BOUNDARIES OF GROMOV-HYPERBOLIC SPACES SATISFYING COARSE POINCARÉ DUALITY

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Abstract. Our main theorem implies that if X is a Gromov hyperbolic simplicial complex satisfying a coarse version of n-dimensional Poincaré duality, then with respect to Steenrod homology, the boundary $\partial_\infty X$ is a linearly locally acyclic homology $(n-1)$-manifold with the homology of an $(n-1)$-sphere.

1. Introduction

This is the first in a series of papers [14, 13] where we discuss spaces satisfying *coarse Poincaré duality;* these are spaces which behave homologically, on a large scale, like uniformly contractible manifolds (see section 3 for the precise definition and examples). In the present paper we study the Gromov hyperbolic case, and show that the boundary behaves homologically like a linearly locally contractible sphere.

Definition 1. A metric space W is *linearly locally acyclic* (respectively *linearly locally coacyclic*) with respect to a (co)homology theory if there is a constant $\lambda > 0$ such that for all $p \in W$, $0 \le r \le \mathrm{diam}(W)$, the inclusion $B(p, \lambda r) \to B(p, r)$ (respectively $W \setminus B(p, r) \to W \setminus B(p, \lambda r)$) induces zero in reduced (co)homology.

Let X be a connected simplicial complex[1] whose links contain a uniformly bounded number of simplices. We endow X with a path metric such that each simplex is isometric to a regular Euclidean simplex of side length 1. Henceforth we will refer to an object satisfying these conditions as a *bounded geometry metric simplicial complex* (BGMSC).

The main result of this paper is the following

Theorem 2. *Assume X is a Gromov hyperbolic BGMSC which satisfies n-dimensional coarse Poincaré duality over a commutative ring with unit $\mathcal{R}$; equip the boundary $\partial_\infty X$ with a Gromov-type metric. Then:*

1. With respect to Čech cohomology with coefficients in $\mathcal{R}$, the ideal boundary of X is linearly locally acyclic, linearly locally coacyclic, and has the same cohomology as the $(n-1)$-sphere. When $n = 2$, the boundary $\partial_\infty X$ homeomorphic to the circle.

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[1]We conflate a simplicial complex with its geometric realization.

2. Suppose $\mathcal{R}$ is a hereditary ring. Then with respect to Steenrod homology with coefficients in $\mathcal{R}$, the ideal boundary $\partial_\infty X$ is a linearly locally acyclic and coacyclic $(n-1)$-dimensional homology manifold with the same homology as the $(n-1)$-sphere S^{n-1}. When $n=3$, then $\partial_\infty X$ is homeomorphic to the 2-sphere.

Universal covers of closed aspherical PL-manifolds with Gromov hyperbolic fundamental group are examples of spaces satisfying the hypotheses of Theorem 2. More generally, any uniformly contractible, Gromov hyperbolic BGMSC which is a topological manifold satisfies the hypotheses.

Theorem 2 resembles a theorem of Bestvina [2, Theorem 2.8], but differs in several respects. First, we stick to Gromov hyperbolic spaces, where there is a canonical notion of boundary, and where the boundary comes equipped with more structure than a mere topology – it has a canonical quasisymmetric structure; [2] considers ideal boundaries of more general classes of spaces. Second, we do not require a group action.[2] Third, we formulate the topological conclusions quantitatively, using the quasisymmetric structure. Parts of the setup and proof – in particular the idea of using Steenrod homology – follow [2].

Applying [21], we obtain:

Corollary 3. *Let X be as in Theorem 2, with $n=2$. Then X is quasi-isometric to the hyperbolic plane $\mathbb{H}^2$.*

Bonk and Schramm established this result when X is a triangulation of the plane with bounded valence, see [4]. Earlier Bowers [6] had given conditions on the 1-skeleton of a (Gromov hyperbolic) planar triangulation which guarantee that its boundary is homeomorphic to the circle.

Spaces satisfying the hypotheses of Theorem 2 but which are not manifolds arise naturally in connection with Gromov hyperbolic Poincaré duality groups, and as the "base spaces" of the *coarse fibrations* studied in [13]. In the forthcoming paper [14] we will apply our results to characterize 2-dimensional Poincaré duality groups over commutative rings, settling a conjecture of Dunwoody and Dicks. This group-theoretic application explains why we are working with rather general commutative rings rather than with $\mathbb{Z}$ or $\mathbb{R}$. In [13] we will study coarse fibrations of manifolds by lines; one of our objectives is to prove a coarse analogue of the Seifert fibered space conjecture and give an alternative proof of G. Mess' part of the proof of the original Seifert fibered space conjecture. By working with spaces satisfying coarse 2-dimensional Poincaré duality instead of insisting on honest Riemannian planes (as in [17, 15]) one can side-step some technicalities.

[2] It appears that much of [2], including [2, Theorem 2.8] can probably be recast in group-less form.

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Contents

2. Preliminaries

Notation and conventions. We let $\mathbb{Z}_+ := \{m \in \mathbb{Z} \mid m \geq 0\}$ and $\mathbb{R}_+ := \{x \in \mathbb{R} \mid x \geq 0\}$. All maps between cell complexes will be continuous unless otherwise specified. Given a map f we let $\mathrm{Im}(f)$ denote the image of f.

We fix once and for all a commutative ring $\mathcal{R}$ with a unit. We will be using singular (co)homology with $\mathcal{R}$ coefficients unless we indicate otherwise (at some point we switch to Steenrod homology and Čech cohomology). For each negative integer k we set $H_k(\cdot) = 0$, $H_c^k(\cdot) = 0$, etc.

Metric space notions. A subset $S \subset Z$ of a metric space is called *δ-dense* if each point $z \in Z$ is within distance $\leq \delta$ from S. A subset in Z which is δ-dense from some $\delta < \infty$, is called a *net* in Z.

A metric space X is said to have *bounded geometry* if there is a function $\phi(R)$ such that each ball $B(x, R) \subset X$ (of radius R centered at $x \in X$) has cardinality $\leq \phi(R)$.

The definition relates to the usual concept of a *Riemannian manifold of bounded geometry* as follows. Suppose that M is a complete Riemannian manifold whose sectional curvature is bounded both from above and from below and whose injectivity radius is bounded away from zero; such a manifold M is called a *Riemannian manifold of bounded geometry*. For instance, one can take a covering space of a compact Riemannian manifold with the pull-back metric. Pick a maximal net $X \subset M$ and give it the induced metric from M. Then X is a metric space of bounded geometry in the sense of the above definition.

If $Y \subset X$ is a subset in a metric space, we will use the notation $\mathcal{N}_R(Y)$ to denote the collection of points $x \in X$ which are within distance $\leq R$ from Y. The set $\mathcal{N}_R(Y)$ is called an R-neighborhood of Y.

A metric space is called *proper* if each metric ball in X is compact.

A map $f : X \to X'$ between metric spaces X and X' is an (L, A)-*quasi-isometry* if for all $x_1, x_2 \in X$ we have

$$\frac{1}{L}d(x_1, x_2) - A \leq d(f(x_1), f(x_2)) \leq Ld(x_1, x_2) + A \tag{4}$$

and $d(x', \mathrm{Im}(f)) < A$ for all $x' \in X'$.

An (L, A)-quasi-geodesic segment in X is a map $f : [0, T] \to X$ which satisfies (4). A metric space X is called *quasi-geodesic* if there exists a pair of constants (L, A) such that every pair of points $a, b \in X$ can be joined by an (L, A)-quasi-geodesic segment (i.e. $f(0) = a, f(T) = b$).

Metric simplicial complexes. A *metric simplicial complex* X is a connected simplicial complex endowed with a path-metric such that each simplex in X is isometric to the regular Euclidean simplex with the unit edges. A metric simplicial complex X is said to have *bounded geometry* if there is a number ν such that the star of each vertex of X contains at most ν simplices. Thus the zero-skeleton of X is a metric space of bounded geometry in the sense of the previous section. Basic examples of metric simplicial complexes of bounded geometry (BGMSC) are given by

1. Covering spaces of compact metric simplicial complexes.
2. Suppose that M is a Riemannian manifold of bounded geometry. Then M admits a triangulation which is a metric simplicial complex of bounded geometry.

A metric simplicial complex X is said to be *uniformly contractible* (resp. *uniformly acyclic*) if there is an increasing function $\psi(R)$ such that for each R-ball $B(x, R) \subset X$ the map $B(x, R) \hookrightarrow B(x, \psi(R))$ induces a zero map on the homotopy groups (resp. reduced homology groups).

An important class of metric simplicial complexes is given by *Rips complexes* of bounded geometry metric spaces. Let Z be such a metric space and $D \in \mathbb{R}_+$. We define the D-*Rips complex* $\mathrm{Rips}_D(Z)$ to be the simplicial complex whose vertex set is Z, where distinct points $x_0, ..., x_n \in Z$ span an n-simplex in $\mathrm{Rips}_D(Z)$ iff $d(x_i, x_j) \leq D$ for all $0 \leq i, j \leq n$. Thus we get a direct system of Rips complexes $\mathrm{Rips}_D(Z)$ with the inclusion morphisms $\mathrm{Rips}_D(Z) \to \mathrm{Rips}_{D'}(Z)$ for $D \leq D'$.

We metrize each connected component of a D-Rips complex by introducing a path-metric such that each simplex in X is isometric to the regular Euclidean simplex with the unit edges. We will be mainly interested in *coarsely connected* bounded geometry metric spaces Z, i.e. metric spaces such that $\mathrm{Rips}_D(Z)$ is connected for sufficiently large D. Note that each quasi-geodesic metric space is coarsely connected. If Z has bounded geometry, so does $\mathrm{Rips}_R(Z)$ for each $R < \infty$.

Let X be a metric simplicial complex. If $V \subset X$ and $R \in \mathbb{Z}_+$, we let $N_R(V)$ denote the smallest metric simplicial subcomplex in X which is contained in the R-neighborhood of V in X. Note that if $Y \subset X$ is a subcomplex then

1. $N_0(Y) = Y$.

2. If Y, $Y' \subset X$ are subcomplexes and $R+R' < d(Y, Y')$, then $N_R(Y) \cap N_{R'}(Y') = \emptyset$.

Although $N_R(Y)$ is not the same as the metric R-neighborhood $\mathcal{N}_R(Y)$ defined in the previous section, these subsets are not that much different:

There exists a function $R = R(r)$ such that for each subcomplex $Y \subset X$,

$$N_{R(r)}(Y) \subset \mathcal{N}_r(Y), \text{ and } \mathcal{N}_{R(r)}(Y) \subset N_r(Y).$$

Gromov hyperbolic spaces. Let Z be a geodesic metric space. A geodesic triangle $\Delta \subset Z$ is called *R-thin* if every side of Δ is contained in the R-neighborhood of the union of two other sides. An *R-fat triangle* is a geodesic triangle which is not R-thin. A geodesic metric space Z is called *δ-hyperbolic in the sense of Rips* (Rips was the first to introduce this definition) if each geodesic triangle in Z is δ-thin.

Let X be a metric space (which is no longer required to be geodesic). For each base-point $p \in X$ define a number $\delta_p \in [0, \infty]$ as follows. For each $x \in X$ set $|x|_p := d(x, p)$ and define the *Gromov product*

$$(x, y)_p := \frac{1}{2}(|x|_p + |y|_p - d(x, y)).$$

Then

$$\delta_p := \inf_{\delta \in [0,\infty]} \{\delta | \forall x, y, z \in X, (x, y)_p \geq \min((x, z)_p, (y, z)_p) - \delta\}.$$

We say that X is *δ-hyperbolic in the sense of Gromov*, if $\infty > \delta \geq \delta_p$ for some $p \in X$. We note that if X a geodesic metric space which is δ-hyperbolic in Gromov's sense then X is 4δ-hyperbolic in the sense of Rips and vice-versa (see [11, 6.3C]).

A metric space Z is *Gromov hyperbolic* if it is δ-hyperbolic for some $\delta < \infty$. If Z is coarsely connected, has bounded geometry, and is δ-hyperbolic then there exists D such that for each $R \geq D$, the complex $\mathrm{Rips}_R(Z)$ is uniformly contractible. To prove it note that under the above assumptions, for all sufficiently large R, the space $Rips_R(Z)$ is a geodesic metric space and the inclusion $\iota : Z \to Rips_R(Z)$ is a quasi-isometry. Thus the image $\iota(Z)$ is a net in $Rips_R(Z)$ and therefore the assertion follows from [11, Lemma 17.A], see also [7, Propositon 3.23].

Recall that a subset $Y \subset X$ is *c-quasi-convex* if every geodesic segment $\gamma \subset X$ with endpoints in Y is contained in $\mathcal{N}_c(Z)$.

If $Y \subset X$ is a c-quasi-convex subset in a δ-hyperbolic metric space X, one defines the *nearest-point projection*

$$p : X \to Y$$

by sending each point in X to a point $y = p(x) \in Y$ such that for each point $y' \in Y$

$$d(x, y') \geq d(x, y) - 1.$$

Such point y is typically non-unique. However there is a function $R = R(\delta, c)$ such that the subset $Y' \subset Y$, consisting of points $y' \in Y$ satisfying the above inequality, has diameter $\leq R$.

Suppose that X is a δ-hyperbolic proper geodesic metric space. Then the *ideal boundary* of X is the set of equivalence classes of geodesic rays in X, emanating from the same point $p \in X$, where rays ρ and ρ' are said to be equivalent if they are within finite Hausdorff distance from each other. The boundary $\partial_\infty X$ of X is topologized as follows: Given a geodesic ray ρ, its basis of neighborhoods consists of the sets $U(\rho, T)$:

$$U(\rho, T) := \{\rho' : \rho'(0) = p \text{ and } \forall t \in [0, T], \quad d(\rho(t), \rho'(t)) \leq 2\delta + 1\}.$$

A sequence (x_i) in X is said to converge to a point $\eta \in \partial_\infty X$ if each subsequence in the sequence of geodesic segments $\overline{px_i}$ converges to a ray ρ' equivalent to ρ, where the convergence is uniform on compacts in $[0, \infty)$.

The topology on $\partial_\infty X$ is metrized as follows (see for instance [7]). We first extend the Gromov product to $\partial_\infty X$:

$$(\xi, \eta)_p := \sup \lim \inf_{i,j \to \infty} (x_i, y_j)_p,$$

where supremum is taken over all sequences $(x_i), (y_j)$ which converge to ξ, η respectively. Then $(\xi, \eta)_p < \infty$ if $\xi \neq \eta$ and $(\xi, \eta)_p = \infty$ otherwise. Now, given a constant $c > 0$ define the function ρ_c on $(\partial_\infty X)^2$:

$$\rho_c(\xi, \eta) := \exp(-c(\xi, \eta)_p),$$

if $\xi \neq \eta$ and $\rho_c(\xi, \xi) := 0$. The function ρ_c is not a metric, however it determines a metric d_c on $\partial_\infty X$ by

$$d_c(\xi, \eta) := \inf \sum_{i=1}^{n} \rho_c(\xi_i, \xi_{i+1}),$$

where the infimum is taken over all finite chains $\xi = \xi_1, \xi_2, ..., \xi_{n-1} = \eta$ "connecting" ξ to η.

Theorem 5. *(See* [3]*.) If X is a contractible Rips complex of a bounded geometry δ-hyperbolic quasi-geodesic metric space Y then $X \cup \partial_\infty X$ is a $\mathcal{Z}$-set compactification of X.*

We refer the reader to [2] for the precise definition of a $\mathcal{Z}$-set compactifications. Intuitively speaking, this compactification of a metric space X homologically looks "like" a compact manifold with boundary, whose interior is X.

Steenrod homology. We refer the reader to [18] and [10] for the detailed account of the Steenrod homology theory.

Let Z be a compact metrizable space. Choose a cofinal sequence $\{\mathcal{U}_i\}_{i\geq 0}$ of finite open covers so that $\mathcal{U}_0$ is the trivial cover $\{Z\}$, and $\mathcal{U}_{i+1}$ refines $\mathcal{U}_i$ for all $i \geq 0$. Let W be the mapping telescope associated with the induced diagram of nerves:

$$\ldots \leftarrow \mathrm{Nerve}(\mathcal{U}_i) \leftarrow \mathrm{Nerve}(\mathcal{U}_{i+1}) \leftarrow \ldots.$$

Then the proper homotopy type of W is independent of the choice of covers, which leads to:

Definition 6. The (reduced) Steenrod homology of Z is defined as the locally finite homology of W:

$$\tilde{H}^{st}_*(Z) := H^{LF}_{*+1}(W).$$

Relative Steenrod homology is defined similarly for compact metrizable pairs. If Z is compact metric space and $U \subset Z$ is open, we define the Steenrod homology of the pair (Z, U) to be the direct limit $\lim \tilde{H}^{st}_*(Z, K)$ where K ranges over compact subsets of U.

If $Z \subset X$ is a $\mathcal{Z}$-set embedding of Z into a compact, finite dimensional, absolute retract X, then $X \setminus Z$ has the same proper homotopy type as the mapping telescope W, and hence $\tilde{H}^{st}(Z) \simeq H^{LF}_{*+1}(X \setminus Z)$. In particular, by applying this definition to the pair (X, Z), where X is a contractible Rips complex of a bounded geometry δ-hyperbolic coarsely connected metric space and $Z = \partial_\infty X$, we compute the Steenrod homology of Z via the locally finite homology of X.

3. Coarse Poincaré duality

We first recall the usual Poincaré duality theorem:

Theorem 7. *Suppose that X is a BGMSC homeomorphic to an n-dimensional manifold. Then for each closed subset $W \subset X$ and $k \in \mathbb{Z}$ there is an isomorphism*

$$P_{W,k} : \check{H}^k_c(W) \to H_{n-k}(X, X \setminus W)$$

which is local in the following sense: For each $\tau \in Z^k_c(W)$ and any open set U containing the support of τ, there is a representative of $P_{W,k}(\tau)$ supported in U ; here $\check{H}^*_c(\cdot)$ denotes Čech cohomology. The family $\{P_{W,k}\}$ is compatible with homomorphisms induced by inclusions.

The *coarse Poincaré duality* is a coarse analogue of the above property. Let $\mathcal{R}$ be a commutative ring with the unit. In the following definition all homology and cohomology groups are taken with coefficients in $\mathcal{R}$.

Definition 8. Let X be a uniformly acyclic BGMSC. We say that X satisfies *coarse n-dimensional Poincaré duality* over $\mathcal{R}$ if the following holds. There is a constant $D \geq 0$ so that for each $k \in \mathbb{Z}$ there is a system of homomorphisms $\{P_K\}$ defined for subcomplexes $K \subset X$:

$$P_K : H_c^k(N_D(K)) \to H_{n-k}(X, X \setminus K),$$

which are compatible with homomorphisms induced by inclusions, and which determine an *approximate isomorphism*[3] in the sense that the homomorphisms α and β in the following commutative diagram are zero:

$$\begin{array}{ccccccc} ker(P_{N_D(K)}) & \to & H_c^k(N_{2D}(K)) & \xrightarrow{P_{N_D(K)}} & H_{n-k}(X, V_D(K)) & \to & coker(P_{N_D(K)}) \\ \alpha \downarrow & & \downarrow & & \downarrow & & \beta \downarrow \\ ker(P_K) & \to & H_c^k(N_D(K)) & \xrightarrow{P_K} & H_{n-k}(X, V_0(K)) & \to & coker(P_K). \end{array} \tag{9}$$

Here and in what follows $V_\rho(K) := \overline{X - N_\rho(K)}$. The homomorphisms P_K are local in the following sense: If $[\sigma] \in H_c^k(N_D(K))$ is represented by a cocycle $\sigma \in Z_c^k(N_D(K))$, then $P_K(\sigma)$ can be represented by a relative cycle τ supported in $N_D(\mathrm{Supp}(\sigma))$.

Note that (9) implies that $H_c^*(X, \mathcal{R}) \simeq H_c^*(\mathbb{R}^n, \mathcal{R})$.

Lemma 10. *Suppose that $n \geq 2$ and X is a BGMSC satisfying coarse n-dimensional Poincaré duality over $\mathcal{R}$. Then X is 1-ended.*

Proof. This follows directly from the fact that $H_c^1(X) = \{0\}$. □

We note that in [12] we have proven a number of coarse versions of Jordan separation theorem for complexes satisfying coarse Poincaré duality. In particular:

Proposition 11. *Suppose that Z is a metric simplicial complex satisfying n-dimensional Poincaré duality, M is a uniformly acyclic BGMSC which is homeomorphic to an n-manifold. Then for each uniform proper map $f : M \to Z$, the image $f(M)$ is a net in Z.*

We next observe that (unlike the usual Poincaré duality) coarse Poincaré duality is a quasi-isometry invariant property:

Lemma 12. *(See* [13]*.) Suppose Y and Y' are quasi-isometric uniformly acyclic BGMSC's. Then Y satisfies coarse n-dimensional Poincaré duality iff Y' does.*

Proof. We first note that if Y satisfies n-dimensional Poincaré duality with the constant D then it also satisfies n-dimensional Poincaré duality with the constant $\tilde{D}$ for any $\tilde{D} \geq D$. Next, the quasi-isometry $f^{(0)} : Y^{(0)} \to Y'^{(0)}$ extends to a Lipschitz mapping $f : Y \to Y'$ which admits a Lipschitz proper homotopy-inverse $\tilde{f}$ such that:

[3]Vanishing of α means the *approximate monomorphism* and vanishing of β means the *approximate epimorphism*.

1. Both f and $\tilde{f}$ are (L, A)-quasi-isometries.

2. The tracks of the homotopies $\tilde{f} \circ f$ and $f \circ \tilde{f}$ have diameter not exceeding certain constant $A' \geq A$.

To construct the duality operators $P'_{K'}$ for $K' \subset Y'$ we take $K := f^{-1}(K')$ and consider the composition $P' = f_* \circ P_{N_R(K)} \circ f^*$:

$$H_c^m(N_{D'}(K')) \xrightarrow{f^*} H_c^m(N_{R+D}(K)) \xrightarrow{P_{N_R(K)}}$$
$$\longrightarrow H_{n-k}(Y, V_0(N_R(K))) \xrightarrow{f_*} H_{n-k}(Y', V_0(K'))$$

for appropriately chosen D' and $R = \frac{D'}{L} - A'$. We have to choose D' so that P' would satisfy the *approximate isomorphism* property. We consider the approximate epimorphism property and leave the approximate monomorphism part to the reader. Consider the following commutative diagram:

$$\begin{array}{ccccccc}
 & & & & & & H_{n-k}(Y', V_{3\rho}(K')) \\
 & & & & & & \downarrow \\
 & & & & H_{n-k}(Y, V_{2d}(K)) & \xrightarrow{f_*} & H_{n-k}(Y', V_{2\rho}(K')) \\
 & & & & \downarrow & & \downarrow \\
 & & H_c^k(N_{2d}(K)) & \xrightarrow{P} & H_{n-k}(Y, V_d(K)) & \xrightarrow{f_*} & H_{n-k}(Y', V_\rho(K')) \\
 & & \downarrow & & \downarrow & & \downarrow \\
H_c^k(N_{3\rho}(K')) & \xrightarrow{f^*} & H_c^k(N_d(K)) & \xrightarrow{P} & H_{n-k}(Y, V_0(K)) & \xrightarrow{f_*} & H_{n-k}(Y', V_0(K'))
\end{array}$$

Here $d \geq D$ is chosen sufficiently large so that

$$A' \leq \rho \leq \frac{d}{L} - A \text{ and } 3\rho \leq \frac{2}{L}(d - 2A').$$

Set $D' = 3\rho$. Consider a homology class $\alpha \in H_{n-k}(Y', V_0(K'))$ which lifts into $\bar{\alpha} \in H_{n-k}(Y', V_{3\rho}(K'))$ with the intermediate lifts $\tilde{\alpha} \in H_{n-k}(Y', V_\rho(K'))$, and $\hat{\alpha} \in H_{n-k}(Y', V_{2\rho}(K'))$. Then using the map $\tilde{f}_*$ one can lift the classes $\tilde{\alpha}, \hat{\alpha}, \alpha$ to classes $\tilde{\beta}, \hat{\beta}, \beta$ which belong to $H_{n-k}(Y, V_{2d}(K))$, $H_{n-k}(Y, V_d(K)), H_{n-k}(Y, V_0(K))$ respectively. Next, use the *approximate epimorphism* property of P to lift $\hat{\beta}$ and β to classes $\hat{\gamma}, \gamma$ in the groups $H_c^k(N_{2d}(K))$, $H_c^k(N_d(K))$ respectively. Finally, using the projection of $\tilde{f}^*(\hat{\gamma})$ to $H_c^k(N_r(K'))$ we lift γ to a class $\epsilon \in H_c^k(N_{3\rho}(K'))$. This establishes the *approximate epimorphism* property for P' with the constant D' above. $\square$

Lemma 13. *(See* [12]*.) The following BGMSC satisfy coarse n-dimensional Poincaré duality:*

1. An acyclic metric simplicial complex X which admits a free, simplicial, cocompact action by a $PD(n)$ group.

2. An n-dimensional, bounded geometry metric simplicial complex X, with an augmentation $\alpha : C_c^n(X) \to \mathbb{Z}$ for the compactly supported simplicial cochain complex, so that $(C_c^*(X), \alpha)$ is uniformly acyclic.

3. A uniformly acyclic, bounded geometry metric simplicial complex X which is a topological n-manifold.

Remark 14. Here an augmentation is a homomorphism α which vanishes on all n-dimensional coboundaries.

Other examples are obtained as total spaces of *coarse fibrations*, see [13].

4. Gromov hyperbolic spaces satisfying coarse Poincaré duality

In this section we prove the main theorem of this paper.

Let $\mathcal{R}$ be a commutative ring with unit. For the rest of this section all (co)homology groups will be with coefficients in $\mathcal{R}$. We recall that a commutative ring $\mathcal{R}$ is called *hereditary* if every ideal $I \subset \mathcal{R}$ is projective as an $\mathcal{R}$-module, see [9]. Every PID and many other rings are hereditary. An example of a commutative ring which is not hereditary is a free Boolean ring on uncountable set of generators, see [1]. (A Boolean ring is a ring of characteristic 2.)

Let X be a Gromov hyperbolic BGMSC satisfying n-dimensional coarse Poincaré duality over $\mathcal{R}$. Since coarse Poincaré duality is quasi-isometry invariant we can replace X with $\mathrm{Rips}_s(X^{(0)})$ (for sufficiently large s): We are primarily interested in the topology of the ideal boundary of X, which of course quasi-isometry invariant. We will retain the notation X for this Rips complex, and let δ denote the hyperbolicity constant of X. We endow $\partial_\infty X$ with a Gromov-type metric as it is done in section 2.

Remark 15. By an argument similar to the one that proves the contractibility of Rips complexes of nets in geodesic Gromov-hyperbolic spaces, one can show that for every C there is an $R_0 = R_0(C)$ such that every C-quasi-convex subcomplex $Y \subset X$, the inclusion $Y \to N_{R_0}(Y)$ is a null-homotopic map and for all $R \geq R_0$ the complex $\mathrm{Rips}_R(Y^{(0)})$ is uniformly contractible with (linear) contractibility function depending only on C and R.

Given a subset $W \subset X \cup \partial_\infty X$, we define the *hull of* W, $\mathrm{Hull}(W)$, to be the union of the geodesics (segments/rays) with (ideal) endpoints in W. Hulls are 10δ-quasi-convex.

Theorem 16 (Linear local (co)acyclicity)**.** *Assume in addition that the ring $\mathcal{R}$ is hereditary. Then $\partial_\infty X$ is linearly acyclic and coacyclic with respect to the Steenrod homology. More precisely: There is a constant $1 > \lambda > 0$ such that for all $\xi \in \partial_\infty X$ and all $r \leq \mathrm{diam}(\partial_\infty X)$, the inclusions $B(\xi, \lambda r) \to B(\xi, r)$ and $\overline{\partial_\infty X \setminus B(\xi, \lambda r)} \leftarrow \overline{\partial_\infty X \setminus B(\xi, r)}$ induce zero in reduced Steenrod homology. The boundary $\partial_\infty X$ is a linearly locally acyclic (Steenrod) homology $(n-1)$-sphere over $\mathcal{R}$.*

Proof. We begin the proof with several technical lemmas.

Following [2], for each C-quasi-convex subset $Y \subset X$ and $R \geq R_0(C)$, we may identify the (reduced) Steenrod homology $\tilde{H}^{st}_*(\partial_\infty Y)$ with the locally finite homology $H^{LF}_{*+1}(\mathrm{Rips}_R(Y^{(0)}))$. This identification is compatible with the inclusions $Y^{(0)} \to Y'^{(0)}$ (i.e. the functors $\tilde{H}^{st}_*(\partial_\infty \cdot)$ and $H^{LF}_{*+1}(\mathrm{Rips}_\cdot(\cdot))$ on the category of quasi-convex subsets of X, are naturally equivalent).

The first lemma relates locally finite homology of Rips complexes of quasi-convex subcomplexes Y in X to the locally finite homology of metric neighborhoods of Y in X. Although these homologies are not isomorphic, they are *approximately isomorphic* in the appropriate sense: If the metric neighborhoods $N_D(Y^{(0)})$ of $Y^{(0)}$ were contractible, we would get an actual isomorphism. We get an *approximate isomorphism* by using uniform contractibility of the ambient space X.

Lemma 17. *For every C, and $R \geq R_0(C)$, there are constants $D_2 = D_2(C)$, $D_3 = D_3(C)$ such that if $Y \subset X$ is a subcomplex so that $Y \subset X$ is C-quasi-convex, then $H^{LF}_*(\mathrm{Rips}_R(Y^{(0)}))$ is canonically isomorphic to*

$$\mathrm{Im}\left(H^{LF}_*(N_{D_2}(Y)) \to H^{LF}_*(N_{D'_3}(Y))\right)$$

for any $D'_3 \geq D_3$. These isomorphisms are compatible with inclusions, i.e. if $Y \subset Y'$ are both C-quasi-convex, then the isomorphisms are compatible with the induced homomorphisms $H^{LF}_*(\mathrm{Rips}_R(Y^{(0)})) \to H^{LF}_*(\mathrm{Rips}_R(Y'^{(0)}))$ and

$$\mathrm{Im}\left(H^{LF}_*(N_{D_2}(Y)) \to H^{LF}_*(N_{D'_3}(Y))\right) \to$$
$$\to \mathrm{Im}\left(H^{LF}_*(N_{D_2}(Y')) \to H^{LF}_*(N_{D'_3}(Y'))\right).$$

Proof. Using the uniform contractibility and the finite dimensionality of X, we get an L-Lipschitz map $i : \mathrm{Rips}_R(Y^{(0)}) \to X$ extending the inclusion

$$\mathrm{Rips}_R(Y^{(0)}) \supset Y^{(0)} \cap X^{(0)} \to X,$$

where $L = L(C, R)$. Similarly, extending a nearest point projection $X^{(0)} \to Y^{(0)}$, we get a map $r : X \to \mathrm{Rips}_R(Y)$ which for each t restricts to an L'-Lipschitz map on $N_t(Y)$, where $L' = L'(C, R, t)$, and which satisfies $d(i \circ r(x), x) \leq C(1 + d(x, Y^{(0)}))$. Using uniform contractibility again, we find a homotopy from $r \circ i$ to $\mathrm{id}_{\mathrm{Rips}_R(Y^{(0)})}$ with tracks of diameter $\leq D = D(C, R)$ and a homotopy $i \circ r \sim \mathrm{id}_X$ whose track at $x \in X$ has diameter bounded by $\phi(d(x, Y^{(0)}))$, where $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ depends only on C and R. The lemma follows. $\square$

Suppose $\xi \in \partial_\infty X$, and $\gamma : \mathbb{R}_+ \to X$ is a unit speed geodesic ray asymptotic to ξ. Define $\rho : X \to \mathbb{R}_+$ so that $\gamma \circ \rho : X \to X$ is a nearest point projection to $\mathrm{Im}(\gamma)$.

For $t \in \mathbb{R}_+$ set $\hat{U}_t := \rho^{-1}([t, \infty))$, $\hat{V}_t := \rho^{-1}([0, t])$, $U_t := \overline{\mathrm{Hull}(\hat{U}_t)}$, and $V_t := \overline{\mathrm{Hull}(\hat{V}_t)}$.

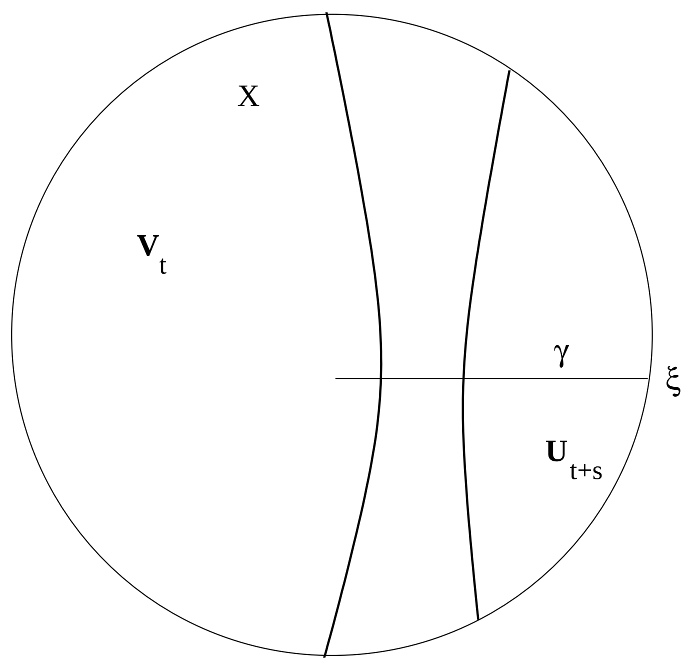

FIGURE 1. *Sets U_{t+s} and V_t.*

Loosely speaking, the subsets U_t and V_t are analogues of geodesic half-spaces in $\mathbb{H}^n$ and of their complements. Such spaces (in the case of $\mathbb{H}^n$) are convex, so taking convex hulls is unnecessary in this situation. In the case of $\mathbb{H}^n$, the ideal boundaries of U_t and V_t are complementary round balls in $S^{n-1} = \partial_\infty \mathbb{H}^n$. The next lemma shows that, to some degree, the same holds for arbitrary Gromov-hyperbolic spaces.

Lemma 18. *There are constants a, b, c and a function $L : \mathbb{R}_+ \to \mathbb{R}_+$ independent of ξ, such that $\lim_{s\to\infty} L(s) = \infty$ and for all $t, s \in \mathbb{R}_+$:*

1. $d(U_{t+s}, V_t) \geq L(s)$.
2. $X \setminus V_{t+c} \subset U_t$ *and* $U_{t+c} \subset X \setminus V_t$.
3. *If $\gamma(0) = \star$ (the base-point in X used to define the metric on $\partial_\infty X$), then*

$$B(\xi, \frac{1}{b}e^{-at}) \subset \partial_\infty U_t \subset B(\xi, be^{-at})$$

and

$$X \setminus B(\xi, be^{-at}) \subset \partial_\infty V_t \subset X \setminus B(\xi, \frac{1}{b}e^{-at}).$$

Proof. Pick u and v such that $100\delta \leq u \leq v \leq \infty$. Applying the δ-thinness condition repeatedly, one gets $\mathrm{Hull}(\rho^{-1}([u, v])) \subset \rho^{-1}([u - 100\delta, v + 100\delta])$. With this observation, 1 and 2 follow from the properties of the nearest-point projection in the hyperbolic spaces, see for instance [8, Lemma 8.4.24]. Claim 3 follows directly from the definition of the Gromov type metric on $\partial_\infty X$. $\square$

Proof of Theorem 16. In view of Lemma 18 (part 3), it suffices to prove that there is a constant c_0 such that for all unit speed rays γ and all $t \in \mathbb{R}_+$, the inclusions $\partial_\infty U_{t+c_0} \to \partial_\infty U_t$ and $\partial_\infty V_t \to \partial_\infty V_{t+c_0}$ induce zero on reduced Steenrod homology. Lemma 17 and the remark preceding it, show that we need only prove that when c_0 is sufficiently large, the inclusions $N_{D_3}(U_{t+c_0}) \to N_{D_3}(U_t)$ and $N_{D_3}(V_t) \to N_{D_3}(V_{t+c_0})$ induce zero on locally finite reduced homology. By the universal coefficient theorem [9] applied to the compactly supported cochain complexes, cf. [2, Remark 1.9], it suffices to show that when c_0 is sufficiently large these inclusions induce zero on the compactly supported cohomology.

Recall that for $C = 10\delta$, the sets U_t and V_t are C-quasi-convex for all t; pick $D_3' \geq \max(D, D_3(C))$ where D is the constant from the Definition 8 and $D_3 = D_3(C)$ is the function from Lemma 17. Let c be a constant as in Lemma 18 and $R_0 = R_0(C)$ be as in Remark 15. By Lemma 18 (parts 1 and 2), for $t \in \mathbb{R}_+$ and c' sufficiently large, we have the inclusions

$$N_{D_3+2D}(U_{t+c'}) \subset X \setminus (N_{R_0}(V_{t+c})) \subset X \setminus V_{t+c} \subset U_t \subset N_{D_3}(U_t).$$

The inclusion $V_{t+c} \to N_{R_0}(V_{t+c})$ induces zero on the reduced homology since V_{t+c} is C-quasi-convex (see Remark 15). Therefore the composition of coarse Poincaré duality

$$P_{D_3(U_t)} : H_c^*(N_{D_3}(U_t)) \to H_{n-*}(X, \overline{X \setminus U_t})$$

with the map

$$H_{n-*}(X, \overline{X \setminus U_t}) \overset{\approx}{\to} \tilde{H}_{n-*-1}(\overline{X \setminus U_t}) \to \tilde{H}_{n-*-1}(X \setminus N_{D_3+D}(U_{t+c'}))$$

is zero. But this composition is the same as the composition of the restriction

$$H_c^*(N_{D_3}(U_t)) \to H_c^*(N_{D_3+2D}(U_{t+c'}))$$

with the coarse Poincaré duality

$$H_c^*(N_{D_3+2D}(U_{t+c'})) \to H_{n-*}(X, \overline{X \setminus N_{D_3+D}(U_{t+c'})}) \overset{\approx}{\to} \tilde{H}_{n-*-1}(\overline{X \setminus N_{D_3+D}(U_{t+c'})}).$$

By the *approximate monomorphism* part of the Definition 8, we have:

$$\begin{aligned} &\operatorname{Ker}\Big(H_c^*(N_{D_3+2D}(U_{t+c'})) \to \tilde{H}_{n-*-1}(\overline{X \setminus N_{D_3+D}(U_{t+c'})})\Big) \\ &\qquad \subset \operatorname{Ker}\Big(H_c^*(N_{D_3+2D}(U_{t+c'})) \to H_c^*(N_{D_3}(U_{t+c'}))\Big), \end{aligned}$$

which implies that the composition of the restriction homomorphisms

$$H_c^*(N_{D_3}(U_t)) \to H_c^*(N_{D_3+2D}(U_{t+c'})) \to H_c^*(N_{D_3}(U_{t+c'}))$$

is zero. Thus the inclusions $N_{D_3}(U_{t+c'}) \to N_{D_3}(U_t)$ induce zero on the locally finite reduced homology groups and hence the maps $\tilde{H}_*^{st}(\partial_\infty U_{t+c'}) \to \tilde{H}_*^{st}(\partial_\infty U_t)$ are zero.

The proof that $\tilde{H}^{st}_*(\partial_\infty V_t) \to \tilde{H}^{st}_*(\partial_\infty V_{t+c'})$ is zero is similar. Thus we have proved that $\partial_\infty X$ is linearly locally acyclic and coacyclic (with respect to Steenrod homology).

We note now that $H^{st}_*(\partial_\infty X) \cong H^{LF}_{*+1}(X) \cong H_{*+1}(\mathbb{R}^n) \cong H_*(S^n)$. Hence $\partial_\infty X$ is a homology $(n-1)$-manifold over $\mathcal{R}$ which is also a homology $(n-1)$-sphere over $\mathcal{R}$. This concludes the proof of Theorem 16. $\square$

In the proof of Theorem 16, the assumption that $\mathcal{R}$ was hereditary was only used when we used the universal coefficient theorem to relate locally finite homology with compactly supported cohomology. Hence we get the following variant of the above theorem for arbitrary rings $\mathcal{R}$:

Theorem 19 (Linear local (co)acyclicity for Čech cohomology)**.** *There is a constant $1 > \lambda > 0$ such that for all $\xi \in \partial_\infty X$ and all $r < \mathrm{diam}(\partial_\infty X)$, the inclusions $B(\xi, \lambda r) \to B(\xi, r)$ and $\overline{\partial_\infty X \setminus B(\xi, \lambda r)} \leftarrow \overline{\partial_\infty X \setminus B(\xi, r)}$ induce zero in reduced Čech cohomology. The boundary $\partial_\infty X$ has the same Čech cohomology over $\mathcal{R}$ as the $(n-1)$-sphere.*

Proof. Following [3], when $Y \subset X$ is C-quasi-convex and $R \geq R_0(C)$, we may identify the reduced Čech cohomology $\tilde{H}^*_c(\partial_\infty Y)$ with the compactly supported cohomology $H^{*+1}_c(\mathrm{Rips}_R(Y^{(0)}))$. (This follows from tautness of Čech cohomology for the subset $\partial_\infty Y \subset \overline{\mathrm{Rips}_R(Y^{(0)})}$, and excision.) This identification is compatible with the inclusions $Y^{(0)} \to Y'^{(0)}$.

With this identification, we can repeat the argument of Proposition 16, replacing locally finite homology with compactly supported cohomology everywhere. $\square$

Corollary 20. *Let X be a Gromov hyperbolic BGMSC.*

1. (Cf. [4]) If X satisfies coarse 2-dimensional Poincaré duality over a ring $\mathcal{R}$, then X is quasi-isometric to $\mathbb{H}^2$.

2. If X satisfies coarse 3-dimensional Poincaré duality over a hereditary ring $\mathcal{R}$, then the boundary $\partial_\infty X$ is a linearly locally contractible 2-sphere.

Proof. 1. We first show that $Z := \partial_\infty X$ is homeomorphic to S^1. We will do this by proving that Z is connected, has no cut points, and is separated by any pair of distinct points, see [23]. Note that Z is connected since $\check{H}^0(Z, \mathcal{R}) \simeq H^0(S^1, \mathcal{R})$. The co-acyclicity implies that there are no cut points [4]:

Given a point $\xi \in Z$ consider the exhaustion of its complement $Z \setminus \{\xi\}$ by the subsets $\overline{Z \setminus B(\xi, r)}$, $r > 0$. By coacyclicity of Z we have zero restriction maps

$$\tilde{H}^0\left(\overline{Z \setminus B(\xi, \lambda r)}, \mathcal{R}\right) \to \tilde{H}^0\left(\overline{Z \setminus B(\xi, r)}, \mathcal{R}\right).$$

[4] When X is quasi-isometric to a group, the fact that Z is locally connected and has no cut points follows from [5, 20]; however their argument doesn't apply when there is nondiscrete cocompact quasi-action.

By taking the inverse limit we conclude that $\tilde{H}^0(Z \setminus \{\xi\}, \mathcal{R}) = 0$ and thus $Z \setminus \{\xi\}$ is connected.

It remains to show that every pair of points separates.

First consider the case when $\mathcal{R}$ is hereditary. Then $\partial_\infty X \setminus \{p\}$ has trivial Steenrod homology for any $p \in \partial_\infty X$. Hence the Meyer-Vietoris sequence shows that each pair of points separates $\partial_\infty X$ into precisely two components. In the general case, let $p, p' \in \partial_\infty X$ be distinct points. By Theorem 19, we can exhaust $\partial_\infty X \setminus \{p\}$ (resp. $\partial_\infty X \setminus \{p'\}$) by a nested sequence of open sets $\{U_i\}$ (resp. $\{U_i'\}$) such that the inclusions $U_i \to U_{i+1}$ (resp. $U_i' \to U_{i+1}'$) induce zero on reduced Čech cohomology. For large i, we obtain a sequence of coverings (U_i, U_i') of $\partial_\infty X$. The nested family $\{U_i \cap U_i'\}$ of intersections exhausts $\partial_\infty X \setminus \{p, p'\}$. Applying Meyer-Vietoris sequence, we obtain a compatible system of monomorphisms $\mathcal{R} \simeq \check{H}^1(\partial_\infty X, \mathcal{R}) \to \widetilde{\check{H}^0}(U_i \cap U_i', \mathcal{R})$. Since $\cup_i U_i \cap U_i' = \partial_\infty X \setminus \{p, p'\}$ can be exhausted by compact sets, this implies that $\partial_\infty X \setminus \{p, p'\}$ is not connected.

Hence $\partial_\infty X = \partial_\infty X$ is a topological circle which is linearly locally connected. Thus $\partial_\infty X$ is quasisymmetrically homeomorphic to the standard circle, see [21]. Then [19] implies that there is a quasiconvex subcomplex $Z \subset X$ which is quasi-isometric to $\mathbb{H}^2$. It then follows from Proposition 11 that $Z^{(0)}$ is a net in $X^{(0)}$. This proves (1).

2. Note that by Proposition 16, over a hereditary ring $\mathcal{R}$, the compact $\partial_\infty X$ is a homology 2-manifold with the homology of a 2-sphere (with respect to Steenrod homology). Thus, $\partial_\infty X$ is connected, locally connected and has no global cut-points. In particular, $\partial_\infty X$ is not a dendroid and therefore contains an embedded topological circle. To conclude that $\partial_\infty X$ is homeomorphic to S^2 it remains to show that each embedded topological circle $S \subset \partial_\infty X$, separates $\partial_\infty X$, see [22], i.e. $\partial_\infty X$ satisfies Jordan curve separation theorem. To prove the latter we repeat the proof of Jordan separation theorem given in [16, Ch. III, Corollary 6.4], using Steenrod homology with $\mathcal{R}$-coefficients instead of the singular homology with the integer coefficients. The inductive proof of [16, Ch. III, Corollary 6.4] is based on vanishing of $\tilde{H}_*(\partial_\infty X \setminus p)$ for each point $p \in \partial_\infty X$. In our case this property is satisfied since $\partial_\infty X$ is a Steenrod homology sphere over $\mathcal{R}$. $\square$

Remark 21. In the case when $X^{(1)}$ is a planar graph, the part 1 of the above corollary was proven in [6].

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