{B^3}$ for each $i = 1, 2$. Denote $\hat{p'} = \hat{h}_2 \circ \hat{p} \circ \hat{h}_1^{-1}$. Then $\hat{p'} : \overline{B^3} \to \overline{B^3}$ is a branched covering which is equivalent to $\hat{p}$. It is also an extension of p' since $\hat{p'}|_{\mathbb{S}^2} = h_2 \circ p \circ h_1^{-1} = p'$. This completes the proof. □

Corollary 3.9 *Any simple branched covering $p : \mathbb{S}^2 \to \mathbb{S}^2$ extends to a simple branched covering $\hat{p} : \overline{B^3} \to \overline{B^3}$.*

Proof. Suppose that p is a simple branched covering of degree d. Let $\gamma : \overline{B^3} \to \overline{B^3}$ be the rotation about the third coordinate axis by $\frac{2\pi}{d}$. Let Γ denote the group generated by γ. Consider the quotient map $g : \overline{B^3} \to \overline{B^3}/\Gamma \cong \overline{B^3}$. Then g is a branched covering of degree d whose singular locus Σ_g is the rotation axis as in Figure 3.2.

Modify g to the simple branched covering $\hat{g} : \overline{B^3} \to \overline{B^3}$ with $d - 1$ components of singular locus $\Sigma_{\hat{g}}$ of index 2 each as in Example 3.5 and Figure 3.2.

The restriction $\hat{g}|_{\mathbb{S}^2} : \mathbb{S}^2 \to \mathbb{S}^2$ is a simple branched covering of degree d. By Luroth's Theorem 3.6, p is equivalent to $\hat{g}|_{\mathbb{S}^2}$ which extends to a simple branched covering $\hat{g} : \overline{B^3} \to \overline{B^3}$. Therefore p extends to a branched covering $\hat{p} : \overline{B^3} \to \overline{B^3}$ which is simple. It follows from Lemma 3.8. □

Remark 3.10 *Let $p : \mathbb{S}^2 \to \mathbb{S}^2$ be a simple branched covering. The number of singular points of p is even, applying Hurwitz's formula. In Corollary 3.9 the sigular locus of $\hat{p}$ is a union of disjoint arcs connecting two singular points of p.*

Theorem 3.11 (Alexander) *Every closed orientable 3-manifold is a branched covering of $\mathbb{S}^3$.*

Proof. Let M be a closed oriented 3-manifold and let $V_1, V_2, \ldots, V_n$ be vertices of a triangulation of M. Pick n points $P_1, P_2, \ldots, P_n$ in $\mathbb{R}^3$ such that they are in general

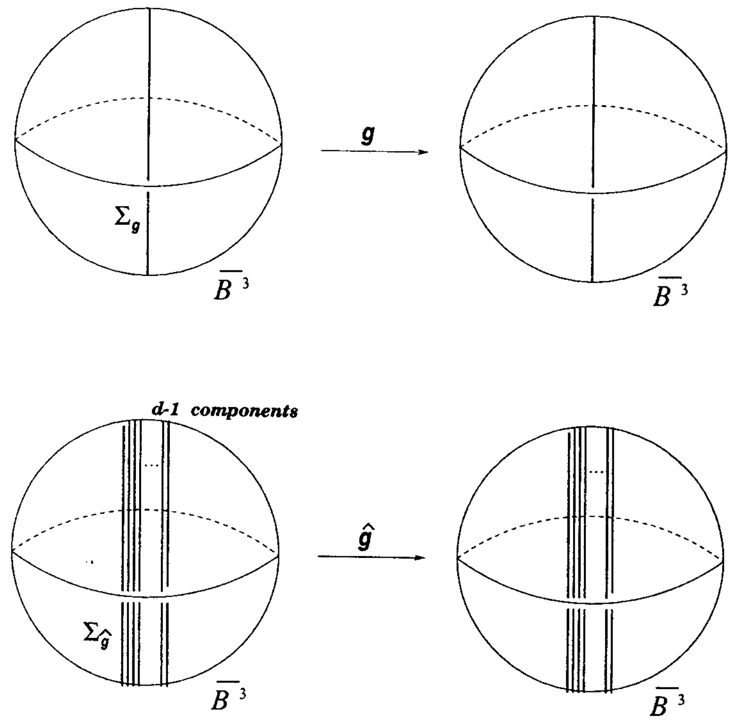

Figure 3.2. Modification around the singular locus Σ_g of index d

position. (No four points are coplanar.) Set $f(V_i) = P_i$ for all $i = 1, 2, \ldots, n$. After taking an orientation for $\overline{\mathbb{R}^3} = \mathbb{S}^3$, we extend $f : \{ V_1, \ldots, V_n \} \to \mathbb{S}^3$ to a map $f : M \to \mathbb{S}^3$ as follows.

Let $[P_iP_j]$ be the shortest geodesic in $\mathbb{R}^3$ for each edge $[V_iV_j]$ of the triangulation. It is uniquely determined by the choice of the points P_k's. Set $f([V_iV_j]) = [P_iP_j]$ and $f([V_iV_jV_k]) = [P_iP_jP_k]$ using affine extensions in $\mathbb{R}^3 \subset \mathbb{S}^3$, where $[V_iV_jV_k]$ is a face in the triangulation. Now, consider 3-simplices in the triangulation of M. Suppose $[V_1V_2V_3V_4]$ is a 3-simplex in the triangulation of M. First set $f([V_1V_2V_3V_4]) = [P_1P_2P_3P_4]$ by the affine extension. If the restriction of f to $[V_1V_2V_3V_4]$ is orientation-preserving, then we must keep such correspondence. If it is orientation-reversing, postcompose an inversion with respect to the boundary $\partial[P_1P_2P_3P_4]$ of $[P_1P_2P_3P_4]$. Retain the notation f for the resulting map. In this case $f([V_1V_2V_3V_4]) = \mathbb{S}^3 - int([P_1P_2P_3P_4])$. We note that an inversion with respect to $\partial[P_1P_2P_3P_4]$ is a topological inversion as follows : Consider a round shpere S

in $[P_1P_2P_3P_4]$ and a homeomorphism h of $\mathbb{S}^3$ which maps $\partial[P_1P_2P_3P_4]$ to S. Let i_S denote the inversion in S. Then the inversion with respect to the boundary $\partial[P_1P_2P_3P_4]$ is the composition $h^{-1} \circ i_S \circ h$.

Do the same construction on each 3-simplex in M. We obtain an orientation-preserving map $f : M \to \mathbb{S}^3$. Note that the restriction of f to the interior of each 3-simplex is a homeomorphism onto its image. Furthermore, $f|_{M-M^{(1)}}$ is a local homeomorphism.

We will modify the map f to a simple branched covering from M to $\mathbb{S}^3$. Let $M^{(1)}$ be the 1-skeleton of M. It is a graph. So we modify the map f until the singular set is a link. The singular set of f is contained in $M^{(1)}$ by the construction of the map f. We take a tubular neighborhood of $M^{(1)}$ which is the union of disjoint balls $\overline{B}_i$ with centers V_i and disjoint cylindrical neighborhoods C_{ij} with axis in the edge $[V_iV_j]$ as Figure 3.3. Moreover, $\overline{B}_i \cap C_{jk}$ is a disk if and only if $i \in \{ j, k \}$.

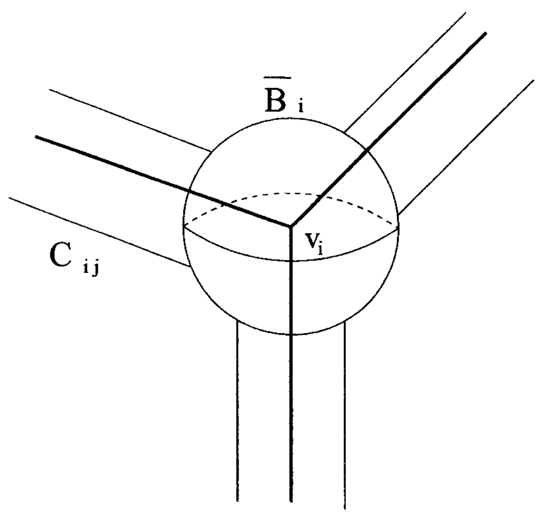

Figure 3.3. A tubular neighborhood of $M^{(1)}$

First, we modify the map f inside of each cylindrical neighborhood C_{ij} of $[V_iV_j]$ as follows. The map f is locally equivalent to $p_{n_{ij}} \times id : B^2 \times I \to B^2 \times I$ as in Definition 3.7, where $p_{n_{ij}}(z) = z^{n_{ij}}$ and $\{ 0 \} \times I$ corresponds to a portion of $[V_iV_j]$.

Modify $p_{n_{ij}}$ to $p'_{n_{ij}}$ so that there are $n_{ij}-1$ components of singular locus of index 2 each, as in Figure 3.4. We retain the notation f for the resulting map.

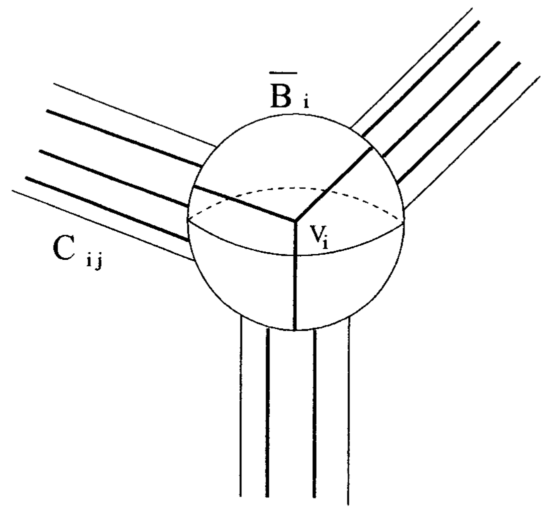

Figure 3.4. Modification inside cylindrical neighborhoods of edges in $M^{(1)}$

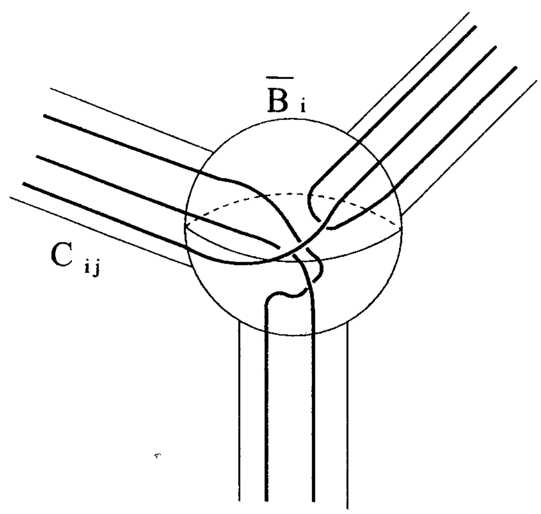

Figure 3.5. Modification inside a ball neighborhood B_i of a vertex V_i in $M^{(1)}$

Second, we modify the map f inside of each ball neighborhood B_i of V_i. Since the modified map f restricted to the boundary sphere of B_i is a simple branched covering onto its image 2-sphere, we obtain that such restriction $f|_{\partial \overline{B_i}}$ extends, by Corollary 3.9, to $\overline{f|_{\partial \overline{B_i}}} : \overline{B_i} \to \overline{B_i}$ which is a simple branched covering as in Figure 3.5.

Retain the notation f for such modified map. Then $f : M \to \mathbb{S}^3$ is a simple branched covering. □

Remark 3.12 ([5]) *We have proved in the proof of Theorem 3.11 that every closed orientable 3-manifold is a simple branched covering whose singular locus is a link.*

Theorem 3.13 *Let N be a compact orientable 3-manifold with boundary and $f : N \to \mathbb{S}^3$ a PL local injection which is generic on the boundary. If M is a closed orientable 3-manifold containing N, then f extends to a simple branched covering $\hat{f} : M \to \mathbb{S}^3$ up to a small perturbation of f near the boundary of N.*

Proof. Consider M as a finite oriented simplicial complex with subcomplex N. Let $V_1, \ldots, V_t, V_{t+1}, \ldots, V_n$ be vertices in $M - int(N)$, where $V_1, \ldots, V_t \in \partial N$ and $V_{t+1}, \ldots, V_n \in M - N$ for some $t < n$. Pick $P_{t+1}, \ldots, P_n \in \mathbb{R}^3 \subset \mathbb{S}^3$ in general position. Suppose that f is orientation-preserving. Let $P_i = \hat{f}(V_i)$, $i = t+1, t+2, \ldots, n$ after taking $\hat{f}(x) = f(x)$ for $x \in N$. Put $\hat{f}([V_iV_j]) = [P_iP_j]$, $\hat{f}([V_iV_jV_k]) = [P_iP_jP_k]$ and $\hat{f}([V_iV_jV_kV_l]) =$ either $[P_iP_jP_kP_l]$ or $inv([P_iP_jP_kP_l])$ where inv is an inversion with respect to $\partial[P_iP_jP_kP_l]$, so that $\hat{f} : M \to \mathbb{S}^3$ is orientation-preserving. $\hat{f}$ is a local homeomorphism except $M^{(1)} - int(N)$. So the singular set of $\hat{f} : M \to \mathbb{S}^3$ is contained in $M^{(1)} - int(N)$. Modify $\hat{f}$ inside a tubular neighborhood of $M^{(1)} - int(N)$ which is the union of balls and cylindrical neighborhoods as in the proof of Theorem 3.11, to get a simple branched covering from M to $\mathbb{S}^3$. Retain the notation $\hat{f}$ for the modified map. By the modification, we obtained that $\hat{f}|_{N-Nbd(\partial N)} = f|_{N-Nbd(\partial N)}$, where $Nbd(\partial N)$ is a regular neighborhood of ∂N in N. Hence $\hat{f}$ is an extension of f up to a small perturbation of f near the boundary ∂N. □

Theorem 3.14 (Whitehead) ([12]) *If N is an open (noncompact without boundary) orientable 3-manifold, then N can be immersed in $\mathbb{R}^3$.*

We obtain a relative version of Alexander's theorem as follows.

Corollary 3.15 *If M is a closed orientable 3-manifold, then there exists a simple branched covering of $\mathbb{S}^3$ whose singular locus is a link contained in a 3-ball in M.*

Proof. Let D be the interior of a 3-simplex in M. Take $N = M - D$. N is a compact orientable 3-manifold whose boundary is the 2-sphere. Let $N' = N \cup (D - \{x\}) = M - \{x\}$, $x \in D$. Then N' is an open orientable manifold. By Whitehead's Theorem 3.14, N' can be immersed in $\mathbb{R}^3$. We call $g : N' \to \mathbb{R}^3$ such an immersion. Let $h = g|_N : N \to \mathbb{R}^3$. h is a PL local injection. By Theorem 3.13, there exists a simple branched covering $f : M \to \mathbb{S}^3$ such that $f|_{N-Nbd(\partial N)} = h|_{N-Nbd(\partial N)}$. Here $Nbd(\partial N)$ is a regular neighborhood of ∂N in M. The singular locus of f is a link contained in $D \cup Nbd(\partial N)$ which is a 3-ball in M. □

Remark 3.16 *If M is a closed oriented 3-manifold, then there exists a simple branched covering $f : M \to \mathbb{S}^3$ such that its singular locus Σ_f is a link in a 3-ball B in M. A regular neighborhood $Nbd(\Sigma_f)$ of Σ_f is a disjoint union of solid tori in B. Let $M_1 = M - int(Nbd(\Sigma_f))$. The restriction of f to M_1 is a local injection. Put the pull-back structure on M_1 of the canonical Möbius structure of $\mathbb{S}^3$ by f. Then M_1 is a Möbius manifold whose boundary is a disjoint union of tori and $f : M_1 \to \mathbb{S}^3$ is a Möbius morphism.*

CHAPTER 4

CONSTRUCTION IN A FUCHSIAN CASE

The main theorem in this thesis is the following :

Let M be a closed oriented 3-manifold. Then there exists a 3-manifold N so that the connected sum of M and N admits a Möbius structure.

We have discussed the strategy of proving this in the previous two chapters. Suppose that M is a closed oriented 3-manifold. By Corollary 3.9, there exists a simple branched covering $f : M \to \mathbb{S}^3$ whose singular locus Σ_f is a link contained in a 3-ball B in M. (It follows from Whitehead's Theorem and Alexander's Theorem.) Since f is simple, we observe that for each $y \in B_f = f(\Sigma_f)$ there exists a unique $x \in \Sigma_f$ such that $y = f(x)$. We obtain that the branch locus B_f doesn't have a self-intersection point in $\mathbb{S}^3$. So B_f is a link in $\mathbb{S}^3$. The map $f : M \to \mathbb{S}^3$ determines a Möbius structure on $M - int(Nbd(\Sigma_f))$. Here $Nbd(\Sigma_f) \subset B$ is a disjoint union of solid tori. Each connected component of the boundary of $M - int(Nbd(\Sigma_f))$ is a torus. We will use Theorem 2.6 to glue the Möbius structure on $M - int(Nbd(\Sigma_f))$ to appropriate Möbius manifolds with toral boundary. The resulting Möbius manifold will be a connected sum because ∂B is a separating 2-sphere in that manifold.

On the above procedure, the simplest case is that the image under f of each connected component of Σ_f is a trivial knot in $\mathbb{S}^3$. A trivial knot is isotopic to a round circle. We will prove the main theorem for the rest of this chapter only in the case where $K = \Sigma_f$ is connected (i.e. K is a knot in $B \subset M$) with its image $f(K)$ a round circle in $\mathbb{S}^3$. We postpone a discussion of the general case until chapter 5.

Remark 4.1 ([9]) *Each Möbius transformation γ acting on $\overline{\mathbb{R}^n}$ has a natural extension to a Möbius transformation acting on $\overline{\mathbb{R}^{n+1}}$ as follows. Let σ be a reflection in S, an $(n-1)$-sphere or an $(n-1)$-plane in $\mathbb{R}^n$. Note that a reflection in a sphere means the inversion in that sphere. There exists a unique sphere or plane $\bar{S}$ in $\mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{n+1}$ such that $\bar{S} \cap \mathbb{R}^n = S$ and $\bar{S}$ is orthogonal to the hyperplane $\mathbb{R}^n \times \{0\}$ which contains S. Denote by $\bar{\sigma}$ the reflection in $\bar{S}$. Since a Möbius transformation γ can be expressed as a finite composition of reflections, $\gamma = \sigma_m \circ \cdots \circ \sigma_1$ for some reflections σ_j. Set $\bar{\gamma} = \bar{\sigma}_m \circ \cdots \circ \bar{\sigma}_1$. It is an extension of γ acting as a Möbius transformation on $\overline{\mathbb{R}^{n+1}}$.*

Definition 4.2 *The* Poncaré extension *of γ in $M\ddot{o}b(\mathbb{S}^n)$ is the Möbius transformation $\bar{\gamma}$ in $M\ddot{o}b(\mathbb{S}^{n+1})$ as defined above.*

We observe that the map $\gamma \mapsto \bar{\gamma}$ is a monomorphism from $M\ddot{o}b(\mathbb{S}^n)$ into $M\ddot{o}b(\mathbb{S}^{n+1})$. The Poincaré extension $\bar{\gamma}$ of any γ in $M\ddot{o}b(\mathbb{S}^n)$ is also an isometry of the hyperbolic space $\mathbb{H}^{n+1}$. Furthermore, $M\ddot{o}b(\mathbb{S}^n) = Isom(\mathbb{H}^{n+1})$, the full group of isometries of $\mathbb{H}^{n+1}$ ([9]).

Let $Isom^+(\mathbb{H}^{n+1})$ be the subgroup of $Isom(\mathbb{H}^{n+1})$ consisting of all orientation-preserving isometries of $\mathbb{H}^{n+1}$. A subgroup Γ of $Isom^+(\mathbb{H}^{n+1})$ is called *discrete* if it is discrete as a topological subspace. Note that $Isom^+(\mathbb{H}^{n+1})$ is a Lie group. From now on, Γ is a discrete subgroup of $Isom^+(\mathbb{H}^{n+1})$.

Definition 4.3 *Denote by $\Lambda(\Gamma) = \overline{\Gamma p} \cap \mathbb{S}^n$, where $p \in \mathbb{H}^{n+1}$ and $\partial\mathbb{H}^{n+1} = \mathbb{S}^n$, the* limit set *of Γ. Each element of $\Lambda(\Gamma)$ is called a* limit point *of Γ.*

Remark 4.4 *$\Lambda(\Gamma)$ is independent of choices of $p \in \mathbb{H}^{n+1}$.*

Definition 4.5 *$x \in \mathbb{S}^n = \partial\mathbb{H}^{n+1}$ is called a* point of discontinuity *if there exists a neighborhood U of x such that $\gamma(U) \cap U = \varnothing$ for all but finitely many γ's in Γ. Denote by $\Omega(\Gamma)$ the set of all such points, called the* domain of discontinuity *of Γ.*

We now summarize some properties of $\Lambda(\Gamma)$ and $\Omega(\Gamma)$ in $\mathbb{S}^n$ as follows. We refer the reader to [9] for more details on such materials.

Proposition 4.6 ([9]) *Let Γ be a discrete subgroup of $Isom^+(\mathbb{H}^{n+1})$. Then:*

(1) The limit set $\Lambda(\Gamma)$ of Γ is closed and invariant under Γ.

(2) The domain $\Omega(\Gamma)$ of discontinuity of Γ is open and invariant under Γ.

(3) $\mathbb{S}^n - \Lambda(\Gamma) = \Omega(\Gamma)$, that is, $\Lambda(\Gamma) \sqcup \Omega(\Gamma) = \mathbb{S}^n$.

(4) Γ acts properly discontinuously *on $\Omega(\Gamma)$, that is, for each compact subset C of $\Omega(\Gamma)$, $\gamma(C) \cap C = \varnothing$ for all but finitely many γ's in Γ. So $\Omega(\Gamma)/\Gamma$ is Hausdorff. Moreover it is a manifold if Γ act freely on $\Omega(\Gamma)$.*

Let $M\ddot{o}b^+(\mathbb{S}^3)$ be the subgroup of $M\ddot{o}b(\mathbb{S}^3)$ consisting of all orientation-preserving Möbius transformations of $\mathbb{S}^3$. On the other hand, $M\ddot{o}b^+(\mathbb{S}^3) = Isom^+(\mathbb{H}^4)$, the group of all orientation-preserving isometries of the hyperbolic space $\mathbb{H}^4$. Let $Stab_G^+(\mathbb{H}^2)$ denote the stabilizer of $\mathbb{H}^2$ in $G = Isom^+(\mathbb{H}^4)$ whose elements act on $\mathbb{H}^2$ as orientation-preserving isometries.

Lemma 4.7 *$Stab_G^+(\mathbb{H}^2) = Isom^+(\mathbb{H}^2) \times SO(2)$, where $G = Isom^+(\mathbb{H}^4)$.*

Proof. Let $r : Stab_G^+(\mathbb{H}^2) \to Isom^+(\mathbb{H}^2)$ be the homomorphism given by restrictions to $\mathbb{H}^2$ and let $k(r)$ be the kernel of r. $k(r) \cong SO(2)$ since each isometry of $k(r)$ fixes $\mathbb{H}^2$ pointwise and $Stab_G^+(0) \cong SO(4)$ where $0 \in \mathbb{H}^2 \subset \mathbb{H}^4$ in the ball model of hyperbolic space $\mathbb{H}^4$. Note that r is an epimorphism due to the Poincaré extensions of $Isom^+(\mathbb{H}^2)$ to G.

We obtain that $k(r) \hookrightarrow Stab_G^+(\mathbb{H}^2) \xrightarrow{r} Isom^+(\mathbb{H}^2)$ is a short exact sequence. Let $e : Isom^+(\mathbb{H}^2) \to Stab_G^+(\mathbb{H}^2) \subset G$ be the Poincaré extension. It's a homomorphism and obviously $r \circ e = id$. Hence $Stab_G^+(\mathbb{H}^2) = Isom^+(\mathbb{H}^2) \ltimes SO(2)$, the semi-direct product of $Isom^+(\mathbb{H}^2)$ and $SO(2)$.

It suffices to show that $Isom^+(\mathbb{H}^2)$ commutes with $SO(2)$. Let $f \in Isom^+(\mathbb{H}^2) = PSL(2, \mathbb{R})$ be represented by $f(z) = \frac{az+b}{cz+d}$ where $a, b, c, d \in \mathbb{R}$ with $ad - bc = 1$ and $z = x + iy$, $y > 0$. It is well defined on $\overline{\mathbb{R}^2}$ as a Möbius transformation. So we retain the same notation f. We will find an explicit formula for the Möbius transformation $e(f) : \mathbb{S}^3 \to \mathbb{S}^3$. We have the map $f : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ given by $f(x) = \frac{ax+b}{cx+d}$. Consider the map $e(f)$ restricted to $\mathbb{R}^3 - \mathbb{R}$. Take $(x, y, t) \in \mathbb{R}^3 - \mathbb{R}$ in the rectangular coordinates system. These coordinates are expressed in terms of the cylindrical coordinates (x', y', θ) by $x = x', y = y' \cos\theta$ and $t = y' \sin\theta$. Here $y' \geq 0$, $y' = 0$ only on the x-axis and $0 \leq \theta < 2\pi$. Note that, for $y' > 0$, $(x', y', \theta) \in \mathbb{H}^2 \times \mathbb{S}^1 = \mathbb{R}^3 - \mathbb{R}$ and $(x', y', 0) = (x, y, 0) = (z, 0) \in \mathbb{H}^2 \times \{0\}$. Let $\Pi_\tau = \{(x', y', \theta) \in \mathbb{H}^2 \times \mathbb{S}^1 | \theta = \tau\}$ be the half plane for $0 \leq \tau < 2\pi$. In addition, $\Pi_0 = \mathbb{H}^2$ and $\Pi_\tau = R_\tau(\mathbb{H}^2)$ where $R_\tau \in SO(2)$ is the rotation about x-axis with the angle τ. Since $e(f)$ is a finite composition of reflections in 2-spheres or planes orthogonal to the x-axis and $\mathbb{R}^2 \times \{0\}$, then $e(f)(\Pi_\tau) = \Pi_\tau$ for each τ. Each reflection restricted to $\Pi_{\tau,\tau\pm\pi} = \Pi_\tau \cup \mathbb{R} \cup \Pi_{\tau\pm\pi}$ is also the reflection acting on $\Pi_{\tau,\tau\pm\pi}$ with respect to the intersection circle or line with the corresponding sphere or plane. So we obtain that $e(f)(z, \theta) = (\frac{az+b}{cz+d}, \theta)$, since $e(f)(z, 0) = (f(z), 0) = (\frac{az+b}{cz+d}, 0)$. Hence $e(f)(z, \theta) = (f(z), \theta)$ for $f \in Isom^+(\mathbb{H}^2)$. Then

$$\begin{aligned}(e(f) \circ R_\tau)(z, \theta) &= e(f)(z, \theta + \tau) \\ &= (f(z), \theta + \tau) \\ &= R_\tau(f(z), \theta) = (R_\tau \circ e(f))(z, \theta), \text{ where } \theta + \tau \in [0, 2\pi) \pmod{2\pi}.\end{aligned}$$

This completes the proof. □

Definition 4.8 *A discrete subgroup F of $M\ddot{o}b^+(\mathbb{S}^2)$ with an invariant round disk is called a* Fuchsian *group.*

We may assume that the upper half plane $\mathbb{H}^2$ is invariant under F and so a Fuchsian group F is a discrete subgroup of $Isom^+(\mathbb{H}^2) = PSL(2, \mathbb{R})$.

Suppose that S_g is a closed surface of genus $g \geq 2$. Let $F = \pi_1(S_g)$ and then F is a Fuchsian group with $\Lambda(F) = \mathbb{S}^1$. Denote by $e(F)$ the group of the Poincaré extensions of F to $\mathbb{H}^4$ in $M\ddot{o}b^+(\mathbb{S}^3)$. We still have $\Lambda(e(F)) = \Lambda(F) = \mathbb{S}^1$. So $\Omega(e(F)) = \mathbb{S}^3 - \mathbb{S}^1 = \mathbb{H}^2 \times \mathbb{S}^1$ by Proposition 4.6. Note that the action of $e(F)$ preserves the product structure of $\mathbb{H}^2 \times \mathbb{S}^1$ established by Lemma 4.7. Hence

$$\frac{\Omega(e(F))}{e(F)} = \frac{\mathbb{H}^2 \times \mathbb{S}^1}{e(F)} = \frac{\mathbb{H}^2}{F} \times \mathbb{S}^1 = S_g \times \mathbb{S}^1.$$

We observe that the quotient map $\Omega(e(F)) = \mathbb{S}^3 - \mathbb{S}^1 \to S_g \times \mathbb{S}^1$ is a covering projection since $e(F)$ acts freely and properly discontinuously on $\Omega(e(F))$. $S_g \times \mathbb{S}^1$

admits a Möbius structure and its developing map is a multi-valued map from $S_g \times \mathbb{S}^1$ into $\mathbb{S}^3$ which is the inverse of the covering projection.

Suppose that a group Γ acts properly discontinuously and freely on a topological space X. Then a subset $\mathcal{F}$ of X is called a *fundamental set* for Γ if the orbit $\Gamma\mathcal{F}$ is equal to X and $\gamma(\mathcal{F}) \cap \mathcal{F} = \varnothing$ for each $\gamma \in \Gamma - \{1\}$.

Definition 4.9 *Let X be $\mathbb{H}^4$, $\Omega(\Gamma)$ or $\mathbb{H}^4 \cup \Omega(\Gamma)$. A* fundamental domain Φ *for a discrete group $\Gamma < Isom^+(\mathbb{H}^4)$ acting freely on X is a codimension zero piecewise-smooth submanifold of X such that :*

(1) there is a fundamental set $\mathcal{F}$ so that $int(\Phi) \subset \mathcal{F} \subset \overline{\Phi}$

(2) $\overline{int(\Phi)} = \overline{\Phi}$ and the boundary of Φ in X can be represented as a union of piecewise-smooth codimension one submanifolds S_i so that for each S_i there are another S_j and $\gamma \in \Gamma - \{1\}$ with $\gamma(S_i) = S_j$.

(3) the orbit $\Gamma\Phi$ is locally finite in X, i.e. each compact set in X intersects only finitely many members of $\{\, \gamma\Phi \mid \gamma \in \Gamma \,\}$.

A *polyhedron* Ψ is the intersection of finitely many closed half-spaces in $\mathbb{H}^4$. The codimension one faces are called *sides*. We say that the sides of Ψ are paired by elements of $Isom^+(\mathbb{H}^4)$ if for every side s there exist a side s' and an element $g_s \in Isom^+(\mathbb{H}^4)$ with $g_s(s) = s'$. The element g_s is called a *side-pairing transformation.* Then $g_{s'} = g_s^{-1}$ and $(s')' = s$.

We describe *cycle transformations* and *infinite cycle transformations* in the following remark.

Remark 4.10 ([9]) *Start with a codimension two face $e = e_1$. Suppose that the sides of the polyhedron Ψ are paired by elements of $Isom^+(\mathbb{H}^4)$ and that e_1 lies on the boundary of a side s_1. Then there are a side s_1' and a side-pairing transformation g_1 with $g_1(s_1) = s_1'$. Let $e_2 = g_1(e_1)$. Suppose that e_2 lies on the boundary of s_1' and the other side, say s_2. Again, there are a side s_2' and a side-pairing transformation g_2 with $g_2(s_2) = s_2'$. Continuing in this manner, we generate sequences $\{e_m\}$, $\{g_m\}$ and $\{(s_m, s_m')\}$. Let k denote the least period such that all three sequences are periodic* *with period k. We observe that $g_k \circ \cdots \circ g_1(e_1) = e_1$. The* cycle transformation *$h_e = g_k \circ \cdots \circ g_1$ keeps e_1 invariant. Note that there is the other side with the boundary e_1 and that if we choose it then we obtain h_e^{-1} as the cycle transformation. Let $\theta(e_i)$ denote the angle measured from inside Ψ at the codimension two face e_i.*

Suppose that the sides of the polyhedron Ψ are paired by elements of $Isom^+(\mathbb{H}^4)$. We might have two sides that are tangent at a point $x = x_1$ on the sphere $\mathbb{S}^3$ at infinity. Call one of these sides s_1. Suppose that g_1 is the side-pairing transformation with $g_1(s_1) = s_1'$. Let $x_2 = g_1(x_1)$. If x_2 is not a point of tangency between two faces, then we stop. Otherwise, let s_2 be the other side tangent to s_1' at x_2 and find a side-pairing transformation g_2 with $g_2(s_2) = s_2'$, let $x_3 = g_2(x_2)$ and continue. We stop either if x_{k+1} is not a point of tangency or if $x_1 = x_{k+1} = g_k(x_k)$. If the latter occurs, we find side-pairing transformations $g_1, \cdots, g_k$ with $g_k \circ \cdots \circ g_1(x) = x$ Denote $h_x = g_k \circ \cdots \circ g_1$ and call h_x the infinite cycle transformation *at x.*

We describe conditions under which a polyhedron in $\mathbb{H}^4$ is a fundamental domain for the group generated by side-pairing transformations.

Theorem 4.11 (Poincaré's Fundamental Polyhedron Theorem) ([9]) *Let Ψ be a polyhedron in $\mathbb{H}^4$. Suppose that the sides s of Ψ are paired by side-pairing transformations $g_s \in Isom^+(\mathbb{H}^4)$ and Ψ satisfies the following :*

(1) $g_s(s) = s'$ and $g_{s'} = g_s^{-1}$.

(2) $g_s(int(\Psi)) \cap int(\Psi) = \varnothing$.

(3) for each codimension two face e, there is a positive integer t such that $h_e^t = 1$ and $\theta(e_1) + \cdots + \theta(e_k) = \frac{2\pi}{t}$, where h_e is the cycle transformation at $e = e_1$.

(4) each infinite cycle transformation is parabolic, i.e. it has exactly one fixed point on $\mathbb{S}^3$.

Then G, the group generated by the side-pairing transformations, is discrete, Ψ is a fundamental domain for G and the cycle relations $h_e^t = 1$ form a complete set of relations for G.

Suppose that there are finitely many closed round isometric balls in $\mathbb{R}^3 \subset \mathbb{S}^3$ such that the balls have transverse pairwise intersections and no three balls intersect. Let Φ denote the closure of the intersection of the exterior of such balls, i.e. the union of the balls is $\mathbb{S}^3 - int(\Phi)$. Let Ψ be the smallest closed convex set in $\mathbb{H}^4$ whose ideal boundary $\partial_\infty \Psi$ is Φ, denoted $\Psi = Hull(\Phi)$. Then there is a one-to-one correspondence between codimension one faces s_i of Ψ and codimension one faces f_i of Φ with $Hull(f_i) = s_i$ and $\partial_\infty s_i = f_i$, i.e. $s_i = Hull(\partial_\infty s_i)$. Furthermore, $Hull(f_i \cap f_j) = s_i \cap s_j$ and $\partial_\infty(s_i \cap s_j) = f_i \cap f_j$. If the sides of Ψ are paired by side-pairing transformations and Ψ satisfies (1), (2) and (3) in Theorem 4.11, then Φ is a fundamental domain for $G < M\ddot{o}b^+(\mathbb{S}^3)$ acting on $\mathbb{S}^3$. Note that no two sides of Ψ are tangent at a point in the sphere $\mathbb{S}^3$ at infinity because the balls in $\mathbb{S}^3$ have transverse pairwise intersections.

Proposition 4.12 *If P is a fundamental domain for $F = \pi_1(S_g)$ in $\mathbb{H}^2$, then $SO(2) \cdot P = \Phi$ is a fundamental domain for $e(F)$ in $\mathbb{S}^3 - \mathbb{S}^1$, where $SO(2)$ is the group of all rotations about $\overline{\mathbb{R}} = \mathbb{S}^1$.*

Proof. We claim that $e(F) \cdot \Phi$ covers $\Omega(e(F)) = \mathbb{S}^3 - \mathbb{S}^1 = \mathbb{H}^2 \times \mathbb{S}^1$. Let $(z, \theta) \in \mathbb{H}^2 \times \mathbb{S}^1$ be given. For such $z \in \mathbb{H}^2$, there exists $f \in F$ such that $f(z) \in P$. Recall that $e(f)(z, \theta) = (f(z), \theta)$ from Lemma 4.7. So we have $e(f)(z, \theta) \in \Phi$.

Since P is a fundamental domain for F, for each $z \in \mathbb{H}^2$ there exists at most one $f \in F$ such that $f(z) \in int(P)$. By Lemma 4.7, there also exists at most one point $e(f)(z, \theta) \in int(\Phi)$ for each $(z, \theta) \in \mathbb{H}^2 \times \mathbb{S}^1$.

To deal with other conditions of a fundamental domain for $e(F)$, we consider those of the fundamental domain P for F and apply the fact that $e(f)(z, \theta) = (f(z), \theta)$. □

We call each transformation of the above $SO(2)$ a *Möbius rotation* about the round circle $\mathbb{S}^1$.

Recall that in this chapter we assume that $f : M - Nbd(K) \to \mathbb{S}^3$ is a single-valued Möbius morphism and the image of the knot K under $f : M \to \mathbb{S}^3$ is a round circle. Denote $M_1 = M - int(Nbd(K))$ and $T_1 = \partial M_1$. Here the restriction $f : T_1 \to T_1'$ is a 2-fold covering because $f : M \to \mathbb{S}^3$ is a simple branched covering with $\Sigma_f = K$, $T_1 = \partial Nbd(K)$ and $T_1' = \partial f(Nbd(K))$. We continue the discussion of $N = S_g \times \mathbb{S}^1$ and recall that N is a Möbius manifold and its developing map $dev : N \to \mathbb{H}^2 \times \mathbb{S}^1 \subset \mathbb{S}^3$ is multi-valued. Let $q : \mathbb{S}^1 \to \mathbb{S}^1$ be a 2-fold covering. Consider the 2-fold covering $p : N' \to N$ where $p = id \times q$. We remark that N' is homeomorphic to N but has different Möbius structure from N. Next, we observe that the manifold $N_o = (S_g - D) \times \mathbb{S}^1$ has toral boundary $T = \partial D \times \mathbb{S}^1$, where D is an open disk in S_g. Let $M_2 = p^{-1}(N_o)$ which is homeomorphic to N_o. Then M_2 is a Möbius manifold with toral boundary $T_2 = \partial M_2$ and its multi-valued developing map is $h = dev|_{N_o} \circ p|_{M_2} : M_2 \to \mathbb{H}^2 \times \mathbb{S}^1$. Next, we consider the multi-valued map $dev|_T : T \to \mathbb{H}^2 \times \mathbb{S}^1$.

Remark 4.13 *We can take a branch of such $dev|_T$ to be a single-valued map as follows. Let $\alpha = \partial D \times \{y\}$ for $y \in \mathbb{S}^1$ and let $\tilde{\alpha}$ be a connected component of $dev(\alpha)$. Because of the quotient map $\mathbb{H}^2 \times \mathbb{S}^1 \to S_g \times \mathbb{S}^1$ by $e(F)$, the image $dev(\alpha)$ is contained in $\mathbb{H}^2 \times \{y\}$. We see that $dev(\alpha) = F \cdot \tilde{\alpha}$ and the multi-valued map $dev|_T : T \to F \cdot \tilde{\alpha} \times \mathbb{S}^1$. Take the branch of $dev|_T$ from T to $\tilde{\alpha} \times \mathbb{S}^1$, denoted $br(dev|_T)$. Letting $T_2' = \tilde{\alpha} \times \mathbb{S}^1$, this map $br(dev|_T) : T \to T_2'$ is a homeomorphism.*

In view of Remark 4.13, $br(dev|_T) \circ p|_{T_2} : T_2 \to T_2'$ is a 2-fold covering since it is a composition of the homeomorphism $br(dev|_T)$ with the 2-fold covering $p|_{T_2}$. Let $br(h)$ denote the above map. Finally, we have a developing map $h : M_2 \to \mathbb{S}^3$ and a 2-fold covering $br(h) : T_2 \to T_2'$. We already got $f : M_1 \to \mathbb{S}^3$ and $f : T_1 \to T_1'$ from the simple branched covering $f : M \to \mathbb{S}^3$. We are now ready to apply the Theorem 2.6 to glue two Möbius manifolds with toral boundary together. However, we wish to find a Möbius transformation $g \in M\ddot{o}b(\mathbb{S}^3)$ such that $g(T_1') = T_2'$, $g : T_1' \to T_2'$ reverses orientations (induced from M_i for $i = 1, 2$) and $g_*(f_*(\pi_1(T_1))) = br(h)_*(\pi_1(T_2))$.

We have a specific construction of a Fuchsian group Γ as follows. Suppose that $f : M \to \mathbb{S}^3$ is a simple branched covering for a oriented closed 3-manifold M and the image $f(K)$ under f of the connected singular locus K is a round circle in $\mathbb{S}^3$. Let $C = f(K)$ be the round circle on the plane Π in $\mathbb{R}^3$ centered at O and let

$N(C) = f(N(K))$ where $N(K)$ is a tubular neighborhood of K in M. We may assume that C is the unit circle and $d(y, \partial N(C))$ is a constant for each $y \in C$, that is, $N(C)$ is the solid torus of revolution with the core C. There exists a positive integer n such that $d(y, \partial N(C)) > \frac{\pi}{4n}$. Consider the rays $R_1, R_2, \cdots, R_{16n}$ from O on Π such that $\measuredangle(R_i, R_{i+1}) = \frac{\pi}{8n}$ at O. Denote by $\bar{R}_i$ the half closed plane containig R_i which is orthogonal to Π and $\partial \bar{R}_i = R$, the line orthogonal to Π passing through O. For each i, there exists 2-sphere $\bar{S}_i(0)$ of center $O_i \in \Pi$ so that $\bar{S}_i(0)$ is tangent to $\bar{R}_i$ and $\bar{R}_{i+1}$ at $C \cap R_i$ and $C \cap R_{i+1}$, respectively. Ler r_0 be the radius of each sphere $\bar{S}_i(0)$ and let $\bar{S}_i(t)$ denote the concentric sphere with $\bar{S}_i(0)$ of radius $r_0 + t$, $t \geq 0$. We note that $\bar{S}_{i+1}(0)$ (resp. $\bar{S}_{i+1}(t)$) is the image of $\bar{S}_i(0)$ (resp. $\bar{S}_i(t)$) under the rotation by $\frac{\pi}{8n}$ about R. It follows that $r_0 < \frac{\pi}{16n}$. For each $t \geq 0$ and i, the dihedral angle $\vartheta_i(t)$ between $\bar{S}_i(t)$ and $\bar{S}_{i+1}(t)$ is $\vartheta_i(t) = 2\sec^{-1}(\frac{r_0+t}{r_0})$. Let $\vartheta_i(t) = \vartheta(t)$. Then $\vartheta(t)$ is increasing and $0 \leq \vartheta(t) < \pi$. For given $t \geq 0$, let $\Theta(t)$ denote the sum of all dihedral angles $\vartheta_i(t)$. So $\Theta(t) = 16n\vartheta(t)$ which is also increasing. There exists $t_0 > 0$ such that $\Theta(t_0) = 2\pi$. Now, fix the spheres $\bar{S}_1(t_0), \bar{S}_2(t_0), \cdots, \bar{S}_{16n}(t_0)$.

We claim that $\bar{S}_i(t_0) \cap \bar{S}_k(t_0) \neq \varnothing$, $i \neq k$, if and only if two spheres $\bar{S}_i(t_0)$ and $\bar{S}_k(t_0)$ are adjacent. Assume that $\bar{S}_1(t_0) \cap \bar{S}_3(t_0) \neq \varnothing$. Then $t_0 \geq r_0$. We have $\vartheta(t_0) \geq 2\sec^{-1}(2)$, which imlpies that $\vartheta(t_0) \geq \frac{\pi}{3}$. This contradicts to the fact that $\Theta(t_0) = 2\pi = 16n\vartheta(t_0)$. So the claim is completed. It follows that $r_0 + t_0 < 2r_0 < \frac{\pi}{8n}$. Remark that all $\bar{S}_i(t_0)$'s are contained in the solid torus $N(C)$ because $d(y, \partial N(C)) > \frac{\pi}{4n}$.

Denote by $C' \subset \Pi$ the concenric circle with C that is orthogonal to $\bar{S}_i(t_0)$ for all $i = 1, 2, \cdots, 16n$. Let λ be the radius of C'. Then we have the equation $\lambda^2 + (r_0 + t_0)^2 = 1 + r_0^2$ by the orthôgonality of C' and $\bar{S}_i(t_0)$, C and $\bar{S}_i(0)$. Obviously $\lambda < 1$. We show that C' is contained in $\bigcup_{i=1}^{16n} \bar{B}_i$, where $\bar{B}_i$'s are the open balls such thst $\partial \bar{B}_i = \bar{S}_i(t_0)$. It is enough to show that $\lambda > 1 - \sqrt{2r_0t_0 + {t_0}^2}$ since the radius of intersection circle $\bar{S}_i \cap \bar{S}_{i+1}$ is $\sqrt{(r_0 + t_0)^2 - r_0^2}$. From the equation above, $\lambda^2 = 1 - (2r_0t_0 + {t_0}^2)$. So we obtain $1 > \lambda > 1 - \sqrt{2r_0t_0 + {t_0}^2}$.

Now there are the circle C' on Π and the spheres $\bar{S}_i(t_0)$'s which are orthogonal

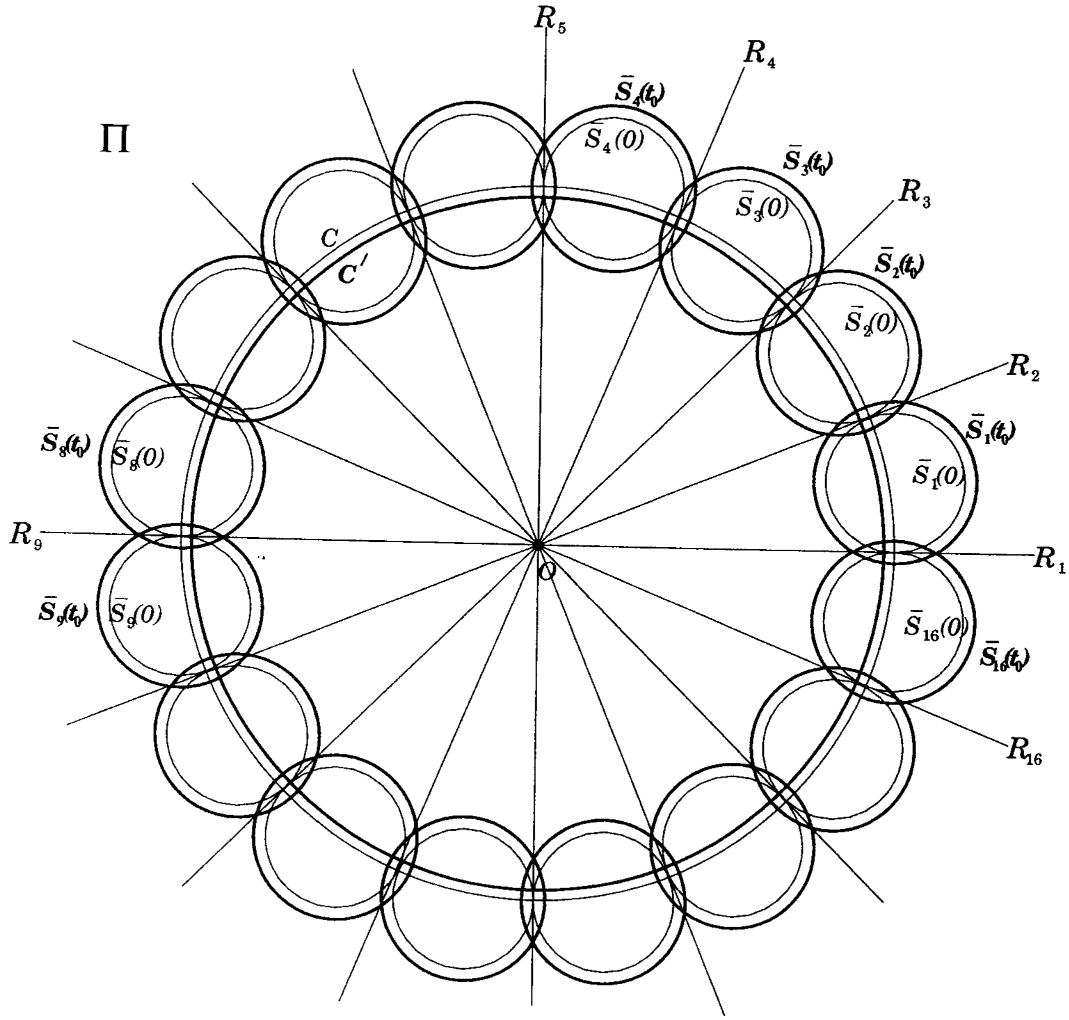

Figure 4.1. Construction with 16 rays on Π

to C' as in Figure 4.1. We remark that C' is isotopic to $C = f(K)$. Take a solid torus $Nbd(C')$ such that $\cup\bar{B}_i \subset Nbd(C') \subset N(C)$ and $Nbd(C')$ is invariant under the group of Möbius rotations about C'. We see $\cup\bar{B}_i$ is also invariant under the group of Möbius rotations about C', since each $\partial\bar{B}_i = \bar{S}(t_0)$ is orthogonal to C'. Recall that $Nbd(C') \subset N(C) = f(N(K))$. Let $Nbd(K') = (f|_{N(K)})^{-1}(Nbd(C'))$ where K' is the inverse image of C' by $(f|_{N(K)})^{-1}$ and let $M_1 = M - Nbd(K')$. $K \subset Nbd(K') \subset N(K)$ because $C \subset Nbd(C') \subset N(C)$. We denote the tori $T_1 = \partial M_1$ and $T_0 = \partial Nbd(C')$. So $f(T_1) = T_0$

We use $Nbd(C') - \bigcup_{i=1}^{16n} \bar{B}_i$ to construct a Möbius manifold M_2 with toral boundary, which is homeomorphic to $(S_g - D) \times \mathbb{S}^1$. In chapter 5, we will call $Nbd(C') - \cup \bar{B}_i$ the *truncated fundamental domain.* Consider spherical faces of $Nbd(C') - \cup \bar{B}_i$. Let D' be the disk on Π with the boundary circle C'. $D' \cap (Nbd(C') - \cup \bar{B}_i)$ is the union of spherical arcs a_j, b'_j, a'_j and b_j for $j = 1, 2, \cdots, 4n$. Here, $a_j \subset \bar{S}_{4j-3}, b'_j \subset \bar{S}_{4j-2}, a'_j \subset \bar{S}_{4j-1}$ and $b_j \subset \bar{S}_{4j}$. We denote by $\bar{a}_j$ (resp. $\bar{b}'_j, \bar{a}'_j, \bar{b}_j$) the $SO(2)$-orbits of a_j (resp. b'_j, a'_j, b_j), where $SO(2)$ is the group of all Möbius rotations about the circle C'. By Lemma 4.7, the $16n$ spherical faces of $Nbd(C') - \cup \bar{B}_i$ are exactly $\bar{a}_j, \bar{b}'_j, \bar{a}'_j$ and $\bar{b}_j$ for $j = 1, 2, \cdots, 4n$.

We define orientation-preserving face-pairing Möbius transformations α_j and β_j, $j = 1, 2, \cdots, 4n$, as follows. Denote by $\bar{P}_i$ the plane containing O and O_i and orthogonal to Π for $i = 1, 2, \cdots, 16n$. Define the inversion in the sphere $\bar{S}_i(t_0)$ by $I(\bar{S}_i(t_0))$ and the reflection in the plane $\bar{P}_i$ by $J(\bar{P}_i)$. Let, for $j = 1, 2, \cdots, 4n$,

$$\alpha_j = J(\bar{P}_{4j-2}) \circ I(\bar{S}_{4j-3}(t_0)), \quad \beta_j = J(\bar{P}_{4j-1}) \circ I(\bar{S}_{4j}(t_0)).$$

Then $\alpha_j, \beta_j \in M\ddot{o}b^+(\mathbb{S}^3)$. Furthermore, $\alpha_j(\bar{a}_j) = \bar{a}'_j$ and $\beta_j(\bar{b}_j) = \bar{b}'_j$ as in Figure 4.2.

Let Γ denote the group generated by α_j and β_j, $j = 1, 2, \cdots, 4n$. Γ is a subgroup of $M\ddot{o}b^+(\mathbb{S}^3)$. We note that α_j and β_j preserve the disk D' on Π. It follows from the fact that $\bar{P}_i$ and $\bar{S}_i(t_0)$ are all symmetric with respect to the plane Π and they are all orthogonal to $\partial D' = C'$. The group Γ is Fuchsian if Γ is discrete.

Let $g = 4n$. We show that $(Nbd(C') - \cup \bar{B}_i)/\Gamma$ is a Mobius manifold which is homeomorphic to $(S_g - D) \times \mathbb{S}^1$ where D is an open disk in S_g. Denote $\Phi = \mathbb{S}^3 - \bigcup_{i=1}^{16n} \bar{B}_i$. By the construction of face-pairing Möbius transformations α_j and β_j, we obtain that $\gamma(int(\Phi)) \cap int(\Phi) = \varnothing$ for $\gamma \in \{\alpha_j, \beta_j\}_{j=1,2,\cdots,4n=g}$. The sum of angles at edges measured from inside Φ is $\Theta(t_0) = 2\pi$. Let $[\alpha_j, \beta_j] = \beta_j^{-1} \circ \alpha_j^{-1} \circ \beta_j \circ \alpha_j$ and $\prod_{j=1}^{g} [\alpha_j, \beta_j] = [\alpha_g, \beta_g] \circ \cdots \circ [\alpha_1, \beta_1]$. We claim that $\prod_{j=1}^{g} [\alpha_j, \beta_j] = id$. It suffices to show that $\prod_{j=1}^{g} [\alpha_j, \beta_j](x) = x$ where $x \in R_1 \cap \bar{S}_1(t_0)$ in D'. It follows from the fact that $\prod_{j=1}^{g} [\alpha_j, \beta_j]$ preserves the sphere $\bar{S}_1(t_0)$ and the disk D'. By Poincaré's Fundamental Polyhedron Theorem, we obtain that Γ is discrete, Φ is a fundamental domain for Γ and $\Gamma = \langle \alpha_j, \beta_j \mid \prod_{j=1}^{g} [\alpha_j, \beta_j] \rangle$. We note that Γ acts on D' as the Fuchsian

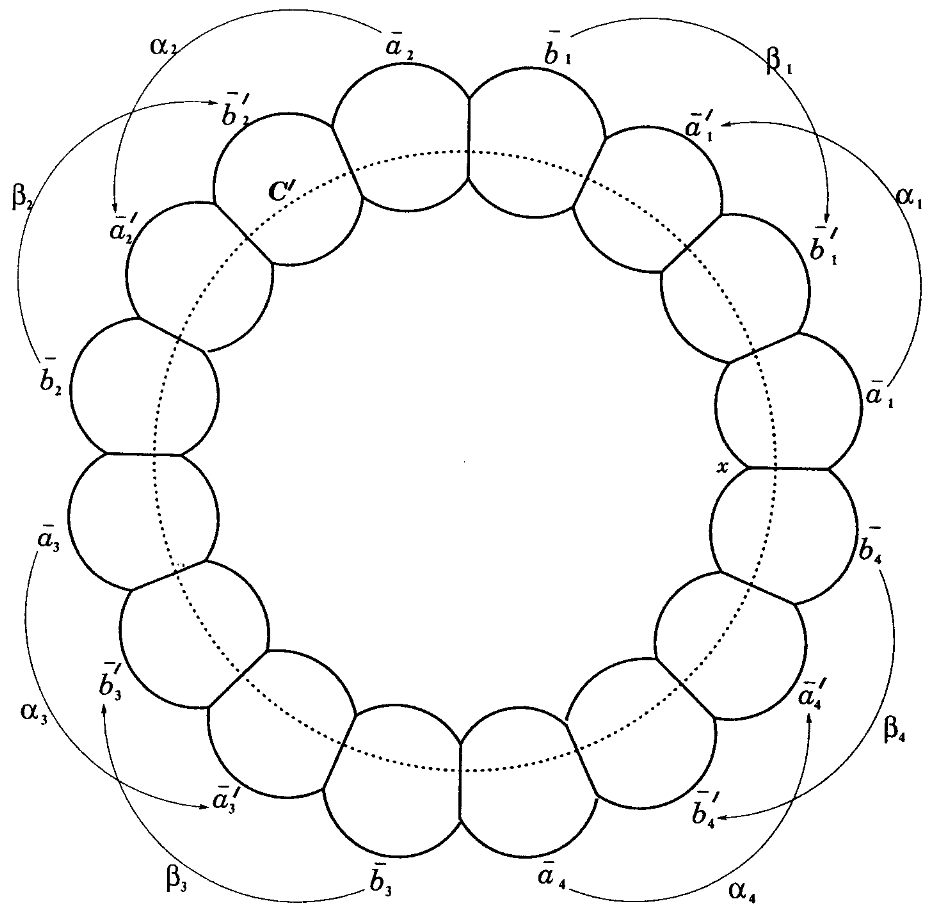

Figure 4.2. face-pairing transformations of $\partial(\cup \bar{B}_i)$

group $\pi_1(S_g)$. So, $\Phi/\Gamma = (\mathbb{H}^2 \times \mathbb{S}^1)/\Gamma = \left(\mathbb{H}^2/\pi_1(S_g)\right) \times \mathbb{S}^1 = S_g \times \mathbb{S}^1 = N$. Let $\hat{D} = D' - Nbd(C')$ and $D = pr(\hat{D})$, where $pr : \mathbb{H}^2 \times \mathbb{S}^1 \to S_g \times \mathbb{S}^1$ is a covering projection. By Lemma 4.7, $\mathbb{S}^3 - Nbd(C') = SO(2) \cdot \overline{D}$. We obtain that $\left(Nbd(C') - \cup\bar{B}_i\right)/\Gamma = (S_g - D) \times \mathbb{S}^1$, denoted N_0.

Let $M_2 = p^{-1}(N_0)$, where $p = id \times q : S_g \times \mathbb{S}^1 \to S_g \times \mathbb{S}^1$ and q is a 2-fold covering of $\mathbb{S}^1$. We have the multi-valued developing map $dev : N \to \mathbb{S}^3$ of $N = S_g \times \mathbb{S}^1$ and recall that M_2 is the Möbius manifold with toral boundary $T_2 = \partial M_2$ and its multi-valued developing map is $h = dev|_{N_o} \circ p|_{M_2} : M_2 \to \mathbb{H}^2 \times \mathbb{S}^1$. We observe the 2-fold covering $br(h) = br(dev|_T) \circ p|_{T_2} : T_2 \to T_2' \subset \mathbb{H}^2 \times \mathbb{S}^1$ as described in Remark 4.13.

A solid torus W is a space which is homeomorphic to $\overline{D^2} \times S^1$, where $\overline{D^2} =$

$\{x \in \mathbb{R}^2 : |x| \leq 1\}$ and $S^1 = \{|x| = 1\}$. A homeomorphic image of $\partial \overline{D^2} \times \{*\}$ (resp. $\{*\} \times S^1$) on ∂W is called a *meridian* (resp. *longitude*) of W or ∂W. Denote by m_k and l_k, for $k = 0, 1, 2$, a meridian and a longitude of the torus T_k. Recall that $T_0 = \partial Nbd(C')$, $T_1 = \partial M_1 = \partial Nbd(K')$ and $T_2 = \partial M_2$. and we have two 2-fold coverings $f : T_1 \to T_0$ and $br(h) : T_2 \to T_0$. So, by the construction above, we obtain that $f_*(m_1) \simeq 2m_0 \simeq br(h)_*(m_2)$ and $f_*(l_1) \simeq l_0 \simeq br(h)_*(l_2)$. Therefore, $f_*\big(\pi_1(\partial M_1)\big) = br(h)_*\big(\pi_1(\partial M_2)\big)$.

There exists a lifting $\tilde{f} : \partial M_1 \to \partial M_2$ of f such that the following diagram commutes:

$$\begin{array}{ccc} \partial M_1 & \xrightarrow{\tilde{f}} & \partial M_2 \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle br(h)} \\ T_0 & = & T_0 \end{array}$$

Note that $\tilde{f}$ is the lifting of $id|_{T_0}$ where $id \in M\ddot{o}b^+(\mathbb{S}^3)$ and that $id|_{T_0}$ reverses orientations (induced from M_1 and M_2). By Theorem 2.6, the attaching manifold $M_1 \cup_{\tilde{f}} M_2$ admits the Möbius structure which extends the Möbius structures of M_1 and M_2. Topologically it is a connected sum of M and $S_g \times \mathbb{S}^1$. We have proved the main theorem in the case that Σ_f is connected and its image $B_f = f(\Sigma_f)$ is a round circle, where $f : M \to \mathbb{S}^3$ is a simple branched covering.

Remark 4.14 *Indeed, we have proved the main theorem in the case that Σ_f is connected and its image B_f is unknotted, since this trivial knot B_f is isotopic to a round circle.*

CHAPTER 5

PROOF OF THE MAIN THEOREM

Recall that we have constructed a Fuchsian group with a fundamental domain whose complement in $\mathbb{S}^3$ is isotopic to a tubular neighborhood of the circular branch locus.

In general, the branch locus B_f of $f : M \to \mathbb{S}^3$ is a link in $\mathbb{S}^3$, where f is a simple branched covering. We may assume each component of the link $B_f = f(\Sigma_f)$ is a polygonal knot. A discrete subgroup of $M\ddot{o}b^+(\mathbb{S}^3)$ whose limit set is a topological circle is called a *quasi-Fuchsian* group. To deal with the general case, we need to construct quasi-Fuchsian groups with prescribed fundamental domains whose complements are isotopic to regular neighborhoods of the given polygonal knots in $\mathbb{S}^3$. It suffices to construct a quasi-Fuchsian group whose fundamental domain has the complementary region isotopic to a regular neighborhood of a given polygonal knot.

Theorem 5.1 *For a given polygonal knot L_0 in $\mathbb{R}^3$, there exist a quasi-Fuchsian group G and a compact fundamental domain Φ for G acting on $\mathbb{S}^3$ such that $\overline{\mathbb{S}^3 - \Phi}$ is isotopic to a regular neighborhood $Nbd(L_0)$ of L_0.*

Proof. Let L_0 be a polygonal knot in $\mathbb{R}^3$. L_0 is isotopic to a right-angled polygonal knot which lies on a plane Π except bridges at its crossings. We may assume that each bridge is of the same height from Π and is contained in an orthogonal plane to the base plane Π. We retain the same notation L_0 for such a polygonal knot.

Give L_0 an orientation and we have a finite set $V(L_0)$ of consecutive vertices whose order is consistent with the orientation of L_0, say $V(L_0) = \{v_1, v_2, \cdots, v_m\}$. Let $e_i = [v_i, v_{i+1}]$ for $i = 1, 2, \cdots, m-1$, $e_m = [v_m, v_1]$ be oriented edges of L_0.

Then e_i is orthogonal to its adjacent edges e_{i-1} and e_{i+1}. We may assume that each edge e_i has rational length l_i, where $l_i = \frac{s_i}{t_i}$ for even integers s_i and t_i. Put $r_0 = (t_1 t_2 \cdots t_m)^{-1}$.

For each $i = 1, 2, \cdots, m$, let $n_i = \frac{l_i}{r_0} = s_i t_1 t_2 \cdots \hat{t_i} \cdots t_m$. We note that each n_i is divisible by $2^4 = 16$ since $m \geq 4$. Cover each e_i by $\frac{n_i}{2}$ closed balls of radius r_0 so that two endpoints v_i and v_{i+1} are centers. By the construction of r_0 and n_i, the center of each ball lies in L_0 and two adjacent balls are tangent to each other at a point in L_0.

We hope that such two balls intersect if and only if they are adjacent. Suppose two non-adjacent balls intersect. Then construct a new cover for L_0 consisting of closed balls of radius $\frac{1}{3}r_0$ with the property that each v_i is the center of such a ball.

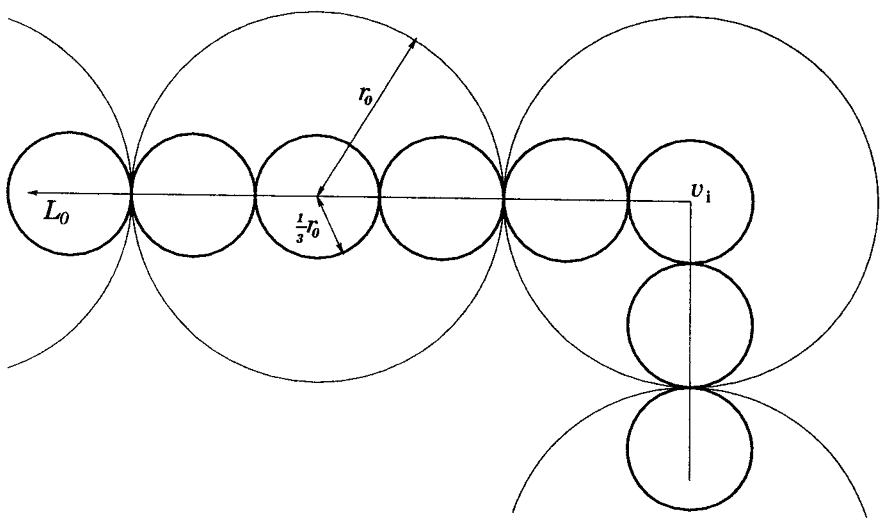

Figure 5.1. A cover for L_0 with balls of radius $\frac{1}{3}r_0$

Keep doing the same procedure as above until we get a cover for L_0 consisting of closed balls of radius $(\frac{1}{3})^n r_0$ for some $n \geq 0$ such that two balls intersect if and only if they are adjacent.

Go two more steps to get the cover for L_0 that consists of closed balls of radius $(\frac{1}{3})^{n+2} r_0$. This will be used for modifying the polygonal knot L_0 inside a certain tubular neighborhood of L_0 containing the union of balls of radius $(\frac{1}{3})^{n+1} r_0$. Now there are $\frac{1}{2}(n_1 + \cdots + n_m)3^{n+2}$ balls of radius $(\frac{1}{3})^{n+2}r_0$ to cover L_0. After rescaling, we may assume that such balls are all of radius 1.

We modify the polygonal knot L_0 into L as follows. Let $L \cap \overline{B_1(v_i)} = \{ v_i^-, v_i^+ \}$. First, modify L_0 inside each ball centered at a vertex v_i using a quarter $\widehat{v_i^- v_i^+}$ of a unit circle centered at O_i as in Figure 5.2.

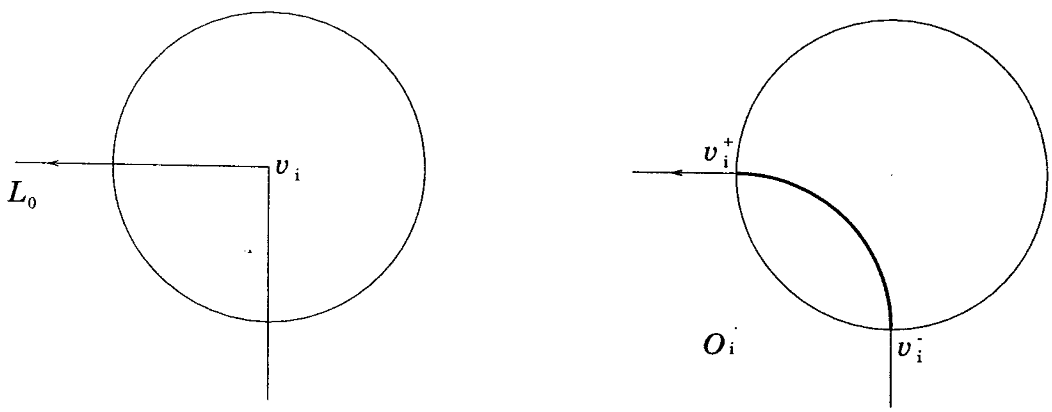

Figure 5.2. Modification of L_0 around each vertex

The second modification occurs on the base plane Π or the bridge planes which are orthogonal to Π. Consider the rest of the segment $\overline{v_i v_{i+1}}$, denoted by $\overline{v_i^+ v_{i+1}^-}$.

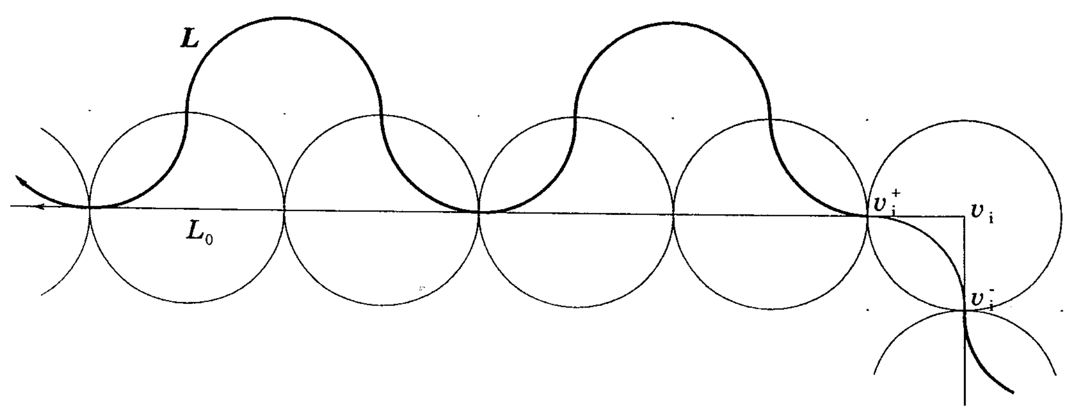

Figure 5.3. Modification of L_0 around each edge

We choose a half plane bounded by $\overleftrightarrow{v_i v_{i+1}}$, in Π or corresponding bridge plane, so that the induced orientation for $\overleftrightarrow{v_i v_{i+1}}$ from the half plane coincide with the orientation for the edge $[v_i, v_{i+1}]$ of L_0. Modify the segment $\overline{v_i^+ v_{i+1}^-}$ in the half plane bounded by $\overleftrightarrow{v_i v_{i+1}}$ using quarters of unit circles as in Figure 5.3. Then the second modification is uniquely determined for each $\overline{v_i^+ v_{i+1}^-}$. Thus we get a modified knot L because there are even number of balls between v_i^+ and v_{i+1}^-.

We will find a cover for L consisting of closed balls of the same size with the property that two adjacent balls are tangent to each other at a point in L and two balls intersect only if they are adjacent.

Recall that L is the union of quarters C_i of unit circles $\tilde{C}_i$ whose order is consistent with the orientation of L for $i = 1, 2, \cdots, g = (n_1 + \cdots + n_m)3^{n+2}$. Denote by $C_i(\pm)$ two endpoints of C_i. such that $C_i(+) = C_{i+1}(-)$. We first consider a quarter C_i of a unit circle $\tilde{C}_i$ because it is a building block for the modified knot L. Let Π_i denote the plane containing C_i. Construct metric balls $\bar{B}(i)_j$, $j = 1, 2, 3, 4$, for each building block C_i as indicated in Figure 5.4.

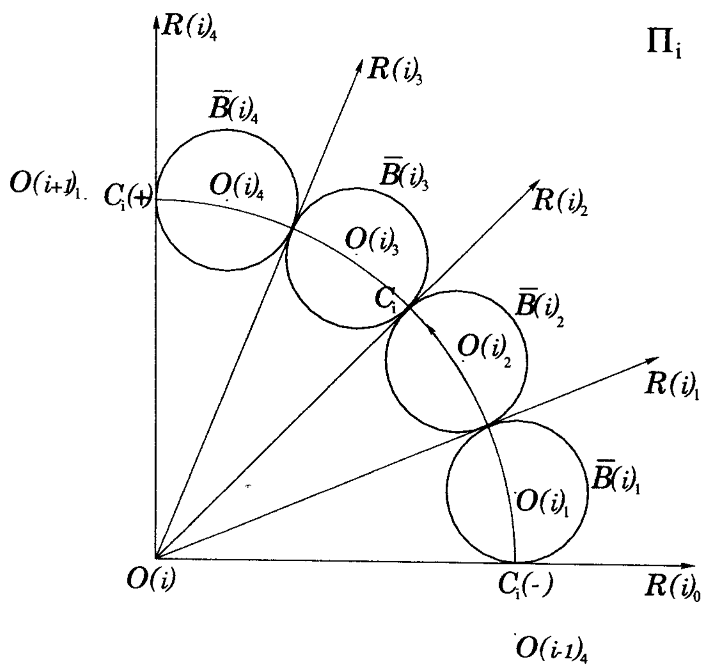

Figure 5.4. A building block with four tangent balls $\bar{B}(i)_j$

Let $O(i)$ be the center of the unit circle $\tilde{C}_i$. Denote by $R(i)_0$ and $R(i)_4$ the rays from $O(i)$ passing through $C_i(-)$ and $C_i(+)$, respectively. Let $R(i)_j$ denote the rays from $O(i)$ on the plane Π_i with $\measuredangle(R(i)_j, R(i)_{j+1}) = \frac{\pi}{8}$ for $j = 0, 1, 2, 3$. There exist four metric balls $\bar{B}(i)_1, \bar{B}(i)_2, \bar{B}(i)_3$ and $\bar{B}(i)_4$ so that each $\bar{B}(i)_j$ is centered on Π_i and tangent to both $R(i)_{j-1}$ and $R(i)_j$ at a point in C_i. Then $C_i \subset \bigcup_{j=1}^{4} \bar{B}(i)_j$ for each i and the spheres $\partial\bar{B}(i)_j$ are orthogonal to the circle $\tilde{C}_i$. Let $O(i)_j$ denote the center of $\bar{B}(i)_j$ for $j = 1, 2, 3, 4$. We note that all $4g$ metric balls $\bar{B}(i)_j$ are isometric for $i = 1, 2, \cdots, g$ and $j = 1, 2, 3, 4$.

Let γ_i denote the rotation by $\frac{\pi}{8}$ about $O(i)$ on Π_i such that $\gamma_i(R(i)_0) = R(i)_1$ and let $e(\gamma_i)$ be its Poincaré extension to $\mathbb{H}^4$. By the construction of the rays, we obtain that $\gamma_i(O(i)_j) = O(i)_{j+1}$ for $j = 1, 2, 3$. So $e(\gamma_i)(\bar{B}(i)_j) = \bar{B}(i)_{j+1}$ for the Poincaré extension $e(\gamma_i)$ of γ_i. We claim that $\gamma_i(O(i)_4) = O(i+1)_1$. Let $\tilde{C}_i$ be the full circle containing C_i on Π_i and let $T_i(\pm)$ be the tangent line at $C_i(\pm)$ to $\tilde{C}_i$ on Π_i. Then we see that two lines $T_i(+)$ and $T_{i+1}(-)$ are identical. So $O(i+1)_1 \in T_i(+)$. Since $O(i)_4, O(i+1)_1 \in T_i(+)$, $\measuredangle(O(i)_4 O(i) O(i+1)_1) = \frac{\pi}{8}$ and the midpoint of $O(i)_4$ and $O(i+1)_1$ is the point $C_i(+)$, we obtain that $\gamma_i(O(i)_4) = O(i+1)_1$. This completes the proof of the claim. It follows that $e(\gamma_i)(\bar{B}(i)_4) = \bar{B}(i+1)_1$.

Now we have a cover for L consisting of isometric balls each of which is tangent to its adjacent balls as in Figure 5.5

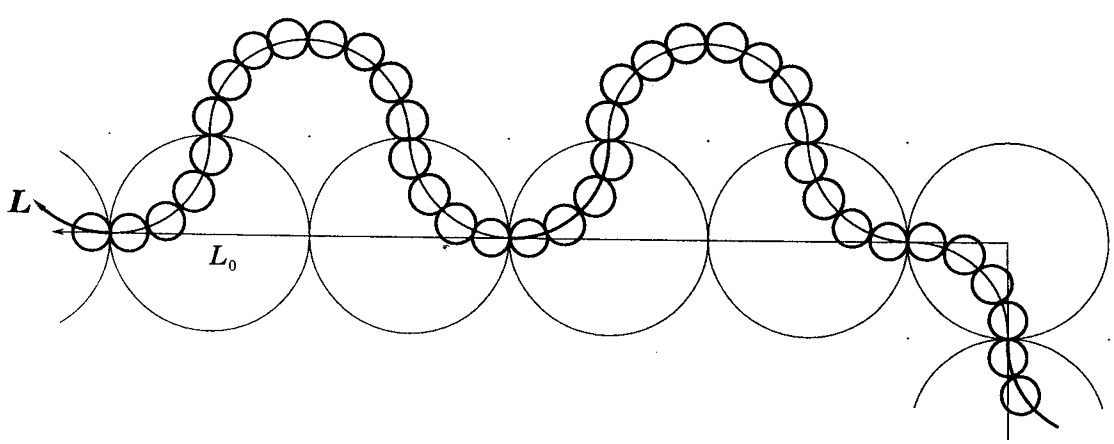

Figure 5.5. A cover for L with $4g$ tangent balls $\bar{B}(i)_j$

Recall that the number of the isometric balls $\bar{B}(i)_j$'s in Figure 5.5 is $4g$ and that $e(\gamma_i)(\bar{B}(i)_j) = \bar{B}(i)_{j+1}$ for all $j = 1,2,3$. and $e(\gamma_i)(\bar{B}(i)_4) = \bar{B}(i+1)_1$. For each $i = 1,2,\cdots,g$, there exist four isometric balls $\bar{B}(i)'_j$ such that $\bar{B}(i)'_j$ is concentric with $\bar{B}(i)_j$ for each $j = 1,2,3,4$ and the dihedral angle between $\bar{B}(i)'_j$ and $\bar{B}(i)'_{j+1}$ is $\frac{\pi}{2g}$ for each $j = 1,2,3$. It comes from a similar construction in Figure 4.1. We also obtain that, for all $i = 1,2,\cdots,g$, the metric balls $\bar{B}(i)'_j$ are isometric for all $j = 1,2,3,4$. Furthermore, $e(\gamma_i)(\bar{B}(i)'_j) = \bar{B}(i)'_{j+1}$ for all $j = 1,2,3$ and $e(\gamma_i)(\bar{B}(i)'_4) = \bar{B}(i+1)'_1$ because $\bar{B}(i)'_j$ is concentric with $\bar{B}(i)_j$.

We claim that the dihedral angle between $\bar{B}(i)_4$ and $\bar{B}(i+1)_1$ is also equal to $\frac{\pi}{2g}$ for each $i = 1,2,\cdots,g$ where $\bar{B}(g+1)_1 = \bar{B}(1)_1$. We recall that $e(\gamma_i)(\bar{B}(i)'_4) = \bar{B}(i+1)'_1$. Since all balls $\bar{B}(i)'_j$ are isometric and the dihedral angles between $\bar{B}(i)'_j$ and $\bar{B}(i)'_{j+1} = e(\gamma_i)(\bar{B}(i)'_j)$ are equal to $\frac{\pi}{2g}$, we have that the dihedral angle between $\bar{B}(i)'_4$ and $\bar{B}(i+1)'_1$ is also $\frac{\pi}{2g}$ for each i. Thus the sum of all dihedral angles between two adjacent balls is equal to $4g \cdot \frac{\pi}{2g} = 2\pi$.

Now we have constructed the solid torus $\bigcup_{i=1}^{g}\bigcup_{j=1}^{4} \bar{B}(i)'_j$ whose interior is isotopic to a regular neighborhood of a polygonal knot L_0 in $\mathbb{R}^3$. Let Φ denote the closure of the complement of $\bigcup_{i=1}^{g}\bigcup_{j=1}^{4} \bar{B}(i)'_j$ in $\mathbb{S}^3$. We will define the face-pairing Möbius transformations α_i and β_i for $i = 1,2,\cdots,g$ and show that Φ is the fundamental domain for the quasi-Fuchsian group generated by these α_i and β_i using the Poincaré's Fundamental Polyhedron Theorem 4.11.

Denote by $\bar{O}(i)_2$ (resp. $\bar{O}(i)_3$) the orthogonal plane to Π_i containing $O(i)$ and $O(i)_2$ (resp. $O(i)_3$) and by $J(i)_2$ (resp. $J(i)_3$) the reflection in $\bar{O}(i)_2$ (resp. $\bar{O}(i)_3$). Let $I(i)_1$ (resp. $I(i)_4$) denote the inversion in $\partial\bar{B}(i)'_1$ (resp. $\partial\bar{B}(i)'_4$) as indicated in Figure 5.6.

Put, for $i = 1,2,\cdots,g$, $\alpha_i = J(i)_2 \circ I(i)_1$, $\beta_i = J(i)_3 \circ I(i)_4$.

We have the spherical faces of Φ as follows.

$\bar{a}_i = \partial\bar{B}(i)'_1 \cap \left(\bar{B}(i-1)'_4 \cup \bar{B}(i)'_2\right)^c$, $\bar{b}'_i = \partial\bar{B}(i)'_2 \cap \left(\bar{B}(i)'_1 \cup \bar{B}(i)'_3\right)^c$,

$\bar{a}'_i = \partial\bar{B}(i)'_3 \cap \left(\bar{B}(i)'_2 \cup \bar{B}(i)'_4\right)^c$, $\bar{b}_i = \partial\bar{B}(i)'_4 \cap \left(\bar{B}(i)'_3 \cup \bar{B}(i+1)'_1\right)^c$,

for $i = 1,2,\cdots,g$

where c stands for the complement in $\mathbb{S}^3$. (See Figure 5.7.)

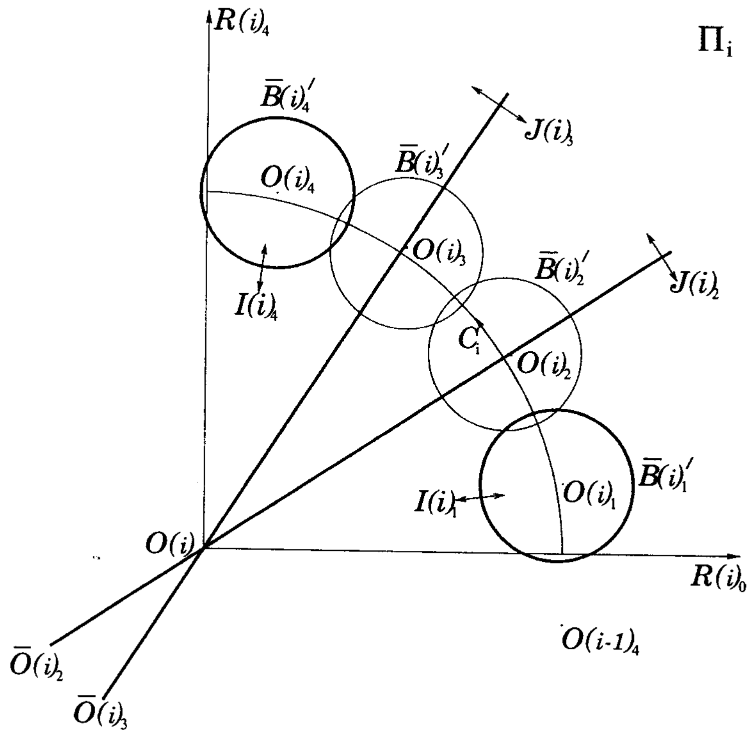

Figure 5.6. Construction of reflections $J(i)_2, J(i)_3$ and inversions $I(i)_1, I(i)_4$

We will show the following :

(0) $\alpha_i,\ \beta_i \in M\ddot{o}b^+(\mathbb{S}^3)$.

(1) $\alpha_i(\partial\bar{B}(i)'_1) = \partial\bar{B}(i)'_3$ and $\beta_i(\partial\bar{B}(i)'_4) = \partial\bar{B}(i)'_2$.

(2) $\alpha_i^{\pm1}(int(\Phi)) \cap int(\Phi) = \varnothing$ and $\beta_i^{\pm1}(int(\Phi)) \cap int(\Phi) = \varnothing$.

(3) (face-pairing Möbius transformations) $\alpha_i(\bar{a}_i) = \bar{a}'_i$, and $\beta_i(\bar{b}_i) = \bar{b}'_i$.

(4) (cycle relation) $\prod_{i=1}^{g}[\alpha_i, \beta_i] = id$.

(0): Since α_i is the composite of the reflection $J(i)_2$ with the inversion $I(i)_1$, α_i is an orientation-preserving Möbius transformation. We note that $\alpha_i^{-1} = I(i)_1 \circ J(i)_2$ and $\beta_i^{-1} = I(i)_4 \circ J(i)_3$.

(1): Since $J(i)_2(R(i)_0) = R(i)_3$ and $J(i)_2(R(i)_1) = R(i)_2$, we have $J(i)_2(O(i)_1) = O(i)_3$, that is, $J(i)_2(\bar{B}(i)'_1) = \bar{B}(i)'_3$. Note that the inversion $I(i)_1$ fixes $\partial\bar{B}(i)_1$ pointwise. Hence, $\alpha_i(\partial\bar{B}(i)'_1) = (J(i)_2 \circ I(i)_1)(\partial\bar{B}(i)'_1) = J(i)_2(\partial\bar{B}(i)'_1) = \partial\bar{B}(i)'_3$.

(2): Recall that $\alpha_i = J(i)_2 \circ I(i)_1$. Since $I(i)_1(int(\Phi)) \subset \bar{B}(i)'_1$ and $J(i)_2(\bar{B}(i)'_1) =$

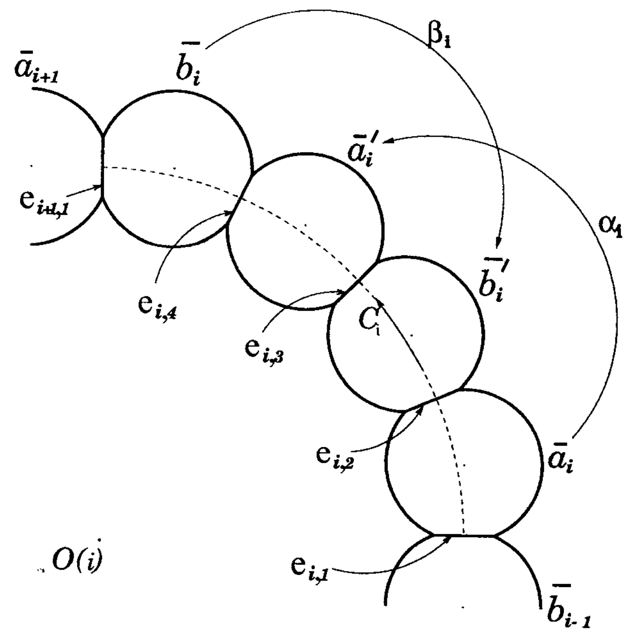

Figure 5.7. Spherical faces of Φ

$\bar{B}(i)_3'$, we obtain that $\alpha_i(int(\Phi)) \subset \bar{B}(i)_3'$. However, $int(\Phi)$ is contained in the complement of $\bar{B}(i)_3'$ in $\mathbb{S}^3$. Hence, $\alpha_i(int(\Phi)) \cap int(\Phi) = \varnothing$.

(3): The reflection $J(i)_2$ takes $O(i)_1$ to $O(i)_3$ and $O(i-1)_4$ to $O(i)_4$ and fixes $O(i)_2$. It follows that $J(i)_2$ takes $\partial\bar{B}(i)_1'$ to $\partial\bar{B}(i)_3'$ and $\bar{B}(i-1)_4'$ to $\bar{B}(i)_4'$ and $\bar{B}(i)_2'$ to $\bar{B}(i)_2'$. Since $\bar{a}_i = \partial\bar{B}(i)_1' \cap (\bar{B}(i-1)_4' \cup \bar{B}(i)_2')^c$, we have that $\alpha_i(\bar{a}_i) = \partial\bar{B}(i)_3' \cap (\bar{B}(i)_4' \cup \bar{B}(i)_2')^c = \bar{a}_i'$. Similarly, we get face-pairing Möbius transformations $\alpha_i^{-1}(\bar{a}_i') = \bar{a}_i$, $\beta_i(\bar{b}_i) = \bar{b}_i'$ and $\beta_i^{-1}(\bar{b}_i') = \bar{b}_i$ as indicated in Figure 5.7.

(4): We show that $\prod_{i=1}^{g}[\alpha_i, \beta_i] = id$. Recall that the cover for $L = \bigcup_{i=1}^{g} C_i$ consists of $4g$ balls $\bar{B}(i)_j'$ and Φ is the complement of the union of $int(\bar{B}(i)_j')$ in $\mathbb{S}^3$. The $4g$ edges of Φ are denoted by $e_{i,1} = \partial\bar{B}(i-1)_4' \cap \partial\bar{B}(i)_1'$ and $e_{i,k} = \partial\bar{B}(i)_{k-1}' \cap \partial\bar{B}(i)_k'$ for $k = 2, 3, 4$. Note that the round circle $e_{i,j}$ is orthogonal to Π_i and contained in the plane whose intersection with Π_i is the line containing the ray $R(i)_{j-1}$. Since $\alpha_i = J(i)_2 \circ I(i)_1$ and $\beta_i = J(i)_3 \circ I(i)_4$, we obtain, as in Figure 5.8,

$$\alpha_i(e_{i,1}) = (J(i)_2 \circ I(i)_1)(e_{i,1}) = J(i)_2(e_{i,1}) = e_{i,4}\,,$$
$$\beta_i(e_{i,4}) = (J(i)_3 \circ I(i)_4)(e_{i,4}) = J(i)_3(e_{i,4}) = e_{i,3}\,,$$

$$\alpha_i^{-1}(e_{i,3}) = \big(I(i)_1 \circ J(i)_2\big)(e_{i,3}) = I(i)_1(e_{i,2}) = e_{i,2}\,,$$
$$\beta_i^{-1}(e_{i,2}) = \big(I(i)_4 \circ J(i)_3\big)(e_{i,2}) = I(i)_4(e_{i+1,1}) = e_{i+1,1}\,.$$

It follows that $[\alpha_i, \beta_i](e_{i,1}) = e_{i+1,1}$. Hence, $\prod_{i=1}^{g}[\alpha_i, \beta_i](e_{1,1}) = e_{1,1}$ where $e_{g+1,1}$ is denoted by $e_{1,1}$. However, it does not complete the proof of the assertion (4) because the restriction $\rho|_{e_{1,1}} : e_{1,1} \to e_{1,1}$ could be a rotation where $\rho = \prod_{i=1}^{g}[\alpha_i, \beta_i]$. It remains only to show that $\prod_{i=1}^{g}[\alpha_i, \beta_i](x_1) = x_1$ for some $x_1 \in e_{1,1}$.

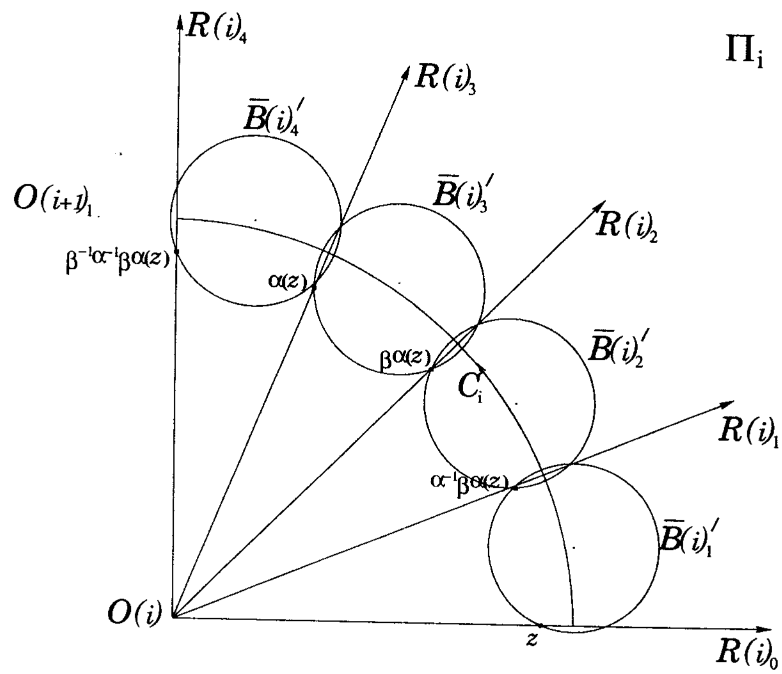

Figure 5.8. Image of z under $[\alpha_i, \beta_i]$

Suppose that Π_{s-1} is the base plane Π and Π_s is a bridge plane which is orthogonal to Π. Let $\Pi^{\perp} = \Pi_s$ and let t denote the smallest integer such that $\Pi_{t+1} = \Pi$ and $s < t$. We see that $\Pi^{\perp} = \Pi_s = \Pi_{s+1} = \cdots = \Pi_t$ is the bridge plane containing the modified bridge $\bigcup_{i=s}^{t} C_i$ where $\bigcup_{i=s}^{t} C_i \cap \Pi = \{\, C_s(-),\, C_t(+) \,\}$ as in Figure 5.9. The set $e_{s,1} \cap \Pi$ consists of two points. We choose a point x_s out of these two points. Consider the line segments $\overline{C_s(-)C_t(+)}$ and $\overline{C_s(-)x_s}$ on Π which are perpendicular to each other. We call $\overline{C_s(-)C_t(+)}$ the *bridge line* in Π corresponding to the bridge $\bigcup_{i=s}^{t} C_i$ as in Figure 5.9. Let x denote the point in Π

such that the parallel transport of the line segment $\overline{C_s(-)x_s}$ along the bridge line $\overline{C_s(-)C_t(+)}$ on Π is the segment $\overline{C_t(+)x}$ in Π. We note that both $\overline{C_s(-)x_s}$ and $\overline{C_t(+)x}$ are orthogonal to $\overline{C_s(-)C_t(+)}$.

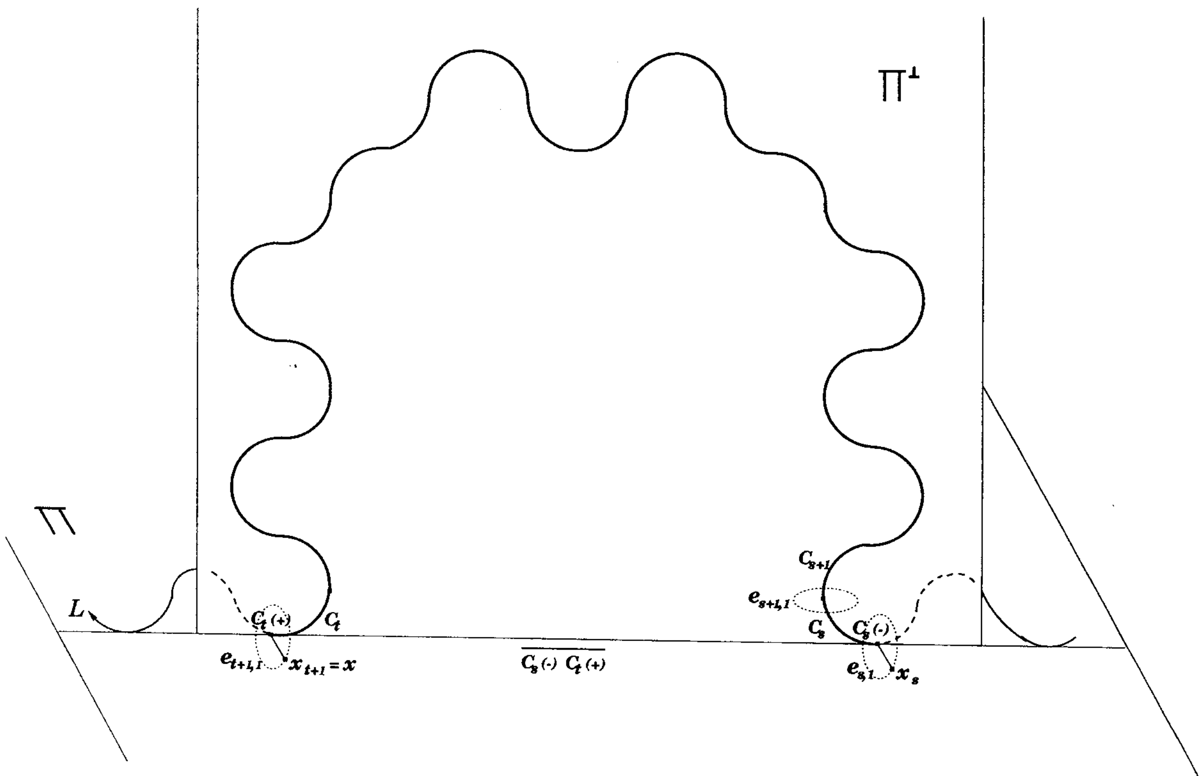

Figure 5.9. A modified bridge in $\Pi^\perp$

We claim that the point x is equal to $\prod_{i=s}^{t}[\alpha_i, \beta_i](x_s)$. Denote $x_{t+1} = \prod_{i=s}^{t}[\alpha_i, \beta_i](x_s)$ and we show that $x_{t+1} = x$. Let $\mathcal{H}$ denote the half 3-space containing x_s and bounded by $\Pi^\perp = \Pi_s$. We recall that $[\alpha_s, \beta_s](e_{s,1}) = e_{s+1,1}$ and $\{ x_s \} = e_{s,1} \cap \Pi \cap \mathcal{H}$. Denote by x_{s+1} the image of x_s under $[\alpha_s, \beta_s]$. So $x_{s+1} \in e_{s+1,1}$. We again observe that

$$\alpha_s(e_{s,1}) = \big(J(s)_2 \circ I(s)_1\big)(e_{s,1}) = J(s)_2(e_{s,1}) = e_{s,4}\,,$$
$$\beta_s(e_{s,4}) = \big(J(s)_3 \circ I(s)_4\big)(e_{s,4}) = J(s)_3(e_{s,4}) = e_{s,3}\,,$$
$$\alpha_s^{-1}(e_{s,3}) = \big(I(s)_1 \circ J(s)_2\big)(e_{s,3}) = I(s)_1(e_{s,2}) = e_{s,2}\,,$$
$$\beta_s^{-1}(e_{s,2}) = \big(I(s)_4 \circ J(s)_3\big)(e_{s,2}) = I(s)_4(e_{s+1,1}) = e_{s+1,1}\,.$$

It follows that $[\alpha_s, \beta_s](e_{s,1}) = [J(s)_2, J(s)_3](e_{s,1})$, since $I(s)_1$ fixes $e_{s,1}$, $e_{s,2}$ pointwise and $I(s)_4$ fixes $e_{s,4}$, $e_{s+1,1}$ pointwise. So $x_{s+1} = [J(s)_2, J(s)_3](x_s)$. We note that $J(s)_2(\mathcal{H}) = \mathcal{H}$ and $J(s)_3(\mathcal{H}) = \mathcal{H}$ because both reflection planes corresponding reflections $J(s)_2$ and $J(s)_3$ are orthogonal to $\Pi^\perp = \partial\mathcal{H}$. The point x_s can be considered as the highest point of the semicircle $e_{s,1} \cap \mathcal{H}$ from $\Pi^\perp$. So $x_{s+1} = [J(s)_2, J(s)_3](x_s)$ is also the highest point of the semicircle $e_{s+1,1} \cap \mathcal{H}$. It follows from the fact that both restrictions $J(s)_2|_{\mathcal{H}}$, $J(s)_3|_{\mathcal{H}} : \mathcal{H} \to \mathcal{H}$ preserve the height from $\Pi^\perp = \partial\mathcal{H}$. Since $x_{t+1} = \prod_{i=s}^{t}[\alpha_i, \beta_i](x_s) = \prod_{i=s}^{t}[J(i)_2, J(i)_3](x_s)$, the point x_{t+1} is also the highest point of the semicircle $e_{t+1,1} \cap \mathcal{H}$ from $\Pi^\perp$. Because the circle $e_{t+1,1}$ is centered at $C_t(+) \in \Pi \cap \Pi^\perp$ and orthogonal to $\Pi^\perp$, the segment $\overline{C_t(+)x_{t+1}}$ is orthogonal to $\Pi^\perp$. Recall that $\overline{C_t(+)x}$ is orthogonal to $\Pi^\perp$. So $\overline{C_t(+)x_{t+1}}$ coincides with $\overline{C_t(+)x}$ since they have the same length. We obtain that $x_{t+1} = x$. This completes the proof of the claim.

We may assume that $C_1 \subset \Pi$ where $L = \bigcup_{i=1}^{g} C_i$ and let $e_{1,1} \cap \Pi = \{x_1, y_1\}$. By the above claim, we can ignore each bridge to prove $\prod_{i=1}^{g}[\alpha_i, \beta_i](x_1) = x_1$. So we have that $\prod_{i=1}^{g}[\alpha_i, \beta_i](x_1) \in e_{1,1} \cap \Pi$, that is, $\prod_{i=1}^{g}[\alpha_i, \beta_i](x_1)$ equals either x_1 or y_1. Let L_Π denote the union of $L \cap \Pi$ and all bridge lines corresponding the bridges of L. This L_Π is a smooth closed curve in Π and has the induced orientation from L. Consider the parallel transport of $\overline{C_1(-)x_1}$ along L_Π on the base plane Π. The result of the parallel transport is $\overline{C_1(-)x_1}$ since Π is orientable. Thus, we obtain that $\prod_{i=1}^{g}[\alpha_i, \beta_i](x_1) = x_1$. It completes the proof of the assertion (4).

Let G denote the group generated by α_i and β_i, $i = 1, 2, \cdots, g$. By the Poincaré's Fundamental Polyhedron Theorem 4.11, Φ is a fundamental domain for G and G is a discrete subgroup of $M\ddot{o}b^+(\mathbb{S}^3)$. By the construction of Φ, $\overline{\mathbb{S}^3 - \Phi}$ is isotopic to a regular neighborhood $Nbd(L_0)$ of the polygonal knot L_0. This completes the proof. □

Remark 5.2 *To complete the proof of Theorem 5.1, we need to show that the limit set of G is a topological circle. We see that $G \cong \pi_1(S)$, where S is a closed surface. Let $F = \pi_1(S)$. This is a Fuchsian group. The limit set $\Lambda(F)$ of F is* *a round circle. Suppose that $\varphi : F \to G$ is an isomorphism. The following is given by Tukia in [11]. If the groups G and F are convex cocompact, i.e. have fundamental polyhedrons in $\mathbb{H}^4$ with finitely many sides and without cusps, then there is a homeomorpfism $\Lambda(F) \to \Lambda(F)$ which induces φ. So the limit set $\Lambda(G)$ is a topological circle.*

The main theorem in this thesis is the following :

Theorem 5.3 *Let M be a closed oriented 3-manifold. Then there exists a 3-manifold N so that the connected sum of M and N admits a Möbius structure.*

Proof. Suppose that M is a closed oriented 3-manifold. Then there is a simple branched covering $f : M \to \mathbb{S}^3$ such that the singular locus Σ_f of f is a link which is contained a 3-ball B in M. It follows from Corollary 3.9. Consider only the case that Σ_f is connected, that is, a knot in B. So the branch locus B_f of f is a knot in $\mathbb{S}^3$.

Denote $M_1 = M - int(Nbd(\Sigma_f))$ and $T_1 = \partial Nbd(\Sigma_f)$. Let $Nbd(B_f) = f(Nbd(\Sigma_f))$ and $T_1' = \partial Nbd(B_f)$. In this case, $Nbd(\Sigma_f)$ and $Nbd(B_f)$ are solid tori with boundary T_1 and T_1', respectively. We consider the restriction of f to M_1. Since $f : M_1 \to \mathbb{S}^3$ is a local injection, we give M_i the pull-back structure of the canonical structure on $\mathbb{S}^3$ by f. Then M_1 is a Möbius manifold with toral boundary and $f : M_1 \to \mathbb{S}^3$ is a Möbius morphism. Let m_1 (resp. l_1) be a meridian (resp. longitude) of T_1 and let m_1' (resp. l_1') be a meridian (resp. longitude) of T_1'. We note that the restriction of f to the boundary torus T_1, $f : T_1 \to T_1'$, is a 2-fold covering such that $f_*(m_1) = 2m_1'$ and $f_*(l_1) = l_1'$ up to homotopy. It follows from the fact that the index of σ_f is 2.

On the other hand, for a knot B_f there exist a quasi-Fuchsian group G and a compact fundamental domain Φ for G acting on $\mathbb{S}^3$ so that $\overline{\mathbb{S}^3 - \Phi}$ is isotopic to a regular neighborhood of B_f. It follows from Theorem 5.1. Denote $\widehat{T} = \overline{\mathbb{S}^3 - \Phi}$ which is a solid torus in $\mathbb{S}^3$. Then there exists a homeomorphism $h : \mathbb{S}^3 \to \mathbb{S}^3$ such that $h(Nbd(\Sigma_f)) = Nbd(\widehat{T})$, where $Nbd(\widehat{T})$ is a regular neighborhood of the solid torus $\widehat{T}$. Denote $T_2' = \partial Nbd(\widehat{T})$. We consider $h \circ f : M_1 \to \mathbb{S}^3$, denoted f_1. Then $f_1 : M_1 \to \mathbb{S}^3$ is a Möbius morphism. We note that the restriction of f_1 to the boundary torus T_1, $f_1 : T_1 \to T_2'$, is a 2-fold covering such that $f_*(m_1) = 2m_2'$ and $f_*(l_1) = l_2'$ up to homotopy, where m_2' (resp. l_2') be a meridian (resp. longitude) of T_2'.

Let $\Phi' = \Phi - Nbd(\widehat{T})$. We call Φ' the *truncated fundamental domain.* Since $\Phi' = Nbd(\widehat{T}) - int(\widehat{T})$, the truncated fundamental domain Φ' is homeomorphic to a solid torus removed an open neighborhood of the core. Note that $\partial\Phi' = \partial\Phi \sqcup T_2'$. We remark that Φ' has a product structure as follows. Φ' is homeomorphic to $T^2 \times I$, where T^2 is the torus and $I = [0,1]$. Furthermore, $\partial\Phi'$ and T_2' correspond to $T^2 \times \{0\}$ and $T^2 \times \{1\}$, respectively.

Let $p : T^2 \to T^2$ be a 2-fold covering such that $p_*(m) = 2m$ and $p_*(l) = l$ up to homotopy, where m and l are a meridian and a longitude of the torus T^2. Denote by $\hat{p} : \Phi' \to \Phi'$ the 2-fold covering with $\hat{p} = p \times id : T^2 \times I \to T^2 \times I$. We note that we have 2-fold covering $\hat{p} : \Phi_2' \to \Phi'$, $\Phi_2' = \Phi'$ and Φ_2' has a different Möbius structure from Φ'. Consider the quotient map $q_2 : \Phi_2' \to \Phi_2'/G$. We note that face-pairing transformations induce an equivalence relation on Φ_2' and each point in $\Phi_2' - \partial\Phi$ is equivalent only to itself. Denote $M_2 = \Phi_2'/G$ and $T_2 = \partial M_2$.

We claim that M_2 is a Möbius manifold with toral boundary T_2. Consider the quotient map $q : \Omega(G) \to \Omega(G)/G$. Then $\Phi'/G = \Omega(G)/G - q(\Phi - Nbd(\widehat{T}))$, denote M_2'. The Möbius manifold M_2' is homeomorphic to $S^* \times \mathbb{S}^1$, where S^* is a surface with boundary gotten by removing an open disk from a closed surface. Then the manifold M_2 is also homeomorphic to $S^* \times \mathbb{S}^1$ and M_2 is a 2-fold cover of M_2' since $S^* \times \mathbb{S}^1 \to S^* \times \mathbb{S}^1$ is given by $(x, e^{it}) \mapsto (x, e^{2it})$. Hence M_2 is the Möbius manifold with boundary torus T_2. The claim is completed.

Now we consider a regular neighborhood $Nbd(T_2)$ in M_2. Recall that $\partial\Phi' = \partial\Phi \sqcup T_2'$. Since there is a one-to one correspondence between $\Phi_2' - \partial\Phi'$ and $M_2 - q_2(\partial\Phi)$ and $\hat{p} : \Phi_2' \to \Phi$ is a 2-fold covering, we obtain the 2-fold covering $f_2 : Nbd(T_2) \to Nbd(T_2')$, where $Nbd(T_2')$ is a regular neighborhood of T_2' in Φ. The single-valued map f_2 is a Möbius morphism. We note that the restriction of f_2 to the boundary torus T_2, $f_2 : T_2 \to T_2'$, is a 2-fold covering such that $f_*(m_2) = 2m_2'$ and $f_*(l_2) = l_2'$ up to homotopy, where m_2 (resp. l_2) be a meridian (resp. longitude) of T_2.

Consider the identity map $id : T_2' \to T_2'$ which reverses orientations (induced form M_1 and M_2). Note that $f_{1*}\big(\pi_1(T_1)\big) = f_{2*}\big(\pi_1(T_2)\big)$. There exists a lifting $\widetilde{id} : \partial M_1 = T_1 \to T_2 = \partial M_2$ of $id : T_2' \to T_2'$ such that the following diagram commutes:

$$\begin{array}{ccc} T_1 & \xrightarrow{\tilde{id}} & T_2 \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_2} \\ T_2' & = & T_2' \end{array}$$

By Theorem 2.6, the attaching manifold $M_1 \cup_{\tilde{id}} M_2$ admits the Möbius structure which extends the Möbius structures of M_1 and M_2. Let $Q = M_1 \cup_{\tilde{id}} M_2$. We recall that $int\big(Nbd(\Sigma_f)\big) \subset B \subset M$. The Möbius manifold Q is the connected sum of M and $N = \big(\mathbb{S}^3 - int\big(Nbd(\Sigma_f)\big)\big) \cup_{\tilde{id}} M_2$. This completes the proof of the case that the singular locus Σ_f is a knot in B.

It is generalized for the case that the singular locus Σ_f is a link in B. Let $\Sigma_f = K_1 \sqcup \cdots \sqcup K_m$, where K_i is a connected component of the link Σ_f and $Nbd(\Sigma_f) = Nbd(K_1) \sqcup \cdots \sqcup Nbd(K_m)$. Suppose that $S_i^* \times \mathbb{S}^1$ are glued to $M - int\big(Nbd(\Sigma_f)\big)$ using homeomorphisms $g_i : \partial S_i^* \times \mathbb{S}^1 \to \partial Nbd(K_i)$ so that the resulting manifold Q admits a Möbius structure. For each i, S_i^* is a surface with boundary gotten by removing an open disk from a closed surface. Denote $X = \sqcup(S_i^* \times \mathbb{S}^1)$ and $g = \cup g_i$. Then the Möbius manifold Q is the connected sum of M and N, where $N = \big(\mathbb{S}^3 - int\big(Nbd(\Sigma_f)\big)\big) \cup_g X$. $\square$

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