Quasi-isometries and the de Rham decomposition

Michael Kapovich,* Bruce Kleiner† and Bernhard Leeb‡

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Abstract

We study quasi-isometries $\Phi : \prod X_i \to \prod Y_j$ of product spaces and find conditions on the X_i, Y_j which guarantee that the product structure is preserved. The main result applies to universal covers of compact Riemannian manifolds with nonpositive sectional curvature. We introduce a quasi-isometry invariant notion of coarse rank for metric spaces which coincides with the geometric rank for universal covers of closed nonpositively curved manifolds. This shows that the geometric rank is a quasi-isometry invariant.

1 Introduction

In this paper we will prove that under suitable assumptions, quasi-isometries preserve product structure. Earlier papers have considered – either implicitly or explicitly – the problem of showing that quasi-isometries preserve prominent geometric structure: [7, 9] show that a natural decomposition of the universal cover of certain Haken manifolds is preserved by any quasi-isometry; [14] uses coarse topology to prove that boundary components are preserved; [10] shows that quasi-isometries of symmetric spaces and Euclidean buildings preserve maximal flats; [12, 10] show that quasi-isometries are equivalent to isometries. It is known [15, 6] that splittability over finite groups is a quasi-isometry invariant property of finitely generated groups; and [3] proves quasi-isometry invariance of splitting of 1-ended hyperbolic groups over virtually $\mathbb{Z}$ subgroups.

We first formulate a version of our main result for Riemannian manifolds with nonpositive sectional curvature:

Theorem A *Suppose M, N are closed nonpositively curved Riemannian manifolds, and consider the de Rham decompositions of their universal covers $\tilde{M} = \mathbb{E}^m \times \prod_{i=1}^k M_i$, $\tilde{N} = \mathbb{E}^n \times \prod_{i=1}^\ell N_i$. Then for every $L \geq 1, A \geq 0$ there is a constant D so that for each (L, A)-quasi-isometry $\phi : \tilde{M} \to \tilde{N}$ we have: $k = \ell$, $m = n$ and after reindexing the factors N_j there are quasi-isometries $\phi_i : M_i \to N_i$ such that for*

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every i the diagram

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\phi} & \tilde{N} \\ \downarrow & & \downarrow \\ M_i & \xrightarrow{\phi_i} & N_i \end{array}$$

commutes up to error at most D.

Remark 1.1 *The first paper to consider the effect of quasi-isometries on product structure was [13], which studied quasi-isometries of* $\mathbb{H}^2 \times \mathbb{R}$.

Another goal of this paper is to prove that two closed nonpositively curved manifolds with quasi-isometric universal covers have the same geometric rank (see [1, p. 73] for the definition). To prove this we define a quasi-isometry invariant for general metric spaces, and then verify that it coincides with the geometric rank in the case of universal covers of closed nonpositively curved manifolds.

Definition 1.2 *If X is a topological space, then the* **topological rank** *of X is*

$$trank(X) := \inf\{k \mid \exists p \in X \text{ so that } H_k(X, X - \{p\}) \neq \{0\}\}.$$

If M is a metric space, then the **coarse rank** *of M is*

$$crank(M) := \inf\{trank(X_\omega) \mid X_\omega \text{ is an asymptotic cone of } X\}.$$

A quasi-isometry $M \to M'$ induces bi-Lipschitz homeomorphisms between corresponding asymptotic cones, so the coarse rank is manifestly a quasi-isometry invariant.

Theorem 1.3 *If $\tilde{M}$ is the universal cover of a closed nonpositively curved manifold*[1] *then the coarse rank of $\tilde{M}$ coincides with the geometric rank of $\tilde{M}$. In particular if two closed nonpositively curved Riemannian manifolds have quasi-isometric universal covers, then they have the same geometric rank.*

The proof relies on the structure theory of nonpositively curved manifolds [1] and computations of local homology groups of asymptotic cones of Hadamard spaces of rank 1 and of Euclidean buildings.

Theorem A is a consequence of a more general fact, see section 3 for necessary definitions:

Theorem B *Suppose $M = Z \times \prod_{i=1}^{k} M_i$ and $N = W \times \prod_{i=1}^{\ell} N_i$ are geodesic metric spaces such that the asymptotic cones of Z and W are homeomorphic to $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and the components M_i, N_j are of* **coarse type**[2] *I and II. Then for every $L \geq 1, A \geq 0$ there is a constant D so that for each (L, A)-quasi-isometry $\phi : M \to N$*

[1] Or a piecewise Riemannian 2-complex with nonpositive curvature admitting a discrete cocompact isometric action, [2].

[2] See definition 3.5.

we have: $k = \ell$, $n = m$ *and after reindexing the factors* N_j *there are quasi-isometries* $\phi_i : M_i \to N_i$ *such that for every* i *the diagram*

$$\begin{array}{ccc} M & \xrightarrow{\phi} & N \\ \downarrow & & \downarrow \\ M_i & \xrightarrow{\phi_i} & N_i \end{array}$$

commutes up to error at most D.

Remark 1.4 *If* $n = m \geq 1$ *then in the theorems above one cannot assert that the quasi-isometry* ϕ *is uniformly close to a product of quasi-isometries.*

The class of metric spaces of coarse type I and II contains universal covers of compact nonpositively curved Riemannian manifolds (which are irreducible and nonflat), Euclidean buildings, certain piecewise Euclidean complexes with nonpositive curvature [2], δ-hyperbolic metric spaces, nonuniform lattices in rank 1 Lie groups, etc. The factors Z, W may be simply connected nilpotent Lie groups with left invariant metrics [11]. The universal covers of closed Riemannian manifolds of nonpositive curvature satisfy the assumptions of Theorem B, see proposition 4.7.

We prove theorem B by using the topology of asymptotic cones of M and N. An important step is the following topological analogue:

Theorem C *Suppose that* X_i, Y_j *is a collection of geodesic metric spaces of types*[3] *I and II. Let* $X := \mathbb{R}^n \times \prod_{i=1}^{k} X_i$ *and* $Y := \mathbb{R}^m \times \prod_{j=1}^{\ell} Y_j$. *Suppose* $f : X \to Y$ *is a homeomorphism. Then* $\ell = k$, $m = n$ *and after reindexing the factors* Y_j *there are homeomorphisms* $f_i : X_i \to Y_i$ *so that the following diagram commutes for every* i*:*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ X_i & \xrightarrow{f_i} & Y_i \end{array}$$

The proof of theorem C makes use of topologically defined decompositions of X and Y into products of metric trees and Euclidean buildings. We show that f must respect these decompositions, and then we reduce theorem C to topological rigidity of homeomorphisms between products of Euclidean buildings, see [10]. Theorem C implies that the homeomorphism of asymptotic cones induced by the mapping ϕ (in Theorem B) must respect the product structure. The proof of the implication (Theorem C $\Longrightarrow$ Theorem B) requires only a certain "nontranslatability" property of the factors M_i and N_j, see definition 2.3 and propositions 2.6, 2.8. If a pair of metric spaces (X, Y) is nontranslatable, then there is a function $D(L, A)$ so that for any pair of (L, A)-quasi-isometries $f, g : X \to Y$ which are within finite distance (in the sup-metric) from each other, we have: the distance between f and g is at most $D(L, A)$. This property is interesting by itself: it allows one to define ineffective kernels of quasi-actions[4] in a natural way. Our results imply that the universal cover

[3] See definitions 3.3 and 3.4.

[4] An (L, A) quasi-action of a group Γ on a metric space X is a map $\rho : \Gamma \times X \to X$ so that $\rho(g, \cdot) : X \to X$ is an (L, A) quasi-isometry for every $g \in \Gamma$, $d(\rho(g_1, \rho(g_2, x)), \rho(g_1 g_2, x)) < A$ for every $g_1, g_2 \in \Gamma$, $x \in X$, and $d(\rho(e, x), x) < A$ for every $x \in X$.

of any closed nonpositively curved Riemannian manifold, which doesn't not have flat factors, is nontranslatable.

The proofs of theorems A, B and 1.3 are in section 6. Theorem C is proven in section 5.

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Contents

2 Quasi-isometries, asymptotic cones and product structures

In this section, we recall some basic definitions. We also give conditions which allow us to derive product splitting theorems for quasi-isometries from splitting theorems for asymptotic cones.

Definition 2.1 *A map $f : X \to Y$ between two metric spaces is (L, A)-***Lipschitz** *if $L \geq 1, A \geq 0$ and:*

$$d(f(x), f(x')) \leq Ld(x, x') + A$$

for all $x, x' \in X$. A map is **coarse Lipschitz** *if it is (L, A)-Lipschitz for some $L, A \in \mathbb{R}$. Note that coarse Lipschitz maps needn't be continuous. A coarse Lipschitz mapping f is an (L, A)-***quasi-isometric embedding** *if it is (L, A)-Lipschitz and*

$$d(f(x), f(x')) \geq L^{-1}d(x, x') - A$$

*for all $x, x' \in X$. Finally, f is an (L, A)-***quasi-isometry** *if it is an (L, A)-quasi-isometric embedding and the space Y lies in the A-neighborhood of the image of f.*

In this paper we will use ultralimits and asymptotic cones of metric spaces and maps between them. We refer to [7] and [10] for precise definitions. We recall however that ultralimits and asymptotic cones depend on the choice of: (1) an ultrafilter, (2) a sequence of base-points, (3) a sequence of scale factors. When referring to an *asymptotic cone* of a metric space X we will mean an asymptotic cone defined using a suitable choice of these data. We shall use the notation X_ω for an asymptotic cone of the metric space X and f_ω for an ultralimit of the mapping f between two metric spaces.

Definition 2.2 *Let (X, Y) be a pair of metric spaces. We say that (X, Y) is* **topologically nontranslatable** *if any two homeomorphisms $X \to Y$ at finite distance (with respect to the sup-metric) coincide. Note that this condition is vacuous unless X and Y are homeomorphic.*

Definition 2.3 *Let (X, Y) be a pair of metric spaces. (X, Y) is* **nontranslatable** *if for every pair of asymptotic cones X_ω, Y_ω, the pair (X_ω, Y_ω) is topologically nontranslatable.*

Lemma 2.4 *Let (X, Y) be a nontranslatable pair of metric spaces. Then there is a function $D(L, A)$ such that for any $L \geq 1$, $A \geq 0$, any two (L, A)-quasi-isometries $X \to Y$ at finite distance have distance $\leq D(L, A)$.*

Proof. Otherwise there are pairs of (L, A) quasi-isometries $\phi_i, \psi_i : X \longrightarrow Y$ with $d(\phi_i, \psi_i) = D_i \to \infty$. Choose points $\star_i \in X$ with $y_i = \phi_i(\star_i), z_i = \psi_i(\star_i)$ such that

$$D_i \geq d(y_i, z_i) \geq \frac{D_i}{2} \quad ,$$

an ultrafilter ω, and scale factors D_i^{-1}. We get the sequence of quasi-isometries

$$\phi_i : (D_i^{-1}X, \star_i) \longrightarrow (D_i^{-1}Y, y_i), \;\; \psi_i : (D_i^{-1}X, \star_i) \longrightarrow (D_i^{-1}Y, z_i)$$

Passing to the ultralimit we obtain two different homeomorphisms

$$\phi_\omega, \psi_\omega : X_\omega \longrightarrow Y_\omega$$

at distance ≤ 1, contradicting the fact that X_ω and Y_ω are nontranslatable. □

Lemma 2.5 *Suppose that X, Y are metric spaces and $h : X \to Y$ is a coarse Lipschitz mapping. Then h is a quasi-isometric embedding iff every ultralimit $h_\omega : X_\omega \to Y_\omega$ of h is injective.*

Proof. The implication (quasi-isometric embedding ⇒ coarse Lipschitz map) is clear, so we prove the converse. We need to prove that for any sequence $x_n, x_n' \in X$ such that $d(x_n, x_n') \to \infty$ the limit

$$\lim_{n\to\infty} \frac{d(h(x_n), h(x_n'))}{d(x_n, x_n')} > 0$$

If not, then there is a sequence so that the limit is equal to zero. Set $y_n := h(y_n)$. Let $r_n := d(x_n, x'_n)$, consider the sequence

$$h : (r_n^{-1}X, x_n) \to (r_n^{-1}Y, y_n)$$

The ultralimit of this sequence of mappings is a map

$$h_\omega : X_\omega \to Y_\omega$$

and $h_\omega(x_\omega) = h_\omega(x'_\omega)$, but $x_\omega \neq x'_\omega$. This contradicts the assumption that h_ω is injective. □

Consider direct products[5] of geodesic metric spaces $X := \prod_{i=1}^m X_i$, $Y := \prod_{j=1}^m Y_j$. We shall denote by π_{X_i}, π_{Y_j} the projections from X, Y to the factors X_i, Y_j respectively. The following two theorems provide the main tool for proving quasi-isometry invariance of product decompositions of geodesic metric spaces.

Proposition 2.6 *Suppose $\phi : X = \prod_{i=1}^m X_i \to Y = \prod_{j=1}^m Y_j$ is an (L, A)-quasi-isometry such that*

(a) All ultralimits ϕ_ω of ϕ preserve the product structures of the asymptotic cones of X and Y;

(b) Each pair (X_i, Y_j) is nontranslatable.

Then there is a function $D(L, A)$ so that the mapping ϕ is at distance $< D(L, A)$ from a product of (L, A')-quasi-isometries, where A' depends only on (L, A).

Proof. We call a pair of points $x, x' \in X$ *i-horizontal* iff $\pi_{X_k}(x) = \pi_{X_k}(x')$ for all $k \neq i$. Fix $L_1 > L$ and $\epsilon \in (0, L_1^{-1})$. We call the i-horizontal pair (x, x') *j-compressed* if

$$\frac{d(\pi_{Y_j}(\phi(x)), \pi_{Y_j}(\phi(x')))}{d(x, x')} < \epsilon$$

and j-uncompressed if

$$L_1^{-1} < \frac{d(\pi_{Y_j}(\phi(x)), \pi_{Y_j}(\phi(x')))}{d(x, x')} < L_1.$$

If d is a positive number, Z is a metric space, and $z, z' \in Z$, then z, z' are called **d-separated** if $d(z, z') \geq d$.

Lemma 2.7 *There exists d_0 such that for all i, j either all d_0-separated i-horizontal pairs are j-compressed or all d_0-separated i-horizontal pairs are j-uncompressed.*

Proof. We first observe that there is a positive number d_0 such that for all $d \geq d_0$ and every $10d$-metric ball $B \subset X$ we have:

All d-separated i-horizontal pairs $x, x' \in B$ are simultaneously either j-compressed or j-uncompressed.

Indeed, otherwise we could find a sequence $d_k \to \infty$, and balls $B_{*_k}(d_k)$ which contain a d_k-separated i-horizontal pair which is j-compressed and a d_k-separated i-horizontal pair which is j-uncompressed. Then the ultralimit of the sequence

$$\pi_{Y_j} \circ \phi : d_k^{-1}X \longrightarrow d_k^{-1}Y_j$$

[5] The distance for points in the product space is given by the Pythagorean formula.

with the basepoint $*_k \in d_k^{-1}X$ is the mapping

$$\phi_\omega : X_\omega \to Y_{j\omega}$$

which is neither L-biLipschitz nor constant on the intersection of the unit ball B_ω with an i-horizontal copy of $X_{i\omega}$. This contradicts assumptions of the proposition.

Now pick two i-horizontal d_0-separated pairs x, x' and y, y' in X. Since X is a geodesic metric space, we can find a chain of d_0-separated i-horizontal pairs connecting $\overline{xx'}$ to $\overline{yy'}$ with the following property: any two successive pairs w, w' and z, z' are contained in a ball of radius $10\min(d(w,w'), d(z,z'))$. Hence the pairs x, x' and y, y' are either both i-compressed or both i-uncompressed. $\square$

¿From the above lemma and the fact that all ultralimits of ϕ respect the product structure of X_ω, Y_ω, we see that for each i there is a unique j so that all d_0-separated i-horizontal pairs (x, x') are j-uncompressed. We can reindex the factors Y_j so that for all i every d_0-separated i-horizontal pair is i-uncompressed and j-compressed for every $j \neq i$. Hence the family of maps

$$\{\phi_{x,i} | x \in \prod_{k \neq i} X_k\}, \quad \phi_{x,i} : X_i \longrightarrow Y_i$$

given by

$$\phi_{x,i}(\bar{x}) = \pi_{Y_i}(\phi(x_1, \dots, x_{i-1}, \bar{x}, x_{i+1}, \dots, x_m))$$

consists of quasi-isometries with uniform constants at pairwise finite distance. By assumption, the pair (X_i, Y_i) is non-translatable and hence the quasi-isometries $\phi_{x,i}$ for fixed i have uniformly bounded distance from one quasi-isometry $\phi_i : X_i \longrightarrow Y_i$ by lemma 2.4. Therefore the product quasi-isometry $\prod \phi_i$ lies at bounded distance from ϕ. The distance between these quasi-isometries is uniformly bounded in terms of (L, A) because each pair (X_i, Y_j) is nontranslatable. $\square$

We will need a modified version of the above result.

Proposition 2.8 *Consider metric products $X = \bar{X} \times Z, Y = \bar{Y} \times W$ of geodesic metric spaces. We assume that: (a) the pair $(\bar{X}, \bar{Y})$ is nontranslatable and (b) that $\phi : X \to Y$ is a (L, A)-quasi-isometry such that each ultralimit ϕ_ω of ϕ preserves the decompositions of $\bar{X}_\omega \times Z_\omega$ and $\bar{Y}_\omega \times W_\omega$ by the Z_ω, W_ω-factors, i.e. there exists a homeomorphism $\psi_\omega : \bar{X}_\omega \to \bar{Y}_\omega$ such that the diagram*

$$\begin{array}{ccc} X_\omega & \xrightarrow{\phi_\omega} & Y_\omega \\ \downarrow & & \downarrow \\ \bar{X}_\omega & \xrightarrow{\psi_\omega} & \bar{Y}_\omega \end{array}$$

commutes. Then there is a quasi-isometry $\bar{\phi} : \bar{X} \to \bar{Y}$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{\phi} & Y \\ \downarrow & & \downarrow \\ \bar{X} & \xrightarrow{\bar{\phi}} & \bar{Y} \end{array}$$

commutes up to a finite error bounded in terms of (L, A).

Proof. For each $z \in Z$ we consider the mappings

$$\phi_z : \bar{X} \longrightarrow \bar{Y}, \phi_z(\bar{x}) = \pi_{\bar{Y}} \circ \phi(\bar{x}, z)$$

All ultralimits of ϕ_z are L-bi-Lipschitz homeomorphisms $\bar{X}_\omega \longrightarrow \bar{Y}_\omega$. By repeating the arguments of proposition 2.6 the family $\{\phi_z : z \in Z\}$ consists of quasi-isometries with uniform constants. Since the pair $(\bar{X}, \bar{Y})$ is nontranslatable we conclude that all the mappings ϕ_z are within a uniformly bounded distance from a quasi-isometry $\bar{\phi} : \bar{X} \longrightarrow \bar{Y}$. □

3 Tree-like decompositions of metric spaces and topology of irreducible buildings

The goal of this section is to study the topology of factors of the space $\bar{X}$ of theorem C. We introduce two types of geodesic metric spaces and establish the nontranslatability properties for these spaces which will be used in proving theorem C. In the end of the section we prove a vanishing theorem for local homology groups of type II spaces.

Let X be a geodesic metric space[6]. We decompose X into maximal "immovable" trees as follows: say that the points $p, q \in X$ satisfy the relation $p \sim q$ if and only if there is a continuous path $\gamma : I \longrightarrow X$ from p to q whose image is contained in the image of any other continuous path from p to q. This decomposition is *topologically invariant.* If $p, q \in X$, $p \sim q$, and γ is as above, then $Im(\gamma)$ is contained in any geodesic segment $\overline{pq}$ from p to q; hence by the connectedness of $Im(\gamma)$ we have $Im(\gamma) = \overline{pq}$. It follows that there is a unique arc joining p to q, and a unique geodesic segment from p to q. If $p \sim q$ and $s \in \overline{pq}$, then clearly $s \sim p$ and $s \sim q$. Also, if $p \sim q$ and $p' \sim q'$, then $\overline{qp} \cap \overline{p'q'}$ is a closed subsegment of $\overline{pq}$ and $\overline{p'q'}$.

Lemma 3.1 *$\sim$ is an equivalence relation. The equivalence classes are metric trees.*

Proof. The relation $\sim$ is obviously reflexive and symmetric.

To show that $\sim$ is also transitive, assume that $p \sim q$ and $q \sim r$. Then $\overline{qs} = \overline{pq} \cap \overline{qr}$ for some $s \in \overline{pq} \cap \overline{qr}$; so $\overline{ps} \cup \overline{sr}$ is an arc joining p to r. The image of any path γ from p to r must contain this arc, for otherwise γ will contradict $s \sim p$, $s \sim r$. So $p \sim r$. □

We will denote the decomposition of X into $\sim$-equivalence classes by $\mathcal{D}(X)$ and refer to the cosets as *leaves.*

Lemma 3.2 *If X is a geodesic metric space then the leaves of $\mathcal{D}(X)$ are closed convex subsets of X.*

Proof. It is obvious that the equivalence classes are convex. Let $T \subset X$ be the leaf of $p \in X$, and let q be a point on the frontier of T. If $p_i \in T$ and $\lim_{i\to\infty} p_i = q$, then $\lim_{k,l\to\infty} Diam(\overline{p_k q} \Delta \overline{p_l q}) \to 0$ (where Δ denotes the symmetric difference). If γ is a path from p to q, $Im(\gamma)$ must contain $\overline{pp_i} \setminus \overline{qp_i}$; in particular $\overline{pp_i} \setminus \overline{qp_i} \subset \overline{pq}$ for

[6] Our definitions and proofs actually apply to a more general situation, namely when X is a topological space where any two points can be connected by a topologically embedded interval.

every geodesic segment $\overline{pq}$ from p to q. This implies $\gamma \supset \cup_i(\overline{pp_i} \setminus \overline{qp_i}) = \overline{pq} \setminus \{q\}$ and $\gamma \supset \overline{pq}$. So $q \sim p$. □

The leaves of $\mathcal{D}(X)$ may be single points, and may have inextendible geodesic segments. If X is a connected complete Riemannian manifold of dimension $\dim X \neq 1$ then all equivalence classes are points. The simplest example where the leaves of this decomposition are not points is when X is a metric tree; in this case any two points are equivalent. An example where some leaves are proper subsets and non-degenerate trees can be obtained as follows: Take the disjoint union of two metric trees T_1, T_2 and the plane $\mathbb{R}^2$. Pick two distinct points $x_1, x_2 \in \mathbb{R}^2$ and a pair of points $y_1 \in T_1, y_2 \in T_2$, then identify x_1 and y_1, x_2 and y_2. Let X denote the resulting metric space. The leaves of $\mathcal{D}(X)$ are:

(1) the trees T_1, T_2,

(2) the one-point sets $\{z\}$ for $z \in \mathbb{R}^2 - \{y_1, y_2\}$.

More generally we can take a continuum of trees T_α and attach them to $\mathbb{R}^2$ at all points $\alpha \in \mathbb{R}^2$. Then the leaves of $\mathcal{D}(X)$ are the trees T_α.

Definition 3.3 *A geodesic metric space X is said to be of* **type I** *if all leaves of $\mathcal{D}(X)$ are geodesically complete trees which branch everywhere.*

Recall that a point in a tree is a branch point if it separates the tree into at least 3 components. We remark that for the purposes of this paper it would suffice to require that T has a dense set of branch points.

The spaces of type I considered in this paper arise as asymptotic cones. Examples of spaces all of whose asymptotic cones are of type I are:

- periodic δ-hyperbolic spaces whose ideal boundary has at least 3 points.
- periodic locally compact Hadamard spaces which are not quasi-isometric to $\mathbb{R}$ and which contain periodic rank 1 geodesics (see proposition 4.7).

Definition 3.4 *A geodesic metric space X is said to be of* **type II** *if it is a thick, irreducible Euclidean building with transitive affine Weyl group and rank $r \geq 2$ (see [10] for definitions).*

Examples of such spaces are asymptotic cones of symmetric spaces and of thick irreducible Euclidean buildings of rank ≥ 2 with cocompact affine Weyl group, see [10].

Definition 3.5 *Let M be a geodesic metric space. We say that M has* **coarse type I** *if every aymptotic cone of M has type I; M has* **coarse type II** *if every asymptotic cone of M has type II.*

As we shall see in proposition 4.8 the universal cover M of any closed Riemannian nonpositively curved manifold has either coarse type I or type II unless M is reducible or flat.

Lemma 3.6 *Suppose that L is a metric tree, $T \subset L$ is a nondegenerate tripod with the central point c and the terminal points x, y, z. Suppose that $T' \subset L$ is another tripod with the terminal points x', y', z' so that*

$$d(x,x') \leq \frac{1}{2} d(x,c), \quad d(y,y') \leq \frac{1}{2} d(y,c), \quad d(z,z') \leq \frac{1}{2} d(z,c)$$

Then the central point c' of T' coincides with the central point c of the tripod T.

Proof. Consider the segment $\overline{xx'}$. Its intersection with T has the length at most $\frac{1}{2}$ of $d(x,c)$, the same is true for the intersections $\overline{yy'} \cap T$, $\overline{zz'} \cap T$. Hence the geodesic segments

$$\overline{x'y'}, \quad \overline{x'z'}, \quad \overline{z'y'}$$

contain the central point c of the tripod T. Thus $c = c'$. □

Lemma 3.7 *Suppose T is a metric tree with dense set of branch-points, A is a path-connected topological space, and Y is a metric space of type I. Assume that the map $g : T \times A \to Y$ is continuous and for each $a \in A$ the mapping $g(\cdot, a) : T \to Y$ is an embedding into a leaf of $\mathcal{D}(Y)$. Then for each $t \in T$ the mapping $g(t, \cdot)$ is constant.*

Proof. We first prove that the leaf L_a of $\mathcal{D}(Y)$ which contains $g(T,a)$ is constant as a function of a. Fix a and pick two distinct points $t, s \in T$ and a sufficiently small neighborhood U of a in A such that

$$d(x,y) \geq 4 \cdot diam(g(x,U)) + 4 \cdot diam(g(y,U))$$

where we let $x = g(t,a)$ and $y = g(s,a)$. For $b \in U$ we denote furthermore $x' = g(t,b)$ and $y' = g(s,b)$. Then $x' \sim y'$ implies that the path $\overline{x'x} \cup \overline{xy} \cup \overline{yy'}$ covers $\overline{x'y'}$. So the intersection $\overline{xy} \cap \overline{x'y'}$ is not empty and therefore $L_a = L_b$. This shows that L_a is locally constant and hence constant as a function of a. The image of g is thus contained in a metric tree and lemma 3.6 implies that the mapping $g(t, \cdot) : A \to Y$ is locally constant for every branch point $t \in T$. The claim follows because T has a dense set of branch points. □

In what follows we will use singular homology groups with integer coefficients and $\tilde{H}_*$ will denote the reduced homology groups.

Definition 3.8 *Let X be an acyclic Hausdorff topological space (i.e. $\tilde{H}_*(X) = 0$), $D = D^d \hookrightarrow X$ is a topological embedding of the open d-dimensional disk. We call D* **essential** *if the inclusion $D \hookrightarrow X$ induces monomorphisms of the local homology groups*

$$H_*(D, D - x) \longrightarrow H_*(X, X - x)$$

for all $x \in D$.

Note that since X is acyclic, the disk D is essential iff we have monomorphisms

$$\tilde{H}_*(D - x) \longrightarrow \tilde{H}_*(X - x)$$

Moreover, the only nonzero local homology groups $H_q(D, D - x)$ occur for $q = d$ and $H_d(D, D - x) \cong \mathbb{Z} \cong \tilde{H}_{d-1}(D - x)$.

Lemma 3.9 *Let X be an acyclic geodesic metric space and let $J \subset X$ be a homeomorphic image of an open interval. Then J lies in a single leaf of $\mathcal{D}(X)$ iff J is essential.*

Proof. Suppose that J is essential, $p, q \in J$, let $\overline{pq}$ be the subinterval between p and q on J. Let's prove that $\overline{pq}$ is contained in the image of any path α connecting p to q. Suppose not, and take $x \in \overline{pq} - Im(\alpha)$. Then the 0-cycle $q - p$ is not a boundary in $J - x$, but $\partial\alpha = q - p$ and α is a chain in $X - x$. This contradicts the assumption that J is essential.

If we reverse this argument then we conclude that:

$$J \text{ lies in a leaf of } \mathcal{D}(X) \iff J \text{ is essential}$$

□

Corollary 3.10 *For every space X of type I we have: $trank(X) = 1$.*

Lemma 3.11 *Suppose that X is a metric space of type II and rank $d \geq 2$. Then any open d-dimensional embedded disk D^d is essential in X.*

Proof. This is proven in lemma 6.2.1 of [10]. □

Lemma 3.12 *Let Y, Y' be metric spaces of type II, and let A be a connected topological space.*

(1) Then the pair (Y, Y') is topologically nontranslatable.

(2) Assume furthermore that a map $g : Y \times A \to Y'$ is continuous and for each $a \in A$ the mapping $g(\cdot, a) : Y \to Y'$ is a homeomorphism. Then for each $y \in Y$ the mapping $g(y, \cdot) : A \to Y'$ is constant.

Proof. Recall that according to [10] the only homeomorphisms between type II metric spaces are homotheties. If two top-dimensional flats in Y' have finite Hausdorff distance then they coincide; therefore if $f, h : Y \to Y'$ are homeomorphisms at finite distance from one another, then for each top-dimensional flat $F \subset Y$ we have $f(F) = h(F)$. Since Y, Y' are thick buildings with transitive affine Weyl group, this implies that $f = g$ and hence the pair (Y, Y') is topologically nontranslatable.

We use similar arguments to verify (2). For a continuous map $f : \mathbb{R}^d \times A \to Y'$, so that $f(\cdot, a)$ are homothetic embeddings, and a point $y' \in Y'$ we consider the subset $S(y') \subseteq A$ consisting of all parameters $a \in A$ so that $y' \in f(\mathbb{R}^d, a)$. This set is closed by trivial reasons. It is open because top-dimensional flats in Euclidean buildings are essential. Thus $S(y') = A$ and the images of all mappings $f(\cdot, a)$ coincide for all $a \in A$. In particular, for every Weyl chamber $F \subset Y$ the sets $g(F, a)$ coincide for all $a \in A$. The lemma follows. □

Lemma 3.13 *Suppose that Y is a complete geodesic metric space of type I and L', L'' are leaves of $\mathcal{D}(Y)$ which are Hausdorff-close and have infinite diameter. Then $L' = L''$.*

Proof. Suppose $L' \neq L''$. Since L', L'' are unbounded we can find a pair of points $x' \in L', y' \in L'$ so that $3d(y', L'') \leq d(x', y') \geq 3d(x', L'')$. Let $x'' \in L'', y'' \in Y''$ be points such that $d(x', x'') \leq d(x', y')/3, d(y', y'') \leq d(x', y')/3$. Then the curve $\overline{x'x''} \cup \overline{x''y''} \cup \overline{y''y'}$ connecting x' to y' doesn't contain $\overline{x'y'}$ since the segments $\overline{x'y'}$, $\overline{x''y''}$ are disjoint. Contradiction. □

Lemma 3.14 *Suppose that X, Y is a pair of metric spaces of type I or II. Then the pair (X, Y) is* **topologically nontranslatable**.

Proof. If X is homeomorphic to Y then they have the same type, since any two points in a type II space lie in an r-flat (where $r \geq 2$). Suppose that X, Y have type II. Then the assertion was proven in lemma 3.12. Now consider the case of the spaces of type I, $f, g : X \to Y$ are Hausdorff-close homeomorphisms. These mappings must carry the decomposition $\mathcal{D}(X)$ to the decomposition $\mathcal{D}(Y)$. Let $L \in \mathcal{D}(X)$ be a leaf. Then by lemma 3.13 the leaves $f(L), g(L)$ coincide with a leaf $T \subset Y$. The mappings $f, g : L \to T$ must be equal by Lemma 3.6. □

Proposition 3.15 *Suppose that X is a metric space of the type II and rank d (as a Euclidean building). Then for any point $p \in X$ the local homology groups $H_i(X, X-p)$ vanish for all $i \neq d$.*

Proof. We will use the notation and terminology from [10].

$H_0(X, X - \{p\}) = \{0\}$ since $rank(X) > 0$. As X is contractible, $\partial : H_k(X, X - \{p\}) \longrightarrow \tilde{H}_{k-1}(X - \{p\})$ is an isomorphism when $k \geq 1$; so it suffices to prove that $\tilde{H}_{k-1}(X - \{p\}) = \{0\}$ unless $k = d$. Consider the logarithm map $\log_{\Sigma_p X} : X - \{p\} \longrightarrow \Sigma_p X$. We will show that $H_*(\log_{\Sigma_p X}) : H_*(X - \{p\}) \longrightarrow H_*(\log_{\Sigma_p X})$ is an isomorphism. In the case when the affine Weyl group of X is discrete, $B_p(r)$ is isometric to a truncated metric cone over $\Sigma_p X$ for sufficiently small $r > 0$ ([10, proposition 4.5.1]); hence $H_*(\log_{\Sigma_p X})$ is an isomorpism because $B_p(r) - \{p\} \longrightarrow X - \{p\}$ is a homotopy equivalence (use geodesic segments to contract X to $B_p(r)$). In the general case we will need the following facts:

1. ([10, Corollary 4.4.3]) Let $S \subset \Sigma_p X$ be a finite union of apartments, and let $CS \subset C_p X$ be the corresponding metric cone in the tangent cone at p. Then there is a subset $Y \subset X$ so that for sufficiently small r, $Y \cap B_p(r)$ is mapped isometrically by $\log_{C_p X}$ to $CS \cap B_o(r)$, $o \in C_p X$ denotes the vertex of the cone $C_p X$. Furthermore, any two subsets Y, Y' with this property satisfy $Y \cap B_p(\bar{r}) = Y' \cap B_p(\bar{r})$ for sufficiently small $\bar{r} > 0$.

2. (a) If $[\alpha] \in H_k(\Sigma_p X)$, there is a finite union of apartments $S \subset \Sigma_p X$ so that $[\alpha] \in Im(H_k(S) \longrightarrow H_k(\Sigma_p X))$, and (b) If S is a finite union of apartments in $\Sigma_p X$ and $[\alpha] \in H_k(S)$ is in $Ker(H_k(S) \longrightarrow H_k(\Sigma_p X))$, then $[\alpha] \in Ker(H_k(S) \longrightarrow H_k(S'))$ for some finite union of apartments $S' \supset S$.

Surjectivity of $\log_{\Sigma_p X}$. Pick $[\alpha_0] \in H_k(\Sigma_p X)$. By fact 2a there is a finite union of apartments $S \subset \Sigma_p X$ and $[\alpha_1] \in H_k(S)$ so that $[\alpha_0] = (i_S)_*([\alpha_1])$ where $i_S : S \longrightarrow \Sigma_p X$ is the inclusion. By fact 1, we have a subset $Y \subset X$ which is mapped isometrically by $\log_{C_pX}$ to $CS \cap B_o(r)$. But then the inverse of this isometry can be used to push $[\alpha_1]$ to $[\alpha_2] \in H_k(Y - \{p\})$. Clearly $(\log_{\Sigma_pX})_*([\alpha_2]) = [\alpha_0]$.

Injectivity of $\log_{\Sigma_pX}$. Pick $[\alpha_0] \in H_k(X - \{p\})$. By the simplex straightening argument of [10, section 6.1], there is a finite union of apartments $\mathcal{P} \subset X$ and $[\alpha_1] \in H_k(\mathcal{P} - \{p\})$ so that $[\alpha_0] = (i_{\mathcal{P}-\{p\}})_*[\alpha_1]$. Moreover by [10, corollary 4.6.8] we may assume that for every $x \in \mathcal{P}$, the segment $\overline{px} \subset \mathcal{P}$. $\mathcal{P}$ determines a finite union of apartments $S \subset \Sigma_pX$, and $\log_{\Sigma_pX}$ maps $[\alpha_1]$ to a cycle α_2 in $S \subset \Sigma_pX$. If $[\alpha_0] \in Ker(\log_{\Sigma_pX})$ then $[\alpha_2] \in H_k(S)$ is in $Ker(H_k(S) \longrightarrow H_k(\Sigma_pX))$. By fact 2b, we have a finite union of apartments $S' \subset \Sigma_pX$ so that $[\alpha_2] \in Ker(H_k(S) \longrightarrow H_k(S'))$. Applying fact 1 to S', we get $Y' \subset X$ and an $r > 0$ so that $\log_{C_pX}$ induces an isometry

$$Y' \cap B_p(r) \longrightarrow CS' \cap B_o(r)$$

and

$$\mathcal{P} \cap B_p(r) = Y' \cap \log^{-1}_{C_pX}(CS) \cap B_p(r).$$

As we may homotope $[\alpha_0]$ radially until it lies in $\mathcal{P} \cap B_p(r)$, we clearly have $[\alpha_0] = 0$.

We now claim that $\tilde{H}_{k-1}(\Sigma_pX) = \{0\}$ unless $k = d$. To see this note[7] that if $v \in \Sigma_pX$ is a regular point and $V \subset \Sigma_pX$ is the set of antipodes of v, then V is discrete, $\Sigma_pX - V$ is contractible since it is the open π-ball centered at v, and each $v' \in V$ has a neighborhood homeomorphic to $\mathbb{R}^{d-1}$. The assertion follows by applying excision and the exact sequence of the pair (Σ_pX, U) where U is the complement of an appropriate neighborhood of V. This proves the lemma. □

Corollary 3.16 *Suppose that X is a space of type II which has rank r (as a building). Then the rank of X equals its topological rank: $trank(X) = r$. So the topological rank of every asymptotic cone of an irreducible symmetric space coincides with its geometric rank.*

4 Examples of spaces of coarse type I

Definition 4.1 *A metric space X is called* **periodic** *if the action of the isometry group $Isom(X)$ on X is* **cobounded***, i.e. there is a metric ball in X whose orbit under $Isom(X)$ equals X (we do not require this action to be properly discontinuous). A geodesic γ in a metric space X is called* **periodic** *if the action on γ of its stabilizer in $Isom(X)$ is cobounded.*

We recall the definition of the *divergence* of a complete minimizing geodesic $\gamma : \mathbb{R} \to X$ in a geodesic metric space X (see [5], [8]). All geodesics will be assumed to be nonconstant. Consider the complement of the open metric R-ball $B(R)$ centered at $\gamma(0)$ equipped with the path metric $d_{X \setminus B_{\gamma(0)}(R)}$. For each $R > 0$, we measure the distance $div(R)$ between the points $\gamma(\pm R) \in X \setminus B_{\gamma(0)}(R)$ using $d_{X \setminus B_{\gamma(0)}(R)}$. The growth rate of the function div is called the *divergence* of γ. (Recall that if $f(t), g(t)$ are positive functions on $\mathbb{R}_+$ then the growth rate of f is less than the growth rate of g iff $\limsup_{t\to\infty} f(t)/g(t) = 0$.)

The following proposition explains why this notion can be useful for proving that certain spaces have coarse type I.

[7] Spherical buildings (with the topology induced from the $CAT(1)$ metric) are homotopy equivalent to a wedge of spheres, as follows from the argument in [4, p. 94].

Proposition 4.2 *Suppose that $\gamma : \mathbb{R} \to X$ is a distance-minimizing periodic geodesic in a geodesic metric space X which has superlinear divergence. Then for every asymptotic cone X_ω of X taken with basepoints in $Im(\gamma)$, the ultralimit $\gamma_\omega : \mathbb{R} \to X_\omega$ has image in a single leaf of the decomposition $\mathcal{D}(X_\omega)$.*

Remark 4.3 *There are examples of spaces where the conclusion of this proposition fails for certain* **nonperiodic** *geodesics which have superlinear divergence.*

Proof. Suppose that the conclusion of the proposition fails. Then we can find a real number t, a piecewise-geodesic path $P_\omega := \overline{x_{1\omega}x_{2\omega}} \cup ... \cup \overline{x_{(m-1)\omega}x_{m\omega}}$ in X_ω between distinct points $x_{1\omega}, x_{m\omega} \in L_\omega$ so that P_ω is disjoint from $\gamma_\omega(t)$ and the points $x_{1\omega}, x_{m\omega}$ are equidistant from $\gamma_\omega(t)$. Since γ is periodic we can assume that $t = 0$. Therefore we can represent the path P_ω by a sequence of picewise-geodesic paths P_n in X connecting points $x_n^+, x_n^- = \gamma(\pm R)$ so that for ω-all n the path P_n lies outside of the metric ball $B_{\gamma(0)}(R/c)$ for the positive constant

$$c := 2d(\gamma_\omega(0), P_\omega)^{-1}$$

The length of P_n grows as a linear function of R which contradicts the assumption about superlinear divergence of γ. □

Definition 4.4 *A* **Hadamard space** *is a complete (not necessarily locally compact) simply-connected geodesic metric space which has nonpositive curvature in the sense of triangle comparisons [1, 10].*

The following proposition was proven in [8], we repeat the proof for convenience of the reader.

Proposition 4.5 *Let X be a locally-compact Hadamard space and let γ be a periodic geodesic in X which doesn't bound a flat half-plane. Then γ has at least quadratic divergence.*

Proof. Suppose that divergence of γ is subquadratic. Pick $\delta > 0$. Let α_R denote a curve in $X \setminus B_R(\gamma(0))$ connecting $\gamma(-R)$ to $\gamma(R)$ so that the length of α_R is $\leq div(R) + \delta$. Subquadratic divergence means that the length of α_R equals $\epsilon_R \cdot R^2$ where $\lim_{R\to\infty} \epsilon_R = 0$. Fix $h > 0$. Denote by $\pi : X \to \gamma(\mathbb{R})$ the nearest-point-projection. For sufficiently large R, we can find a subsegment $\overline{a_1 a_2} \subset \gamma(-R/2, R/2)$ of length h so that the portion of α_R which projects on $\overline{a_1 a_2}$ via π has length at most $\epsilon_R h R$. Pick points $b_i \in \alpha_R$ with $\pi(b_i) = a_i$. Let $\rho_i : [0, L_i] \to X$ be the unit speed geodesic joining $a_i = \rho_i(0)$ to b_i. We have $L_i \geq R/2$. The function $\psi(t) := d(\rho_1(t), \rho_2(t))$ is convex, monotonically increasing on $[0, R/2]$ and satisfies

$$\psi(0) = h, \qquad \psi(R/2) \leq \epsilon_R R h.$$

Therefore

$$h \leq \psi(h) \leq (1 + 2\epsilon_R h) \cdot h.$$

The quadrilateral with vertices a_i and $\rho_i(h)$ has three sides of length h, one side of length $\leq (1 + 2\epsilon_R h) \cdot h$ and angles $\geq \pi/2$ at a_i. We have a family of such quadrilaterals Q_R parametrized by R. Using the translations along $\gamma(\mathbb{R})$, we transport the quadrilaterals Q_R to a fixed compact subset of X. The Hausdorff limit (as R tends to infinity) of a convergent subsequence of the translated quadrilaterals is isometric to a square of the side-length h in $\mathbb{R}^2$. Hence for each h, we obtain a flat square of side-length h in X adjacent to γ. The local compactness of X implies existence of a flat half-plane bounded by γ. □

Corollary 4.6 *Let X be a locally-compact Hadamard space and let γ be a periodic geodesic which doesn't bound a flat half-plane. Then for every asymptotic cone X_ω of X taken with basepoints in $Im(\gamma)$, the ultralimit $\gamma_\omega : \mathbb{R} \to X_\omega$ has image in a single leaf of the decomposition $\mathcal{D}(X_\omega)$.*

Proposition 4.7 *Let X be a locally-compact Hadamard space containing a periodic geodesic which doesn't bound a flat half-plane. Suppose also that X is periodic and not quasi-isometric to $\mathbb{R}$. Then X has coarse type I.*

Proof. Since X is periodic, the isometry type of the asymptotic cones X_ω is independent of the sequence of base points and we may choose it to be constant. Furthermore, X_ω is a homogeneous metric space and it therefore suffices to check that one leaf of the decomposition $\mathcal{D}(X_\omega)$ contains a complete geodesic and a branch point.

Let $\gamma(\mathbb{R}) = L \subset X$ be a nonconstant periodic geodesic which doesn't bound a flat half-plane. If there is an isometry $g \in Isom(X)$ for which L and gL are not parallel then we may proceed as follows: The distance $d(\gamma(t), gL)$ is a convex unbounded function of t and, after reversing the sign of t if necessary, the limit $\lim_{t\to\infty} d(\gamma(t), gL)/t$ is strictly positive. This implies that L_ω and $(gL)_\omega$ are different complete geodesics in the same leaf of $\mathcal{D}(X_\omega)$. Thus the leaves of $\mathcal{D}(X_\omega)$ are geodesically complete trees which branch everywhere and X has coarse type I.

Suppose now that the geodesics L and gL are parallel for all isometries $g \in Isom(X)$. Then $Isom(X)$ preserves the parallel set $P(L)$ of L which is the union of all geodesics parallel to L. The periodicity of X implies that $P(L)$ has finite Hausdorff distance from X. $P(L)$ in turn has bounded Hausdorff distance from L because L does not bound a flat half-plane. This contradicts our assumption that X is not quasi-isometric to $\mathbb{R}$. □

Proposition 4.8 *Let M be a periodic Hadamard manifold, and assume that the isometry group of M satisfies the duality condition*[8] *([1, p. 5-6]). Then every nonflat de Rham factor of M has coarse type I or coarse type II.*

Proof. As the periodicity and duality conditions project to de Rham factors, we may assume that M is de Rham indecomposable. By [1, Theorems B,C], M is either an irreducible symmetric space of noncompact type of rank at least two, or M contains a periodic geodesic which doesn't bound a flat half-plane. In the former case M has coarse type II by [10]; in the latter M has coarse type I by proposition 4.7. □

[8]This will be true, for example, if M admits a discrete cocompact group of isometries.

5 Topological splitting

The goal of this section is to prove the following result about the invariance of product splittings under homeomorphisms:

Theorem 5.1 *Suppose X_i and Y_j are geodesic metric spaces of types I and II. Let $X := \mathbb{R}^n \times \prod_{i=1}^k X_i$ and $Y := \mathbb{R}^m \times \prod_{j=1}^{\ell} Y_j$. Suppose $f : X \to Y$ is a homeomorphism. Then $\ell = k$, $m = n$ and after reindexing the factors Y_j there are homeomorphisms $f_i : X_i \to Y_i$ so that the following diagrams commute:*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ X_i & \xrightarrow{f_i} & Y_i \end{array}$$

The rigidity theorem for homeomorphisms of Euclidean buildings proven in [10, Theorem 1.2.2] covers theorem 5.1 when all type I factors are trees:

Theorem 5.2 *Suppose that X_i, Y_j is a collection of geodesic metric spaces of types I and II, where all type I factors are trees. Let $X := \mathbb{R}^n \times \prod_{i=1}^k X_i$ and $Y := \mathbb{R}^m \times \prod_{j=1}^{\ell} Y_j$. Suppose that $f : X \to Y$ is a homeomorphism. Then $\ell = k$, $m = n$ and after reindexing the factors Y_j there are homeomorphisms $f_i : X_i \to Y_i$ so that the following diagrams commute:*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ X_i & \xrightarrow{f_i} & Y_i \end{array}$$

To apply this result to the general case, we will construct a topologically invariant decomposition of the spaces X and Y into cosets which are homeomorphic to Euclidean buildings. This is done as follows:

Let X be as in theorem 5.1, i.e. let $X = \mathbb{R}^n \times \prod_{i=1}^k X_i$ be a product of metric spaces where each X_i has either type I or II, and let $d_i := 1$ if X_i is a type I space, $d_i := rank(X_i)$ if X_i is type II. We let $X_0 := \mathbb{R}^n$ and $\bar{X} := \prod_{i=1}^k X_i$. We define the decomposition $\mathcal{F}(X)$ of X as follows:

Definition 5.3 *The* **leaves** *of $\mathcal{F}(X)$ are product subspaces $\mathbb{R}^n \times \prod_{i=1}^k T_i \subseteq \mathbb{R}^n \times \prod_{i=1}^k X_i$ where T_i is a leaf of the decomposition $\mathcal{D}(X_i)$ if X_i has type I and $T_i = X_i$ otherwise.*

To prove that the decomposition $\mathcal{F}(X)$ is topologically invariant, we characterize the leaves of $\mathcal{F}(X)$ using essential disks (cf. definition 3.8). The following observation shows that any two points in the same leaf of $\mathcal{F}(X)$ lie in an essential disk:

Lemma 5.4 *In each type I factor X_i pick an open interval I_i contained in a leaf of the decomposition $\mathcal{D}(X_i)$. In each type II factor X_j of rank $d_j \geq 2$ pick an embedded open d_j-disk D_j. Finally take an open disk $D_0 \subseteq \mathbb{R}^n$. Then the product of these disks is an essential disk in X.*

Proof. The 1-disks I_i contained in leaves of the type I factors X_i are essential due to lemma 3.9. According to lemma 6.2.1 of [10], the disks D_j are essential in X_j. The Künneth formula implies the assertion of the lemma:

$$\begin{array}{ccc} H_d(D, D-p) & \xrightarrow{\cong} & \oplus_{|\alpha|=d} \ \otimes_j H_{\alpha_j}(D_j, D_j - p_j) \\ \downarrow & & \downarrow \\ H_d(X, X-p) & \xrightarrow{\cong} & \oplus_{|\alpha|=d} \ \otimes_j H_{\alpha_j}(X_j, X_j - p_j) \end{array}$$

where $p = (p_j) \in X$ is a point contained in D. □

The next result shows that, conversely, any essential disk lies in a leaf of $\mathcal{F}(X)$.

Proposition 5.5 *Suppose $D^d \hookrightarrow X$ is an essential disk. Then:*

(1) The projection of each compact subdisk $\hat{D}^d \subset D^d$ to every type I factor X_i of X is contained in a finite number of geodesic segments lying within a single leaf of the decomposition $\mathcal{D}(X_i)$.

(2) The projection of each compact subdisk $\hat{D}^d \subset D^d$ to every type II factor in X is contained in a finite number of top-dimensional flats.

(3) The projection of D^d to the factor $\mathbb{R}^n$ is an open map.

Proof. Consider an essential d-disk $D \hookrightarrow X$ and $p \in D$. Choose a compact subdisk $\hat{D} \subset D$ containing p in its interior and choose a closed metric polydisk $B := \prod_i B_i \subset X$ centered at p with $\partial\hat{D} \cap B = \emptyset$. The relative fundamental class of $(\hat{D}, \partial\hat{D})$ determines an element $[\hat{D}]$ of $H_d(X, X - B)$. Then by the Kunneth formula we have

$$H_d(X, X-B) \cong \oplus_{|\alpha|=d} \otimes_j H_{\alpha_j}(X_i, X_i - B_i)$$

where $\alpha = (\alpha_1, \alpha_2, ...)$ is a multiindex. By the dimension assumption

$$H_k(X_i, X_i - p_i) = 0, \quad \text{for all } \ k < d_i$$

(see proposition 3.15). Hence after shrinking (if necessary) the polydisk B we get:

$$[\hat{D}] = \sum \otimes_i [\beta_i] \in \otimes_i H_{d_i}(X_i, X_i - B_i)$$

where $[\beta_i] \in H_{d_i}(X_i, X_i - B_i)$.

Using an approximation argument we may take each relative cycle β_i to be a linear combination of geodesic segments for each type I factor X_i, a PL-chain in X_0 and for each of remaining factors X_j we may represent β_j by a singular chain contained in a finite number of flats, cf. [10]. Since D is an essential disk we conclude that $D \cap B \subset P := \prod_i P_i$, where each P_i is a finite union of d_i-flats in X_i. In particular each P_i is a polyhedron (see [10]).

We are already done as far as type II factors of X are concerned. Now consider factors of type I. By restricting ourselves to a smaller polydisk B we may assume that for each type I factor X_i we have $\pi_{X_i}(D \cap B) \subset P_i$, where P_i is a collection of radial segments emanating from $\pi_{X_i}(p) = p_i$. If $q \in D \cap B$ and $\pi_{X_i}(q)$ is not the vertex of P_i then the Kunneth formula applied to $(D \cap B, (D-q) \cap B) \hookrightarrow (P, P-q) \hookrightarrow (X, X-q)$ implies that the interior of each radial segment of $\pi_{X_i}(D \cap B)$ is essential. By Lemma

3.9 it follows that each of these segments lies in a single leaf of the decomposition $\mathcal{D}(X_i)$. Since leaves of $\mathcal{D}(X_i)$ are closed and disjoint we conclude that $\pi_{X_i}(D \cap B)$ is contained in a single leaf of $\mathcal{D}(X_i)$. Finally we note that the compact subdisk $\hat{D}$ is covered by a finite number of the intersections with small polydisks $B \cap D$.

Similar arguments applied to the factor X_0 imply that the projection of D^d to X_0 is open. □

Lemma 5.4 and proposition 5.5 yield:

Corollary 5.6 *The relation*

$$x \cong y \Leftrightarrow x, y \text{ lie in an open essential disk.}$$

is an equivalence relation, and the equivalence classes are the leaves of $\mathcal{F}(X)$.

Since essential disks are defined purely topologically, the previous corollary provides a topological characterization for the leaves of the decomposition $\mathcal{F}(X)$ and shows that they are preserved under homeomorphisms:

Corollary 5.7 *Let $f : X \to Y$ be a homeomorphism where X, Y are product spaces as in theorem 5.5. Then f carries the decomposition $\mathcal{F}(X)$ to the decomposition $\mathcal{F}(Y)$.*

Proof of Theorem 5.1. By Corollary 5.7, f carries a leaf $F = \mathbb{R}^n \times \prod_{i=1}^k U_i$ of the decomposition $\mathcal{F}(X)$ to a leaf $G = \mathbb{R}^m \times \prod_{j=1}^l V_j$ of $\mathcal{F}(Y)$. Theorem 5.2 applies to the restricted homeomorphism $f|_F : F \to G$ and we see that $m = n$ and $k = l$.

Set $f_j := \pi_{Y_j} \circ f$. For each $p \in \hat{X}_i := \mathbb{R}^n \times X_1 \times \cdots \times X_{i-1} \times X_{i+1} \times \cdots \times X_k$ there is a unique index $j(U_i, p)$ such that

$$f_j|_{U_i \times \{p\}} : p_0 \times \dots p_{i-1} \times U_i \times p_{i+1} \times \cdots \times p_n \to Y_{j(U_i,p)}$$

is a homeomorphism onto a leaf $L_{U_i,p}$ of $\mathcal{F}(Y_{j(U_i,p)})$ and $f_j|_{U_i \times \{p\}}$ is constant for $j \neq j(U_i, p)$. The sets

$$S_j := \{p \in \hat{X}_i : f_j|_{U_i \times \{p\}} \text{ is not constant}\}$$

are open and disjoint subsets of $\hat{X}_i$. Since $\hat{X}_i$ is connected we conclude that $j(U_i, p)$ does not depend on the point p: $j(U_i, p) = j(U_i)$. The sequence of indices $j(U_1), \dots,$ $j(U_k)$ forms a permutation of $1, \dots, k$. This implies that, if we exchange one of the leaf factors U_i by U_i', we have $j(U_i) = j(U_i')$. Hence $j(U_i)$ depends only on i and, after rearranging the factors Y_j, we can assume that $j(U_i) = i$ and X_i, Y_i have the same type for every i. We apply Lemmas 3.7 and 3.12 (with $A := \hat{X}_i$) to conclude that for each $x_i \in U_i$ the mapping $f_i(\dots, \cdot, x_i, \cdot, \dots) : \hat{X}_i \to Y_i$ is constant. Hence $f_i(x_1, \dots, x_k)$ depends only on x_i and f_i descends to a homeomorphism $f_i : X_i \to Y_i$ as desired. □

Corollary 5.8 *Suppose X and Y are metric spaces which are products of finitely many geodesic metric spaces of types I and II. Then the pair (X, Y) is* **topologically nontranslatable**.

Proof. Let $f, g : X \to Y$ are homeomorphisms. They must be product homeomorphisms by theorem 5.1. Now the assertion follows from Lemma 3.14. $\square$

As another application we get

Corollary 5.9 *Suppose X and Y are metric spaces which are products of finitely many of metric spaces of coarse types I and II. Then the pair (X, Y) is* **nontranslatable**.

6 Geometric splitting

In this section we prove the main results of this paper (theorems A and B from the introduction).

Proof of Theorem B. We let $\bar{M} := \prod_{i=1}^{k} M_i$, $\bar{N} := \prod_{i=1}^{\ell} N_i$. According to theorem 5.1 all ultralimits ϕ_ω of ϕ preserve the foliations of the asymptotic cones M_ω, N_ω by copies of Z_ω and W_ω factors. The pair $(\bar{M}, \bar{N})$ is nontranslatable according to Corollary 5.9. Therefore we apply theorem 5.1 and proposition 2.8 to conclude that $m = n$, $k = \ell$, and there is a quasi-isometry $\bar{\phi} : \bar{M} \longrightarrow \bar{N}$ such that the diagram

$$\begin{array}{ccc} M & \stackrel{\phi}{\longrightarrow} & N \\ \downarrow & & \downarrow \\ \bar{M} & \stackrel{\bar{\phi}}{\longrightarrow} & \bar{N} \end{array}$$

commutes up to a finite error bounded in terms of (L, A). Now we apply theorem 5.1 and proposition 2.6 to the quasi-isometry $\bar{\phi}$ to conclude the proof of the theorem. $\square$

As a direct corollary of the above theorem and proposition 4.8 we obtain theorem A about quasi-isometry invariance of de Rham decomposition of universal covers of nonpositively curved Riemannian manifolds.

Now we prove the equality between coarse rank and geometric rank for universal covers of compact nonpositively curved Riemannian manifolds.

Proposition 6.1 *Let $X = \prod_i X_i$ be a finite product of Hausdorff topological spaces. Then*

$$trank(X) = \sum_i trank(X_i)$$

Proof. Directly follows from Kunneth formula (as in proposition 5.5). $\square$

Proof of Theorem 1.3. Let $\tilde{M} = \mathbb{R}^n \times \prod M_i$ be the de Rham decomposition of the universal cover of a closed Riemannian manifold of nonpositive curvature. By proposition 4.7 each factor M_i with geometric rank 1 has coarse type I, and each factor with geometric rank $r > 1$ is an irreducible symmetric space of rank r. Since the topological rank of every asymptotic cone of a space with coarse type I is 1 by corollary 3.10, and the topological rank of every asymptotic cone of any rank r symmetric space is r (corollary 3.16), proposition 6.1 implies the theorem. $\square$

References

[1] W. Ballmann, *Lectures of spaces of nonpositive curvature*, DMV Seminar Band 25, Birkhäuser, 1995.

[2] W. Ballmann, M. Brin, *Orbihedra of nonpositive curvature*, Publ. IHES, vol. **82**, (1995) 169–209.

[3] B. Bowditch, *Connectedness properties of limit sets*, Preprint, 1996.

[4] K. Brown, *Buildings*, Springer Verlag, 1989.

[5] S. Gersten, *Quadratic divergence of geodesics in CAT(0)-spaces*, Geometric Analysis and Functional Analysis, vol. **4**, no. 1 (1994) 37– 51.

[6] E. Ghys, P. De la Harpe, *Sur les Groupes Hyperbolique d'apres Mikhael Gromov*, Progress in Mathematics, vol. **83**, Birkhäuser, 1990.

[7] M. Kapovich and B. Leeb, *On asymptotic cones and quasi–isometry classes of fundamental groups of 3-manifolds*, Geometric Analysis and Functional Analysis, vol. **5**, no. 3 (1995) 582–603.

[8] M. Kapovich and B. Leeb, *3-manifold groups and nonpositive curvature,* IHES Preprint 1996.

[9] M. Kapovich and B. Leeb, *Quasi-isometries preserve the geometric decomposition of Haken manifolds,* Preprint 1995, to appear in Inventiones.

[10] B. Kleiner and B. Leeb, *Rigidity of quasi-isometries for symmetric spaces and Euclidean buildings,* Preprint, 1996.

[11] P. Pansu, *Croissance des boules et des géodesiques fermées dans les nilvariétés*, Ergod. Th. Dynam. Sys., vol. **3**, (1983) 415–445.

[12] P. Pansu, *Metriques de Carnot-Caratheodory et quasiisometries des spaces symetriques de rung un,* Ann. of Math., vol. **129** (1989) 1–60.

[13] E. Rieffel, *Groups coarse quasi-isometric to* $\mathbb{H}^2 \times \mathbb{R}$, PhD Thesis, UCLA, 1993.

[14] R. Schwartz, *The quasi-isometry classification of rank one lattices,* Publ. of IHES, vol. **82** (1995) 133– 168.

[15] J. Stallings, *3-dimensional topology and group theory,* Yale mathematical monographs, 1970.

Michael Kapovich, Department of Mathematics, University of Utah, Salt Lake City, UT 84112, USA; kapovich@math.utah.edu

Bruce Kleiner, Department of Mathematics, University of Pennsylvania, Philadelphia, PA 19104-6395; bkleiner@math.upenn.edu

Bernhard Leeb, Mathematisches Institut, Universität Bonn, Beringstr. 1, 53115 Bonn, Germany; leeb@rhein.iam.uni-bonn.de