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On the Absence of Sullivan's Cusp Finiteness Theorem in Higher Dimensions

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ABSTRACT. We prove the existence of a discontinuous finitely generated free conformal group K_3 acting on $\mathbb{R}^3$ such that the number of rank 1 cusps of K_3 is infinite. Small deformations of K_3 provide a finitely generated Kleinian group with infinitely many conjugacy classes of finite order elements.

§1. Introduction

1.1. In this paper we continue the discussion **[K-P 1]** of the failure of Ahlfors's finiteness theorem for Kleinian groups in dimensions ≥ 3. Here we establish the failure of Sullivan's cusp finiteness theorem. Namely

THEOREM. *There exists a finitely generated free Kleinian group $K_3 \subset \mathrm{Mob}(S^3)$ such that*

(a) *The number of conjugacy classes of maximal parabolic subgroups of K_3 is infinite.*
(b) *The fixed points of these parabolic subgroups $\langle u_i \rangle \subset K_3$ have pairwise disjoint horoball neighborhoods $U_i \subset H^4$ which are precisely invariant under $\langle u_i \rangle$ in K_3.*
(c) *If $K_n \subset \mathrm{Mob}(S^n)$ is the conformal (Poincaré) extension of K_3 to S^n $(n \geq 3)$, then*

$$\operatorname{rank}\Big(H_{n-1}\big(\Omega(K_n)/K_n, \mathbb{Q}\big)\Big) = \infty.$$

Here and below $\Omega(\ )$ denotes the discontinuity domain for a Kleinian group.

COROLLARY 1. *For each $n \geq 3$ there exists a finitely generated Kleinian group $K_n \subset \mathrm{Mob}(S^n)$ such that the quotient manifold $M(K_n) = \Omega(K_n)/K_n$ has infinite homotopy type.*

So, the analog of Ahlfors's finiteness theorem fails for every dimension $n \geq 3$.

COROLLARY 2. *For all but finitely many $q \in \mathbb{Z}$ there exist:*
1) *an integer r, the rank of free group $F_r = \langle x_1, \dots, x_r \rangle$,*

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2) *an automorphism* θ: $F_r \to F_r$ *such that the group*

$$F_{r,q} = \left\langle x_1, \dots, x_r : \ (\theta^n(x_1))^q = 1, \ n = \pm 1, \pm 2, \dots \right\rangle$$

has infinitely many conjugacy classes of finite order elements $[\theta^n(x_1)]$ *and* $F_{r,q}$ *admits a discrete faithful representation in* $\mathrm{Mob}(S^3)$.

REMARK. According to Selberg's lemma **[Se]** $F_{r,q}$ has a torsion-free finite index subgroup (as a finitely generated linear group).

The manifold $\Omega(K_3)/K_3$ is homeomorphic to an open handlebody X_r from which some ∞-component link L_∞ is removed. Each component of the link is a knot representing a free generator of $\pi_1(X_r)$.

The underlying space of the orbifold $M(F_{r,q})$ is homeomorphic to X_r, where the singular set is L_∞(see above).

Historical remarks.

1.2. The first version of the cusp finiteness theorem for Kleinian subgroups of $PSL(2, \mathbb{C})$ was given by Ahlfors **[Ah]** (see also **[Kr 1]**). In particular, he proved that for a finitely generated Kleinian group $K \subset PSL(2, \mathbb{C})$ the surface $\Omega(K)/K$ has only finitely many punctures. Combined with the Leutbecher-Shimizu lemma, this implies that K can have only finitely many conjugacy classes of maximal parabolic subgroups of rank 1. Ahlfors's proof was based on analytic function theory and fails in higher dimensions. Fifteen years later Sullivan **[Su]** proved that any finitely generated discrete subgroup of $PSL(2, \mathbb{C})$ has only finitely many cusps (i.e., conjugacy classes of maximal parabolic subgroups). Sullivan also gave a numerical estimate for the number of cusps (in terms of the number of generators of the group K). That estimate was improved by Kulkarni and Shalen **[K-S, K]** and by Abikoff **[Ab 1]**, who used topological considerations based on existence of a compact "Scott core" for 3-manifolds with finitely generated fundamental group. Certainly, a "Scott core" does not exist in higher dimensions. In contrast, Sullivan exploited Eisenstein series associated with cusps of K and elements of the finite-dimensional space $H^1(K, sl(2, \mathbb{C}))$ (these arguments were elaborated in **[Kr 2]**). The idea of Eisenstein series remains meaningful in higher-dimensional hyperbolic spaces; hence there was a hope of generalizing Sullivan's proof to the case of discrete subgroups in $\mathrm{Mob}(S^n)$ ($n > 2$). This is probably possible under some restrictions on K (see the Conjecture below). However, the main theorem above shows that in general the cusp finiteness theorem fails in higher dimensions.

CONJECTURE. *Let* Γ *be a finitely generated discrete subgroup of* $\mathrm{Mob}(S^n)$ *such that*:

(1) *any cusp of* Γ *has "maximal" rank* $= n$;
(2) *every cusp of* Γ *corresponds to a "cusp end" of* H^{n+1}/Γ.
Then Γ *has only finitely many cusps.*

1.3. Evidently, every geometrically finite discrete subgroup of $\mathrm{Mob}(S^n)$ has only finitely many cusps.

1.4. We now discuss Corollary 2. Every finitely generated discrete subgroup of $\mathrm{Mob}(S^2)$ has only finitely many conjugacy classes of finite order elements (briefly CCFE) (the author could not find an exact reference; this statement will be proved in §7). Every geometrically finite subgroup of $\mathrm{Mob}(S^n)$ has only a finite number of

CCFE. As a generalization of this fact, Gromov **[Gr]** proved that any word hyperbolic group has only finitely many CCFE. There is also an old lemma of Selberg **[Se]** on this matter. Selberg showed that if in every finitely generated subgroup of $SL(n, \mathbb{C})$ the finite subgroups have bounded order, then there are only finitely many $SL(n, \mathbb{C})$–conjugacy classes of finite order elements. Probably, examples like Corollary 2 are known to algebraists for the case of arbitrary finitely generated subgroups of $SL(n, \mathbb{C})$.

1.5. Corollary 1 for $n = 3$ was proved by a different example in the joint paper of the author and Potyagailo **[K-P 1]**. For $n \geq 5$, Corollary 1 cannot be deduced from **[K-P 1]**, in view of Potyagailo's remark that the Kleinian group constructed there is not finitely presented.

1.6. The results of the present paper were announced in **[Ka 1, Ka 2]**. Corollary 2 was announced in **[K-P 2]**. In this current article we simplify the original proof of the main theorem first given in **[K-P 2]**.

§2. Outline of the proofs

2.1. Consider the configuration of four Euclidean spheres $\Sigma_1, \Theta_2, \Theta_3, \Sigma_4 \subset \mathbb{R}^3$ drawn in Figure 1. We shall construct Kleinian groups $\Gamma_1', \Gamma_2', \Gamma_3', \Gamma_4'$ whose limit sets are the spheres $\Sigma_1, \Theta_2, \Theta_3, \Sigma_4$, respectively. All the groups Γ_i' (up to finite index) are conjugate in $\mathrm{Mob}(S^3)$ to the groups Γ_i of our previous paper **[K-P 1]**. The groups Γ_i' possess the following properties:

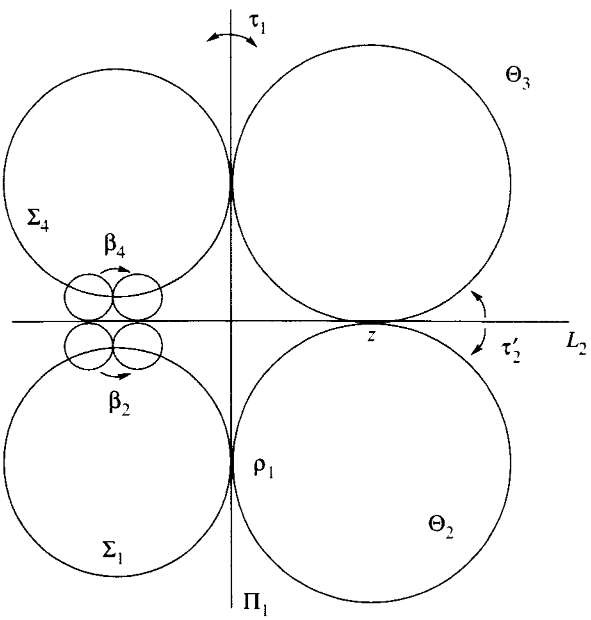

FIGURE 1

(1) The group $G' = \langle \Gamma'_1, \Gamma'_2, \Gamma'_3, \Gamma'_4 \rangle$ is Kleinian and contains a finitely generated free normal subgroup F' such that $G'/F' \simeq \mathbb{Z}$.
(2) If τ'_2 is the reflection in the symmetry plane L_2, then $\tau'_2 \Gamma'_1 \tau'_2 = \Gamma'_4$, $\tau'_2 \Gamma'_2 \tau'_2 = \Gamma'_3$.
(3) The groups $\Gamma'_1 \cap F'$ and $\Gamma'_4 \cap F'$ contain parabolic elements β_2 and $\beta_4 = \tau'_2 \beta_2 \tau'_2$, respectively, such that the isometric spheres $I(\beta_2)$, $I(\beta_2^{-1})$ touch L_2 at some point $x = I(\beta_2) \cap L_2$.
(4) The point x is a fixed point of the parabolic transformation $u = (\beta_4)^{-1} \circ \beta_2$. This point has a cusp neighborhood, precisely invariant under $\langle u \rangle$ in G'.

Then the group F' is the group K_3 we need. Indeed, consider an element $t_2 \in G'$ such that $\langle F', t_2 \rangle = G'$. The elements u and $u_m = t_2^m u t_2^{-m}$ $(m \in \mathbb{Z})$ are parabolic. Property (4) implies that the groups $\langle u_m \rangle$ are maximal nonconjugate parabolic subgroups of F'. So, Property (a) holds. Property (b) also follows from (4). Moreover, according to (3), the manifold $M(K_n)$ has infinitely many cusp ends, giving an infinite system of independent cocycles in $H_{n-1}(M(K_n), \mathbb{Q})$. This consideration completes the proof of the theorem.

2.2. The proof of Corollary 2 proceeds as follows. We slightly enlarge the spheres Σ_1 and Σ_4 so that the isometric sphere $I(\beta_2)$ intersects L_2 at an angle $\pi/2q$, $q \in \mathbb{Z}\backslash\{0\}$. Next, repeating the previous construction, we obtain representations $\rho_q : K_3 \to \mathrm{Mob}(S^3)$, $\lim_{q \to \infty} \rho_q = \mathrm{id}$. Then the elements $\rho_q(u_m)$ become elliptic of order q. These elliptic elements are not conjugate in $\rho_q(K_3)$ which happens to be Kleinian (if q is sufficiently large).

§3. Proof of the theorem

3.0. Notation. Below, we shall denote by $P(K)$ an isometric fundamental domain for a Kleinian group K and by $\Lambda(K)$ its limit set. If S is a closed surface in $\mathbb{R}^3$, then $\mathrm{int}(S)$ will denote the interior of the compact component of $\mathbb{R}^3 \backslash S$ and $\mathrm{ext}(S) = S^3 \backslash \mathrm{cl}(\mathrm{int}(S))$.

Recall that a subset S of S^n is called *precisely invariant* under a subgroup H of a discrete group $G \subset \mathrm{Mob}(S^n)$, if $h(S) = S$ for every $h \in H$ and $g(S) \cap S = \varnothing$ for any $g \in G \backslash H$.

3.1. First we recall several constructions of our previous paper **[K-P 1]**. Let us consider the unit sphere $\Sigma_1 \subset \mathbb{R}^3$ centered at zero; $p_1 = (0, 1, 0)$, $p_2 = (0, 0, 1)$. Let Π_i be the extended Euclidean planes tangent to Σ at the points p_i (see Figure 1) and let Π_i^- be the component of $\mathbb{R}^3 \backslash \Pi_i$ such that $\Pi_i^- \cap \Sigma = \varnothing$ $(i = 1, 2)$.

In **[K-P 1**, Lemma 1 and 2] we constructed a Kleinian group Γ_1 with the following properties:

(1) Γ_1 contains maximal parabolic subgroups $\widetilde{H}_1$, $\widetilde{H}_2$ with limit points p_1, p_2, respectively.
(2) Γ_1 contains a free finitely generated normal subgroup F_1 such that: $\Gamma_1/F_1 \cong \mathbb{Z}$, $\langle F_1, \tilde{t}_2 \rangle = \Gamma_1$ for some $\tilde{t}_2 \in \widetilde{H}_2$; $\langle \tilde{t}_2 \rangle \oplus \langle \beta_2 \rangle = \widetilde{H}_2$.
(3) The group Γ_1 has a fundamental set P such that $P \cap \mathrm{cl}(\Pi_i^-)$ is an isometric fundamental domain for the action of the group $\widetilde{H}_i$ on $\mathrm{cl}(\Pi_i^-)$.
(4) The group Γ_1 has the S-RF property for every geometrically finite subgroup $S \subset \Gamma_1$; i.e., for any element $g \in \Gamma_1 \backslash S$ there is a finite-index subgroup $\Gamma_1^0 \subset \Gamma_1$ that contains S but not g.

REMARK. Without loss of generality, we can assume that the center of $I(\beta_2)$ has coordinates $(0, a_2, a_3)$.

Denote by L_2 the plane $\{(x_1, \lambda, x_3) : (x_1, x_3) \in \mathbb{R}^3\}$ such that $\lambda > 1$ and L_2 is tangent to the isometric spheres $I(\beta_2)$, $I(\beta_2^{-1})$ (see Figure 2). Let $\overline{L}_2 = L_2 \cup \{\infty\}$, $L_2^- = \{(x_1, x_2, x_3) : x_2 > \lambda_2\}$. Denote by τ_1 the reflection in the plane Π_1 and by τ_2' the reflection in the plane L_2. Put $L_2^+ = \tau_2'(L_2^-)$, $\Pi_i^+ = \tau_1(\Pi_1^-)$.

LEMMA 1. *These exists a finite-index subgroup $\Gamma_1' \subset \Gamma_1$ possessing the following properties*:

(a) $\beta_2 \in \Gamma_1'$;

(b) $\Gamma_1' = \langle F_1', t_2 = (\tilde{t}_2)^n \rangle$ *for some finitely generated free normal subgroup* $F_1' \subset \Gamma_1'$ *and* $n \in \mathbb{Z}$;

(c) *if* $\gamma \in \Gamma_1' \backslash \{\beta_2, \beta_2^{-1}, 1\}$, *then* $I(\gamma) \cap L_2' = \varnothing$;

(d) *let* $H_i' = \widetilde{H}_i \cap \Gamma_1'$ $(i = 1, 2)$. *If* $\gamma \in \Gamma_1' \backslash H_1'$ *then* $I(\gamma) \cap (\Pi_1 \backslash P(H_1')) = \varnothing$.

PROOF. Note that $\overline{L}_2'$ and $\mathrm{cl}(\Pi_1 \backslash P(H_1'))$ are compact subsets of $H^3 = \mathrm{ext}\, \Sigma_1$. Then there exist only finitely many elements $\{\gamma_1, \dots, \gamma_p\} \subset \Gamma_1$ such that $I(\gamma_i) \cap \overline{L}_2' \neq \varnothing$. The group Γ_1 has the S-RF property for geometrically finite subgroups S. Hence we can find a finite-index subgroup $\Gamma_1'' = \langle F_1'', t_2 = (\tilde{t}_2)^n \rangle < \Gamma_1$ such that $\{\gamma_1, \dots, \gamma_p\} \cap \Gamma_1'' \backslash \{\beta_2, \beta_2^{-1}, 1\} = \varnothing$. Let $H_2' = \Gamma_1'' \cap \widetilde{H}_2$. Then there are only finitely many elements $\{\gamma_1', \dots, \gamma_k'\} \subset \Gamma_1'' \backslash H_2'$ such that $I(\gamma_i') \cap \mathrm{cl}(\Pi_1 \backslash P(H_1')) \neq \varnothing$. Property (3) of the group Γ_1 implies that $(\widetilde{H}_2 * \widetilde{H}_1) \backslash H_2' \cap \{\gamma_1', \dots, \gamma_k'\} = \varnothing$. The Schottky-type group $H_2' * H_1'$ is geometrically finite. Hence Γ_1'' has the $H_2' * H_1'$–RF property and we can find a finite-index subgroup $\Gamma_1' \subset \Gamma_1''$ such that $H_2' \cup H_1' \subset \Gamma_1'$, $\Gamma_1' \cap \{\gamma_1', \dots, \gamma_q'\} = \varnothing$. The group Γ_1' has all the desired properties.

Lemma 1 is proved. □

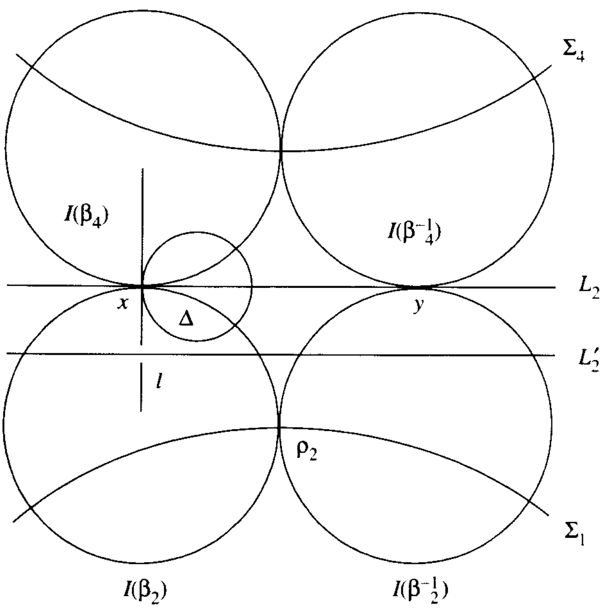

FIGURE 2

3.2. Let Θ_2 be a sphere that touches Σ_1 at the point p_1 and touches L_2 at some point $z \in \Pi_1^-$. Next, let $x = L_2 \cap I(\beta_2)$ and $y = L_2 \cap I(\beta_2^{-1})$. There exists a unique orientation-reversing Möbius transformation T that maps p_2 to z and commutes with every element of H_1'. It is easy to see that $T^{-1}(L_2^-) \subset \Pi_2^-$ and that $T^{-1}(\bar{L}_2^-)$ touches Σ_1 at the point p_2. Hence, it follows from property (3) above that L_2^- is precisely invariant under $H_3' = T H_2' T^{-1}$ in the group $\Gamma_2' = T\Gamma_1 T^{-1}$. Also, for the groups $\Gamma_1' = \Gamma_1$, Γ_2' and the amalgamated subgroup $\widetilde{H}_1$ the conditions of the first Maskit Combination Theorem are fulfilled.

3.3. LEMMA 2. *The group $G_1' = \langle \Gamma_1', \Gamma_2' \rangle$ is Kleinian and has a simply connected invariant component $\Omega_1' \ni \infty$. The manifold Ω_1'/G_1' is obtained by gluing together two hyperbolic components homeomorphic to* $(\mathrm{ext}\ \Sigma_1 \backslash \Gamma_1'(\Pi_1^-))/\Gamma_1'$. *The manifold Ω_1'/G_1' fibers over S^1. The group $\Phi_1' = \langle F_1', T F_1' T^{-1} \rangle$ is a finitely generated free normal subgroup of G_1' corresponding to the fiber $G_1'/\Phi_1' \cong \mathbb{Z}$. There is a fundamental set D_1 for the action of G_1' on Ω_1' such that:*

(i) $(D_1 \cap \mathrm{cl}\ L_2^-) \cup \{y\}$ *is a fundamental set for the action of H_3' on* $\mathrm{cl}(L_2^-)$;

(ii) *for some $\lambda' < \lambda$ and plane $L_2' = \{(x_1, \lambda', x_3) : (x_1, x_3) \in \mathbb{R}^2\}$ the intersection $L_2' \cap D_1 \cap \tau_1(\Pi_1^-)$ coincides with $L_2' \cap P(\langle \beta_2 \rangle) \cap \tau_1(\Pi_1^-)$.*

PROOF. All statements, except (i) and (ii), easily follow from the Maskit Combination Theorem (see the proof of Lemma 3 in **[K-P 1]**). Let $D_1 = (P \backslash \Pi_1^-) \cup T(P \backslash \Pi_1^-)$. Then statements (i), (ii) follow from the properties of the group Γ_1' listed above. □

3.4. Introduce the following notation: $J = H_3'$, $X_2 = L_2^- \cup \left(\bar{L}_2 \backslash J(\{x, y, z\})\right)$, $X_1 = \tau_2'(X_2)$, $G_2' = \tau_2' G_1' \tau_2'$, $\Phi_2' = \tau_2' \Phi_1' \tau_2'$, $D_2 = \tau_2'(D_1)$.

Direct considerations based on Lemma 1 imply that the triple (G_1', G_2', J) is proper interactive **[Mk,** Chapter VII] for the pair of sets (X_1, X_2). Moreover, $D_i \cap X_i \subset \bar{L}_2$ for $i = 1, 2$.

3.5. LEMMA 3.

(1) *The groups G_1', G_2', J satisfy the conditions of the weak Maskit Combination Theorem* **[Mk,** Chapter VII, Theorem A.15].

(2) *The group $G' = \langle G_1', G_2' \rangle$ is isomorphic to $G_1' *_J G_2'$.*

(3) *The set $D = (D_1 \cap X_1) \cup (D_2 \cap X_1)$ does not contain points equivalent under the action of G'.*

(4) $\mathrm{int}(D) \subset \Omega(G')$.

PROOF. The first statement follows from item 3.4; the remaining statements follow from the weak Maskit Combination Theorem. □

3.6. Denote by Ω' the component of $\Omega(G')$ containing the point ∞. It is easy to see that $G'(\Omega') = \Omega'$. Let $\beta_4 = \tau_2' \beta_2 \tau_2'$; then the element $u = (\beta_4)^{-1} \circ \beta_2$ is composition of the inversions in the spheres $I(\beta_2)$, $I(\beta_4)$. Hence u is a parabolic transformation conjugate to Euclidean translation; $u(x) = x$.

3.7. LEMMA 4. *Let G^n be the conformal extension of G' to the space $\mathbb{R}^n$ $(n \geq 3)$. Then the point x is a parabolic cusp point for the group G^n. If $g \in G^n$ stabilizes the point x, then $g \in \langle u \rangle$.*

PROOF. First we construct a cusp-neighborhood of the point x in $\mathbb{R}^3$. Let l be the straight line which passes through x and is orthogonal to Π_2. Let Δ be a closed disk with the following properties (see Figure 2):

(i) $\Delta \subset \{(0, x_2, x_3)\colon (x_2, x_3) \in \mathbb{R}^2\}$;
(ii) $\partial\Delta$ touches l at x;
(iii) the diameter of Δ is less than the radius of $I(\beta_2)$;
(iv) $\Delta \cap L_2' = \varnothing$.

Then we define $\mathcal{O}$ to be the open body in $\mathbb{R}^3$ obtained by rotating $\operatorname{int}(\Delta)$ around l. Obviously we have $u(\mathcal{O}) = \mathcal{O}$. Next we shall prove $\mathcal{O}$ to be precisely invariant under $\langle u\rangle$ in G'.

By Lemmas 2 and 3, the intersection $\mathcal{O}_- = \mathcal{O} \cap \operatorname{cl}\operatorname{ext} I(\beta_2) \cap \operatorname{cl}\operatorname{ext} I(\beta_4)$ lies in D and contains no G'-equivalent points.

Let $w\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the translation $w\colon A \mapsto A + (y - x)$. Then the set $w(\mathcal{O}_-)\backslash(I(\beta_2^{-1}) \cup I(\beta_4^{-1})) = \mathcal{O}_-'$ is also contained in D.

Hence, $\mathcal{O}_- \cup \beta_2^{-1}(\mathcal{O}_-') = \mathcal{O}_+$ contains no G'-equivalent point; this set is a fundamental domain for the action of $\langle u\rangle$ in $\mathcal{O}$. Thus $\mathcal{O}$ is precisely invariant under $\langle u\rangle$, x is a cusp point for G', and $\langle u\rangle$ is a maximal elementary subgroup of G'.

It remains to verify the statement concerning the conformal extension G^n $(n \geq 4)$ of the group $G^3 = G'$. The parabolic transformation $u \in G^n$ is conjugate to a Euclidean translation. Then the existence of a precisely invariant cusp-neighborhood $\mathcal{O}_n$ of the point x (with respect to G^n, $n \geq 4$) easily follows from the properties of $\mathcal{O}$ and [W]. Lemma 4 is proved. □

3.8. Let $F' = \langle\Phi_1', \Phi_2'\rangle$ (see 3.4). Then Lemma 3 implies that F' is normal in G' and $G'/F' \cong \mathbb{Z}$. Also, we have that F' is a finitely generated free group; $G' = \langle F', t_2\rangle$, $\{\beta_2, \beta_4\} \subset F'$. Hence, the element $u = \beta_2 \circ (\beta_4)^{-1}$ is contained in F'.

Next, we put $u_m = t_2^m\, u\, t_2^{-m}$, $m \in \mathbb{Z}$. Every u_m belongs to F'. Also, if $g'(x) = x$, $g' \in G'$, then $g' \in \langle u\rangle$. Suppose for a moment that $g\, u_m\, g^{-1} = u_k$, $g \in F'$. Then $(t_2^{-k} g\, t_2^m) u (t_2^{-m} g^{-1}\, t_2^k) = u$ and $t_2^{-k} g\, t_2^m(x) = x$. Hence $t_2^{-k} g\, t_2^m = u^r$ and $t_2^{m-k} \in F'$. However, it now follows that $t_2^{m-k} = 1$, $m = k$, and $u_m = u_k$.

Thus, the parabolic groups $\langle u_k\rangle$, $k \in \mathbb{Z}$, are maximal nonconjugate parabolic subgroups of $F' = K_3$ and we obtain property (a) (see the main theorem).

3.9. The point x admits a precisely invariant cusp-neighborhood $\mathcal{O} \subset \mathbb{R}^3$ with respect to the group G'. Hence [W] implies that x has a precisely invariant horoball neighborhood U_0 in $\mathbb{H}^4$. The desired horoball neighborhoods $U_i \subset \mathbb{H}^4$ of the points $t_2^i(x)$ are equal to $t_2^i(U_0)$. Thus, we have proved property (b) of the group $K_3 = F'$.

REMARK. In fact, the configuration in Figure 1 is familiar from the theory of planar b-groups; the element u can be considered to be an accidental parabolic element. The only unusual thing is that the action $\operatorname{Ad}(t)$ does not preserve the F'-conjugacy class of $\langle u\rangle$.

3.10. Proof of assertion (c) of the theorem. Every point $x_i = t_2^i(x)$ $(i \in \mathbb{Z})$ has a precisely invariant cusp-neighborhood $O_{n,i} = t_2^i(O_n) \subset \mathbb{R}^n$ such that $O_{n,i} \cap O_{m,i} = \varnothing$ if $i \neq j$. Each neighborhood $O_{n,i}$ is conformally equivalent to $\left(\mathbb{R}^n\backslash\text{solid cylinder}\right) = \left(\mathbb{R}^{n-1}\backslash\text{unit ball}\right) \times \mathbb{R} = [0, \infty) \times S^{n-2} \times \mathbb{R}$. The projection $E(n, i)$ of $O_{n,i}$ to the manifold $M(K_n) = \Omega(K_n)/K_n$ is homeomorphic to $[0, \infty) \times S^{n-2} \times S^1$. Hence, $\partial E(n, i) = S^{n-2} \times S^1$, $E(n, i)$ represents one end $[E(n, i)]$ of the manifold $M(K_n)$,

$i \in \mathbb{Z}$; $[E(n,i)] \neq [E(n,j)]$ if $i \neq j$. Hence, the cycles $[\partial E(n,i)] \in H_{n-1}(M(K_n), \mathbb{Q})$ are independent and $\operatorname{rank}\left(\mathbb{H}_{n-1}(\Omega(K_n)/K_n, \mathbb{Q})\right) = \infty$. So the theorem is completely proved.

§4. Proof of Corollary 1

Corollary 1 directly follows from part (c) of the theorem.

§5. Proof of Corollary 2

5.1. First we construct a sequence of representations $\rho_q : K_3 \to \operatorname{Mob}(S^3)$ such that $\lim_{q\to\infty} \rho_q = \rho_\infty = \mathrm{id}$ and $\operatorname{order}(\rho_q(u_m)) = q$.

Let $\Sigma_1(s)$ be the family of Euclidean spheres tangent to each other at the point p_1, $\operatorname{radius}(\Sigma_1(s)) = (\lambda - 1)s + 1 = r(s)$, $-1 \le s \le 1$. Define $p_2(s)$ to be the point of $\Sigma_1(s)$ with coordinates $(0, r(s), *)$. Choose a parabolic transformation ξ_s that commutes with H_1' and maps p_2 to $p_2(s)$. Let $\beta_2(s) = \xi_s\beta_2\xi_s^{-1}$. It is easy to see that the isometric spheres $I(\beta_2(s))$, $I(\beta_2^{-1}(s))$ intersect L_2 at equal angles $\varphi(s)$; $\varphi(0) = 0$, $\varphi(1) = \pi/2$, φ is continuous function. Let $0 \le s(q) \le 1$ be a sequence of numbers such that $\varphi(s(q)) = \pi/2q$.

Let $\rho_q\colon \Gamma_1' \to \operatorname{Mob}(S^3)$ be the representation given by $\rho_q(\gamma) = \xi_{s(q)}\,\gamma\,\xi_{s(q)}^{-1}$. Hence, $\rho_q|_{H_1'} = \mathrm{id}$. Define $\rho_q\colon G_1' \to \operatorname{Mob}(S^3)$ to be a homomorphism such that $\rho_q|_{\Gamma_2'} = \mathrm{id}$. Next, $\rho_q\colon G_2' \to \operatorname{Mob}(S^3)$ is given by the formula $\tau_2'\rho_q(\tau_2' g \tau_2')\tau_2' = \rho_q(g)$, $g \in G_2'$.

Clearly, ρ_q is a homomorphism and $\lim_{q\to\infty} \rho_q = \rho_\infty = \mathrm{id}$.

The element $\rho_q(u)$ is the composition of inversions in the two spheres $I(\rho_q(\beta_2))$, $I(\rho_q(\beta_4))$, which intersect at the angle π/q. Hence $\rho_q(u)$ is a rotation of order q around the circle $\mathscr{L}_q = I(\rho_q(\beta_2)) \cap I(\rho_q(\beta_4))$.

5.2. In this subsection we consider the geometric and algebraic properties of the group $\langle \rho_q(\Gamma_1'), \rho_q(\Gamma_4')\rangle$ for large q.

Define Γ_{14} to be the group $\langle \Gamma_1', \Gamma_4'\rangle$, $\Gamma_i(q) = \rho_q\Gamma_i'$, $\Gamma_{14}(q) = \langle \Gamma_1(q), \Gamma_4(q)\rangle$.

There exists a number $q_0 \in \mathbb{N}$ such that for every $q \ge q_0$ $\partial P(\Gamma_1(q)) \cap \partial P(\Gamma_4(q)) = \partial P\left(\langle \rho_q(\beta_2)\rangle\right) \cap \partial P\left(\langle \rho_q(\beta_4)\rangle\right) \subset L_2$ (this follows directly from Lemma 1). Define $Q_{14}(q) = P\left(\langle \rho_q(\Gamma_1')\rangle\right) \cap P\left(\langle \rho_q(\Gamma_4')\rangle\right)$, $q_0 \le q \le \infty$.

Lemma 5. *For every $q \ge q_0$ we have*

(i) *The polyhedron $Q_{14}(q)$ is fundamental for the group $\Gamma_{14}(q)$;*

(ii) *The circle $\mathscr{L}_q$ has a regular neighborhood $\mathscr{N}(\mathscr{L}_q)$ which is precisely invariant under $\langle \rho_q(u)\rangle$ in $\Gamma_{14}(q)$ $(q < \infty)$;*

(iii) *The group $\Gamma_{14}(q)$ is isomorphic to*

$$(\Gamma_{14}(\infty) = \Gamma_1' * \Gamma_4')/\langle\langle u^q\rangle\rangle.$$

Proof. Statement (i) follows from the Poincaré theorem on fundamental polyhedra **[Mk]**. Statement (i) implies (ii) in the same way as the properties of G' imply Lemma 4.

Consider (iii). Every relation in $\Gamma_{14}(q)$ follows from relations corresponding to edge cycles on $\partial Q_{14}(q)$. Let $c = \{e_1, \dots, e_p\}$ be any edge cycle on $\partial Q_{14}(q)$. Then we have three possibilities:

(a) $c \subset \partial P(\Gamma_1(q))$,
(b) $c \subset \partial P(\Gamma_4(q))$,
(c) $c = \mathscr{L}_q \cup \rho(\beta_4)(\mathscr{L}_q)$ (if $q < \infty$).

In the cases (a),(b) the relation $R(c)$ follows from the relations of the groups Γ_1', Γ_4', respectively. In the case (c) we have the relation $(\rho_q(u)^q)$.

The lemma is proved. □

5.3. Lemma 6. *The half-space $\Pi_1 \cup \Pi_1^- = V_1$ is precisely invariant in the group $\Gamma_{14}(q)$ under the subgroup*

$$H(q) = H_1' * \rho_q(\tau_2' H_1' \tau_2').$$

Proof. First consider the plane domain

$$\mathscr{R}_q = \Big(P(H(q)) = P(H') \cap P\big(\rho_q(\tau_2' H_1' \tau_2')\big)\Big) \cap \Pi_1.$$

The Klein Combination Theorem implies that $\mathscr{R}_q$ is an isometric fundamental domain for the action of $H(q)$ in Π_1. It follows from Lemma 1 and Lemma 2 that the domain $\mathscr{R}_q$ is contained in $Q_{14}(q)$. Hence, the sphere Π_1 is precisely invariant under $H(q)$ (cf. **[K-P 1]**). The projection of $\mathscr{R}_q$ to $M(\Gamma_{14}(q)) = \Omega(\Gamma_{14}(q))/\Gamma_{14}(q)$ is a regular compact surface $\widehat{\mathscr{R}}_q$. There are two cases:

(a) $\widehat{\mathscr{R}}_q$ divides $M(\Gamma_{14}(q))$,
(b) $\widehat{\mathscr{R}}_q$ does not divide $M(\Gamma_{14}(q))$.

Consider (a). Then there exists an element $\gamma \in \Gamma_{14}(q)$ such that $\gamma(\Pi_1^+) \subset V_1$, where $\Pi_1^+ = \tau_1(\Pi_1^-)$. Note that $\Pi_1^+ \supset \operatorname{int}\Lambda(\Gamma_1(q)) \cup \operatorname{int}\Lambda(\Gamma_4(q))$. It is easy to see that $g(\Pi_1^-) \cap \Big(\operatorname{int}\Lambda(\Gamma_1(q)) \cup \operatorname{int}\Lambda(\Gamma_4(q))\Big) = \varnothing$ for every $g \in \Gamma_{14}(q)$. Hence, for the element γ we have $\gamma(\Pi_1^+) \supset \operatorname{int}\Lambda(\Gamma_1(q)) \cup \operatorname{int}\Lambda(\Gamma_4(q))$, so that $\gamma(\Pi_1^+) \cap \Pi_1^+ \neq \varnothing$. This contradiction shows that the case (a) does not hold.

Consider (b). Then there is an element $\gamma \in \Gamma_{14}(q)$ such that $\gamma(\Pi_1^-)$ is the complement to an open ball in ext $\Lambda(\Gamma_1(q)) \cap$ ext $\Lambda(\Gamma_4(q))$. Then $\gamma(\Pi_1^-) \supset \operatorname{int}\Lambda(\Gamma_1(q)) \cap \operatorname{int}\Lambda(\Gamma_4(q))$ and we get a contradiction as above.

The lemma is proved. □

5.4. Lemma 7.

(i) *The pair of groups $(\Gamma_{14}(q), \langle\Gamma_2', \Gamma_3'\rangle)$ with amalgamated subgroup $H(q)$ satisfies the conditions of the first Maskit Combination Theorem.*
(ii) *The group $G(q) = \rho_q(G')$ is isomorphic to $G'/\langle\langle u^q \rangle\rangle$.*
(iii) *The regular neighborhood $\mathscr{N}(\mathscr{L}_q)$ of $\mathscr{L}_q$ is precisely invariant in $G(q)$ under $\langle \rho_q(u)^q \rangle$.*

Proof. As we have proved in Lemma 6, the half-space V_1 is precisely invariant in $\Gamma_{14}(q)$ under $H(q)$. Consider the group $\Gamma_{23}(q) = \Gamma_{23} = \langle \Gamma_2', \Gamma_3' \rangle$. The sets $P_2 = T(P)$ and $\tau' T(P) = P_3$ are fundamental for the groups Γ_2', Γ_3', respectively (see 3.2). We have $P_2 \cap L_2' = P_3 \cap L_2' = P(J) \cap L_2'$. Hence, Γ_{23} results from the Maskit combination of the groups Γ_2', Γ_3' along the subgroup J. The domain $Q_{23} = P_2 \cap P_3$ is fundamental for the group Γ_{23}. The property (d) of the group Γ_1' implies that

$$\Pi_1 \cap Q_{23} = \Pi_1 \cap P(H(q)).$$

Similarly to Lemma 6, we see that $W_1 = \Pi_1 \cup \Pi_1^+$ is precisely invariant under $H(q)$ in Γ_{23}. Then assertion (1) is true. As a fundamental set for the group

$$G(q) = \Gamma_{14}(q) *_{H(q)} \Gamma_{23}$$

we shall use $Q_{23} \cap Q_{14}(q) = Q(q)$. Then Lemma 5 implies assertion (iii).

Next we note that

$$\Gamma(q) \cong (\Gamma_1 * \Gamma_4/\langle\langle u^q \rangle\rangle) *_{H(q)} \Gamma_{23} \quad \text{and} \quad G' \cong \Gamma_1 * \Gamma_4 *_{H(q)} \Gamma_{23}.$$

Hence $G(q) \cong G'/\langle\langle u^q \rangle\rangle$.

The lemma is proved. □

5.5. Now, using assertion (ii) of the previous lemma, arguing analogously as in subsection 3.8 we deduce that the elliptic elements $\rho_q(u_k)$, $\rho_q(u_m)$ are not conjugate for any $q_0 \leq q < \infty$, $\quad m, k \in \mathbb{Z}, m \neq k$. So the finitely generated Kleinian group $\rho_q(F')$ contains infinitely many conjugacy classes of finite order elements.

Consider the representation $\psi_q = \rho_q|_{F'}$. We know that $\mathrm{Ker}(\rho_q) : G' \to G(q)$ is the normal closure of $\langle u_q \rangle$ in G'. Since $\langle u \rangle \subset F'$ and F' is normal in G', we obtain that $\mathrm{Ker}(\psi_q)$ is the normal closure of $\bigcup_{m\in\mathbb{Z}} t_2^m \langle u^q \rangle t_2^{-m}$ in F'. So the group $\psi_q(F')$ has the presentation

$$\langle x_1, \dots, x_r : u_m^q, \ m \in \mathbb{Z} \rangle .$$

It is easy to see that the elements β_2, β_4 can be included in a system of free generators of the free group F'. Hence, the same is true for the element $u = \beta_4^{-1}\beta_2$. Then the group $\psi_q(F')$ has the desired representation

$$\langle x_1, \dots, x_r : \theta^m(x_1^q) = 1, \ m \in \mathbb{Z} \rangle \cong F_{r,q},$$

where θ is the automorphism of free group F_r induced by $\mathrm{Ad}(t_2)$. Corollary 2 is proved. □

§6. Description of the quotient manifolds

In this section we describe briefly the topology of the manifolds $M(K_3)$ and $O(F_{r,q}) = \Omega(F_{r,q})/F_{r,q}$, $q \geq q_0$.

6.1. We start with the sphere $\Sigma_i(s)$ for some $s > 0$, $\quad s < 1/(\lambda - 1)$. Then we construct a representation $\rho_s : \ G' \to G(s) \subset \mathrm{Mob}(S^3)$ in the same manner as in subsection 5.1. A fundamental polyhedron $Q(s)$ for the group $G(s)$ is equal to $P_2 \cap P_3 \cap P(\xi_s \Gamma_1' \xi_s^{-1}) \cap P(\tau_2' \xi_s \Gamma_4' \xi_s^{-1} \tau_2')$. As in Lemma 7, we conclude that $G(s)$ is a Kleinian group. Let $\Omega(s)$ be the infinite component of $\Omega(G(s))$. Then, according to the Maskit Combination Theorem, the manifold $M(s) = \Omega(G(s))/G(s)$ is homeomorphic to a fiber bundle over S^1 (cf. Lemma 3 in **[K-P 1]**). Let $x_2(s)$, $y_2(s) = \rho_s(\beta_2)(x_2(s))$ be distinct points of $\partial P(\rho_s(\Gamma_1')) \cap \partial P(\langle \rho_s(\beta_2) \rangle)$; $x_4(s) = \tau_2'(x_2(s))$ and $y_2(s) = \tau_2'(y_2(s))$. Join the pair of points $x_2(s)$ and $x_4(s)$ by a Euclidean segment I_s. Join the pair of points $y_2(s)$ and $y_4(s)$ by a Euclidean segment I_s'.

The union $I_s \cup I_s'$ projects to a circle C in $M(s)$.

6.2. Proposition 1. *The manifold Ω'/G' is homeomorphic to $M(s) \backslash C$; $\pi_1(\Omega')$ $= 1$.*

Proof. Note that the removal of $I_s \cup I_s'$ from $Q(s)$ is equivalent to deleting $\{x, y\}$ from $\mathrm{cl}(Q(\infty) = P(G'))$. Then the proof is concluded as in **[K-P 1, Lemma 3]**.

6.3. Let $\mathfrak{F}$ be a fiber of the manifold $M(s)$. The cyclic covering $\eta_s : \Omega(s)/\rho_s F' \to \Omega(s)/G'(s)$ is induced by the embedding $\pi_1(\mathfrak{F}) \to \pi_1(M(s))$. The manifold $X_r = \Omega(s)/\rho_s F'$ is homeomorphic to the handlebody $\mathfrak{F} \times \mathbb{R}$ and $\pi_1(\mathfrak{F})$ is a free group of rank r. Then the manifold Ω'/F' is homeomorphic to $X_r \backslash (\eta_s^{-1}(C) = L_\infty)$, which is the complement to the infinite-component link in the handlebody.

Let $\mathcal{N}(C)$ be an open regular neighborhood of the knot $C, M^-(s) = M(s) \backslash \mathcal{N}(C)$.

6.4. PROPOSITION 2. *Let Ω_q be the infinite component of $\Omega(G(q))$. Then the orbifold $\Omega_q/G(q)$ is homeomorphic to $M^-(s) \bigcup_{\partial \mathcal{N}(C)} D(q) \times S^1$, where $D(q)$ is 2-disc with one singular point of order q.*

PROOF. Evident (see Lemmas 5–7 and the considerations above). □

6.5. Thus, the quotient orbifold $\Omega_q/F(q)$ is supported by an open handlebody X_r and the singular set is the infinite-component link L_∞ of order q.

REMARK. Every quotient manifold $\Omega(F(q))/F(q)$ $(q_0 \le q \le \infty)$ has also four components besides $\Omega_q/F(q)$. These components are homeomorphic to the handlebody $\operatorname{int}\big(\Lambda(F(q))\big)/(F(q) \cap \Gamma_i(q))$, $i = 1, \dots, 4$.

§7. Finite order elements in discrete subgroups of $PSL(2, \mathbb{C})$

PROPOSITION 3. *Let Γ be any discrete finitely generated subgroup of $PSL(2, \mathbb{C})$. Then Γ contains only a finite number of conjugacy classes of finite order elements.*

PROOF. Denote by $M(G)$ the factor orbifold for a discrete group G in $PSL(2, \mathbb{C})$. Let $\Gamma_0 < \Gamma$ be a torsion-free finite index normal subgroup in Γ (which exists according to Selberg's lemma). Then the finite group $F = \Gamma/\Gamma_0$ acts on the manifold $M(\Gamma_0)$, preserving the peripheral structure of the fundamental group $\pi_1(M(\Gamma_0)) \cong \Gamma_0$. Then it is easy to see that the manifold $M(\Gamma_0)$ has a compact Scott core $C(\Gamma_0)$ **[Sc]** invariant under F. The compact orbifold $O(\Gamma) = C(\Gamma_0)/F$ has the singular set $\Sigma(\Gamma)$, which is a finite graph. Vertices of $\Sigma(\Gamma)$ and vertex-free components of it are in one-to-one correspondence to the Γ-conjugacy classes of maximal finite order subgroups of Γ. Hence, Γ contains only finitely many conjugacy classes of finite order subgroups. The proposition is proved. □

§8. On the Abikoff conjecture about noncone limit points for finitely generated discrete subgroups of $PSL(2, \mathbb{C})$

8.1. Let $\Gamma \subset \operatorname{Mob}(S^n)$ be a discrete group. Then there is a hierarchy of limit points $x \in \Lambda(\Gamma)$ according to the way in which they can be approximated by an orbit $\Gamma(z)$, $z \in H^{n+1}$. A point x is said to be an approximation or cone limit point if up to an infinite subsequence the family $\Gamma(z)$ lies in some Euclidean cone $K \subset H^{n+1}$. This may be considered as the best (fastest) approximation. The set of cone limit points is denoted by $\Lambda_c(\Gamma)$. The dynamics of Γ near an approximation point is similar to the dynamics near a fixed point of a loxodromic element. In contrast, parabolic fixed points cannot be cone limit points. Unlike cone limit points, nonapproximation limit points are a much more intriguing matter. In a very instructive survey **[Ab 2]** Abikoff proposed the following

CONJECTURE. *Let $\Gamma \subset PSL(2, \mathbb{C})$ be any finitely generated discrete group. Then $(\Lambda(\Gamma) \backslash \Lambda_c(\Gamma))/\Gamma$ is a finite set.*

REMARK. This conjecture was motivated by Sullivan's finiteness theorem (parabolic fixed points modulo Γ form a finite set).

8.2. The main aim of this section is to disprove the above conjecture. There are two types of counterexamples. In fact, both are from Abikoff's paper.

8.3. Next, we recall some constructions from **[Ab 2]**. Consider a Fuchsian torsion-free group $F \subset PSL(2,\mathbb{R})$ such that $S = H^2/F$ is a compact surface. Let $h : F \to F$ be a pseudo-Anosov homomorphism with attractive and repulsive foliations $\mathcal{L}$ and $\mathcal{L}^*$ respectively. We realize H^2 as the unit disk $\Delta \subset \overline{\mathbb{C}}$, $\Delta^* = \text{ext}(\Delta)$. Lift the foliations $\mathcal{L}$, $\mathcal{L}^*$ to foliations $\mathcal{N} \subset \Delta$, $\mathcal{N}^* \subset \Delta^*$ invariant under F. Add to any geodesic in $\mathcal{N}$, $\mathcal{N}^*$ its end-points; the resulting "foliations" will be denoted by $\mathcal{J}$, $\mathcal{J}^*$. Consider two equivalence relations on $\overline{\mathbb{C}}$:

(1) $x \sim y$, if x and y belong to a path-connected component of $\mathcal{J}$ or to the closure in $\mathbb{C}$ of some component of $\Delta \backslash \mathcal{J}$;

(2) $x \approx y$, if x and y belong to a path-connected component of $\mathcal{J} \cup \mathcal{J}^*$ or to the closure in $\overline{\mathbb{C}}$ of some component of $\Delta \backslash (\mathcal{J} \cup \mathcal{J}^*)$.

These equivalence relations are invariant under the action of F. Hence, the action of F descends to $\overline{\mathbb{C}}/\sim$ and $\overline{\mathbb{C}}/\approx$. Thus, we obtain topological models for the action on $\overline{\mathbb{C}}$ of singly (case 1) and doubly (case 2) degenerate groups **[Ab 2]**. Let us denote the corresponding discrete subgroups of $PSL(2,\mathbb{C})$ by F_1 and F_2; let ζ_1 be the projection: $\overline{\mathbb{C}} \to \overline{\mathbb{C}}/\sim$, $\zeta_2 : \overline{\mathbb{C}} \to \overline{\mathbb{C}}/\approx$.

PROPOSITION 4. *The sets* $\Lambda_0(F_i) = \Lambda(F_i)\backslash\Lambda_c(F_i)$ *contain continua* $\zeta_i(\overline{\mathbb{C}}\backslash\partial\Delta) \cap \Lambda(F_i)$.

REMARK. Indeed, $\zeta_1(\Delta) = \Lambda(F_1)\backslash\Lambda_c(F_1)$ consists of the end-points of the tree $\Lambda(F_1)$.

PROOF OF PROPOSITION 4. We shall discuss only case (1); the second case is essentially similar. □

8.4. First we recall another definition of approximation points **[B-M]**.

Let $\Gamma \subset \text{Mob}(S^n)$ be a discrete group, $x \in \Lambda(\Gamma)$. Then $x \in \Lambda_c(\Gamma)$ if and only if there exists an infinite sequence $\{\gamma_m\} \subset \Gamma$ such that for every point $y \in S^n$

(a) the limit $\lim_{m\to\infty} \gamma_m(y) = y^*$ exists,

(b) $y^* \neq x^*$ for all $y \neq x$, and

(c) the point $z = y^*$ is one and the same for all $y \neq x$.

8.5. Suppose that $x \in \Lambda_c(F_1) \cap \zeta_1(\Delta)$. Hence $x = \zeta_1(l)$, where l is some geodesic in $\mathcal{J}$ and x admits a sequence $\{f_m\}$ with the properties (a)–(c) above. We shall denote by $\{f_m\}$ the corresponding sequence in F. Let $\{\alpha, \beta\}$ be the set of end-points of l. The sequence $\{f_m\}$ is not relatively compact in $PSL(2,\mathbb{C})$; hence there exists a subsequence $\{f_{m_s}\} \subset \{f_m\}$ and points v, $w \in \Lambda(F)$ such that:

$$\lim_{s\to\infty} \{f_{m_s}(z)\} = w \in \Lambda(F) \quad \text{for every } z \in \overline{\mathbb{C}}\backslash\{v\}.$$

Then for point α or β (say α) we have that $w = \lim_{s\to\infty}\{f_{m_s}(z)\} = \lim_{s\to\infty}\{f_{m_s}(\alpha)\}$ for all but one point $z \in \overline{\mathbb{C}}$. Consequently $\lim_{s\to\infty}\{f_{m_s}\zeta_1(x)\} = \lim_{s\to\infty}\{f_{m_s}\zeta_1(\alpha)\} = \lim_{s\to\infty}\{f_{m_s}\zeta_1(z)\}$. This contradiction proves that x is not an approximation limit point.

8.6. Now we prove that every $\widehat{x} \in \Lambda(F_1)\backslash\zeta_1(\Delta)$ is an approximation point. Let $x = \zeta_1^{-1}(\widehat{x})$; clearly, this is not an end-point of any geodesic from $\mathcal{F}$. The group F is geometrically finite and it is easy to construct a sequence $\{f_m\} \subset F$ possessing the properties (a)–(b) with respect to x, such that

(d) y^* is not an end-point of geodesics from the foliation $\mathcal{F}$.

Then x^* is not equivalent to the point y^*. So, the sequence $\{f_m\} \subset F_1$ has the properties (a)–(c) with respect to $\widehat{x}$.

The proposition is proved. □

REMARK. The results of 8.5 can be proved more geometrically without passing to a Fuchsian group, as in **[Ab 2]**. However that proof cannot be generalized to the case of doubly degenerate groups.

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