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Lectures on quasi-isometric rigidity

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Introduction: What is Geometric Group Theory?

Historically (in the 19th century), groups appeared as automorphism groups of certain structures:

- Polynomials (field extensions) — Galois groups.
- Vector spaces, possibly equipped with a bilinear form — Matrix groups.
- Complex analysis, complex ODEs — Monodromy groups.
- Partial differential equations — Lie groups (possibly infinite-dimensional ones)
- Various geometries — Isometry groups of metric spaces, both discrete and nondiscrete.

A goal of Geometric Group Theory (which I will abbreviate as GGT) is to study finitely-generated groups G as automorphism groups (symmetry groups) of metric spaces X.

Accordingly, the central question of GGT is: *How are the algebraic properties of a group G reflected in the geometric properties of a metric space X and, conversely, how is the geometry of X reflected in the algebraic structure of G?*

This interaction between groups and geometry is a fruitful two-way road. An inspiration for this viewpoint is the following (essentially) bijective correspondence (established by E. Cartan):

Simple noncompact connected Lie groups $\longleftrightarrow$ Irreducible symmetric spaces of noncompact type.

Here the correspondence is between algebraic objects (Lie groups of a certain type) and geometric objects (certain symmetric spaces). Namely, given a Lie group G one constructs a symmetric space $X = G/K$ (K is a maximal compact subgroup of G) and, conversely, every symmetric space corresponds to a Lie group G (its isometry group) and this group is unique.

Imitating this correspondence is an (unreachable) goal of GGT.

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LECTURE 1

Groups and spaces

Convention. *Throughout these lectures, I will be working in ZFC: Zermelo–Fraenkel Axioms of set theory + Axiom of Choice.*

1. Cayley graphs and other metric spaces

Recall that we are looking for a correspondence:

$$\textbf{Groups} \longleftrightarrow \textbf{Metric Spaces}$$

The first step is to associate with a finitely-generated group G a metric space X. Let G be a group with a finite generating set $S = \{s_1, ..., s_k\}$. Then we construct a graph X, whose vertex set $V(X)$ is the group G itself and whose edges are

$$[g, gs_i], s_i \in S, g \in G.$$

(If $gs_i = gs_j$, i.e., $s_i = s_j$, then we treat $[g, gs_i], [g, gs_j]$ as distinct edges, but this is not very important.) We do not orient edges.

The resulting graph $X = \Gamma_{G,S}$ is called a *Cayley graph* of the group G with respect to the generating set S. Then the group G acts (by multiplication on the left) on X: Every $g \in G$ defines a map

$$g(x) = gx, \quad x \in V(X) = G.$$

Clearly, edges are preserved by this action. Since S is a generating set of G, the graph X is connected.

We now define a metric on the graph $X = \Gamma_{G,S}$. If X is any connected graph, then we declare every edge of X to have unit length. Then we have a well-defined notion of length of a path in X. The distance between vertices in X is the length of the shortest edge-path in X connecting these points.

Exercise 1.1. *Shortest edge-paths always exist.*

One can also think of the graph X as a cell complex, which we then conflate with its geometric realization (a topological space). Then, one can talk about points in X which lie in the interiors of edges. We then identify each edge with the unit interval and extend the above metric to all of X. As we will see, later, this distinction between the metric on $V(X)$ and the metric on X is not very important. The metric on $G = V(X)$ is called a *word-metric* on G. Here is why:

Example 1.2. *Let X be a Cayley graph of a group G. The distance $d(1, g)$ from $1 \in G$ to $g \in G$ is the same thing as the "norm" $|g|$ of g, the minimal number m of symbols in the decomposition (a "word in the alphabet $S \cup S^{-1}$")*

$$g = s_{i_1}^{\pm 1} s_{i_2}^{\pm 1} ... s_{i_m}^{\pm 1}$$

of g as a product of generators and their inverses. Note: If $g = 1$ then $m := 0$.

Thus, we have a correspondence: **Groups** $\longrightarrow$ **Metric spaces**,

$$\text{Cayley} : G \to X = \text{A Cayley graph of } G.$$

Is this the only correspondence? Is this map "Cayley" well defined?

We will see that both questions have negative answer and our first goal will be to deal with this issue. (Note that infinite finitely-generated groups G have infinitely many distinct finite generating sets.)

Definition 1.3. *Let X be a metric space and G be a group acting on X. The action $G \curvearrowright X$ is called* geometric *if:*

1. G acts isometrically on X.

2. G acts properly discontinuously on X, i.e., $\forall$ compact $C \subset X$, the set $\{g \in G : gC \cap C \neq \emptyset\}$ is finite.

3. G acts cocompactly on X: X/G is compact.

Informally, a group G is a group of (discrete) symmetries of X if G acts geometrically on X. (Note that in some natural situations in GGT one considers non-geometric actions of groups on metric spaces, but we will not address this in these lectures.)

Example 1.4. *Suppose G is a finitely-generated group and X is its Cayley graph. Then the action of G on X is geometric. Question: What is the quotient graph X/G?*

Other metric spaces which appear naturally in GGT are connected Riemannian manifolds (M, ds^2). In this case, the distance between points is

$$d(x,y) = \inf\{\int_p ds = \int_0^T |p'(t)|dt\}$$

where the infimum is taken over all paths p connecting x to y. When dealing with connected Riemannian manifolds we will always implicitly assume that they are equipped with the above distance function.

Example 1.5. *Suppose that M is a compact connected Riemannian manifold with the fundamental group $\pi = \pi_1(M)$, $X = \tilde{M}$ is the universal cover of M (with lifted Riemannian metric), π acts on X as the group of covering transformations for the covering $X \to M$. Then $\pi \curvearrowright X$ is a geometric action.*

More generally, let $\phi : \pi \to G$ be an epimorphism, $X \to M$ be the covering corresponding to $Ker(\phi)$. Then the group of covering transformations of $X \to M$ is isomorphic to G and, thus, G acts geometrically on X.

Note: For every finitely-generated group G there exists a compact Riemannian manifold M (of every dimension ≥ 2) with an epimorphism $\pi_1(M) \to G$. Thus, we get another correspondence **Groups** $\longrightarrow$ **Metric Spaces**:

$$\text{Riemann} : G \to X = \text{a covering space of some } M \text{ as above.}$$

Thus, we have a problem on our hands: We have too many candidates for the correspondence **Groups** $\to$ **Spaces** and these correspondences are not well-defined. What do different spaces on which G acts geometrically have in common?

2. Quasi-isometries

Definition 1.6. *a. Let X, X' be metric spaces. A map $f : X \to X'$ is called an (L, A)-*quasi-isometry *if:*

1. f is (L, A)-coarse Lipschitz:

$$d(f(x), f(y)) \leq Ld(x,y) + A.$$

2. There exists an (L, A)-coarse Lipschitz map $\bar{f} : X' \to X$, which is "quasi-inverse" to f:

$$d(\bar{f}f(x), x) \leq A, \quad d(f\bar{f}(x'), x') \leq A.$$

When the constants L, A are not important, we will simply say that f is a quasi-isometry.

b. Spaces X, X' are quasi-isometric *to each other if there exists a quasi-isometry $X \to X'$.*

Note that every $(L, 0)$-quasi-isometry f is a bilipschitz homeomorphism; if, in addition, $L = 1$, then f is an isometry.

Example 1.7. *1. Every bounded metric space is QI to a point.*

2. $\mathbb{R}$ is QI to $\mathbb{Z}$.

3. Every metric space is QI to its metric completion.

Here and in what follows I will abbreviate "quasi-isometry" and "quasi-isometric" to QI.

Exercise 1.8.

- *Every quasi-isometry $f : X \to X'$ is "quasi-surjective":* $\exists C < \infty | \forall x' \in X', \exists x \in X | d(x', f(x)) \leq C$.
- *Show that a map $f : X \to X'$ is a quasi-isometry iff it is quasi-surjective and is a "quasi-isometric embedding": $\exists L, \exists A$ so that $\forall x, y \in X$:*
$$\frac{1}{L} d(x,y) - A \leq d(f(x), f(y)) \leq L d(x,y) + A.$$
- *Composition of quasi-isometries is again a quasi-isometry.*
- *Quasi-isometry of metric spaces is an equivalence relation.*

Exercise 1.9. *1. Let S, S' be two finite generating sets for a group G and d, d' be the corresponding word metrics. Then the identity map $(G,d) \to (G,d')$ is an $(L,0)$-quasi-isometry for some L.*

2. G is QI to its Cayley graph X. The map $G \to X$ is the identity. What is the quasi-inverse?

Given a metric space X, we thus have a semigroup $\widehat{QI}(X)$ consisting of quasi-isometries $X \to X$. This semigroup, of course, is not a group, since quasi-isometries need not be invertible. However, one can form a group using $\widehat{QI}(X)$ as follows. We define the equivalence relation $\sim$ on $\widehat{QI}(X)$ by

$$f \sim g \iff dist(f,g) = \sup\{d(f(x), g(x)) : x \in X\} < \infty.$$

Then the quotient $QI(X) = QI(X)/_\sim$ is a group: If $\bar{f}$ is quasi-inverse to f, then

$$[f]^{-1} = [\bar{f}]$$

where $[h]$ denotes the projection of h to $QI(X)$.

Definition 1.10. *1. A* geodesic *in a metric space X is a distance-preserving map γ of an interval $I \subset \mathbb{R}$ to X. A* geodesic ray *is a geodesic whose domain in a half-line in $\mathbb{R}$. If $I = [a,b]$ then we will use the notation $\overline{pq}$, $p = \gamma(a), q = \gamma(b)$, to denote a geodesic connecting p to q. We will frequently conflate geodesics and their images.*

2. A metric space X is called geodesic *if for every pair of points $x, y \in X$, there exists a geodesic $\gamma : [0,T] \to X$, connecting x to y.*

3. A metric space X is proper *if every closed metric ball in X is compact.*

A subset N of a metric space X is called an *ϵ-separated R-net* if:

(1) For all $x \neq y \in N$, $d(x,y) \geq \epsilon$.
(2) For every $x \in X$ there exists $y \in N$ so that $d(x,y) \leq R$.

Here $\epsilon > 0, R < \infty$.

Exercise 1.11. *1. Let X be a Cayley graph of a group G. Then G is a separated net in X.*

2. Every metric space X admits a separated net. (You need Zorn's lemma to prove this.)

Definition 1.12. *Suppose that X is a proper metric space. A sequence (f_i) of maps $X \to Y$ is said to* coarsely uniformly converge to a map $f : X \to Y$ on compacts, *if:*

There exists a number $R < \infty$ so that for every compact $K \subset X$, there exists an i_K so that for all $i > i_K$,

$$\forall x \in K, \quad d(f_i(x), f(x)) \leq R.$$

To simplify the notation, we will say that $\lim^c_{i\to\infty} f_i = f$.

Note that the usual uniform convergence on compacts implies coarse convergence.

Proposition 1.13 (Arzela–Ascoli theorem for coarsely Lipschitz maps)**.** *Fix real numbers L, A and D and let X, Y be proper metric spaces so that X admits a separated R-net. Let $f_i : X \to Y$ be a sequence of (L_1, A_1)-Lipschitz maps, so that for some points $x_0 \in X, y_0 \in Y$ we have $d(f(x_0), y_0) \le D$. Then there exists a subsequence (f_{i_k}), and a (L_2, A_2)–Lipschitz map $f : X \to Y$, so that $\lim^c_{k\to\infty} f_i = f$. Furthermore, if the maps f_i are (L_1, A_1) quasi-isometries, then f is also an (L_3, A_3)–quasi-isometry for some L_3, A_3.*

Proof. Let $N \subset X$ be a separated net. We can assume that $x_0 \in N$. Then the restrictions $f_i|N$ are L'-Lipschitz maps and, by the usual Arzela–Ascoli theorem, the sequence $(f_i|N)$ subconverges (uniformly on compacts) to an L' -Lipschitz map $f : N \to Y$. We extend f to X by the rule: For $x \in X$ pick $x' \in N$ so that $d(x, x') \le R$ and set $f(x) := f(x')$. Then $f : X \to Y$ is an (L_2, A_2)–Lipschitz map. For a metric ball $B(x_0, r) \subset X, r \ge R$, there exists i_r so that for all $i \ge i_r$ and all $x \in N \cap B(x_0, r)$, we have $d(f_i(x), f(x)) \le 1$. For arbitrary $x \in K$, we find $x' \in N \cap B(x_0, r + R)$ so that $d(x', x) \le R$. Then

$$d(f_i(x), f(x)) \le d(f_i(x'), f(x')) \le L_1(R + 1) + A.$$

This proves coarse convergence. The argument for quasi-isometries is similar. □

Theorem 1.14 (Milnor–Schwarz lemma)**.** *Suppose that G acts geometrically on a proper geodesic metric space X. Then G is finitely generated and ($\forall x \in X$) the orbit map $f : g \mapsto g(x), f : G \to X$, is a quasi-isometry, where G is equipped with a word-metric.*

Proof. Our proof follows [**15**, Proposition 10.9]. Let $B = \bar{B}_R(x_0)$ be the closed R-ball of radius R in X centered at x_0, so that $B_{R-1}(x_0)$ projects onto X/G. Since the action of G is properly discontinuous, there are only finitely many elements $s_i \in G$ such that $B \cap s_iB \ne \emptyset$. Let S be the subset of G which consists of the above elements s_i (it is clear that s_i^{-1} belongs to S iff s_i does). Let

$$r := \inf\{d(B, g(B)), g \in G \setminus S\}.$$

Since B is compact and $B \cap g(B) = \emptyset$ for $g \notin S$, $r > 0$. We claim that S is a generating set of G and that for each $g \in G$

$$|g| \le d(x_0, g(x_0))/r + 1 \tag{1.1}$$

where $|\cdot|$ is the word length on G (with respect to the generating set S). Pick $g \in G$ and connect x_0 to $g(x_0)$ by a shortest geodesic γ in X. Let m be the smallest integer so that $d(x_0, g(x_0)) \le mr + R$. Choose points $x_1, ..., x_{m+1} = g(x_0) \in \gamma$, so that $x_1 \in B$, $d(x_j, x_{j+1}) < r$, $1 \le j \le m$. Then each x_j belongs to $g_j(B)$ for some $g_j \in G$. Let $1 \le j \le m$, then $g_j^{-1}(x_j) \in B$ and

$$d(g_j^{-1}(g_{j+1}(B)), B) \le d(g_j^{-1}(x_j), g_j^{-1}(x_{j+1})) < r.$$

Thus the balls $B, g_j^{-1}(g_{j+1}(B))$ intersect, which means that $g_{j+1} = g_js_{i(j)}$ for some $s_{i(j)} \in S$. Therefore

$$g = s_{i(1)}s_{i(2)}....s_{i(m)}.$$

We conclude that S is indeed a generating set for the group G. Moreover,

$$|g| \le m \le (d(x_0, g(x_0)) - R)/r + 1 \le d(x_0, g(x_0))/r + 1.$$

The word metric on the Cayley graph $\Gamma_{G,S}$ of the group G is left-invariant, thus for each $h \in G$ we have:

$$d(h, hg) = d(1, g) \le \frac{1}{r} d(x_0, g(x_0))/r + 1 = \frac{1}{r} d(h(x_0), hg(x_0)) + 1.$$

Hence for any $g_1, g_2 \in G$

$$d(g_1, g_2) \le \frac{1}{r} d(f(g_1), f(g_2)) + 1.$$

On the other hand, the triangle inequality implies that

$$d(x_0, g(x_0)) \le t|g|$$

where $d(x_0, s(x_0)) \le t \le 2R$ for all $s \in S$. Thus

$$\frac{1}{t} d(f(g_1), f(g_2)) \le d(g_1, g_2).$$

We conclude that the map $f : G \to X$ is a quasi-isometric embedding. Since $f(G)$ is R-dense in X, it follows that f is a quasi-isometry. $\square$

Thus, if instead of isometry classes of metric spaces we use their QI classes then both *Cayley* and *Riemann* correspondences are well-defined and are equal to each other! Now, we have a well-defined map

$$geo : \text{finitely-generated groups} \longrightarrow \text{QI equivalence classes of metric spaces.}$$

Problem: This map is very far from being 1-1, so our challenge is to "estimate" the fibers of this map.

Exercise 1.15. *Show that the half-line is not QI to any Cayley graph. Prove first that every unbounded Cayley graph contains an isometrically embedded copy of $\mathbb{R}$ (hint: use Arzela-Ascoli theorem). Then show that there is no QI embedding $f : \mathbb{R} \to \mathbb{R}_+$. Hint: Replace f with a continuous (actually, piecewise-linear) QI embedding h so that $d(f, h) \le C$ and then use the intermediate value theorem to get a contradiction.*

Example 1.16. *Every finite group is QI to the trivial group.*

In particular, from the QI viewpoint, the entire theory of finite groups (with its 150 year-old history culminating in the classification of finite simple groups) becomes trivial. Is this good news or is this bad news?

This does not sound too good if we were to recover a group from its geometry (up to an isomorphism). Is there a natural *algebraic* equivalence relation on groups which can help us here?

3. Virtual isomorphisms and QI rigidity problem

In view of Milnor-Schwarz lemma, the following provide examples of quasi-isometric groups:

1. If $G' < G$ is a finite-index subgroups then G is QI to G'. (G' acts on a Cayley graph of G isometrically and faithfully so that the quotient is a finite graph.)

2. If $G' = G/F$, where F is a finite group, then G is QI to G'. (G acts isometrically and transitively on a Cayley graph of G' so that the action has finite kernel.)

Combining these two examples we obtain

Definition 1.17. *1. G_1 is* virtually isomorphic *(which will be abbreviated as VI) to G_2 if there exist finite index subgroups $H_i \subset G_i$ and finite normal subgroups $F_i \triangleleft H_i, i = 1, 2$, so that the quotients H_1/F_1 and H_2/F_2 are isomorphic.*

2. A group G is said to be virtually *cyclic if it is VI to a cyclic group. Similarly, one defines virtually abelian groups, virtually free groups, etc.*

Exercise 1.18. *VI is an equivalence relation.*

To summarize: By the Milnor–Schwarz lemma,

$$VI \Rightarrow QI.$$

Thus, if we are to recover a group from its geometry (treated up to QI), then the best we can hope for is to recover the group up to VI. This is bad news for people in finite group theory, but good news for the rest of us.

Remark 1.19. *There are some deep and interesting connections between the theory of finite groups and GGT, but quasi-isometries do not see these.*

Informally, *quasi-isometric rigidity* is the situation when the arrow $VI \Rightarrow QI$ can be reversed.

Definition 1.20. *1. We say that a group G is QI rigid if every group G' which is QI to G, is in fact VI to G.*

2. We say that a class C of group is QI rigid if every group G' which is QI to some $G \in C$, there exists $G'' \in C$ so that G' is VI to G''.

3. A property P of groups is said to be "geometric" or "QI invariant" whenever the class of groups satisfying P is QI rigid.

Note that studying QI rigidity and QI invariants is by no means the only topic of GGT, but this will be the topic of my lectures.

4. Examples and non-examples of QI rigidity

At first glance, any time QI rigidity holds (in any form), it is a minor miracle: How on earth are we supposed to recover precise algebraic information from something as sloppy as a quasi-isometry? Nevertheless, instances of QI rigidity abound. We refer the reader to [**6**] for some of the proofs and further references:

Examples of QI rigid groups/classes/properties (all my groups are finitely-generated, of course):

- Free groups. (J. Stallings; see [**6**].)
- Free abelian groups. (M. Gromov; P. Pansu; see [**6**].)
- Class of nilpotent groups. (M. Gromov; see [**6**].)
- Class of fundamental groups of closed (compact, without boundary) surfaces. (This is a combination of work of P. Tukia; D. Gabai; A. Casson and D. Jungreis, see [**34**], [**14**], [**5**].)
- Class of fundamental groups of closed (compact, without boundary) 3-dimensional manifolds. (This is a combination of work of R. Schwartz [**28**]; M. Kapovich and B. Leeb [**19**]; A. Eskin, D. Fisher and K. Whyte [**11**], and, most importantly, the solution of the geometrization conjecture by G. Perelman.)
- Class of finitely-presentable groups. (M. Gromov; see [**6**].)
- Class of hyperbolic groups. (M. Gromov; see [**6**].)
- Class of amenable groups.
- Class of fundamental groups of closed n-dimensional hyperbolic manifolds. For $n \geq 3$ this result, due to P. Tukia [**33**], will be the central theorem of my lectures.
- Class of uniform lattices in each connected simple noncompact Lie group G (i.e., discrete cocompact subgroups Γ in G). (This is a combination of work of P. Pansu [**23**]; P. Tukia [**33**]; R. Chow [**4**]; B. Kleiner and B. Leeb [**20**]; A. Eskin and B. Farb [**10**].)
- Every non-uniform lattice in a connected simple Lie group (i.e., a discrete subgroup Γ in a simple noncompact Lie group G so that G/Γ has finite volume but noncompact). For instance, every group which is QI to $SL(n, \mathbb{Z})$ is in fact VI to $SL(n, \mathbb{Z})$. (This is a combination of work of R. Schwartz [**28**]; B. Farb and R. Schwartz [**13**]; A. Eskin [**9**]. We also refer to Farb's paper [**12**] for a summary in the uniform and nonuniform case.)
- Solvability of the word problem (say, for finitely-presented groups).
- Cohomological dimension over $\mathbb{Q}$. (R. Sauer, [**26**].)
- Admitting a "geometric" action on a contractible CW-complex (i.e., an action which is cocompact on each skeleton and is properly discontinuous). (See [**6**].)
- Admitting an amalgam decomposition (amalgamated free product or HNN decomposition) over a finite subgroup. (J. Stallings [**30**]; see also [**6**].)
- Admitting an amalgam decomposition over a virtually cyclic subgroup. (P.Papasoglou, [**24**].)

Rule of thumb: The closer a group (or a class of groups) is to a Lie group, the higher are the odds of QI rigidity.

Examples of failure of QI rigidity:

- Suppose that S is a closed oriented surface of genus ≥ 2 and $\pi = \pi_1(S)$. Then $\mathbb{Z} \times \pi$ is QI to any Γ which appears in any central extension
$$1 \to \mathbb{Z} \to \Gamma \to \pi \to 1.$$
The same holds if π is replaced by a nonelementary hyperbolic group. (This was independently observed by D.B.A. Epstein, S. Gersten, G. Mess; see [**6**].) For instance, the fundamental group Γ of the unit tangent bundle of S is realized this way.
- In particular, the property of being the fundamental group of a compact nonpositively curved Riemannian manifold with convex boundary is not QI invariant.

- There are countably many VI classes of groups which act geometrically on hyperbolic 3-space. All these groups are QI to each other by Milnor-Schwarz lemma. Same for all irreducible nonpositively curved symmetric spaces of dimension ≥ 3.
- Class of solvable groups is not QI rigid. (A. Erschler, [**7**].)
- Class of simple groups is not QI rigid: $F_2 \times F_2$ is QI to a simple group. (M. Burger and S. Mozes, [**2**].)
- Class of residually-finite groups is not QI rigid. (M. Burger and S. Mozes, [**2**].)
- Property (T) is not QI invariant. (S. Gersten and M. Ramachandran; see [**6**].)

Few open problems:

- Is the class of fundamental groups of closed aspherical n-dimensional orbifolds QI rigid?
- Is the class of polycyclic groups QI rigid? (Conjecturally, yes.)
- Is the class of elementary amenable groups QI rigid?
- Prove QI rigidity for various classes of Right-Angled Artin Groups (RAAGs): It is known that some of these classes are QI rigid but some are not (e.g., $F_2 \times F_2$).
- Are random finitely-presented groups QI rigid?
- Construct examples of QI rigid hyperbolic groups whose boundaries are homeomorphic to the Menger curve.
- Classify up to a quasi-isometry fundamental groups of compact 3-dimensional manifolds.
- Verify QI invariance of JSJ decomposition (in the sense of Leeb ad Scott [**22**]) of closed nonpositively curved Riemannian manifolds of dimesnion ≥ 4. (Note that the 3-dimensional case was done in [**19**].)
- Is the Haagerup property (see [**3**] for the definition) QI invariant?
- Is the Rapid Decay property (see e.g. [**21**] for the definition) QI invariant?
- Is the property of having uniform exponential growth QI invariant?
- Is the class of hyperbolic free-by-cyclic groups $F_n \rtimes \mathbb{Z}$ QI rigid ($n \geq 3$)?
- Is every group QI to a torsion-free group?
- Prove QI rigidity for classes of lattices (including reducible ones) in every semisimple Lie group G. Open cases include products of nonuniform lattices in rank 1 and higher rank Lie groups, e.g. $SL(2, \mathbb{Z}) \times SL(n, \mathbb{Z}), n \geq 3$.

Where do the tools of GGT come from? From almost everywhere! Here are some examples:

- Group theory (of course)
- Geometry (of course)
- Topology (point-set topology, geometric topology, algebraic topology)
- Lie theory
- Analysis (including PDEs, functional analysis, real analysis, complex analysis, etc.)
- Probability
- Logic
- Dynamical systems
- Homological algebra
- Combinatorics

In these lectures, I will introduce two tools of QI rigidity: Ultralimits (coming from logic) and quasiconformal maps (whose origin is in geometric analysis and complex analysis).

LECTURE 2

Ultralimits and Morse lemma

Motivation: Quasi-isometries are not nice maps, they need not be continuous, etc. We will use ultralimits of metric spaces to convert quasi-isometries into homeomorphisms. Also, in many cases, ultralimits of sequences of metric spaces are simpler than the original spaces. We will use this to prove stability of geodesics in hyperbolic space (Morse Lemma).

1. Ultralimits of sequences in topological spaces.

Definition 2.21. *An* ultrafilter *on the set* $\mathbb{N}$ *of natural numbers is a finitely-additive measure* ω *defined for all subsets of* $\mathbb{N}$ *and taking only the values* 0 *and* 1.

In other words, $\omega : 2^{\mathbb{N}} \to \{0, 1\}$ *is:*

- *Finitely-additive:* $\omega(A \cup B) = \omega(A) + \omega(B) - \omega(A \cap B)$.
- $\omega(\emptyset) = 0$.

We will say that a subset E of $\mathbb{N}$ is ω-large if $\omega(E) = 1$. Similarly, we will say that a property $P(n)$ holds for ω-all natural numbers if $\omega(\{n : P(n) \text{ is true }\}) = 1$.

Trivial examples of ultrafilters are those for which $\omega(\{n\}) = 1$ for some $n \in \mathbb{N}$ (such ultrafilters are called *principal*). I will always assume that ω vanishes on all finite sets, in other words, I will consider only nonprincipal ultrafilters.

Existence of ultrafilters does not follow from the Zermelo-Fraenkel (ZF) axioms of set theory, but it follows from ZFC.

We will use ultrafilters to define limits of sequences:

Definition 2.22. *Let X be a Hausdorff topological space and ω an ultrafilter (on $\mathbb{N}$). Then, for a sequence (x_n) of points $x_n \in X$, we define the ω-limit (ultralimit),* $\lim_\omega x_n$*, to be a point $a \in X$ so that:*

For every neighborhood U of a, the set $\{n \in \mathbb{N} : x_n \in U\}$ is ω-large.

In other words, $x_n \in U$ for ω-all n.

As X is assumed to be Hausdorff, $\lim_\omega x_n$ is unique (if it exists).

Exercise 2.23. *If* $\lim x_n = a$ *(in the usual sense) then* $\lim_\omega x_n = a$ *for every* ω.

I will fix an ultrafilter ω once and for all.

Exercise 2.24. *If X is compact then every sequence in X has ultralimit. Hint: Use a proof by contradiction.*

In particular, every sequence $t_n \in \mathbb{R}_+$ has ultralimit in $[0, \infty]$.

Exercise 2.25. *What is the ultralimit of the sequence $(-1)^n$ in $[-1, 1]$?*

2. Ultralimits of sequences of metric spaces

Our next goal is to define ultralimit for a sequence of metric spaces (X_n, d_n). The definition is similar to the Cauchy completion of a metric space: Elements of the ultralimit will be equivalence classes of sequences $x_n \in X_n$. For every two sequences $x_n \in X_n, y_n \in X_n$ we define

$$d_\omega((x_n),(y_n)) := \lim_\omega d_n(x_n, y_n) \in [0,\infty].$$

Exercise 2.26. *Verify that d_ω is a pseudo-metric. (Use the usual convention $\infty + a = \infty$, for every $a \in \mathbb{R} \cup \{\infty\}$.)*

Of course, some sequences will be within zero distance from each other. As in the definition of Cauchy completion, we will identify such sequences (this is our equivalence relation). After that, d_ω is "almost" a metric: The minor problem is that sometimes d_ω is infinite. To handle this problem, we introduce a sequence of "observers", points $p_n \in X_n$. Then, we define $\lim_\omega X_n = X_\omega$, the ultralimit of the sequence of *pointed* metric spaces (X_n, p_n) to be the set of equivalence classes of sequences $x_n \in X_n$ so that

$$d_\omega((x_n),(p_n)) < \infty.$$

Informally, X_ω consists of equivalence classes of sequences which the "observers" can see.

In case $(X_n, d_n) = (X, d)$, we will refer to $\lim_\omega X_n$ as a *constant* ultralimit.

Exercise 2.27.

- *If X is compact then the constant ultralimit $\lim_\omega X$ is isometric to X (for any sequence of observers).*
- *Suppose that X admits a geometric group action. Then the constant ultralimit $\lim_\omega X$ does not depend on the choice of the observers.*
- *Suppose that X is a proper metric space. Then for every bounded sequence $p_n \in X$ the constant ultralimit $\lim_\omega X$ is isometric to X.*
- *Show that $\lim_\omega \mathbb{R}^k$ is isometric to $\mathbb{R}^k$.*

Let $(X_n, p_n), (Y_n, q_n)$ be pointed metric spaces and $f_n : X_n \to Y_n$ a sequence of isometries, so that

$$\lim_\omega d_{Y_n}(f_n(p_n), q_n) < \infty.$$

Then the sequence (f_n) defines a map

$$f_\omega : X_\omega \to Y_\omega, \quad f_\omega(x_\omega) = ((f_n(x_n))).$$

It is immediate that the map f_ω is well-defined and is an isometry. In particular, the ultralimit of a sequence of geodesic metric spaces is again a geodesic metric space.

3. Ultralimits and CAT(0) metric spaces

Recall that a CAT(0) metric space is a geodesic metric space where triangles are "thinner" than triangles in the plane. One can express this property as a *4-point condition*:

Definition 2.28. *A geodesic metric space X is said to be CAT(0) if the following holds. Let $x, y, z, m \in X$ be points such that $d(x,m) + d(m,y) = d(x,y)$. Let $x', y', z', m' \in \mathbb{R}^2$ be their "comparison" points, i.e.:*

$$d(x,m) = d(x',m'), d(m,y) = d(m',y'), d(x,y) = d(x',y'), d(y,z) = d(y',z'), d(z,x) = d(z',x').$$

(Thus, the triangle with vertices x, y, m is degenerate.) Then $d(z,m) \le d(z',m')$.

For instance, hyperbolic spaces $\mathbb{H}^n$ are CAT(0). The important property of CAT(0) spaces is that they are *uniquely geodesic*, i.e., for any pair of points $x, y \in X$ there is a unique geodesic connecting x to y.

Exercise 2.29. *Ultralimits of sequences of CAT(0) spaces are again CAT(0). Hint: Start with a 4-point configuration $x_\omega, y_\omega, z_\omega, m_\omega \in X_\omega$ with a degenerate triangle with vertices $x_\omega, y_\omega, m_\omega$. Represent the points $x_\omega, y_\omega, z_\omega$ by sequences $x_n, y_n, z_n \in X_n$. Use the CAT(0) property to find $m_n \in X_n$ representing m_ω so that the triangle spanned by x_n, y_n, m_n is degenerate.*

4. Asymptotic cones

The ultralimits that we will be using are not constant. We start with a metric space (X, d) and a sequence of positive scale factors λ_n so that $\lim_\omega \lambda_n = 0$. Then set $d_n := \lambda_n d$. Hence, the sequence (X, d_n) consists of rescaled copies of (X, d).

Definition 2.30. *An* asymptotic cone *of X, denoted $Cone(X)$, is the ultralimit of the sequence of pointed metric spaces: $Cone(X) = \lim_\omega(X_n, \lambda_n d, p_n)$.*

Note that, in general, the asymptotic cone depends on the choices of ω, (λ_n) and (p_n), so the notation $Cone(X)$ is ambiguous, but it will be always implicitly understood that ω, (λ_n) and (p_n) are chosen in the definition of $Cone(X)$.

Exercise 2.31. *Let $G = \mathbb{Z}^k$ be the free abelian group with its standard set of generators. Let $X = G$ with the word metric. Then $Cone(X)$ is isometric to $\mathbb{R}^k$ with the ℓ_1-metric corresponding to the norm*

$$\|(x_1, ..., x_k)\| = |x_1| + ... + |x_k|.$$

Definition 2.32. *A (geodesic) triangle T in a metric space X is a concatenation of three geodesic segments in X: $\overline{xy}, \overline{yz}, \overline{zx}$, where $\overline{pq}$ denotes a geodesic segment connecting p to q. We will use the notation $T = [x, y, z]$ to indicate that x, y, z are the vertices of T. A triangle T is called* δ-thin *if every side of T is contained within distance $\leq \delta$ from the union of the two other sides. A geodesic metric space X is called δ-hyperbolic if every geodesic triangle in X is δ-thin. When we do not want to specify δ, we will simply say that X is Gromov-hyperbolic.*

Lemma 2.33. *Suppose that X is the hyperbolic space $\mathbb{H}^k$, $k \geq 2$. Then every asymptotic cone $X_\omega = Cone(X)$ is a tree. (Note that this tree branches at every point and has infinite (continual) degree of branching at every point x_ω: the cardinality of the number of components of $X_\omega - \{x_\omega\}$ is the cardinality of the continuum.)*

PROOF. We need to verify that every geodesic triangle $T_\omega = [x_\omega, y_\omega, z_\omega] \subset X_\omega$ is 0-thin, i.e., every side is contained in the union of two other sides. First of all, we know, that X_ω is CAT(0) and, hence, uniquely geodesic. Thus, the triangle T_ω appears as an ultralimit of a sequence of geodesic triangles $T_n = [x_n, y_n, z_n]$ in $X_k = (X, \lambda_k d_X)$. Each triangle T_n in (X, d_X) is δ-thin, where $\delta \leq 1$ (see Appendix 1). Therefore, the triangle T_n, regarded as a triangle in X_k, is $\lambda_k\delta$-thin. Since $\lim_\omega \lambda_k \delta = 0$, we conclude that T_ω is 0-thin. $\square$

Exercise 2.34. *Show that every closed geodesic m-gon $[x_1, ..., x_m]$ in a tree T is 0-thin, i.e., the side $[x_m, x_1]$ is contained in the union of the other sides.*

Lemma 2.35. *Suppose that α is a simple topological arc in a tree T. Then α, after a reparameterization, is a geodesic arc.*

PROOF. Let $\alpha : [0, 1] \to T$ be a continuous injective map (a simple topological arc), $x = \alpha(0), y = \alpha(1)$. Let $\alpha^* = [x, y]$ be the geodesic connecting x to y. I claim that the image of α contains the image of α^*. Indeed, we can approximate α by piecewise-geodesic (nonembedded!) arcs

$$\alpha_n = [x_0, x_1] \cup ... \cup [x_{n-1}, x_n], \quad x_0 = x, x_n = y.$$

Then the above exercise shows that α_n contains the image of α^* for every n. Therefore, the image of α also contains the image of α^*. Considering the map $\alpha^{-1} \circ \alpha^*$ and applying the intermediate value theorem, we see that the images of α and α^* are equal. $\square$

5. Quasi-isometries and asymptotic cones

Suppose that $f : X \to X'$ is an (L, A)-quasi-isometric embedding:

$$\frac{1}{L} d(x, y) - A \leq d(f(x), f(y)) \leq L d(x, y) + A.$$

Pick a sequence of scale factors λ_n, a sequence of observers $p_n \in X$ and their images $q_n := f(x_n)$. Then,

$$\frac{\lambda_n}{L} d(x,y) - \lambda_n A \le \lambda_n d(f(x), f(y)) \le L\lambda_n d(x,y) + \lambda_n A.$$

Let $d_{X_n} = \lambda_n d_X, d_{X'_n} = \lambda_n d_{X'}$. Hence:

$$\frac{1}{L} d_{X_n}(x,y) - \lambda_n A \le d_{X'_n}(f(x), f(y)) \le L d_{X_n}(x,y) + \lambda_n A.$$

Thus, after taking the ultralimit:

$$f_\omega : X_\omega \to X'_\omega, \quad f_\omega((x_n)) = (f(x_n)),$$

we get:

$$\frac{1}{L} d_\omega(x,y) \le d_\omega(f_\omega(x), f_\omega(y)) \le L d_\omega(x,y)$$

for all $x, y \in X_\omega$. Therefore, f_ω is a bilipschitz embedding, since the additive constant A is gone! Even better, if f was quasi-surjective then f_ω is surjective. Thus, $f_\omega : X_\omega \to X'_\omega$ is a homeomorphism!

The same observation applies to sequences of quasi-isometric embeddings/quasi-isometries as long as the constants L, A are fixed.

Exercise 2.36. *$\mathbb{R}^n$ is QI to $\mathbb{R}^m$ iff $n = m$.*

Exercise 2.37. *Suppose that $\mathbb{R}^n \to \mathbb{R}^n$ is a QI embedding. Then f is quasi-surjective. Hint: If not, then, taking an appropriate sequence of scaling factors and observers, and passing to asymptotic cones, we get $f_\omega : \mathbb{R}^n \to \mathbb{R}^n$, a bilipschitz embedding which is not onto. This map has to be open by the invariance of domain theorem (since the dimensions of the domain and range are the same), it is also proper since f_ω is bilipschitz. Thus, f_ω is also closed. It follows that f_ω is onto.*

Unfortunately, we cannot distinguish $\mathbb{H}^n$ from $\mathbb{H}^m$ (these are real-hyperbolic spaces of dimensions $n \ge 2$, $m \ge 2$ respectively) using asymptotic cones since all of their asymptotic cones are isometric to the same tree [**8**]!

6. Morse lemma

Let $X = \mathbb{H}^n$ be real-hyperbolic n-space. A *quasi-geodesic* in X is a QI embedding $f : J \to X$, where J is an interval in $\mathbb{R}$ (either finite or infinite).

Theorem 2.38 (Morse Lemma[1])**.** *There exists a function $D(L, A)$ so that every (L, A)-quasi-geodesic α in X is D-Hausdorff close to a geodesic α^*.*

Proof. I will first prove the Morse Lemma in the case of finite quasi-geodesics. Here is the idea behind the proof: If the Morse Lemma fails, a sequence of "counter-examples" α_i to its statement yields a bi-Lipschitz map from an interval to a suitable asymptotic cone X_ω of X. Lemma 2.35 then implies that the image of this arc is a geodesic α_ω in X_ω. On the other hand, the sequence of geodesics α_i^* in X connecting the end-points of α_i also converges to a geodesic arc in X_ω. Since X_ω is uniquely geodesic, the resulting geodesic arcs are equal to α_ω, contradicting the assumption that the distances between quasi-geodesics α_i and geodesics α_i^* diverge to infinity.

Below is the actual proof. For a quasi-geodesic $\alpha : J = [0, a] \to X$, let $\alpha^* : J^* = [0, a^*] \to X$ denote the geodesic connecting $\alpha(0)$ to $\alpha(a)$. Define two numbers:

$$D_\alpha = dist(\alpha, \alpha^*) := \sup_{t \in I} d(\alpha(t), Im(\alpha^*)), \quad D^*_\alpha = dist(\alpha^*, \alpha) := \sup_{t \in J^*} d(\alpha^*(t), Im(\alpha))$$

Recall that Hausdorff distance between α, α^* is $max(D_\alpha, D^*_\alpha)$. I will prove that the quantities D_α are uniformly bounded, since the proof of boundedness of D^*_α is completely analogous.

Suppose that the Morse Lemma fails. Then there exists a sequence $f_i : J_i \to X$ of (L, A)-quasi-geodesics, so that $\lim_i D_{\alpha_i} = \infty$. For each i pick a point $x_i \in \alpha_i(J_i)$ so that $d(x_i, \alpha_i^*)$ is within $\frac{1}{i}$

[1]Maybe this should be called the 2nd Morse lemma, since the 1st, and more famous, Morse lemma appears in the theory of Morse functions.

from $D_{\alpha_i} = D_i$. Now, rescale the metrics on J_i and on X by $\lambda_i = D_i^{-1}$ and take ultralimits of the rescaled intervals and the hyperbolic spaces. Then, quasi-isometric (resp. isometric) embeddings α_i (resp. α_i^*) yield bilipschitz (resp. isometric) embeddings

$$\alpha : J_\omega \to X_\omega = Cone(X), \quad \alpha^* : J_\omega^* \to Cone(X).$$

By our choice of x_i and scaling factors, $dist(\alpha, \alpha^*) = 1$.

Since the maps α_i were (L, A)-quasi-isometric embeddings, it follows that J_ω is finite iff J_ω^* is finite. I first consider the case when J_ω is finite. Then α, α^* have common end-points (since the curves α_i, α_i^* did). Recall that X_ω is a tree. By Lemma 2.35, the images of α, α^* are the same. This contradicts the fact that $dist(\alpha, \alpha^*) = 1$.

Suppose that J_ω is infinite, i.e., $J_\omega = \mathbb{R}_+$. The semi-infinite arcs $\alpha(\mathbb{R}_+), \alpha^*(\mathbb{R}_+)$ are within unit distance from each other. Let $x_\omega = \alpha_\omega(t)$ be the point represented by the sequence (x_i). Let $C = d(\alpha(0), x_\omega)$. There exists a geodesic arc $\beta \subset X_\omega$ of length ≤ 1 connecting points $x = \alpha(s), x^* = \alpha^*(s^*)$ so that $\beta \cap Im(\alpha) = x, \beta \cap Im(\alpha^*) = x^*$ and so that $d(x, \alpha(0)) > C$. Thus, the simple arc $\gamma = \alpha([0, s]) \cup \beta$ connects the end-points of the geodesic segment $\gamma^* = \alpha^*([0, s^*])$. On the other hand, $x_\omega \in \gamma \setminus \gamma^*$. This contradicts Lemma 2.35.

It remains to prove the Morse Lemma for infinite quasi-geodesics. Such quasi-geodesics, say, $\alpha : \mathbb{R} \to X$, can be exhausted by finite quasi-geodesics $\alpha_i : [-i, i] \to X$. Applying the Morse Lemma to quasi-geodesics α_i, we get the desired conclusion for α. $\square$

Morse Lemma also applies to all Gromov-hyperbolic geodesic metric spaces (e.g., Gromov-hyperbolic groups). On the other hand, Morse Lemma fails completely in the case of quasi-geodesics in the Euclidean plane.

Exercise 2.39. *Let $\phi : \mathbb{R} \to \mathbb{R}$ be an L-Lipschitz function. Show that the map $f(x) = (x, \phi(x))$ is a quasi-geodesic in $\mathbb{R}^2$.*

LECTURE 3

Boundary extension and quasi-conformal maps

1. Boundary extension of QI maps of hyperbolic spaces

Suppose that $X = \mathbb{H}^n$ and $f : X \to X$ is a QI map. Then, by Morse Lemma, f sends geodesic rays uniformly close to geodesic rays: $\forall \rho, \exists \rho'$ so that

$$d(f(\rho), \rho') \le D$$

where ρ, ρ' are geodesic rays, $\rho' = (f(\rho))^*$ (where the notation $*$ is taken from Theorem 2.38). Let ξ, ξ' be the limits of the rays ρ, ρ' on the boundary sphere of $\mathbb{H}^n$. Then we set

$$f_\infty(\xi) := \xi'.$$

Here and in what follows, the limit point of a geodesic ray ρ in $\mathbb{H}^n$ is the limit

$$\lim_{t\to\infty} \rho(t) \in S^{n-1} = \partial \mathbb{H}^n.$$

Exercise 3.40. *The point ξ' depends only on the point ξ and not on the choice of a ray ρ that limits to ξ.*

Thus, we obtain the *boundary extension* f_∞ of the quasi-isometry f of $\mathbb{H}^n$ to the boundary sphere S^{n-1}.

Exercise 3.41. $(f \circ g)_\infty = f_\infty \circ g_\infty$ *for all quasi-isometries $f, g : X \to X$.*

Exercise 3.42. *Suppose that $d(f, g) < \infty$, i.e., there exists $C < \infty$ so that*

$$d(f(x), g(x)) \le C$$

for all $x \in X$. Then $f_\infty = g_\infty$. In particular, if $\bar f$ is quasi-inverse of f, then $(\bar f)_\infty$ is inverse of f_∞.

Our next goal is to see that the extensions f_∞ are continuous. Actually, they satisfy some further regularity properties which will be critical for the proof of Tukia's theorem.

Let γ be a geodesic ray in $\mathbb{H}^n$ and $\pi = \pi_\gamma : \mathbb{H}^n \to \gamma$ be the orthogonal projection (the nearest-point projection). Then for all $x \in \gamma$ (except for the initial point), $H_x := \pi^{-1}(x)$ is an $n-1$-dimensional hyperbolic subspace of $\mathbb{H}^n$, which is orthogonal to γ. The projection π extends continuously to a projection

$$\pi : \mathbb{H}^n \cup S^{n-1} \setminus \{\xi\} \to \gamma,$$

where ξ is the limit point of γ.

Clearly, isometries commute with projections π to geodesic rays. The following lemma is a "quasification" of the above observation. We leave the proof of the lemma to the reader, since it amounts to nothing but "chasing triangle inequalities."

Lemma 3.43. *Quasi-isometries quasi-commute with the nearest-point projections. More precisely, let $f : \mathbb{H}^n \to \mathbb{H}^n$ be an (L, A)-quasi-isometry. Let γ be a geodesic ray and γ' be a geodesic ray within distance $\le D(L, A)$ from the quasi-geodesic $f(\gamma)$. Let $\pi : \mathbb{H}^n \to \gamma, \pi' : \mathbb{H}^n \to \gamma'$ be nearest-point projections. Then, for some $C = C(L, A)$, we have:*

$$d(f\pi, \pi' f) \le C,$$

i.e.,

$$\forall x \in \mathbb{H}^n, \quad d(f\pi(x), \pi' f(x)) \le C.$$

Let ξ be the limit point of γ. Then, for $x_i \in \gamma$ converging to ξ, the boundary spheres Σ_i of the subspaces $H_{x_i} = \pi^{-1}(x_i)$, bound round balls $B_i \subset S^{n-1}$ (containing ξ). These balls form a basis of topology at the point $\xi \in S^{n-1}$. Since f is a quasi-isometry, the points $y_i = f(x_i)$ cannot form a bounded sequence in $\mathbb{H}^n$, hence, $\lim y_i = \xi$. Using the above lemma, we see that all the sets $f_\infty(B_i)$ are contained in round balls B_i', whose intersection is the point $\xi' = f_\infty(\xi)$. Thus, f_∞ is continuous and hence, a homeomorphism. We thus obtain

Lemma 3.44. *For every quasi-isometry $f : \mathbb{H}^n \to \mathbb{H}^n$, the boundary extension f_∞ is a homeomorphism.*

Corollary 3.45. *$\mathbb{H}^n$ is QI to $\mathbb{H}^m$ if and only if $n = m$.*

2. Quasi-actions

The notion of an *action* of a group on a space is replaced, in the context of quasi-isometries, by *quasi-action*. Recall that an *action* of a group G on a set X is a homomorphism $\phi : G \to Aut(X)$, where $Aut(X)$ is the group of bijections $X \to X$. Since quasi-isometries are defined only up to "bounded noise", the concept of a homomorphism has to be modified when we use quasi-isometries.

Definition 3.46. *Let G be a group and X be a metric space. An (L, A)–quasi-action of G on X is a map $\phi : G \to \widehat{QI}(X)$, so that:*

- *$\phi(g)$ is an (L, A)-quasi-isometry of X for all $g \in G$.*
- *$d(\phi(1), id_X) \le A$.*
- *$d(\phi(g_1 g_2), \phi(g_1)\phi(g_2)) \le A$ for all $g_1, g_2 \in G$.*

Thus, Parts 2 and 3 say that ϕ is "almost" a homomorphism with the error A.

In particular, every quasi-action determines a natural homomorphism $G \to QI(X)$.

Example 3.47. *Suppose that G is a group and $\phi : G \to \mathbb{R}$ is a function which determines a quasi-action of G on $\mathbb{R}$ by translations ($g \in G$ acts on $\mathbb{R}$ by translation by $\phi(g)$). Such maps ϕ are called* quasi-morphisms *and they appear frequently in GGT. Many interesting groups do not admit nontrivial homomorphisms of $\mathbb{R}$ but admit unbounded quasimorphisms.*

Here is how quasi-actions appear in the context of QI rigidity problems. Suppose that G_1, G_2 are groups acting isometrically on metric spaces X_1, X_2 and $f : X_1 \to X_2$ is a quasi-isometry with quasi-inverse $\bar{f}$. We then define a *conjugate* quasi-action ϕ of G_2 on X_1 by

$$\phi(g) = \bar{f} \circ g \circ f.$$

Exercise 3.48. *Show that ϕ is indeed a quasi-action.*

For instance, suppose that $X_1 = \mathbb{H}^n$, $\psi : G_1 \curvearrowright X$ is a geometric action, and suppose that G_2 is a group which is QI to G_1 (and, hence, by the Milnor-Schwarz Lemma, G_2 is QI to X). We then take $X_2 = G_2$ (with a word metric). Then the quasi-isometry $f : G_1 \to G_2$ yields a quasi-action $\phi_{f,\psi}$ of G_2 on $\mathbb{H}^n$.

We now apply our extension functor (sending quasi-isometries of $\mathbb{H}^n$ to homeomorphisms of the boundary sphere). Then, Exercises 3.40 and 3.41 imply:

Corollary 3.49. *Every quasi-action ϕ of a group G on $\mathbb{H}^n$ extends (by $g \mapsto \phi(g)_\infty$) to an action ϕ_∞ of G on S^{n-1} by homeomorphisms.*

Lemma 3.50. *The kernel for the action ϕ_∞ is finite.*

Proof. The kernel of ϕ_∞ consists of the elements $g \in G$ such that $d(\phi(g), id) < \infty$. Since $\phi(g)$ is an (L, A)-quasi-isometry of $\mathbb{H}^n$, it follows from Morse Lemma that $d(\phi(g), id) \le C = C(L, A)$. Thus, such g (as an isometry $G \to G$) moves every point at most by $C' = C'(L, A)$. However, clearly the set of such elements of G is finite. Hence, $Ker(\phi_\infty)$ is finite as well. $\square$

Geometric quasi-actions. The following three definitions for quasi-actions are direct generalizations of the corresponding definitions for actions.

A quasi-action $\phi : G \curvearrowright X$ of a group G on a metric space X is called *properly discontinuous* if for every bounded subset $B \subset X$ the set

$$\{g \in G : \phi(g)(B) \cap B \neq \emptyset\}$$

is finite. A quasi-action $\phi : G \curvearrowright X$ is *cobounded* if there exists a bounded subset $B \subset X$ so that for every $x \in X$ there exists $g \in G$ so that $\phi(g)(x) \in B$ (this is an analogue of a cocompact isometric action). Finally, we say that a quasi-action $\phi : G \curvearrowright X$ is *geometric* if it is properly discontinuous and cobounded.

Exercise 3.51. *Suppose that $\phi_2 : G \curvearrowright X_2$ is a quasi-action, $f : X_1 \to X_2$ is a quasi-isometry and $\phi_1 : G \curvearrowright X_1$ is the conjugate quasi-action. Then ϕ_2 is properly discontinuous (resp. cobounded, resp. geometric) if and only if ϕ_1 is properly discontinuous (resp. cobounded, resp. geometric).*

3. Conical limit points of quasi-actions

Suppose that ϕ is a quasi-action of a group G on $\mathbb{H}^n$. A point $\xi \in S^{n-1}$ is called a *conical limit point* for the quasi-action ϕ if the following holds:

For some (equivalently every) geodesic ray $\gamma \subset \mathbb{H}^n$ limiting to ξ, and some (equivalently every) point $x \in \mathbb{H}^n$, there exists a constant $R < \infty$ and a sequence $g_i \in G$ so that:

- $\lim_{i\to\infty} \phi(g_i)(x) = \xi$.
- $d(\phi(g_i)(x), \gamma) \leq R$ for all i.

In other words, the sequence $\phi(g_i)(x)$ converges to ξ in a closed cone (contained in $\mathbb{H}^n$) with the tip ξ.

Lemma 3.52. *Suppose that $\psi : G \curvearrowright X = \mathbb{H}^n$ is a cobounded quasi-action. Then every point of the boundary sphere S^{n-1} is a conical limit point for ψ.*

Proof. Consider the sequence $x_i \in X, x_i = \gamma(i)$, where γ is a ray in X limiting to a point $\xi \in S^{n-1}$. Fix a point $x_0 \in X$ and a ball $B = B_R(x_0)$ so that for every $x \in X$ there exists $g \in G$ so that $d(x, \phi(g)(x_0)) \leq R$. Then, by coboundedness of the quasi-action ψ, there exists a sequence $g_i \in G$ so that

$$d(x_i, \phi(g_i)(x_0)) \leq R.$$

Thus, ξ is a conical limit point. □

Corollary 3.53. *Suppose that G is a group and $f : \mathbb{H}^n \to G$ is a quasi-isometry. Then every point of S^{n-1} is a conical limit point for the quasi-action $\psi : G \curvearrowright \mathbb{H}^n$, which is induced by conjugating the action $G \curvearrowright G$ by f.*

Proof. The action $G \curvearrowright G$ by left multiplication is cobounded, hence, the conjugate quasi-action $\psi : G \curvearrowright \mathbb{H}^n$ is also cobounded. □

If ϕ_∞ is a topological action of a group G on S^{n-1} which is obtained by extension of a quasi-action ϕ of G on $\mathbb{H}^n$ then we will say that *conical limit points* of the action $G \curvearrowright S^{n-1}$ are the conical limit points for the quasi-action $G \curvearrowright \mathbb{H}^n$.

4. Quasiconformality of the boundary extension

Can we get a better conclusion than just *a homeomorphism* for the maps f_∞? Let $f : \mathbb{H}^n \to \mathbb{H}^n$ be an (L, A)–quasi-isometry. I will work in the upper half-space model of $\mathbb{H}^n$. After composing f with isometries of $\mathbb{H}^n$, we can (and will) assume that:

- $\xi = 0 \in \mathbb{R}^{n-1}$ and γ is the vertical geodesic above 0.
- $0 = \xi' = f_\infty(\xi) \in \mathbb{R}^{n-1}$.
- $f_\infty(\infty) = \infty$. In particular, the vertical geodesic γ above ξ maps to a quasi-geodesic within bounded distance from the vertical geodesics $\gamma' = \gamma$ above $\xi' = \xi = 0$.

Consider an annulus $\mathbb{A} \subset \mathbb{R}^{n-1}$ given by

$$\mathbb{A} = \{x : R_1 \le |x| \le R_2\}$$

where $0 < R_1 \le R_2 < \infty$. We will refer to the ratio $\frac{R_2}{R_1}$ as the *eccentricity* of $\mathbb{A}$. Then, $\pi_\gamma(\mathbb{A})$ is an interval of hyperbolic length $d = \log(R_2/R_1)$ in γ. Recall that f *quasi-commutes* with the orthogonal projection (Lemma 3.43):

$$d(f \circ \pi_\gamma, \pi_\gamma \circ f) \le C = C(L, A).$$

Thus, $\pi_\gamma(f(\mathbb{A}))$ is an interval of the hyperbolic length $\le c' := 2C + Ld + A$. Hence, $f(\mathbb{A})$ is contained in the Euclidean annulus $\mathbb{A}'$:

$$\mathbb{A}' = \{x : R_1' \le |x| \le R_2'\}, \quad \frac{R_2'}{R_1'} \le e^{c'}.$$

We now define the function

$$\eta(r) = e^{c'}, c' = 2C + L\log(r) + A,$$

$$\eta(r) = r^L e^{2C+A}.$$

Note that $\eta(r), r \ge 1$ is a continuous monotonic function of r so that

$$\lim_{r\to 1} \eta(r) = 1, \quad \lim_{r\to\infty} \eta(r) = \infty.$$

We thus proved,

Lemma 3.54. *The topological annulus $f_\infty(\mathbb{A})$ is contained in an annulus $\mathbb{A}'$, so that eccentricity of $\mathbb{A}'$ is $\le \eta(r)$, where r is the eccentricity of $\mathbb{A}$. In particular, round spheres (corresponding to $r = 1$) map to "quasi–ellipsoids" of eccentricity $\le e^{2C+A}$.*

This leads to the definition:

Definition 3.55. *Let $\eta : [1, \infty) \to [1, \infty)$ be a continuous surjective monotonic function. A homeomorphism $f : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ is called η-quasi-symmetric*[1]*, if for all $x, y, z \in \mathbb{R}^n$ we have*

$$\frac{|f(x) - f(y)|}{|f(x) - f(z)|} \le \eta\left(\frac{|x - y|}{|x - z|}\right) \tag{3.1}$$

A homeomorphism f is c-weakly quasi-symmetric if

$$\frac{|f(x) - f(y)|}{|f(x) - f(z)|} \le c \tag{3.2}$$

for all x, y, z so that $|x - y| = |x - z| > 0$.

Remark 3.56. *It turns out that every weakly quasi-symmetric map is also quasi-symmetric but we will not dwell on this.*

I will now change my notation and will use n to denote the dimension of the boundary sphere of the hyperbolic $n + 1$-dimensional space. I will think of S^n as the 1-point compactification of $\mathbb{R}^n$ and will use letters x, y, z, etc., to denote points on $\mathbb{R}^n$. I will also use the notation f for the maps $\mathbb{R}^n \to \mathbb{R}^n$.

We will think of quasi-symmetric maps as homeomorphisms of $S^n = \mathbb{R}^n \cup \infty$, which send ∞ to itself. The following theorem was first proven by Tukia in the case of hyperbolic spaces and then extended by Paulin in the case of more general Gromov-hyperbolic spaces.

Theorem 3.57 (P.Tukia [**35**], F.Paulin [**25**])**.** *Every η-quasi-symmetric homeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ extends to an $(A(\eta), A(\eta))$-quasi-isometric map F of hyperbolic space $\mathbb{H}^{n+1}$.*

[1] Quasi-symmetric maps can be also defined for general metric spaces.

PROOF. Here is the idea of the proof. Since all ideal triangles in $\mathbb{H}^{n+1}$ are δ-thin, given a triple of distinct points $x, y, z \in S^n$ we have their *center* $c(x, y, z) \in \mathbb{H}^{n+1}$, which is a point within distance $\leq \delta$ from every side of the ideal hyperbolic triangle with the vertices x, y, z. The point c is not uniquely defined, but any two centers are uniformly close to each other. Thus, we can extend the map f to $\mathbb{H}^{n+1}$ via the formula

$$F(c(x, y, z)) = c(F(x), F(y), F(z)).$$

With this definition, however, it is far from clear why F is coarsely well-defined. For maps f which fix the point $z = \infty \in S^n$, it is technically more convenient to work instead with the points $\pi_\alpha(x)$, where α is the hyperbolic geodesic connecting y and z. (The points $\pi_\alpha(x)$ and $c(x, y, z)$ will be uniformly close to each other.) This is the approach that we will use below.

We define the extension F as follows. For every $p \in \mathbb{H}^{n+1}$, let $\alpha = \alpha_p$ be the complete vertical geodesic through p. This geodesic limits to points ∞ and $x = x_p \in \mathbb{R}^n$. Let $y \in \mathbb{R}^n$ be a point so that $\pi_\alpha(y) = p$ (the point y is non-unique, of course). Let $x' := f(x), y' := f(y)$, let $\alpha' \subset \mathbb{H}^{n+1}$ be the vertical geodesic through x' and let $p' := \pi_{\alpha'}(y')$. Lastly, set $F(p) := p'$.

I will prove only that F is an (A, A)–coarse Lipschitz, where $A = A(\eta)$. The quasi-inverse to F will be a map $\bar{F}$ defined via extension of the map f^{-1} following the same procedure. I will leave it as an exercise to verify that $\bar{F}$ is indeed a quasi-inverse to F and to estimate $d(\bar{F} \circ F, id)$.

Suppose that $d(p_1, p_2) \leq 1$. We would like to bound $d(p_1', p_2')$ from above. Without loss of generality, we may assume that $p_1 = e_{n+1} \in \mathbb{H}^{n+1}$. It suffices to consider two cases:

1. Points p_1, p_2 belong to the common vertical geodesic α, $x_1 = x_2 = x$ and $d(p_1, p_2) \leq 1$. I will assume, for concreteness, that $y_1 \leq y_2$. Hence,

$$d(p_1, p_2) = \log\left(\frac{|y_2 - x|}{|y_1 - x|}\right) \leq 1.$$

Since the map f is η-quasi-symmetric,

$$\frac{1}{\eta(e)} \leq \left(\eta\left(\frac{|y_2 - x|}{|y_1 - x|}\right)\right)^{-1} \leq \frac{|y_2' - x'|}{|y_1' - x'|} \leq \eta\left(\frac{|y_2 - x|}{|y_1 - x|}\right) \leq \eta(e).$$

In particular,

$$d(p_1', p_2') \leq C_1 = \log(\eta(e)).$$

2. Suppose that the points p_1, p_2 have the same last coordinate, which equals 1 since $p_1 = e_{n+1}$, and $t = |p_1 - p_2| \leq e$. The points p_1', p_2' belong to vertical lines α_1', α_2' which limit to points $x_1', x_2' \in \mathbb{R}^n$. Without loss of generality (by postcomposing f with an isometry of $\mathbb{H}^{n+1}$) we may assume that $|x_1' - x_2'| = 1$. Let $y_i \in \mathbb{R}^n, y_i' \in \mathbb{R}^n$ be points so that

$$\pi_{\alpha_i}(y_i) = p_i, \pi_{\alpha_i'}(y_i') = p_i'.$$

Then

$$|y_i - x_i| = |p_i - x_i| = R_i = 1, \quad i = 1, 2,$$
$$|y_i' - x_i'| = |p_i' - x_i'| = R_i' \quad i = 1, 2.$$

We can assume that $R_1' \leq R_2'$. Then

$$d(p_1', p_2') \leq \frac{1}{R_1'} + \log(R_2'/R_1'),$$

since we can first travel from p_1' to the line α_2' horizontally (along path of the length $\frac{1}{R_1'}$) and then vertically, along α_2' (along path of the length $\log(R_2'/R_1')$). We then apply the η-quasi-symmetry condition to the triple of points x_1, y_1, x_2 and get:

$$\frac{1}{R_1'} \leq \eta\left(\frac{t}{R_1}\right) \leq \eta(e).$$

Setting $R_3 := |x_1 - y_2|, R_3' := |x_1' - y_2'|$ and applying the η-quasi-symmetry condition to the triple of points x_1, y_1, y_2, we obtain

$$\frac{R_3'}{R_1'} \leq \eta(\frac{R_3}{R_1}) \leq \eta\left(\frac{t+1}{1}\right) \leq \eta(e+1).$$

Since $R_2' \le R_3' + 1$, we get:

$$\frac{R_2'}{R_1'} \le \frac{R_3' + 1}{R_1'} \le \eta(e+1) + \eta(e).$$

Putting it all together, we obtain that in Case 2:

$$d(p_1', p_2') \le \eta(e) + \log(\eta(e+1) + \eta(e)) = C_2.$$

Thus, in general, for $p_1, p_2 \in \mathbb{H}^{n+1}, d(p_1, p_2) \le 1$, we get:

$$d(F(p_1), F(p_2)) \le C_1 + C_2 = A. \quad \square$$

Now, for points $p, q \in \mathbb{H}^{n+1}$, so that $d(p,q) \ge 1$, we find a chain of points $p_0 = p, ..., p_{k+1} = q$, where $k = \lfloor d(p,q) \rfloor$ and $d(p_i, p_{i+1}) \le 1, i = 0, ..., k$. Hence,

$$d(F(p), F(q)) \le A(k+1) \le Ad(p,q) + A.$$

Hence, the map F is (A, A)–coarse Lipschitz, where A depends only on η.

Remark 3.58. *One can prove QI rigidity for groups acting geometrically on $\mathbb{H}^{n+1}$, $n \ge 2$, without using this theorem but the proof would be less clean this way.*

The drawback of the definition of quasi-symmetric maps is that we are restricted to maps of $\mathbb{R}^n$ rather than S^n. In particular, we cannot apply this definition to Moebius transformations.

Definition 3.59. *A homeomorphism of S^n is called* quasi-moebius *if it is a composition of a Moebius transformation with a quasi-symmetric map.*

We thus conclude that every (L, A)-quasi-isometry $\mathbb{H}^{n+1} \to \mathbb{H}^{n+1}$ extends to a quasi-moebius homeomorphism of the boundary sphere. Unfortunately, this definition of quasi-moebius maps is not particularly useful. One can define instead quasi-moebius maps by requiring that they quasi-preserve the cross-ratio, but then the definition becomes quite cumbersome.

What we will do instead is to take the limit in the inequality (3.1) as $r \to 0$. Then for every c-weakly quasi-symmetric map f we obtain:

$$\forall x, \quad H_f(x) := \limsup_{r \to 0} \left(\sup_{y,z} \frac{|f(x) - f(y)|}{|f(x) - f(z)|} \right) \le c. \tag{3.3}$$

Here, for each $r > 0$ the supremum is taken over all points y, z so that $r = |x - y| = |x - z|$.

Definition 3.60. *Let U, U' be domains in $\mathbb{R}^n$. A homeomorphism $f : U \to U'$ is called quasiconformal if $\sup_{x \in U} H_f(x) < \infty$. A quasiconformal map f is said to have* linear dilatation[2] $H = H(f)$, *if*

$$H(f) := ess \sup_{x \in U} H_f(x).$$

I will abbreviate quasiconformal *to* **qc**.

We say that f is 1-quasiconformal if $H(f) = 1$.

Thus, every H-weakly-quasi-symmetric map f is quasiconformal with $H(f) \le H$. The advantage of quasiconformality is that every Moebius map $f : S^n \to S^n$ is 1-quasiconformal on $S^n \setminus f^{-1}(\infty)$. In particular, all quasi-moebius maps are qc.

Proofs of the converse, which is a much harder theorem (that we will not use), can be found for instance, in [**17**] and [**36**].

We can now reformulate Lemma 3.54 as

Lemma 3.61. *Let $f : \mathbb{H}^{n+1} \to \mathbb{H}^{n+1}$ be an (L, A)-quasi-isometry. Then its boundary extension $h = f_\infty : S^n \to S^n$ is quasiconformal with $H(h) \le c(L, A)$.*

[2]Usually one uses a different quantity, $K(f)$, to measure the degree of quasiconformality of f, see Appendix 3. However, we will not use $K(f)$ in these lectures.

Theorem 3.62. *Every quasiconformal map $f : \mathbb{R}^n \to \mathbb{R}^n$ is η-quasi-symmetric for some $\eta = \eta(H(f))$.*

I will assume from now on that $n \geq 2$ since for $n = 1$ the notion of quasiconformality is essentially useless.

Example 3.63. *1. Every Moebius transformation of S^n is 1-quasiconformal.*

2. Every diffeomorphism $f : S^n \to S^n$ is quasiconformal.

Here is a non-smooth example of a quasiconformal map of $\mathbb{R}^2$. Let (r, θ) be the polar coordinates in $\mathbb{R}^2$ and let $\phi(\theta)$ denote diffeomorphisms $\mathbb{R}_+ \to \mathbb{R}_+$ and $S^1 \to S^1$. Then the map $f : \mathbb{R}^2 \to \mathbb{R}^2$, given in polar coordinates by the formula:

$$f(r, \theta) = (r, \phi(\theta)), f(0) = 0,$$

is quasiconformal but is not smooth (unless ϕ is a rotation).

Analytic properties of qc maps. Proofs of the following can be found, for instance, in [**17**] and [**36**].

(1) $H(f \circ g) \leq H(f)H(g)$, $H(f^{-1}) = H(f)$. These two properties follow directly from the definition.
(2) (J.Väisälä) Every qc map f is differentiable a.e. in $\mathbb{R}^n$. Furthermore, its partial derivatives are in $L^n_{loc}(\mathbb{R}^n)$. In particular, they are measurable functions.
(3) (J.Väisälä) The Jacobian J_f of a qc map f is nonzero a.e. in $\mathbb{R}^n$.
(4) Suppose that f is a quasiconformal map. For x where $D_x f$ exists and is invertible, we let $\lambda_1 \leq ... \leq \lambda_n$ denote the singular values of the matrix $D_x f$. Then
$$\frac{\lambda_n}{\lambda_1} = H_f(x)$$
Thus, the image of the unit sphere in the tangent space $T_x S^n$ under $D_x f$ is an ellipsoid of eccentricity $\leq H$. This is the geometric interpretation of qc maps: They map infinitesimal spheres to infinitesimal ellipsoids of uniformly bounded eccentricity.
(5) **QC Liouville's theorem (F. Gehring and Y. Reshetnyak).** 1-quasiconformal maps are conformal. (Here and in what follows we do not require that conformal maps preserve orientation, only that they preserve angles. Thus, from the viewpoint of complex analysis, we allow holomorphic and antiholomorphic maps of the 2-sphere.)
(6) **Convergence property for quasiconformal maps (J.Väisälä).** Let $x, y, z \in S^n$ be three distinct points. A sequence of quasiconformal maps (f_i) is said to be "normalized at $\{x, y, z\}$" if the limits $\lim_i f_i(x), \lim_i f_i(y), \lim f_i(z)$ exist and are all distinct. Then every normalized sequence of quasiconformal maps (f_i) with $H(f_i) \leq H$ contains a subsequence which converges to an quasiconformal map f with $H(f) \leq H$.
(7) **Semicontinuity of linear dilatation (P. Tukia; T. Iwaniec and G. Martin).** Suppose that (f_i) is a convergent sequence of quasiconformal maps with $H(f_i) \leq H$ so that the sequence of functions H_{f_i} converges to a function H in measure:
$$\forall \epsilon > 0, \lim_{i \to \infty} mes(\{x : |H_{f_i}(x) - H(x)| > \epsilon\}) = 0.$$
(Here mes is the Lebesgue measure on S^n.) Then the sequence (f_i) converges to a qc map f so that $H_f(x) \leq H(x)$ a.e..

LECTURE 4

Quasiconformal groups and Tukia's rigidity theorem

1. Quasiconformal groups

Recall that we abbreviate quasiconformal to **qc**.

A group G of quasiconformal homeomorphism of S^n is called (uniformly) quasiconformal if there exists $H < \infty$ so that for every $g \in G$, $H(g) \le H$. We will simply say that such a G is a *qc group.*

Example 4.64. *1. Every conformal (Moebius) group is quasiconformal (take $H = 1$).*

2. Suppose that $f : S^n \to S^n$ is quasiconformal with $H(f) \le H$, and G is a group of conformal transformations of S^n. Then the conjugate group $G_f := fGf^{-1}$ is uniformly quasiconformal. This follows from the inequality:

$$H(fgf^{-1}) \le H(f) \cdot 1 \cdot H = H^2.$$

3. Suppose that ϕ is a quasi-action of a group G on $\mathbb{H}^{n+1}$. Then the extension ϕ_∞ defines an action of G on S^n as a qc group. This follows immediately from Lemma 3.61.

4. Conversely, in view of Theorem 3.57, every qc group action $G \curvearrowright S^n$ extends to a quasi-action $G \curvearrowright \mathbb{H}^{n+1}$.

D. Sullivan [**31**] proved that for $n = 2$, every qc group is qc conjugate to a conformal group. This fails for $n \ge 3$. For instance, there are qc groups acting on S^3 which are not isomorphic to any subgroup of isometries of $\mathbb{H}^4$, see [**32, 16**]. Note that Tukia's examples are solvable and nondiscrete, while Isachenko's examples are discrete and are virtually isomorphic to free products of surface groups.

Our goal is to prove

Theorem 4.65 (P.Tukia, [**33**])**.** *Suppose that G is a (countable) qc group acting on $S^n, n \ge 2$, so that (almost) every point of S^n is a conical limit point of G. Then G is qc conjugate to a group acting conformally on S^n.*

Once we have this theorem, we obtain:

Theorem 4.66. *Suppose that $G = G_2$ is a group QI to a group G_1 acting geometrically on $\mathbb{H}^{n+1}$ ($n \ge 2$). Then G also acts geometrically on $\mathbb{H}^{n+1}$.*

Proof. We already know that a quasi-isometry $G_1 \to G_2$ yields a quasi-action ϕ of G on $\mathbb{H}^{n+1}$. Every boundary point of $\mathbb{H}^{n+1}$ is a conical limit point for this quasi-action. We also have a qc extension of the quasi-action ϕ to a qc group action $G \curvearrowright S^n$. Theorem 4.65 yields a qc map h_∞ conjugating the group action $G \curvearrowright S^n$ to a conformal action $\eta : G \curvearrowright S^n$. Every conformal transformation g of S^n extends to a unique isometry $ext(g)$ of $\mathbb{H}^{n+1}$. Thus, we obtain a homomorphism $\rho : G \to Isom(\mathbb{H}^{n+1}), \rho(g) = \mathrm{ext}(\eta(g))$; the kernel of ρ has to be finite since the kernel of the action $\phi_\infty : G \curvearrowright S^n$ is finite. We need to verify that the action ρ of G on $\mathbb{H}^{n+1}$ is geometric.

Let $h := ext(h_\infty)$ be an extension of h_∞ to quasi-isometry of $\mathbb{H}^{n+1}$. Then

$$g \mapsto \bar{h} \circ \rho(g) \circ h$$

determines a quasi-action ν of G on $\mathbb{H}^{n+1}$ whose extension to S^n is the qc action ϕ_∞. Thus, there exists C so that for every $g \in G$

$$d(\nu(g), \phi(g)) \le C.$$

It follows that h quasi-conjugates the action ρ and the quasi-action ϕ. Since the latter was geometric, the former is geometric as well. ☐

Remark 4.67. *Of course, the action ρ can have nontrivial finite kernel.*

By taking $\bar{G} = \rho(G)$ we obtain:

Corollary 4.68. *Let G be a group QI to $\mathbb{H}^{n+1}$. Then G contains a finite normal subgroup K so that $\bar{G} = G/K$ embeds in $Isom(\mathbb{H}^{n+1})$ as a properly discontinuous cocompact subgroup.*

Thus, our objective now is to prove Theorem 4.65.

2. Invariant measurable conformal structure for qc groups

Let Γ be a group acting conformally on $S^n = \mathbb{R}^n \cup \infty$ and let ds_E^2 be the usual Euclidean metric on $\mathbb{R}^n$. Then conformality of the elements of Γ amounts to saying that for every $g \in \Gamma$, and every $x \in \mathbb{R}^n$ (which does not map to ∞ by g)

$$(D_x g)^T \cdot D_x g$$

is a scalar matrix (scalar multiple of the identity matrix). Here and in what follows, $D_x f$ is the matrix of partial derivatives of f at x. In other words, the product

$$(J_{g,x})^{-\frac{2}{n}} \cdot (D_x g)^T \cdot D_x g$$

is the identity matrix I. Here $J_{g,x} = det(D_x g)$ is the Jacobian of g at x. This equation describes (in terms of calculus) the fact that the transformation g preserves the conformal structure on S^n.

More generally, consider Riemannian metrics ds^2 on S^n (given by symmetric positive-definite matrices A_x depending smoothly on $x \in \mathbb{R}^n$). A *conformal structure* on $\mathbb{R}^n$ is a metric ds^2 on $\mathbb{R}^n$ up to multiplication by a conformal factor. It is convenient to use *normalized* Riemannian metrics ds^2 on $\mathbb{R}^n$, where we require that $det(A_x) = 1$ for every x. Geometrically speaking, this means that the volume of the unit ball in $T_x(\mathbb{R}^n)$ with respect to the metric ds^2 is the same as the volume ω_n of the unit Euclidean n-ball. Normalization for a general metric A_x is given by multiplication by $det(A)^{-1/n}$. We then identify conformal structures on $\mathbb{R}^n$ with smooth matrix-valued functions A_x, where A_x is a positive-definite symmetric matrix with unit determinant.

The *pull-back* $g^*(ds^2)$ of ds^2 under a diffeomorphism $g : S^n \to S^n$ is given by the symmetric matrices

$$M_x = (D_x g)^T A_{gx} D_x g$$

If A_x was normalized, then, in order to have normalized pull-back $g^\bullet(ds^2)$ we again rescale:

$$B_x := (J_{g,x})^{-\frac{1}{2n}} (D_x g)^T \cdot A_{gx} \cdot D_x g.$$

How do we use this in the context of qc maps? Since their partial derivatives are measurable functions on $\mathbb{R}^n$, it makes sense to work with *measurable Riemannian metrics* and *measurable conformal structures* on $\mathbb{R}^n$. (One immediate benefit is that we do not have to worry about the point ∞.) We then work with measurable matrix-valued functions A_x, otherwise, nothing changes. Given a measurable conformal structure μ, we define its *linear dilatation* $H(\mu)$ as the essential supremum of the ratios

$$H(x) := \frac{\sqrt{\lambda_n(x)}}{\sqrt{\lambda_1(x)}},$$

where $\lambda_1(x) \le ... \le \lambda_n(x)$ are the eigenvalues of A_x. Geometrically speaking, if $E_x \subset T_x\mathbb{R}^n$ is the unit ball with respect to A_x, then $H(x)$ is the eccentricity of the ellipsoid E_x (with respect to the standard Euclidean metric on $\mathbb{R}^n$).

A measurable conformal structure μ is said to be *bounded* if $H(\mu) < \infty$.

A measurable conformal structure μ on $\mathbb{R}^n$ is *invariant* under a qc group G if

$$g^\bullet \mu = \mu, \forall g \in G.$$

In detail:

$$\forall g \in G, \quad (J_{g,x})^{-\frac{2}{n}} (D_x g)^T \cdot A_{gx} \cdot D_x g = A_x$$

a.e. in $\mathbb{R}^n$.

Theorem 4.69 (D.Sullivan [**31**], P.Tukia [**33**]). *Every qc group acting on $S^n, n \geq 2$, admits a bounded invariant measurable conformal structure.*

PROOF. The idea is to start with an arbitrary conformal structure μ_0 on $\mathbb{R}^n$ (say, the Euclidean structure) and then "average" it over $g \in G$. I will prove this only for countable groups G (which is all we need since we are interested in f.g. groups). Our proof is somewhat different from the one given by Sullivan and Tukia. Let A_x be the matrix-valued function defining a normalized Riemannian metric on $\mathbb{R}^n$; for instance, we can take $A_x = I$ for all $x \in \mathbb{R}^n$. Then, since G is countable, for a.e. $x \in \mathbb{R}^n$, we have a well-defined matrix-valued function corresponding to $g^\bullet(\mu_0)$ on $T_x\mathbb{R}^n$:

$$A_{g,x} := (J_{g,x})^{-\frac{1}{2n}} (D_x g)^T \cdot D_x g.$$

For such x we let $E_{g,x}$ denote the unit ball in $T_x\mathbb{R}^n$ with respect to $g^\bullet(\mu_0)$. From the Euclidean viewpoint, $E_{g,x}$ is an ellipsoid of volume ω_n. This ellipsoid (up to scaling) is the image of the unit ball under the inverse of the derivative $D_x g$. Since $H(g) \leq H$ for all $g \in G$, the ellipsoids $E_{g,x}$ have uniformly bounded eccentricity, i.e., the ratio of the largest to the smallest axis of this ellipsoid is uniformly bounded independently of x and g. Since the volume of $E_{g,x}$ is fixed, it follows that the diameter of the ellipsoid is uniformly bounded above and below.

Let U_x denote the union of the ellipsoids

$$\bigcup_{g \in G} E_{g,x}.$$

This set has diameter $\leq R$ for some R independent of x. Note also that U_x is symmetric (about 0). Note that the family of sets $\{U_x, x \in \mathbb{R}^n\}$ is invariant under the group G:

$$(J_{g,x})^{-1/n} D_x g(U_x) = U_{g(x)}, \quad \forall g \in G.$$

Lemma 4.70. *Given a bounded symmetric subset U of $\mathbb{R}^n$ with nonempty interior, there exists a unique ellipsoid $E = E_U$ (centered at 0) of smallest volume containing U. The ellipsoid E is called the John–Loewner ellipsoid of U.*

Existence of such an ellipsoid is clear. Uniqueness is not difficult, but not obvious (see Appendix 2). We then let E_x denote the John-Loewner ellipsoid of U_x. This ellipsoid defines a measurable function of x to the space of positive-definite $n \times n$ symmetric matrices. In other words, we obtain a measurable Riemannian metric ν on $\mathbb{R}^n$. Uniqueness of the John-Loewner ellipsoid and G-invariance of the sets U_x imply that the action of G preserves ν_x (up to scaling, of course). One can then get a normalized conformal structure μ by rescaling ν, so that

$$g^\bullet \mu = \mu, \forall g \in G.$$

It remains to show that μ is bounded. Indeed, the length of the major semi-axis of E_x does not exceed R while its volume is $\geq Vol(U_x) \geq \omega_n$ (here we are using the fact that all the matrices $A_{g,x}$ have unit determinant). Thus, the eccentricity of E_x is uniformly bounded. Hence μ is a bounded measurable conformal structure. $\square$

3. Proof of Tukia's theorem

We are now ready to prove Theorem 4.65. As a warm-up, we consider the easiest case, $n = 2$ (the argument in this case is due to D.Sullivan). In the 2-dimensional case, Theorem 4.65 holds without the conical limit points assumption. Let μ be a bounded measurable conformal structure on S^2 invariant under the group G. The *measurable Riemann mapping theorem* for S^2 states that every bounded measurable conformal structure μ on S^2 is quasiconformally equivalent to the standard conformal structure μ_0 on S^2, i.e., there exists a quasiconformal map $f : S^2 \to S^2$ which sends μ_0 to μ:

$$f^\bullet \mu_0 = \mu.$$

(Analytically, this theorem amounts to solvability of the Beltrami equation $\bar{\partial} f = \mu(z)\partial f$ for every measurable Beltrami differential μ on S^2.) Since the quasiconformal group G preserves μ on S^2, it follows that the group $G_f = fgf^{-1}$ preserves the structure μ_0. Thus, G_f acts as a group of conformal automorphisms of the round sphere, which proves the theorem for $n = 2$.

We now consider the case of arbitrary $n \geq 2$.

Definition 4.71. *A function $\eta : \mathbb{R}^n \to \mathbb{R}$ is called* approximately continuous *at a point $x \in \mathbb{R}^n$ if for every $\epsilon > 0$*

$$\lim_{r\to 0} \frac{mes\{y \in B_r(x) : |\eta(x) - \eta(y)| > \epsilon\}}{mes\ B_r(x)} = 0.$$

Here mes stands for the Lebesgue measure and $B_r(x)$ is the r-ball centered at x. In other words, as we "zoom into" the point x, "most" points $y \in B_r(x)$, have value $\eta(y)$ close to $\eta(x)$, i.e., the rescaled functions $\eta_r(x) := \eta(rx)$ converge in measure to the constant function.

We will need the following result from real analysis:

Lemma 4.72 (See Theorem 3.37 in [**1**])**.** *For every L^∞ function η on $\mathbb{R}^n$, a.e. point $x \in \mathbb{R}^n$ is an approximate continuity point of η.*

The functions to which we will apply this lemma are the matrix entries of a (normalized) bounded measurable conformal structure $\mu(x)$ on $\mathbb{R}^n$ (which we will identify with a matrix-valued function A_x). Since μ is bounded and normalized, the matrix entries of $\mu(x)$ will be in L^∞.

We let $\mu(x)$ again denote a bounded normalized measurable conformal structure on $\mathbb{R}^n$ invariant under G. Since a.e. point in $\mathbb{R}^n$ is a conical limit point of G, we will find such a point ξ which is also an approximate continuity point for $\mu(x)$.

Then, without loss of generality, we may assume that the point ξ is the origin in $\mathbb{R}^n$ and that $\mu(0) = \mu_0(0)$ is the standard conformal structure on $\mathbb{R}^n$. We will identify $\mathbb{H}^{n+1}$ with the upper half-space $\mathbb{R}^{n+1}_+$. Let $e = e_{n+1} = (0, ..., 0, 1) \in \mathbb{H}^{n+1}$.

Let $\phi(g)(x)$ denote the quasi-action of the elements $g \in G$ on $\mathbb{H}^{n+1}$. Since 0 is a conical limit point of G, there exists $C < \infty$ and a sequence $g_i \in G$ so that $\lim_{i\to\infty} \phi(g_i)(e) = 0$ and

$$d(\phi(g_i)(e), t_i e) \leq c$$

where d is the hyperbolic metric on $\mathbb{H}^{n+1}$ and $t_i > 0$ is a sequence converging to zero. Let T_i denote the hyperbolic isometry (Euclidean dilation) given by

$$x \mapsto t_i x, x \in \mathbb{H}^{n+1}.$$

Set

$$\tilde{g}_i := \phi(g_i^{-1}) \circ T_i.$$

Then

$$d(\phi(\tilde{g}_i)(e), e) \leq Lc + A$$

for all i. Furthermore, each $\tilde{g}_i$ is an (L, A)-quasi-isometry of $\mathbb{H}^{n+1}$ for fixed L and A. By applying coarse Arzela-Ascoli theorem, we conclude that the sequence $(\tilde{g}_i)$ coarsely subconverges to a quasi-isometry $\tilde{g}$. Thus, the sequence of quasiconformal maps $f_i := (\tilde{g}_i)_\infty$ subconverges to a quasiconformal map $f = (\tilde{g})_\infty$.

We also have:

$$\mu_i := f_i^\bullet(\mu) = (T_i)^\bullet (g_i)^{-1\bullet}(\mu) = (T_i)^\bullet \mu,$$

since $g^\bullet(\mu) = \mu, \forall g \in G$. Thus,

$$\mu_i(x) = \mu(T_i x) = \mu(t_i x),$$

in other words, the measurable conformal structure μ_i is obtained by "zooming into" the point 0. Since x is an approximate continuity point for μ, the functions $\mu_i(x)$ converge (in measure) to the constant function $\mu_0 = \mu(0)$. Thus, we have the diagram:

$$\begin{array}{ccc} \mu & \xrightarrow{f_i} & \mu_i \\ & & \downarrow \\ \mu & \xrightarrow{f} & \mu_0 \end{array}$$

If we knew that the derivatives Df_i subconverge (in measure) to the derivative of Df, then we would conclude that

$$f^{\bullet}\mu = \mu_0.$$

Then f would conjugate the group G (preserving μ) to a group G_f preserving μ_0 and, hence, acting conformally on S^n.

However, derivatives of quasiconformal maps (in general), converge only in the "biting" sense (see [**17**]), which will not suffice for our purposes. Thus, we have to use a less direct argument below.

We restrict to a certain round ball B in $\mathbb{R}^n$. Since μ is approximately continuous at 0, for every $\epsilon \in (0, \frac{1}{2})$,

$$\|\mu_i(x) - \mu(0)\| < \epsilon$$

away from a subset $W_i \subset B$ of measure $< \epsilon_i$, where $\lim_i \epsilon_i = 0$. Thus, for $x \in W_i$,

$$1 - \epsilon < \lambda_1(x) \le ... \le \lambda_n(x) < 1 + \epsilon,$$

where $\lambda_k(x)$ are the eigenvalues of the matrix $A_{i,x}$ of the metric $\mu_i(x)$. Thus,

$$H(\mu_i, x) < \frac{\sqrt{1+\epsilon}}{\sqrt{1-\epsilon}} \le \sqrt{1+4\epsilon} \le 1 + 2\epsilon.$$

away from subsets W_i. For every $g \in G$, each map $\gamma_i := f_i g f_i^{-1}$ is conformal with respect to the structure μ_i and, hence $(1+2\epsilon)$-quasiconformal away from the set W_i. Since $\lim_i mes(W_i) = 0$, we conclude, by the semicontinuity property of qc mappings, that each $\gamma := \lim \gamma_i$ is $(1+2\epsilon)$-quasiconformal. Since this holds for arbitrary $\epsilon > 0$ and arbitrary round ball B, we conclude that each γ is conformal (with respect to the standard conformal structure on S^n).

Thus, the group $\Gamma = fGf^{-1}$ consists of conformal transformations. □

4. QI rigidity for surface groups

The proof of Tukia's theorem mostly fails for groups QI to the hyperbolic plane. The key reason is that quasi-symmetric maps of the circle are differentiable a.e. but are *not absolutely continuous*. Thus, their derivative could (and, in the interesting cases will) vanish a.e. on the circle.

Nevertheless, the same proof yields: If G is a group QI to the hyperbolic plane, then G acts on S^1 by homeomorphisms with finite kernel K, so that the action is "discrete and cocompact" in the following sense:

Let T denote the set of ordered triples of distinct points on S^1. Thus, T is an open 3-dimensional manifold; one can compute its fundamental group and see that it is infinite cyclic, furthermore, T is homeomorphic to $D^2 \times S^1$. The action $G \curvearrowright S^1$, of course, yields an action $G \curvearrowright T$. Then $G \curvearrowright T$ is *properly discontinuous and cocompact*. The only elements of G that can fix a point in T are the elements of K. Thus, $\Gamma = G/K$ acts freely on T and the quotient T/Γ is a closed 3-dimensional manifold M.

It was proven, in a combination of papers by Tukia, Gabai, Casson and Jungreis in 1988—1994, that such a group Γ acts geometrically and faithfully on the hyperbolic plane. Their proof was mostly topological. One can now also derive this result from Peremlan's proof of Thurston's geometrization conjecture as follows. The infinite cyclic group $\pi_1(T)$ will be a normal subgroup of $\pi_1(M)$. Then, you look at the list of closed aspherical 3-dimensional manifolds (given by the Geometrization Conjecture) and see that such an M has to be a *Seifert manifold*, modelled on one of the geometries $\mathbb{H}^2 \times \mathbb{R}$, $SL(2,\mathbb{R})$, Nil, $\mathbb{E}^3$, see [**27**]. In the case of the geometries Nil, $\mathbb{E}^3$, one sees that the quotient of π_1 by normal infinite cyclic subgroup yields a group Γ which is VI to $\mathbb{Z}^2$. Such a group cannot act on S^1 so that $\Gamma \curvearrowright T$ is properly discontinuous and cocompact. On the other hand, in the case of the geometries $\mathbb{H}^2 \times \mathbb{R}$, $SL(2,\mathbb{R})$, the quotient by a normal cyclic subgroup will be VI to a group acting geometrically on $\mathbb{H}^2$.

LECTURE 5

Appendix

1. Appendix 1: Hyperbolic space

The upper half-space model of hyperbolic n-space $\mathbb{H}^n$ is

$$\mathbb{R}^n_+ = \{(x_1, ...x_n) : x_n > 0\}$$

equipped with the Riemannian metric

$$ds^2 = \frac{|dx|^2}{x_n^2}.$$

Thus, the length of a smooth path $p(t), t \in [0, T]$ in $\mathbb{H}^n$ is given by

$$\int_p ds = \int_0^T \frac{|p'(t)|_e}{p_n(t)} dt.$$

Here $|v|_e$ is the Euclidean norm of a vector v and $p_n(t)$ denotes the n-th coordinate of the point $p(t)$.

The (ideal) *boundary sphere* of $\mathbb{H}^n$ is the sphere $S^{n-1} = \mathbb{R}^{n-1} \cup \{\infty\}$, where $\mathbb{R}^{n-1}$ consists of points in $\mathbb{R}^n$ with vanishing last coordinate x_n.

Complete geodesics in $\mathbb{H}^n$ are Euclidean semicircles orthogonal to $\mathbb{R}^{n-1}$ as well as vertical straight lines. For instance, if $p, q \in \mathbb{H}^n$ are points on a common vertical line, then their hyperbolic distance is

$$d(p, q) = |\log(p_n/q_n)|.$$

The group of isometries of $\mathbb{H}^n$ is denoted $Isom(\mathbb{H}^n)$. Every isometry of $\mathbb{H}^n$ extends uniquely to a *Moebius transformation* of the boundary sphere S^{n-1}. The latter are the conformal diffeomorphisms of S^{n-1} in the sense that they preserve (Euclidean) angles. (I do not assume that conformal transformations preserve orientation.) Conversely, every Moebius transformation of S^{n-1} extends to a unique isometry of $\mathbb{H}^n$.

The group Mob_{n-1} of Moebius transformations of S^{n-1} contains all inversions, all Euclidean isometries of $\mathbb{R}^{n-1}$ and all dilations. (Compositions of Euclidean isometries and dilations are called *similarities.*) In fact, a single inversion together with all similarities of $\mathbb{R}^{n-1}$ generate the full group of Moebius transformations. Furthermore, every similarity of $\mathbb{R}^{n-1}$ extends to a similarity of $\mathbb{R}^n_+$

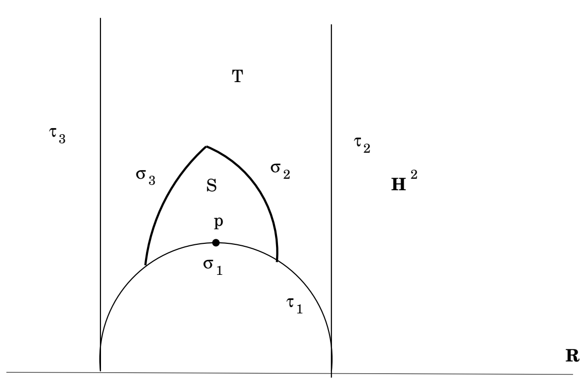

FIGURE 1. Hyperbolic triangles S and T.

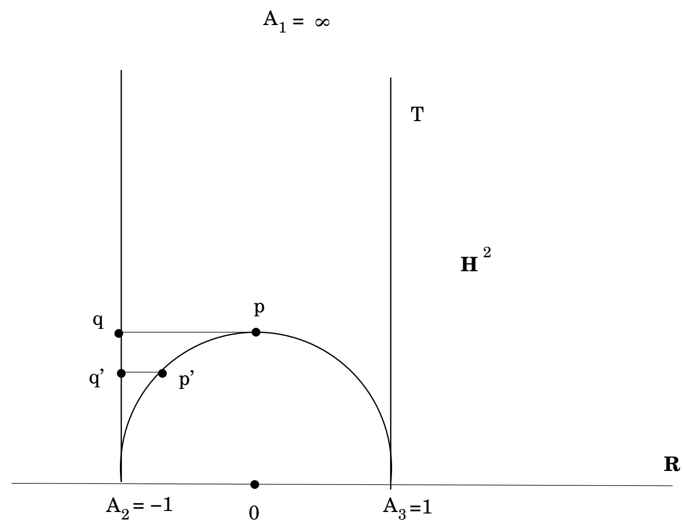

FIGURE 2. Thinness estimate for the ideal hyperbolic triangle T.

in the obvious fashion, so that the extension is an isometry of $\mathbb{H}^n$. Similarly, inversions extend to inversions which are also isometries of $\mathbb{H}^n$.

Exercise 5.73. *Show that the group Mob_{n-1} acts transitively on the set of triples of distinct points in S^{n-1}.*

The key fact of hyperbolic geometry that we will need is that *all triangles in $\mathbb{H}^n$ are δ-thin*, for $\delta \le 1$. Here is an outline of the proof. First, every geodesic triangle in $\mathbb{H}^n$ lies in a 2-dimensional hyperbolic subspace $\mathbb{H}^2 \subset \mathbb{H}^n$, so it suffices to consider the case $n = 2$. Next, consider a geodesic triangle $S \subset \mathbb{H}^2$ with the sides $\sigma_1, \sigma_2, \sigma_3$. Then S can be "enlarged" to an ideal hyperbolic triangle $T \subset \mathbb{H}^2$, i.e., a triangle all whose vertices belong to the boundary circle S^1 of $\mathbb{H}^2$, see Figure 1. For every point p which belongs to a side σ_1 of the triangle S, every geodesic in $\mathbb{H}^2$ connecting p to the union of the two opposite sides $(\tau_2 \cup \tau_3)$ of T, will have to cross $\sigma_2 \cup \sigma_3$. Thus, S is "thinner" than the triangle T, so it suffices to estimate the thinness of T. Since Mob_1 acts transitively on triples of distinct points in S^1, it suffices to consider the case where the ideal vertices of the triangle T are the points $A_1 = \infty, A_2 = -1, A_3 = 1$ in $S^1 = \mathbb{R} \cup \{\infty\}$. Now, consider points on the side τ_1 of T connecting A_2 to A_3. Let p denote the top-most point of the Euclidean semicircle τ_1, i.e., $p = (0, 1)$. Then, considering the horizontal Euclidean segment γ connecting p to the point $q = (-1, 1) \in \tau_3$, we see that hyperbolic length of γ equals 1 and, hence, $d(p, q) \le 1$. Consider points $p' \in \tau_1$, so that the first coordinate of p' is negative. (See Figure 2.)

Exercise 5.74. *The (hyperbolic) length of the horizontal Euclidean segment connecting p' to $q' \in \tau_3$ is < 1.*

The same argument applies to points p' with positive first coordinate. We thus conclude that for every point in τ_1, the distance to $\tau_2 \cup \tau_3$ is ≤ 1. Therefore, every hyperbolic triangle is δ-thin for $\delta \le 1$.

Remark 5.75. *The optimal thinness constant for hyperbolic triangles is* $\operatorname{arccosh}(\sqrt{2})$*, see e.g.* [**6**, Proposition 6.42].

2. Appendix 2: Least volume ellipsoids

Recall that a closed ellipsoid (with nonempty interior) centered at 0 in $\mathbb{R}^n$ can be described as

$$E = E_A = \{x \in \mathbb{R}^n : \varphi_A(x) = x^T A x \le 1\}$$

where A is some positive-definite symmetric $n \times n$ matrix. The volume of such an ellipsoid is given by the formula

$$Vol(E_A) = \omega_n \left(\det(A)\right)^{-1/2}$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$. Recall that a subset $X \subset \mathbb{R}^n$ is *centrally-symmetric* if $X = -X$.

Theorem 5.76 (F. John). *For every compact centrally-symmetric subset $X \subset \mathbb{R}^n$ with nonempty interior, there exists a unique ellipsoid $E(X)$ of least volume containing X. The ellipsoid $E(X)$ is called the* John-Loewner *ellipsoid of X.*

Proof. The existence of $E(X)$ is clear by compactness. We need to prove uniqueness. Consider the function f on the space S_n^+ of positive definite symmetric $n \times n$ matrices, given by

$$f(A) = -\frac{1}{2}\log\det(A).$$

Lemma 5.77. *The function f is strictly convex.*

Proof. Take $A, B \in S_n^+$ and consider the family of matrices $C_t = tA + (1-t)B$, $0 \le t \le 1$. Strict convexity of f is equivalent to strict convexity of f on such line segments of matrices. Since A and B can be simultaneously diagonalized by a matrix M, we obtain:

$$f(D_t) = f(MC_tM^T) = -\log\det(M) - \frac{1}{2}\log\det(C_t) = -\log\det(M) + f(C_t),$$

where D_t is a segment in the space of positive-definite diagonal matrices. Thus, it suffices to prove strict convexity of f on the space of positive-definite diagonal matrices $D = Diag(x_1, ..., x_n)$. Then,

$$f(D) = -\frac{1}{2}\sum_{i=1}^{n}\log(x_i)$$

is strictly convex since log is strictly concave. □

In particular, whenever $V \subset S_n^+$ is a convex subset and $f|V$ is proper, f attains a unique minimum on V. Since log is a strictly increasing function, the same uniqueness assertion holds for the function $\det^{-1/2}$ on S_n^+. Let $V = V_X$ denote the set of matrices $C \in S_n^+$ so that $X \subset E_C$. Since $\varphi_A(x)$ is linear as a function of A for any fixed $x \in X$, it follows that V convex. Thus, the least volume ellipsoid containing X is unique. □

3. Appendix 3: Different measures of quasiconformality

Let M be an $n \times n$ invertible matrix with singular values $\lambda_1 \le ... \le \lambda_n$. Equivalently, these numbers are the square roots of eigenvalues of the matrix MM^T. The singular value decomposition yields:

$$M = UDiag(\lambda_1, ..., \lambda_n)V$$

where U, V are orthogonal matrices.

We define the following *distortion quantities* for the matrix M:

- Linear dilatation:
$$H(M) := \frac{\lambda_n}{\lambda_1} = \|M\| \cdot \|M^{-1}\|,$$
where $\|A\|$ is the operator norm of the $n \times n$ matrix A:
$$\max_{v \in \mathbb{R}^n \setminus 0} \frac{|Av|}{|v|}.$$

- Inner dilatation:
$$H_I(M) := \frac{\lambda_1 .... \lambda_n}{\lambda_1^n} = \frac{|\det(M)|}{\|M^{-1}\|^{-n}}$$
- Outer dilatation
$$H_O(M) := \frac{\lambda_n^n}{\lambda_1 .... \lambda_n} = \frac{\|M\|^n}{|\det(M)|}$$
- Maximal dilatation
$$K(M) := \max(H_I(M), H_O(M)).$$

Exercise 5.78.
$$(H(M))^{n/2} \le K(M) \le (H(M))^{n-1}$$
Hint: It suffices to consider the case when $M = Diag(\lambda_1, ..., \lambda_n)$ is a diagonal matrix.

As we saw, qc homeomorphisms are the ones which send infinitesimal spheres to infinitesimal ellipsoids of uniformly bounded eccentricity. The usual measure of quasiconformality of a qc map f is its *maximal distortion* (or *maximal dilatation*) $K(f)$, defined as
$$K(f) := ess \sup_x K(D_x(f))$$
where the essential supremum is taken over all x in the domain of f. Here $D_x f$ is the derivative of f at x (Jacobian matrix). See e.g. J.Väisälä's book [**36**]. A map f is called K-quasiconformal if $K(f) \le K$.

In contrast, the measure of quasiconformality used in these lectures is:
$$H(f) := ess \sup_x H(D_x f).$$
To relate the two definitions we observe that
$$1 \le (H(f))^{n/2} \le K(f) \le (H(f))^{n-1}.$$
In particular, $K(f) = 1$ if and only if $H(f) = 1$.

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