On sequences of finitely generated discrete groups

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Abstract

We consider sequences of discrete subgroups $\Gamma_i = \rho_i(\Gamma)$ of a rank 1 Lie group G, with Γ finitely generated. We show that, for algebraically convergent sequences (Γ_i), unless Γ_i's are (eventually) elementary or contain normal finite subgroups of arbitrarily high order, their algebraic limit is a discrete nonelementary subgroup of G. In the case of divergent sequences (Γ_i) we show that the resulting action $\Gamma \curvearrowright T$ on a real tree satisfies certain semistability condition, which generalizes the notion of stability introduced by Rips. We then verify that the group Γ splits as an amalgam or HNN extension of finitely generated groups, so that the edge group has an amenable image in $\mathrm{Isom}(T)$.

1 Introduction

One of the basic questions about discrete subgroups of Lie groups is to understand limiting behavior of sequences of such groups. In this paper, we consider finitely generated discrete subgroups of a rank 1 semisimple Lie group G, i.e., generalizations of the classical Kleinian groups. Given a finitely generated group Γ and a sequence of subgroups $\Gamma_i = \rho_i(\Gamma) \subset G$, one says that this sequence *converges algebraically* to a subgroup $\Gamma_\infty \subset G$ if the sequence of homomorphisms $\rho_i : \Gamma \to G$ converges (pointwise) to an epimorphism $\rho_\infty : \Gamma \to \Gamma_\infty \subset G$.

More generally, one can consider algebraic convergence of G–equivalence classes of the representations ρ_i, where ρ_i's are replaced with their projections to the character variety $\mathfrak{X}(\Gamma, G)$. Sequences ρ_i which do not subconverge even in this sense, are called *divergent*. Every divergent sequence (ρ_i) yields a nontrivial action $\Gamma \curvearrowright T$ of the group Γ on a real tree T. One can regard such action as a generalization of the algebraic limit of the sequence Γ_i.

If the groups Γ_i are *discrete* and the representations ρ_i are *faithful* then the limiting behavior is completely understood due to the following theorems:

Theorem 1.1. *Suppose that the group Γ is not virtually nilpotent and the sequence Γ_i converges algebraically to Γ_∞. Then the algebraic limit Γ_∞ is discrete and ρ_∞ is faithful.*

The above theorem is due to V. Chuckrow [7], N. Wielenberg [28], T. Jorgenesen [12], G. Martin [18], in different degrees of generality, see [18] for the most general statement.

Remark 1.2. *Historically, it was H. Poincaré [23] who first (unsuccessfully) tried to prove Theorem 1.1 for Fuchsian subgroups of $SL(2,\mathbb{R})$ as a part of his first attempt on proving the Uniformization Theorem (via the continuity method).*

Theorem 1.3. *Suppose that the group Γ is not virtually nilpotent and the sequence (ρ_i) is divergent (in the character variety). Then the limiting group action on the tree $\Gamma \curvearrowright T$ is such that:*

1. $\Gamma \curvearrowright T$ is small, *i.e. arc stabilizers are virtually solvable.*

2. The action $\Gamma \curvearrowright T$ is stable. *The group Γ splits as $\Gamma = \Gamma_1 *_E \Gamma_2$ or $\Gamma = \Gamma_1 *_E$ with the edge group E amenable.*

The first part is due to J. Morgan and P. Shalen [20], J. Morgan [19], M. Bestvina [4] and F. Paulin [21] in the case when $G = SO(n,1)$. The proof in the case of other rank 1 Lie groups follows, for instance, by repeating the argument using the ultralimits which can be found in [14, Chapter 10]. The second part, for finitely-presented groups, is mostly due to I. Rips; see [25, 5, 10, 22, 14] for the proofs. The theorem was recently extended to the case when Γ is merely finitely generated by V. Guirardel [11].

The main goal of this paper is to analyze the case when the groups Γ_i are discrete but the representations ρ_i are not necessarily faithful. As far as convergent sequences of discrete groups, the best one can hope for is to show that Γ_∞ is discrete and nonelementary, provided that the groups Γ_i are also discrete and nonelementary. This was proven by T. Jorgensen and P. Klein [13] in the case when $G = SL(2,\mathbb{C})$ by methods specific to the 3-dimensional hyperbolic geometry. G. Martin [17] observed that already for the hyperbolic 4-space, discreteness of Γ_∞ can fail. His example consisted of groups $\Gamma_i = \Gamma_i' \times \Phi_i$, where each Γ_i preserves a hyperbolic plane $\mathbb{H}^2 \subset \mathbb{H}^4$ and the groups Φ_i are finite cyclic groups, so that the generators of Φ_i converge to a rotation of infinite order about $\mathbb{H}^2$. Martin proved in [17] for $G = SO(n,1)$ and in [18] for isometry groups of negatively pinched Hadamard manifolds, that Γ_∞ is discrete and nonelementary provided that the groups Γ_i have *uniformly bounded torsion.* (See also [14, Proposition 8.9], and [2] for the proofs of discreteness of *geometric* limits, under the same assumption of uniformly bounded torsion and [27] for another variation on the bounded torsion condition.) The uniform bound on torsion allows one to reduce the arguments to analyzing certain torsion-free elementary subgroups of G; such groups have the following property:

If Λ_1, Λ_2 are torsion-free discrete elementary subgroups of G, so that $\Lambda_1 \cap \Lambda_2$ is nontrivial, then Λ_1, Λ_2 generate an elementary subgroup of G.

It is easy to see that this property fails for subgroups with torsion and this is where the arguments of [17, 18, 14, 2] break down in the presence of unbounded torsion.

Our first result is

Theorem 1.4. *Suppose that Γ_∞ is an algebraic limit of a sequence of discrete nonelementary subgroups $\Gamma_i \subset G$. Then:*

1. Γ_∞ is nonelementary.

2. If Γ_∞ is nondiscrete, then for every sufficiently large i, each Γ_i preserves a proper symmetric subspace $X_i \subset X$. The kernel Φ_i of the restriction map $\Gamma_i \to \mathrm{Isom}(X_i)$ is a finite subgroup whose order D_i diverges to infinity as $i \to \infty$.

3. Every element γ of $\ker(\rho_\infty)$ either belongs to $\ker(\rho_i)$ for all sufficiently large i, or $\rho_i(\gamma) \in \Phi_i$, where Φ_i is as in 2.

Therefore, the example of G. Martin described above is, in a sense, the only way the group Γ_∞ may fail to be discrete. (See remarks in the end of section 5 in [17].) In Corollary 4.4 we generalize Theorem 1.4 to geometric limits of algebraically convergent sequences Γ_i.

Our second result deals with the group actions on trees. Suppose that Γ is finitely generated, the groups Γ_i are discrete and the sequence (ρ_i) diverges in the character variety. In general, there is no reason to expect the action $\Gamma \curvearrowright T$ to be *stable*. In Section 6 we introduce the notion of *semistable* actions to remedy this problem. This notion requires stabilization not of sequences of arc stabilizers

$$\Gamma_{I_1} \subset \Gamma_{I_2} \subset ...$$

(as in the Rips' notion of stability) but stabilization of their *algebraic hulls*

$$\mathbb{A}(\Gamma_{I_1}) \subset \mathbb{A}(\Gamma_{I_2}) \subset ... \tag{1}$$

which are certain solvable subgroups of $\mathrm{Isom}(T)$ canonically attached to Γ_{I_k}. In the case at hand, the subgroups $\mathbb{A}(\Gamma_{I_k})$ are connected algebraic subgroups of a certain nonarchimedean Lie group $\underline{G}(\mathbb{F})$, for which T is the Bruhat-Tits tree. Stabilization of the sequence (1) then comes from the fact that the dimensions of the groups $\mathbb{A}(\Gamma_{I_k})$ eventually stabilize.

Remark 1.5. *M. Dunwoody in his recent preprint [9] proposed another way to eliminate the stability assumption for group actions on trees with* slender *arc stabilizers. (A group is called* slender *if every subgroup is finitely generated.) However, both* slender *assumption is too restrictive (for instance, it forces the kernel of the action $\Gamma \curvearrowright T$ to be slender) and the conclusion that Dunwoody obtains is not as strong as one would like.*

We then verify that semistability is sufficient for the Rips theory to work. As the result we obtain:

Theorem 1.6. *Let $\rho : \Gamma \curvearrowright T$ be the limiting action arising from a divergent sequence (ρ_i). Then:*

1. The action on T of the image group $\bar{\Gamma} := \rho(\Gamma) \subset \mathrm{Isom}(T)$ is small.

2. The action $\rho : \Gamma \curvearrowright T$ is semistable.

3. Assume that Γ is finitely-presented. Then Γ splits as $\Gamma = \Gamma_1 *_E \Gamma_2$ or $\Gamma = \Gamma_1 *_E$, where $\rho(E)$ is a virtually solvable subgroup of $\mathrm{Isom}(T)$, and the groups Γ_1, Γ_2, E are finitely generated.

In Propositions 5.4 and 5.6, we also describe the kernel of the action $\Gamma \curvearrowright T$.

The key technical ingredient in the proof of Theorems 1.4 and 1.6 is the definition of the *algebraic hull* $\mathbb{A}(\Lambda)$ for amenable subgroups $\Lambda \subset \mathbb{G} = \underline{G}(\mathbb{L})$, where $\underline{G}$ is a reductive algebraic group and $\mathbb{L}$ is a field of cardinality continuum and zero characteristic. The group $\mathbb{A}(\Lambda)$ is a (Zariski) connected algebraic solvable subgroup of $\mathbb{G}$ so that the intersection

$$A(\Lambda) := \mathbb{A}(\Lambda) \cap \Lambda$$

is a subgroup of uniformly bounded index in Λ. (The bound depends only on $\mathbb{G}$.)

The results of this paper probably generalize to sequences of isometric group actions $\rho_i : \Gamma \curvearrowright X_i$, where X_i are Hadamard manifolds of fixed dimension with fixed pinching constants. However, at the moment, I am not sure how to establish such a generalization, as the concept of *algebraic hull* is missing in this context.

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2 Preliminaries

For a $CAT(0)$ space X we let $\partial_\infty X$ denote its visual boundary. We let $\overline{xy} \subset X$ denote the geodesic segment between points $x, y \in X$.

Let G be a Lie group. Then there exists a neighborhood U of 1 in G, called *Zassenhaus neighborhood*, so that for every discrete subgroup $\Gamma \subset G$, generated by elements from U, it follows that Γ is nilpotent. See e.g. [24].

Let X be a *negatively pinched* Hadamard manifold, i.e. a complete simply-connected Riemannian manifold whose sectional curvature is bounded by two negative constants:

$$-a^2 \le K_X \le -1.$$

Then there exists a constant μ (called *Margulis constant*) which depends only on a and the dimension of X so that the following holds. For every point $x \in X$ and a discrete subgroup Γ of $\mathrm{Isom}(X)$ generated by elements γ_i so that $d(x, \gamma_i(x)) \le \mu$, it follows that the group Γ is virtually nilpotent. See e.g. [1].

Let X be a negatively pinched Hadamard manifold. A discrete subgroup $\Lambda \subset \mathrm{Isom}(X)$ is called *elementary* if one of the following equivalent conditions is satisfied:

a. Λ is amenable.

b. Λ contains no free nonabelian subgroups.

c. Λ is virtually nilpotent.

d. Λ either fixes a point in $\bar{X} = X \cup \partial_\infty X$, or preserves a geodesic in X.

We refer the reader to [3] for a detailed description of the structure and quotient spaces X/Λ for such subgroups.

3 Amenable subgroups of algebraic groups

Let $G = \underline{G}(\mathbb{C})$ be a connected reductive complex-algebraic Lie group, where $\underline{G}$ is defined over $\mathbb{Q}$. We will consider amenable subgroups $\Lambda \subset G$.

Theorem 3.1. *There exists a number $d = d(G)$ so that the following holds. For every amenable subgroup $\Lambda \subset G$ there exists a canonical (Zariski) connected solvable algebraic subgroup $\mathbb{A}(\Lambda) \subset G$ (the* algebraic hull *of Λ) so that:*

1. $|A(\Lambda) : \Lambda| \le d(G)$, where $A(\Lambda) := \mathbb{A}(\Lambda) \cap \Lambda$.

2. $\mathbb{A}(\Lambda)$ is canonical in the following sense:

a. If $\Lambda_1 \subset \Lambda_2$ *then* $\mathbb{A}(\Lambda_1) \subset \mathbb{A}(\Lambda_2)$.

b. For every automorphism ϕ *of* G *(either algebraic, or coming from* $Gal(\mathbb{C})$*),*

$$\phi\mathbb{A}(\Lambda) = \mathbb{A}(\phi\Lambda).$$

c. For every $g \in G$*, if* $g\mathbb{A}(\Lambda)g^{-1} \subset \mathbb{A}(\Lambda)$*, then* $g\mathbb{A}(\Lambda)g^{-1} = \mathbb{A}(\Lambda)$.

Proof: We first, for the sake of being concrete, define $\mathbb{A}(\Lambda)$ in the case $G = SL(2,\mathbb{C})$. Amenable subgroups $\Lambda \subset G$ are classified as follows:

1. Λ is finite of order ≤ 120; we then let $\mathbb{A}(\Lambda) := \{1\}$.

2. Λ is finite of order > 120; it then contains an abelian subgroup $A(\Lambda)$ of index ≤ 2. The group $A(\Lambda)$ is contained in a unique maximal torus $\mathbb{C}^\times \cong \mathbb{T} \subset G$. We then let $\mathbb{A}(\Lambda) := \mathbb{T}$.

We now assume that Λ is infinite.

3. The subgroup Λ is diagonalizable. Then it is contained in a unique maximal torus $\mathbb{T} \subset G$ (which has to be unique). Set $\mathbb{A}(\Lambda) := \mathbb{T}$.

4. Λ is contained in the index 2 extension of a maximal torus $\mathbb{T} \subset G$. We then let $\mathbb{A}(\Lambda) := \mathbb{T}$.

5. Λ has a unique fixed point ξ in $S^2 = \partial_\infty \mathbb{H}^3$. We then let $\mathbb{A}(\Lambda)$ be the full stabilizer of ξ in G. Up to conjugation, this group consists of upper–triangular matrices in G and is, therefore, solvable.

We now discuss the general case.

1. Let $\Phi \subset G$ be a finite subgroup. Then (up to conjugation) Φ is contained in the maximal compact subgroup $K \subset G$. According to the Jordan Theorem, see e.g. [24, Theorem 8.29], there exists a canonical torus $\mathbb{T} = \mathbb{T}(\Phi) \subset K$, so that the abelian subgroup $T(\Phi) = \Phi \cap \mathbb{T}(\Phi)$ has index $\leq a(G)$ in Φ. We then let $\mathbb{A}(\Phi) \subset G$ be the complexification of the torus $\mathbb{T}$.

Let $\Lambda \subset G$ be an infinite amenable subgroup. Then, by the Tits alternative, the Zariski closure $H := \bar{\Lambda} \subset G$ has to be virtually solvable.

2. Suppose that H is an infinite reductive subgroup of G, i.e., its Zariski component of the identity is a nontrivial torus $\mathbb{T}_H \subset H$. (This torus is not necessarily maximal.) Since H has only finitely many components, the quotient $\Lambda/(\Lambda \cap \mathbb{T}_H)$ is finite. The torus $\mathbb{T}_H$ is contained in the unique smallest torus $\mathbb{T}$ which is the intersection of maximal tori in G. The torus $\mathbb{T}$ corresponds under the exponential map to a face of a Weyl chamber of G. Therefore, the number of conjugacy classes of such tori $\mathbb{T} \subset G$ is finite.

The group Λ is contained in $N(\mathbb{T})$, the normalizer of the torus $\mathbb{T}$ in G. Let $Z(\mathbb{T})$ denote the centralizer of the torus $\mathbb{T}$ in G. Recall that

$$N(\mathbb{T})/Z(\mathbb{T}) = W_{\mathbb{T}}$$

is the Weyl group associated with the torus $\mathbb{T}$. Hence, its order is bounded from above by a constant $b = b(G)$. Therefore, Λ contains a subgroup Λ' of index $\leq b$, so that $\Lambda' \subset Z(\mathbb{T})$. The quotient $\Lambda'/(\Lambda' \cap \mathbb{T})$ is a finite subgroup Φ of the Lie group $Q := Z(\mathbb{T})/\mathbb{T}$. Since the number of conjugacy classes of the tori $\mathbb{T} \subset G$ is finite, the number of components of Q is bounded from above by some $c = c(G)$. Therefore (by Case 1), there exists a canonical torus $\mathbb{A}(\Phi) \subset Q$ so that

$$|\Phi : A(\Phi)| \leq a(Q),$$

where $A(\Phi) = \mathbb{A}(\Phi) \cap \Phi$. The sequence

$$1 \to \mathbb{T} \to Z(\mathbb{T}) \to Q \to 1$$

splits and we obtain

$$\Lambda'' := \Lambda' \cap \mathbb{T} \times \mathbb{A}(\Phi), |\Lambda : \Lambda''| \leq d := a(Q)b(G).$$

We then set $\mathbb{A}(\Lambda) := \mathbb{T} \times \mathbb{A}(\Phi)$.

3. Suppose that $H = \bar{\Lambda}$ is a non-reductive subgroup of G. Let $U \subset H$ be the unipotent radical of H, so $M := H/U$ is reductive. Let $\pi : H \to M$ denote the canonical projection. The subgroup U is solvable and is canonically defined. The Levi subgroup $M \subset G$ is again algebraic. Therefore, we apply Case 2 to the subgroup $\pi(\Lambda) \subset M$. Then we set

$$\mathbb{A}(\Lambda) := \pi^{-1}(\mathbb{A}(\pi(\Lambda)).$$

Since $\mathbb{A}(\Lambda) = U \cdot \mathbb{A}(\pi(\Lambda))$ is the semidirect product of two solvable groups, it is solvable itself.

Lastly, we verify the fact that $\mathbb{A}(\Lambda)$ is canonical. Property (a) and invariance under algebraic automorphisms of G follow from the construction. Consider invariance under the automorphisms ϕ of G induced by $\sigma^{-1} \in Gal(\mathbb{C})$. It suffices to treat the case when G is an affine algebraic group (i.e., $GL(n, \mathbb{C})$). Let $E \subset G$ be a subset and $f \in \mathbb{C}[G]$ be a polynomial function vanishing on E. Then f^σ vanishes on $\phi(E)$. Moreover, if the ideal generated by the functions f determines an algebraic subgroup H of G, the same is true for the ideal generated by the functions f^σ. The subgroup H is solvable and connected iff the corresponding subgroup H^σ is.

To check property (c) note that $g\mathbb{A}(\Lambda)g^{-1} \subset \mathbb{A}(\Lambda)$ implies that the above groups have the same Lie algebra. Then the equality follows from the connectedness of $\mathbb{A}(\Lambda)$. $\square$

Corollary 3.2. *Let $\mathbb{F}$ be a field of characteristic zero and cardinality continuum, $G = \underline{G}(\mathbb{F})$ be an algebraic group. Then there exists a constant $d = d(G)$ so that the following holds. Let $\Lambda \subset G$ be an amenable subgroup. Then there exists a canonical (Zariski) connected solvable algebraic subgroup $\mathbb{A}(\Lambda) \subset G$ so that: $|A(\Lambda) : \Lambda| \leq d(G)$, where $A(\Lambda) := \mathbb{A}(\Lambda) \cap \Lambda$.*

Proof: Let $\bar{\mathbb{F}}$ denote the algebraic closure of $\mathbb{F}$. Then $\mathbb{F} \cong \mathbb{C}$ since both are extensions of $\mathbb{Q}$, algebraically closed and have the same cardinality. Therefore, we may regard Λ as a subgroup of $\underline{G}(\mathbb{C})$. Let $\bar{\mathbb{A}}(\Lambda)$ denote the algebraic hull of $\Lambda \subset \underline{G}(\mathbb{C})$. Then, since $\bar{\mathbb{A}}(\Lambda)$ is canonical, for every $\sigma \in Gal(\mathbb{C}/\mathbb{F})$, we have

$$\sigma(\bar{\mathbb{A}}(\Lambda)) = \bar{\mathbb{A}}(\Lambda)$$

We set

$$\mathbb{A}(\Lambda) := \bar{\mathbb{A}}(\Lambda) \cap G.$$

Then $\mathbb{A}(\Lambda)$ is again solvable and Zariski connected. The rest of the properties follow from Theorem 3.1. $\square$

We will apply the above corollary in the following cases: $\mathbb{F} = \mathbb{R}$ and G is a real Lie group of rank 1; $\mathbb{F}$ is a complete nonarchimedean valued field of zero characteristic and G has rank 1.

4 Algebraic limits of sequences of discrete groups

In this section we prove Theorem 1.4. Let X be a negatively curved symmetric space; its isometry group is isomorphic to a (real) rank 1 algebraic group G defined over $\mathbb{Q}$. For instance, the reader can think of $G = SO(n, 1)$ and $X = \mathbb{H}^n$. Let $\rho_i : \Gamma \to G$ be a sequence of discrete (but not necessarily faithful) representations of a finitely generated group Γ. We let Γ_i denote the image of ρ_i. Suppose that $\lim_i \rho_i = \rho_\infty$ and $\Gamma_\infty = \rho_\infty(\Gamma)$ is the algebraic limit of the sequence (Γ_i). In the "generic" case, the group $\Gamma_\infty := \rho_\infty(\Gamma)$ is a discrete nonelementary subgroup of G. The theorem below describes what happens in the exceptional cases.

Theorem 4.1. *1. If Γ_∞ is discrete and elementary, then for sufficiently large i, each Γ_i is elementary.*

2. If Γ_∞ is nondiscrete, then either:

a. For every sufficiently large i, each Γ_i is elementary, or

b. For every sufficiently large i, each Γ_i preserves a proper symmetric subspace $X_i \subset X$. The kernel Φ_i of the restriction map $\Gamma_i \to \mathrm{Isom}(X_i)$ is a finite subgroup whose order D_i diverges to infinity as $i \to \infty$.

Proof: Let $U \subset G$ denote the Zassenhaus neighborhood of $1 \in G$. Let $g_1, ..., g_m$ denote the generators of Γ. We can assume that Γ is free on the generators $g_1, ..., g_m$. We will need

Lemma 4.2. *Let $\gamma \in \ker(\rho_\infty)$. Then for all but finitely many i either*

(a) $\rho_i(\gamma) = 1$, or

(b) Γ_i is elementary, or

(c) Γ_i preserves a proper symmetric subspace $X_i \subset X$, which is fixed pointwise by $\rho_i(\gamma)$.

Proof: We assume that (a) does not occur. Let $K \subset \Gamma$ denote the normal closure of $\{\gamma\}$. Exhaust K by finitely generated subgroups

$$K_1 \subset K_2 \subset ...$$

so that

$$g_j K_l g_j^{-1} \subset K_{l+1}, \quad \forall l, \forall j = 1, ..., m. \tag{2}$$

Without loss of generality, we may assume that $\gamma \in K_1$. It is standard that if (h_i) is a sequence of nontrivial elements in a Lie group converging to 1, then the orders of h_i (regarded as elements of $\mathbb{N} \cup \{\infty\}$) converge to infinity.

Therefore, since $\rho_i(\gamma) \neq 1$ but $\lim_i \rho_i(\gamma) = 1$, the order of $\rho_i(\gamma)$ diverges to infinity as $i \to \infty$ for each $j = 1, ..., s$. It follows that the order of $\rho_i(K_1)$ diverges to infinity as $i \to \infty$. In particular, without loss of generality, we may assume that for each i, the hull $\mathbb{A}(\rho_i(K_1))$ is a nontrivial connected solvable subgroup of G.

For every $g \in K$, there exists i_g so that for all $i \geq i_g$, $\rho_i(g) \in U$. Therefore, without loss of generality, we may assume that for all i, the groups

$$\rho_i(K_l), l = 1, ..., D = \dim(G),$$

are elementary, where $\dim(G)$ is the dimension of G. Hence, for each i, there exists a pair of groups $\mathbb{A}(\rho_i(K_l)), \mathbb{A}(\rho_i(K_{l+1}))$ (for some $0 \leq l \leq D-1$ depending on i) which have the same dimension, and, hence, are equal. These groups are necessarily nontrivial.

Since $\mathbb{A}_{il} = \mathbb{A}(\rho_i(K_l))$ is canonical, in view of (2) we obtain

$$\rho_i(g_j)\mathbb{A}_{il}\rho_i(g_j)^{-1} = \mathbb{A}_{i(l+1)} = \mathbb{A}_{il}, \forall j = 1, ..., m. \tag{3}$$

If the group $\mathbb{A}_{il}$ is noncompact, then it either has a unique fixed point in $\partial_\infty X$ or an invariant geodesic. This point or a geodesic are invariant under Γ_i according to (3). Therefore, it follows that Γ_i is elementary in this case.

We next assume that $\mathbb{A}_{il}$ is compact for each i, l. By (3), the group Γ_i preserves the fixed–point set $X_i \subset X$ of $\mathbb{A}_{il}$, which is a symmetric subspace in X. Since

$$|\rho_i(K_1) : \mathbb{A}_{i1} \cap \rho_i(K_1)| \leq d(G),$$

we have

$$\rho_i(\gamma)^q \in \mathbb{A}_{i1}$$

for some $1 \leq q \leq d(G)$. Hence, $\rho_i(\gamma)|X_i$ is an element of order $\leq d(G)$ of $\mathrm{Isom}(X_i)$. Since $\rho_i(\gamma)|X_i$ converge to 1 as $i \to \infty$, it follows that $\rho_i(\gamma)$ restrict trivially to X_i for all sufficiently large i, $j = 1, ..., k$. Therefore, X_i is a proper subspace in X invariant under Γ_i. □

We now continue with the proof of Theorem 4.1.

1. Suppose that Γ_∞ is discrete and elementary. Then Γ_∞ is a lattice in a nilpotent Lie group with finitely many components. In particular, Γ_∞ is finitely-presented. It therefore has the presentation

$$\langle g_1, ..., g_m | R_1, ..., R_k \rangle$$

where $R_1, ..., R_k$ are words in the generators $g_1, ..., g_m$. Since Γ is free, we can regard these words as elements of Γ. By Lemma 4.2, for all sufficiently large i one of the following holds:

a. The group Γ_i is elementary.

b. X contains a symmetric subspace X_i invariant under Γ, so that each $\rho_i(R_j), j = 1, ..., k$ restricts trivially to X_i. Therefore, $R_1, ..., R_k$ belong to the kernel of the restriction homomorphism

$$\Gamma \to \Gamma_i \to \mathrm{Isom}(X_i).$$

Therefore the homomorphism $\Gamma \to \mathrm{Isom}(X_i)$ factors through $\Gamma \to \Gamma_\infty$. Thus, its image is an amenable group. Since the kernel of $\Gamma_i \to \mathrm{Isom}(X_i)$ is amenable, it follows that Γ_i is itself amenable and, hence, elementary.

Case 2. Suppose that Γ_∞ is nondiscrete. Our arguments are somewhat similar to the Case 1.

Let $\bar{\Gamma}_\infty$ denote the closure of Γ in G with respect to the classical topology. Then the identity component $\bar{\Gamma}^0_\infty$ of this group is a nontrivial nilpotent group, see e.g. [14, Proposition 8.9] or [2, Lemma 8.8]. In any case, Γ_∞ contains nontrivial elements $\gamma = \rho_\infty(g)$ arbitrarily close to 1. As before, the order of such γ necessarily goes to infinity as γ approaches 1.

Let V be a neighborhood of 1 in G whose closure is contained in the Zassenhaus neighborhood U. By choosing γ sufficiently close to 1, we obtain:

$$\gamma, \rho_\infty(g_j)\gamma\rho_\infty(g_j)^{-1}, ..., \rho_\infty(g_j)^D\gamma\rho_\infty(g_j)^{-D} \in V, \quad j = 1, ..., m,$$

where D can be taken as large as we like. Consider the subgroups

$$K_s := \langle g_j^t g g_j^{-t}, j = 1, ..., m, t = 0, ..., s\rangle \subset \Gamma$$

for $s = 0, ..., D$. Then,

$$K_0 \subset K_1 \subset ... \subset K_D$$

and

$$g_j^t K_s g_j^{-t} \subset K_{s+1}, \forall j = 1, ..., m, s = 0, ..., D-1.$$

As before, we choose D equal the dimension of G. By considering sufficiently large i we can assume that

$$\rho_i(g_j^s g g_j^{-s}) \in U, j = 1, ..., m, s = 0, ..., D.$$

Therefore, the subgroups $\Lambda_{is} = \rho_i(K_s)$ generated by the above elements of Γ_i, are elementary for $s = 0, ..., D$. Since γ can be taken to have arbitrarily high (possibly infinite) order, we can assume that the algebraic hull $\mathbb{A}(\Lambda_{is})$ is nontrivial for each i and s.

We now repeat the arguments from the proof in Case 1. For each i, there exists $0 \le s < D$ so that

$$\mathbb{A}(\Lambda_{is}) = \mathbb{A}(\Lambda_{i(s+1)}).$$

Therefore,

$$\rho_i(g_j)\mathbb{A}(\Lambda_{is})\rho_i(g_j)^{-1} = \mathbb{A}(\Lambda_{is}), j = 1, ..., m. \tag{4}$$

If $\mathbb{A}(\Lambda_{is})$ is noncompact, it follows from (4) that Γ_i is elementary, which contradicts our assumptions. Therefore $\mathbb{A}(\Lambda_{is})$ is compact (a torus in G); this subgroup fixes (pointwise) a proper symmetric subspace $X_i \subset X$. According to (4), this subspace is invariant under the group Γ_i. The kernel Φ_i of the restriction homomorphism $\Gamma_i \to \mathrm{Isom}(X_i)$ contains the abelian subgroup $A(\Lambda_{is}) = \mathbb{A}(\Lambda_{is}) \cap \Lambda_{is}$. By construction, the order of $A(\Lambda_{is})$ diverges to infinity as $i \to \infty$. Therefore, the order D_i of Φ_i also diverges to infinity as $i \to \infty$. $\square$

Corollary 4.3. *Suppose that $G = PSL(2, \mathbb{C})$ and, hence, $X = \mathbb{H}^3$. Then:*

1. Either $\Gamma_\infty := \rho_\infty(\Gamma)$ is discrete and nonelementary, or

2. For each sufficiently large i, the group Γ_i is elementary.

Proof: It suffices to analyze Case 2b of the above theorem. Then Γ_i contains a nontrivial finite normal subgroup Φ_i of rotations about a symmetric subspace $X_i \subset \mathbb{H}^3$; this subspace is either a point or a geodesic. In either case, Γ_i is elementary. $\square$

Corollary 4.4. *Suppose that Γ is a finitely generated group, homomorphisms $\rho_i : \Gamma \to \Gamma_i = \rho_i(\Gamma) \subset G$ converge to $\rho_\infty : \Gamma \to \Gamma_\infty = \rho_\infty(\Gamma) \subset G$ and the groups Γ_i are discrete and nonelementary. Let $\Gamma_\infty^{geo} \subset G$ be the geometric limit of the sequence of groups Γ_i. Then:*

1. Γ_∞^{geo} is nonelementary.

2. If Γ_∞^{geo} is nondiscrete, then each Γ_i contains a finite normal subgroup Φ_i, whose order diverges to infinity as $i \to \infty$.

Proof: Recall that $\Gamma_\infty \subset \Gamma_\infty^{geo}$ (see e.g. [14]). Since Γ_∞ is nonelementary by Theorem 4.1, it follows that Γ_∞^{geo} is nonelementary as well. To prove Part 2, we modify Part 2 of the proof of Theorem 4.1 as follows. Consider an element $\gamma \in \Gamma_\infty^{geo} \setminus \{1\}$ sufficiently close to $1 \in G$. Instead of using a fixed element $g \in \Gamma$ so that $\rho_\infty(g) = \gamma$, we consider a sequence $h_i \in \Gamma$ so that

$$\lim_{i\to\infty} \rho_i(h_i) = \gamma.$$

Instead of the subgroups $K_s \subset \Gamma$ we use

$$K_{s,i} := \langle g_j^t h_i g_j^{-t}, j = 1, ..., m, t = 0, ..., s \rangle \subset \Gamma.$$

With these modifications, the proof of Part 2 of Theorem 4.1 goes through in the context of the geometric limit. $\square$

5 Small actions

In this section we prove the first assertion of Theorem 1.6.

Let $\rho : \Gamma \curvearrowright T$ be an isometric action of a group Γ on a metric tree T. Let $\bar{\Gamma} := \rho(\Gamma) \subset \mathrm{Isom}(T)$ denote the image of Γ in $\mathrm{Isom}(T)$. Given an axial isometry $g \in \Gamma$, let A_g denote the axis of g and $\ell(g)$ the translation length of g. Recall that the action $\bar{\Gamma} \curvearrowright T$ is called *nontrivial* if Γ does not have a global fixed point. This action is called *small* if the arc stabilizers are amenable.

Suppose that (X, d) is a negatively pinched simply-connected complete Riemannian manifold and Γ is a finitely–generated group with the generating set $\{g_1, ..., g_m\}$. Given a representation $\rho : \Gamma \to \mathrm{Isom}(X)$, define

$$b_x(\rho) := \max(d(\rho(g_i)(x), x), i = 1, ..., m),$$

$$b(\rho) := \inf_{x\in X} b_x(\rho).$$

Then a sequence of representations $\rho_i : \Gamma \to \mathrm{Isom}(X)$ is *divergent* if and only if

$$\lim_i b(\rho_i) = \infty.$$

Indeed, if there is a subsequence (ρ_{i_j}) so that $b(\rho_{i_j}) \le C$, then we can conjugate ρ_{i_j} by the elements $h_{i_j} \in G$ which move x_{i_j} to a base-point $o \in X$. Since G is locally, compact, it follows that the new sequence

$$Ad(h_{i_j})\rho_{i_j}$$

converges in $Hom(\Gamma, G)$.

Let ω be a nonprincipal ultrafilter on $\mathbb{N}$. We recall that a divergent sequence yields a nontrivial isometric action $\rho_\omega : \Gamma \curvearrowright T$ of Γ on a metric tree T, well-defined up to scaling (given the choice of ω). The tree T is the ω-ultralimit of the sequence of pointed metric spaces

$$(X, \frac{d}{b(\rho_i)}, o_i)$$

where $o_i \in X$ is the point *nearly* realizing $b(\rho_i)$, i.e.,

$$|b(\rho_i) - b_{o_i}(\rho_i)| \le 1.$$

See e.g. [14, 15] for the details.

We now assume that X is a symmetric space, i.e. its isometry group is a rank 1 algebraic group G.

The following theorem is standard in the case of sequences of discrete *and faithful* representations (ρ_i):

Theorem 5.1. *Let $\rho_i : \Gamma \to \mathrm{Isom}(X)$ be a divergent sequence of representations with discrete images. Let $\rho_\omega : \Gamma \curvearrowright T$ denote the limiting action on a tree and $\bar{\Gamma} := \rho_\omega(\Gamma) \subset \mathrm{Isom}(T)$. Then the action $\bar{\Gamma} \curvearrowright T$ is small.*

Proof: Our proof repeats the arguments of the proof of Theorem 10.24 in [14] with minor modifications. Let $\mu > 0$ denote the *Margulis constant* for X.

For a nondegenerate arc $I \subset T$ let Γ_I denote the stabilizer of I in Γ. Let $\Gamma'_I \subset \Gamma_I$ be the commutator subgroup. Exhaust Γ'_I by an increasing sequence of finitely–generated subgroups $\Lambda_n \subset \Gamma'_I$.

Lemma 5.2. *For each n and ω–all i, the group $\rho_i(\Lambda_n)$ is elementary.*

Proof: The arc I corresponds to a sequence of geodesic arcs $I_i \subset X$. Let $m_i \in I_i$ be the midpoint. Let $h_1, ..., h_l$ be generators of Λ_n. Since each h_j is a product of commutators of elements of Γ_I, the arguments of the proof of Theorem 10.24 in [14] imply that $\rho_i(h_j)$ moves m_i by $\le \mu$ for ω–all i. Therefore, by Kazhdan–Margulis lemma, the group $\rho_i(\Lambda_n)$ is elementary. □

For an elementary subgroup $\Lambda \subset G$, let $\mathbb{A}(\Lambda) \subset G$ denote the algebraic hull of Λ defined in Corollary 3.2 and set $A(\Lambda) := \mathbb{A}(\Lambda) \cap \Lambda$.

Therefore, each group $\Lambda_{in} := \rho_i(\Lambda_n)$ contains a canonical nilpotent subgroup $A_{in} = A(\Lambda_{in})$ of index $\le c$ (where c depends only on G). Since A_{in} is canonical, we have

$$A_{in} \subset A_{i(n+1)}$$

for each n and ω–all i. It follows (by taking the ω-ultralimit) that each $\rho_\omega(\Lambda_n)$ contains a canonical nilpotent subgroup A_n of index $\le c$. Thus, the nilpotent subgroup

$$A := \bigcup_n A_n$$

has index $\le c$ in $\rho_\omega(\Lambda)$. Therefore, the group $\rho_\omega(\Gamma'_I)$ is virtually nilpotent. Hence, the group $\rho_\omega(\Gamma_I)$ fits into the short exact sequence

$$1 \to \rho_\omega(\Gamma'_I) \to \rho_\omega(\Gamma_I) \to B \to 1$$

where B is abelian. Since amenability is stable under group extensions with amenable kernel and quotient, the group $\rho_\omega(\Gamma_I)$ is (elementary) amenable. We proved, therefore, that $\bar{\Gamma} \curvearrowright T$ is *small.* □

Remark 5.3. *The above argument also works for sequences of group actions on negatively pinched Hadamard manifolds of fixed dimensions with fixed pinching constants.*

The following two propositions describe, to a certain degree, the kernel of the action $\Gamma \curvearrowright T$.

Proposition 5.4. *Suppose that each Γ_i is nonelementary and does not preserve a proper symmetric subspace in X. Then for every $g \in Ker(\rho_\omega)$, for ω-all i we have*

$$g \in Ker(\rho_i).$$

Proof: We conjugate the representations ρ_i so that $o_i = o$ for all i. We will need

Lemma 5.5. *For every $g \in Ker(\rho_\omega)$, we have*

$$\omega\text{-}lim\, \rho_i(g) = 1 \in G.$$

Proof: For $g \in Ker(\rho_\omega)$ set $\gamma_i := \rho_i(g)$. Set $R_i := b(\rho_i)$ and let $B_{R_i}(o)$ be the R_i-ball centered at o.

Then we obtain

$$\omega\text{-lim}\, \frac{d(x, \gamma_i(x))}{R_i} = 0, \quad \forall x \in B_{R_i}(o).$$

Therefore, there exists r_i so that:

$$\omega\text{-lim}\, \frac{r_i}{R_i} \in (0, \infty)$$

and for each geodesic segment $\sigma \subset B_{R_i}(o)$ we have

$$dist(\sigma \cap B_{r_i}(o), \gamma_i(\sigma) \cap B_{r_i}(o)) \le \epsilon_i,$$

where

$$\omega\text{-lim}\, \epsilon_i = 0,$$

and *dist* stands for the Hausdorff distance. See Lemma 3.10 in [14].

By applying this to geodesic segments $\sigma, \tau \subset B_{R_i}(o)$ which pass through a given point $p \in B_1(o)$ and are orthogonal to each other, we conclude that

$$\omega\text{-lim}\, d(\gamma_i(p), p) = 0.$$

Therefore,

$$\omega\text{-lim}\, \rho_i(g) = 1. \quad \square$$

Let $g_1, ..., g_m$ be the generators of Γ. Suppose that the assertion of the Proposition fails. Take $\gamma \in \ker(\rho_\omega)$ so that for ω-all i, $\rho_i(\gamma) \neq 1$. By Lemma 5.5,

$$\omega\text{-lim}\, ord(\rho_i(\gamma)) = \infty$$

where *ord* stands for the order of an element of G.

We now repeat the arguments of the proof of Lemma 4.2. Let $g \in K = \ker(\rho_\omega)$. We find finitely-generated subgroups

$$K_1 \subset K_2 \subset ... \subset K,$$

so that $g \in K_1$ and

$$g_j K_l g_j^{-1} \subset K_{l+1}, j = 1, ..., m.$$

Lemma 5.5 implies that for each l, $\rho_i(K_l)$ is an elementary subgroup of G for ω-all i. Set $\mathbb{A}_{il} := \mathbb{A}(\rho_i(K_l))$. As in the proof of Lemma 4.2, the group $\mathbb{A}_{il}$ is nontrivial for ω-all i and each l.

Then, for ω-all i there exist l so that for every $j = 1, ..., m$ we have

$$\rho_i(g_j)\mathbb{A}_{il}\rho_i(g_j)^{-1} = \mathbb{A}_{il}.$$

Therefore, either Γ_i is elementary or preserves a proper symmetric subspace in X (fixed by $\mathbb{A}_{il}$). In either case, we obtain a contradiction with the assumptions of Proposition 5.4. $\square$

The tree T contains a unique subtree T_{min} which is the smallest Γ–invariant subtree, see e.g. [14]. The kernel K of the action $\Gamma \curvearrowright T_{min}$ is, a priori, larger than the kernel of $\Gamma \curvearrowright T$.

Proposition 5.6. *Suppose that the tree T_{min} is not a line and the hypothesis of Proposition 5.4 hold. Then for every $g \in K$, for ω-all i, we have $g \in Ker(\rho_i)$.*

Proof: Since T_{min} is not a line, it contains a nondegenerate triangle $x_\omega y_\omega z_\omega \subset T_{min}$. The vertices $x_\omega, y_\omega, z_\omega$ of this triangle are represented by sequences $(x_i), (y_i), (z_i)$ in X. Let $m_i \in \overline{x_i y_i}$ be a point within distance $\leq \delta$ from the other two sides of the triangle $x_i y_i z_i$, where δ is the hyperbolicity constant of X. For $g \in Ker(\rho_\omega)$ set $\gamma_i := \rho_i(g)$.

Lemma 5.7.

$$\omega\text{-}lim\, d(\gamma_i(m_i), m_i) = 0.$$

Proof: Our argument is similar to that of the proof of Lemma 5.5. We again set $R_i := b(\rho_i)$; then

$$\omega\text{-lim}\, \frac{d(x_i, \gamma_i(x_i))}{R_i} = 0, \quad \omega\text{-lim}\, \frac{d(y_i, \gamma_i(y_i))}{R_i} = 0,$$

$$\omega\text{-lim}\, \frac{d(z_i, \gamma_i(z_i))}{R_i} = 0.$$

As in the proof of Lemma 5.5, the segment $\overline{x_i y_i}$ will contain a subsegment $\sigma_i := \overline{x_i' y_i'}$ of length r_i so that $m_i \in \overline{x_i' y_i'}$,

$$\omega\text{-lim}\, \frac{r_i}{R_i} \in (0, \infty),$$

and

$$\omega\text{-lim}\, d(\gamma_i(x_i'), \sigma_i) = \omega\text{-lim}\, d(\gamma_i(y_i'), \sigma_i) = 0.$$

Define points $p_i, q_i \in \sigma_i$ nearest to $z_i, \gamma_i(z_i)$ respectively. Then

$$\omega\text{-lim}\, d(q_i, \gamma_i(p_i)) = 0.$$

Suppose that the isometries γ_i *shear* along the segments σ_i, i.e.

$$\omega\text{-lim}\, d(\gamma_i(m_i), m_i) \neq 0.$$

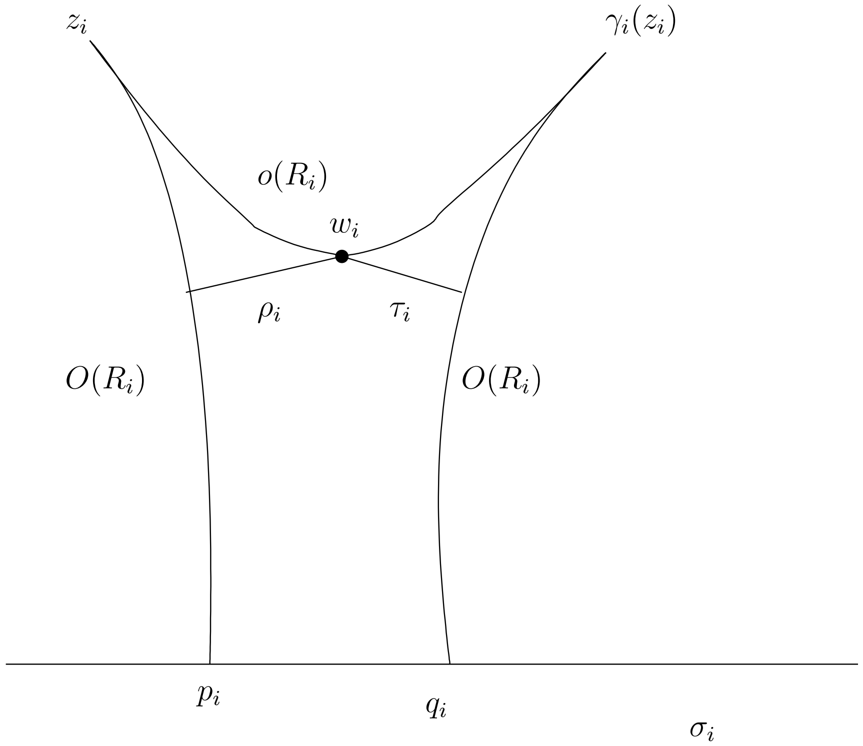

Figure 1:

Then

$$\omega\text{-}\lim d(\gamma_i(p_i), p_i) = \omega\text{-}\lim d(p_i, q_i) = \omega\text{-}\lim d(\gamma_i(m_i), m_i) \neq 0.$$

Since

$$\omega\text{-}\lim \frac{d(z_i, p_i)}{R_i} = d(z_\omega, p_\omega) \neq 0, \quad \omega\text{-}\lim \frac{d(z_i, \gamma_i(z_i))}{R_i} = 0,$$

it follows that there exists a point $w_i \in \overline{z_i\gamma_i(z_i)}$ within distance $\leq 2\delta$ from both

$$\overline{z_ip_i}, \quad \overline{q_i\gamma_i(z_i)}.$$

See Figure 1. Since

$$\omega\text{-}\lim \frac{d(z_i, \gamma_i(z_i))}{R_i} = 0, \quad \omega\text{-}\lim \frac{d(z_i, p_i)}{R_i} \neq 0,$$

we obtain

$$\omega\text{-}\lim d(w_i, \sigma_i) = \infty.$$

Take the shortest segments ρ_i, τ_i from w_i to $\overline{z_ip_i}, \overline{q_i\gamma_i(z_i)}$. The nearest-point projection to σ_i sends $\rho_i \cup \tau_i$ onto $\overline{p_iq_i}$. However, this projection is exponentially contracting and $\omega\text{-}\lim d(w_i, \sigma_i) = \infty$. This contradicts the assumption that

$$\omega\text{-}\lim d(p_i, q_i) \neq 0.$$

Therefore,

$$\omega\text{-}\lim d(\gamma_i(m_i), m_i) = 0. \quad \square$$

Given $g \in K = \ker(\Gamma \to \mathrm{Isom}(T_{min}))$, we define the finitely-generated subgroups $K_l \subset K$ in the same fashion it was done in the proof of Proposition 5.4. By Lemma 5.7, it follows that for every generator $h \in K_l$ and $\eta_i := \rho_i(h)$, we have

$$\omega\text{-}\lim d(\eta_i(m_i), m_i) = 0.$$

Therefore, by Kazhdan–Margulis lemma, for each l and ω-all i, the group $\rho_i(K_l)$ is elementary. Now, the arguments from the proof of Proposition 5.4 go through and we obtain $\rho_i(g) = 1$ for ω-all i. $\square$

6 Semistability

The purpose of this section is to weaken the notion of *stability* used in the Rips' theory, so that the Rips Machine still applies. We recall

Definition 6.1. *Let $\Gamma \curvearrowright T$ be an isometric group action on a tree. A nondegenerate arc $I \subset T$ is called* stable *if for every decreasing sequence of nondegenerate subarcs*

$$I \supset I_1 \supset I_2 \supset \ldots$$

the corresponding sequence of stabilizers

$$\Gamma_I \subset \Gamma_{I_1} \subset \Gamma_{I_2} \subset \ldots$$

is eventually constant. The action $\Gamma \curvearrowright T$ is called stable *if every nondegenerate arc $J \subset T$ contains a stable subarc.*

M. Dunwoody [8] constructed example of a small but unstable action of a finitely generated group Γ on a tree. To remedy this, we introduce the following modification of stability, adapted to the case of actions whose image on $\mathrm{Isom}(T)$ is small:

Definition 6.2. *Suppose that we are given an isometric action of a group on a tree $\rho : \Gamma \curvearrowright T$. We say that this action is* semistable *if it satisfies the following property:*

For every nondegenerate arc $I \subset T$ and its stabilizer $\Gamma_I \subset \Gamma$, there exists a canonical amenable subgroup $\mathbb{A}(\Gamma_I) \subset \mathrm{Isom}(T)$ so that:

1. If $I \supset J$ then $\mathbb{A}(\Gamma_I) \subset \mathbb{A}(\Gamma_J)$.

2. $A(\Gamma) := \mathbb{A}(\Gamma_I) \cap \rho(\Gamma_I)$ has index $\leq c < \infty$ in $\rho(\Gamma_I)$, where $c = c_T$ is a uniform constant.

3. If $\alpha \in \Gamma$ is such that $\alpha\mathbb{A}(\Gamma_I)\alpha^{-1} \subset \mathbb{A}(\Gamma_I)$, then $\alpha\mathbb{A}(\Gamma_I)\alpha^{-1} = \mathbb{A}(\Gamma_I)$.

4. For every nondegenerate arc $J \subset T$, there exists a nondegenerate subarc $I \subset J$ so that the following holds:

If $I \supset I_1 \supset I_2 \supset \ldots$ is a decreasing sequence of nondegenerate arcs, then the sequence of groups

$$\mathbb{A}(\Gamma_1) \subset \mathbb{A}(\Gamma_2) \subset \ldots$$

is eventually constant.

We say that an arc $I \subset T$ is *stabilized* (with respect to the action of Γ) if for every nondegenerate subarc $J \subset I$, we have

$$\mathbb{A}(\Gamma_I) = \mathbb{A}(\Gamma_J).$$

We let $\mathbb{A}_I$ denote $\mathbb{A}(\Gamma_I)$ in this case.

Note that every semistable action is automatically small, since a finite index extension of an amenable group is also amenable.

It is easy to classify the possible amenable groups $\mathbb{A} \subset \mathrm{Isom}(T)$:

1. $\mathbb{A}$ is parabolic, i.e., it fixes a point in $\partial_\infty T$ and does not fix any other points in $T \cup = \partial_\infty T$.

2. $\mathbb{A}$ is hyperbolic, i.e., it has a unique invariant geodesic $T_{\mathbb{A}} \subset T$ and contains a nontrivial translation along this geodesic.

3. $\mathbb{A}$ is elliptic, i.e., it fixes a nonempty subtree $T_{\mathbb{A}} \subset T$.

We now give examples of semistable actions.

Example 1. Consider $\rho : \Gamma \to \mathrm{Isom}(T)$, so that the action of the image group $\bar{\Gamma} = \rho(\Gamma)$ on T is small and stable. Then $\Gamma \curvearrowright T$ is also semistable: take $\mathbb{A}(\Gamma_I) := \rho(\Gamma_I)$.

Example 2. Let $\mathbb{F}$ be a nonarchimedean valued field of zero characteristic and cardinality continuum and $G = \underline{G}(\mathbb{F})$ be a group of rank 1. We then consider the Bruhat–Tits tree T associated with the group G. The quotient group $G/Z(G)$ acts faithfully on T, where $Z(G)$ is the center of G. Let $\Gamma \subset G/Z(G) \subset \mathrm{Isom}(T)$ be a subgroup so that the associated action $\Gamma \curvearrowright T$ is small.

Given an amenable subgroup $\Lambda \subset \Gamma$, consider its lift $\tilde{\Lambda} \subset G$, which is still an amenable subgroup. Let $\mathbb{A}(\Lambda) \subset G/Z(G)$ denote the projection of the hull $\mathbb{A}(\tilde{\Lambda}) \subset G$, defined in Corollary 3.2. It is immediate that $\mathbb{A}(\Lambda)$ satisfies Properties 1–3 of Definition 6.2. Consider Property 4.

For the amenable groups $\Lambda = \Gamma_I$, the algebraic hulls $\mathbb{A}(\Gamma_I)$ are Zariski connected algebraic subgroups of G. Since the dimensions of the groups in the sequence

$$\mathbb{A}(\Gamma_I) \subset \mathbb{A}(\Gamma_{I_1}) \subset \mathbb{A}(\Gamma_{I_2})...$$

are eventually constant, this sequence is eventually constant as well.

Example 3. Let $\rho_\omega : \Gamma \to Isom(T)$ be a group action on a tree associated with a divergent sequence of representations $\rho_i : \Gamma \to \mathrm{Isom}(X)$, where X is a negatively curved symmetric space. The asymptotic cone $T = T_\omega$ of X associated with this sequence is a metric tree. According to [6, 16, 26], the asymptotic cone T is the Bruhat-Tits tree associated with a group $\underline{G}(\mathbb{F})$, where $\mathbb{F}$ is a certain nonarchimedean valued complete field of cardinality continuum and characteristic zero. Moreover, the group Γ maps to $\mathrm{Isom}(T)$ via a homomorphism

$$\rho_\omega : \Gamma \to \underline{G}(\mathbb{F}) \subset \mathrm{Isom}(T).$$

Remark 6.3. *The field $\mathbb{F}$ is a subfield of the field of nonstandard reals, which is the ultraproduct*

$$\mathbb{R}_* = \prod_{i \in \mathbb{N}} \mathbb{R}/\omega.$$

The choice of the subfield and valuation depends on ω and on the divergent sequence $b(\rho_i)$.

In the case $X = \mathbb{H}^3$ and $\underline{G} = SL(2)$, we can use algebraically closed field $\mathbb{F}$, which is a subfield of the ultraproduct

$$\mathbb{C}_* = \prod_{i \in \mathbb{N}} \mathbb{C}/\omega.$$

Therefore, for each amenable subgroup $\Lambda = \rho_\omega(\Gamma_I)$, we can define the algebraic hull $\mathbb{A}(\Lambda)$ using Corollary 3.2 (see Example 2 above). In case $X = \mathbb{H}^3$ and $\mathbb{F}$ algebraically closed, we can use Theorem 3.1, or, rather, the example which appears in the beginning of the proof. In particular, by Example 2, it follows that the action $\rho_\omega : \Gamma \curvearrowright T$ is semistable.

Corollary 6.4. *Part 2 of Theorem 1.6 holds.*

Implications of semistability. We now assume that we are given a semistable action $\Gamma \curvearrowright T$ and the corresponding action $\bar{\Gamma} \curvearrowright T$ of the image of Γ in $\mathrm{Isom}(T)$.

Let $I \subset T$ be a stabilized arc and $\alpha \in \Gamma$ be an axial isometry of T, whose axis contains I, and so that

$$J = I \cap \alpha(I)$$

is nondegenerate. Then

$$\alpha \Gamma_I \alpha^{-1} \subset \Gamma_J.$$

Since I is stabilized, follows that $\alpha \mathbb{A}_I \alpha^{-1} \subset \mathbb{A}_J = \mathbb{A}_I$. Thus $\alpha \mathbb{A}_I \alpha^{-1} = \mathbb{A}_I$ (see Part 3 of Definition 6.2). Suppose that we are given two elements $\alpha, \beta \in \Gamma$ as above, so that

$$\alpha \mathbb{A}_I \alpha^{-1} = \mathbb{A}_I, \quad \beta \mathbb{A}_I \beta^{-1} = \mathbb{A}_I. \tag{5}$$

Case 1. $\mathbb{A}_I$ is parabolic. Then the equalities (5) imply that α, β both fix the unique fixed point at infinity of the group $\mathbb{A}_I$. Since the action $\bar{\Gamma} \curvearrowright T$ is small, it follows that the group $\rho\langle\alpha, \beta\rangle$ generated by $\rho(\alpha), \rho(\beta)$ is virtually solvable, see [14, §10.5].

Case 2. $\mathbb{A}_I$ is hyperbolic. Then the equalities (5) imply that α, β preserve the unique invariant geodesic of the group $\mathbb{A}_I$. Hence, the commutator subgroup of $\langle\alpha, \beta\rangle$ fixes this geodesic pointwise. It again follows that $\rho\langle\alpha, \beta\rangle$ is virtually solvable.

Case 3. $\mathbb{A}_I$ is elliptic. Let $T' \subset T$ denote the subtree fixed by $\mathbb{A}_I$. Then T' is invariant under both α and β. The restrictions of these isometries to T' remain axial.

Recall that

$$|\rho(\Gamma_J) : A(\Gamma_J)| \le c_T$$

for every arc J.

Assumption 6.5. *We now assume in addition that n is a natural number so that*

$$\frac{\mathrm{length}(A_\alpha \cap A_\beta)}{\ell(\alpha) + \ell(\beta)} \ge 2n > 2c_T.$$

Under this assumption, for each $i = 1, .., n$, $[\alpha^i, \beta] \in \Gamma_J \subset \bar{\Gamma}_J$ for some nondegenerate subinterval $J \subset I$. Moreover, there exist $m \neq n$ so that we have the equality of the cosets

$$[\alpha^m, \beta]\mathbb{A}_I = [\alpha^n, \beta]\mathbb{A}_I.$$

Since $\mathbb{A}_I$ fixes T' pointwise, it follows that

$$[\alpha^m, \beta]|_{T'} = [\alpha^n, \beta]|_{T'}.$$

Hence,

$$[\alpha^{m-n}|_{T'}, \beta|_{T'}] = 1.$$

Since $\alpha^{m-n}|_{T'}, \beta|_{T'}$ are commuting nontrivial axial elements, they have to have common axis. Therefore, α, β also have common axis. Now, analogously to the Case 2, it follows that $\rho\langle\alpha, \beta\rangle$ is virtually solvable.

We conclude that in each case (provided that the Assumption 6.5 holds in the elliptic case), we have

Proposition 6.6. *The group $\rho\langle\alpha, \beta\rangle$ is amenable.*

7 Generalization of the Rips theory to the semistable case

In this section we will finish the proof of Theorem 1.6 by verifying Part 3.

Suppose that we are given a semistable nontrivial action

$$\rho : \Gamma \to \mathrm{Isom}(T),$$

of a finitely-presented Γ on a tree T. Then one can apply the arguments of the Rips Theory (see [5] or [14, Chapter 12]) to the action $\Gamma \curvearrowright T$. Note that the only place the *stability* condition is used in the proof of the Rips theorem, is the analysis of the *axial* pure band complex C, see e.g. [14, Proposition 12.69].

In this case one deals with pairs of axial isometries $\alpha, \beta \in \Gamma$, so that the ratio

$$\frac{\mathrm{length}(A_\alpha \cap A_\beta)}{\ell(\alpha) + \ell(\beta)}$$

can be taken as large as one wishes. Therefore, one can choose this ratio to satisfy the Assumption 6.5 as above. The conclusion of the Rips Theory in the Axial case is then that the action of the fundamental group $\pi_1(C)$ of the component C (which is a subgroup of Γ) on the tree T has an invariant geodesic. It then deduced that the action of $\pi_1(C)$ factors through action of a solvable group.

In our case, Proposition 6.6 implies that the action $\pi_1(C) \curvearrowright T$ either has an invariant geodesic or is parabolic; in either case, it factors through action of an amenable group.

Therefore, repeating verbatim the proof of Theorem 12.72 in [14], we obtain

Theorem 7.1. *One of the following holds:*

1. If the action $\Gamma \curvearrowright T$ is not pure *then the group Γ splits nontrivially as $\Gamma = \Gamma_1 *_E \Gamma_2$ or $\Gamma = \Gamma_1 *_E$, over a subgroup E, which fits into a short exact sequence*

$$1 \to K_E \to E \to Q \to 1,$$

where K_E fixes a nondegenerate arc in T and Q is either finite or cyclic. Moreover, the group E fixes a point in T and the groups Γ_1, Γ_2, E are finitely generated.

2. If the action is pure then G belongs to one of the following types:

(a) Surface type.

(b) Axial type.

(c) Thin type.

In either case, Γ *splits nontrivially as* $\Gamma = \Gamma_1 *_E \Gamma_2$ *or* $\Gamma = \Gamma_1 *_E$*, over a subgroup* E*, which fits into a short exact sequence*

$$1 \to K_E \to E \to Q \to 1,$$

where K_E *fixes a nondegenerate arc in* T *and* Q *is abelian. The groups* Γ_1, Γ_2, E *are finitely generated.*

Therefore, the image (in $\mathrm{Isom}(T)$) of the edge subgroup of Γ is amenable.

We now assume that the action $\Gamma \curvearrowright T$ arises from a divergent sequence of discrete but not necessarily faithful representations

$$\rho_i : \Gamma \to \mathrm{Isom}(X)$$

where X is a negatively curved symmetric space. Then we obtain $\Gamma \curvearrowright T$, where T is an asymptotic cone of X, which can be realized as the Bruhat-Tits tree of a rank 1 algebraic group $\underline{G}(F)$. Thus we obtain a homomorphism $\rho_\omega : \Gamma \to \bar{\Gamma} \subset \underline{G}(F) \subset \mathrm{Isom}(T)$. Then, according to Section 5, the action $\bar{\Gamma} \curvearrowright T$ is small. According to Section 6, this action is also semistable. Therefore, Theorem 7.1 applies and we obtain:

Corollary 7.2. *The group* Γ *splits nontrivially as* $\Gamma = \Gamma_1 *_E \Gamma_2$ *or* $\Gamma = \Gamma_1 *_E$*, over a subgroup* E*, so that* $\rho(E)$ *is amenable. The groups* Γ_1, Γ_2, E *are finitely generated.*

Remark 7.3. *M. Dunwoody [9] proved another version of Rips Theorem in the case of* slender *faithful actions of finitely-presented groups on trees without the stability hypothesis. However his main theorem only yields a splitting of* Γ *where each edge group is either slender or fixes a point in the tree. This is not enough to guarantee amenability of the edge groups in the resulting decomposition. Moreover, it appears that the arc stabilizers* Γ_I *for group actions on trees associated with divergent sequences of discrete representations, need not be slender. For instance, it seems that they can contain infinitely generated abelian subgroups.*

Since $\rho(E) \subset \underline{G}(F)$, it follows that this subgroup is virtually solvable. By combining the above results, we obtain Theorem 1.6.

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