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Discreteness is undecidable

Michael Kapovich

Department of Mathematics,
University of California, Davis
CA 95616, USA
kapovich@math.ucdavis.edu

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We prove that the discreteness problem for 2-generated nonelementary subgroups of $SL(2,\mathbb{C})$ is undecidable in the BSS computability model.

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1. Introduction

This paper is motivated by the following basic question:

Question 1.1. Let G be a connected Lie group and let $\mathcal{A} = (A_1, \ldots, A_k)$ be a finite ordered subset of G. Is the discreteness problem for the subgroup $\Gamma_{\mathcal{A}} := \langle A_1, \ldots, A_k \rangle < G$ decidable?

This question, in the case of $G = PSL(2,\mathbb{C})$, was raised, most recently, in the paper [8] by J. Gilman and L. Keen, who noted that "it is a challenging problem that has been investigated for more than a century and is still open." The decidability problem was solved in the case $G = PSL(2,\mathbb{R})$ by R. Riley [20] and, more efficiently, in the case of 2-generated subgroups, by J. Gilman and B. Maskit [9] and Gilman [6], (cf. [7] for a comparison of the two approaches).

To make the general decidability question more precise one has to specify the model of computability. There are several computability models over the real numbers; we refer the reader to [1] and [21] for summaries of these and in-depth treatment of the *BSS* and the *bit-computability* approaches respectively. In this paper we address decidability of the discreteness problem in the *real-RAM* or *BSS* (which stands for Blum–Shub–Smale) computability model as it is the closest in spirit to the papers by Gilman, Maskit and Keen mentioned above as well as Riley's work [20].

We will address decidability of the discreteness problem in the bit-computabulity model in another paper, [13].

Remark 1.2. We refer the reader to the paper by J. Gilman in [7], where several (semi)algorithms for the discreteness problem in $PSL(2,\mathbb{R})$ and $PSL(2,\mathbb{C})$ in different computability models, including the BSS model, are compared.

Briefly, computations in the BSS model over the real numbers are performed by a *BSS machine*, which is an analogue of a Turing machine except that a BSS machine can store finite lists of real numbers and do elementary algebraic and order operation with real numbers: Such a machine can add, subtract, multiply and divide, as well as verify inequalities and equalities $a < b, a = b$ for real numbers. (BSS machines are also defined for computations in other rings, but, in this paper we will use only real numbers.) We refer to [1] for the details.

A subset $E \subset \mathbb{R}^n$ is *BSS-semicomputable* (or the membership problem for E is *BSS-semidecidable*) if E is the *halting set* of a BSS machine: There exists a BSS machine which, given an input vector $x \in \mathbb{R}^n$, stops iff $x \in E$. A membership problem for E is BSS-decidable iff both E and $E^c = \mathbb{R}^n - E$ are BSS-semicomputable. We refer the reader to the book [1] for the details.

Remark 1.3. In our paper, the input for a BSS machine is a tuple $\mathcal{A}$ of 2×2 complex matrices.

The main result about BSS machines needed for our paper is the following theorem due to Blum, Shub and Smale, see [1, Theorem 1, Chapter 2]:

Theorem 1.4. *The halting set for a BSS machine is a (computable) countable union of real semialgebraic subsets of* $\mathbb{R}^n$.

Remark 1.5. We note that the proof of this theorem in [1] actually shows more: Allow a *generalized* BSS machine to do boolean operations with inequalities, as well as to compute not only rational functions, but also *real algebraic functions*, i.e., functions whose graphs are given by finite sets of polynomial equations and inequalities, e.g. $\sqrt{x}$. Then the halting set of such a machine is still a countable union of real semialgebraic subsets.

Before stating our main results, we note that the nondiscreteness problem for 1-generator subgroups of $G = S^1 \subset \mathbb{C}^*$ is *not semidecidable*, since a subgroup $\langle A\rangle < S^1$ is nondiscrete if and only if A has infinite order, i.e., is not a root of unity. The complement in S^1 of the set of roots of unity is clearly not a countable union of arcs, therefore, it cannot be a halting set of a BSS machine. Thus, the discreteness problem, strictly speaking, is undecidable already in $G = PSL(2,\mathbb{R})$. To make it decidable in $G = PSL(2,\mathbb{R})$ one has to exclude from G^k the algebraic subvariety consisting of tuples of matrices generating (virtually) abelian subgroups. Regarding subgroups of $G = PSL(2,\mathbb{C})$ with two (or more) generators, one has to exclude, for a similar reason, dihedral subgroups (both finite and infinite). In line with the work of Gilman, Keen, Maskit and Riley, we will, moreover, exclude from consideration all tuples $\mathcal{A}$ which generate *elementary subgroups* of $G = SL(2,\mathbb{C})$. (This exclusion also allows for a clean discussion of the *character variety*, which is a quotient of $Hom(F_k, G)$ by the group G acting *via* conjugation.) Observe also that Riley's arguments in [20] (based on the Jørgensen inequality) show that the *nondiscreteness problem* for nonelementary subgroups of $PSL(2,\mathbb{C})$ is BSS-semidecidable.

The space G^k of k-tuples of matrices $\mathcal{A}$, $A_j \in G$, is naturally identified with the *representation variety*, which is the algebraic variety $Hom(F_k, G)$, via the map

$$\phi \mapsto (A_1, \ldots, A_k), \quad A_j = \phi(x_j), \quad j = 1, ..., k,$$

where $F_k = \langle x_1, ..., x_k \rangle$ is the free group of rank k. The variety $Hom(F_k, G)$ contains a (closed) real semialgebraic subvariety $Hom_e(F_k, G)$ consisting of representations ϕ whose images are *elementary subgroups* of G, i.e., subgroups which either fix a point in the hyperbolic 3-space or on its ideal boundary sphere or preserve a geodesic in the hyperbolic 3-space. The complement

$$Hom_{ne}(F_k, G) = Hom(F_k, G) - Hom_e(F_k, G)$$

is the space of *nonelementary representations*. This space is the main object of our study. We let

$$Hom_d(F_k, G) \subset Hom_{ne}(F_k, G)$$

denote the subset consisting of nonelementary representations with discrete images. Since elementary representations are excluded, the subset $Hom_d(F_k, G)$ is known to be closed (in the classical topology), see the paper of T. Jorgensen and P. Klein [10], as well as [11].

In this paper we prove:

Theorem 1.6. *The subset $Hom_d(F_2, SL(2,\mathbb{C}))$ is not BSS-semicomputable.*

Thus, at least in the BSS-computability model, the discreteness problem for 2-generated subgroups of $SL(2,\mathbb{C})$ is undecidable. Our proof is modeled on the undecidability result for the Mandelbrot set $\mathbb{M}$: The membership problem for $\mathbb{M}$ is BSS-undecidable according to [1, Chapter 2]. The proof of Theorem 1.6 is not difficult, but it relies upon three deep results:

- Description of BSS-computable sets by Blum, Shub and Smale, see [1].
- Minsky's solution of the ending lamination conjecture for punctured tori [15].[a]
- Miyachi's theorem [17], proving non-smoothness (at the "cusps") of the boundary of the Maskit slice in the character variety of the punctured torus.

[a] Minsky's work used here was one of the many papers leading, eventually, to the solution of the full Ending Lamination Conjecture by Minsky, Brock and Canary, [16,3].

The undecidability theorem in this paper should be contrasted with the *semidecidability result* for *convex-cocompact faithful representations* into $PSL(2,\mathbb{C})$, cf. [8]. We note that a more general semidecidability result for *Morse* (*Anosov*) representations of hyperbolic groups into semisimple Lie groups is proven in the work of the author with B. Leeb and J. Porti [12, section 7.7].

2. Proof of Theorem 1.6

Set $G = SL(2,\mathbb{C})$. We will show that the set in Theorem 1.6 is not a countable union of real semialgebraic subsets of $Hom_{ne}(F_2, G)$, where we regard G as a real algebraic group. First of all, instead of working in $Hom(F_2, G)$, it suffices to work with the character variety $X = X(F_2, G) = Hom(F_2, G)//G$. The reason is that there is a polynomial map $\tau : Hom(F_2, G) \to X$ whose fibers are the *extended* G-orbits in $Hom(F_2, G)$, where G acts via composition of representations $F_2 \to G$ with inner automorphisms of G. Discreteness, of course, is invariant under conjugation. We will avoid discussion of the *extended* orbit equivalence and only note that for representations in $Hom_{ne}(F_2, G)$ the extended orbit equivalence is the same as the orbit equivalence. Therefore, it suffices to work with the character variety. Concretely, the map τ is given by

$$\tau(A, B) = (tr(A), tr(B), tr(AB)) \in \mathbb{C}^3.$$

Our next reduction is to the *Maskit slice* X_M in X, i.e., the complex-algebraic subset given by the following trace conditions:

$$tr([A, B]) = -2, \quad tr(A) = 2.$$

Since the Maskit slice is algebraic, the problem now reduces to the one in the Maskit slice. The Maskit slice of X is complex 1-dimensional, it is biregularly isomorphic to the complex line $\mathbb{C}$ via the map

$$(tr(A), tr(B), tr(AB)) \mapsto tr(B) \in \mathbb{C}.$$

We, therefore, identify X_M with $\mathbb{C}$ via this map. Recall that geometrically finite representations are dense among all discrete and faithful representations $\Gamma \to SL(2,\mathbb{C})$ (for any finitely generated group Γ). This was proven first by Y. Minsky [15] for representations of punctured torus groups, and, hence, in the Maskit slice, which suffices for our purposes. The general case is due to the work of many people, most notably, K. Bromberg [4], J. Brock and K. Bromberg [2], H. Namazi and J. Souto [18], and K. Ohshika [19]. We, thus, have:

Proposition 2.1. *The space $\mathcal{D} \subset \mathbb{C}$ of equivalence classes of discrete representations $[\rho] \in X_M = \mathbb{C}$ has the following structure:*

$$\mathcal{D} = \mathcal{DF} \sqcup C,$$

where C is a countable subset of non-faithful geometrically finite representations and $\mathcal{DF}$ is the set of equivalence classes $[\rho] \in X_M$ such that $\rho : F_2 \to G$ is discrete and faithful.

Thus, it suffices to show that $\mathcal{DF}$ is not a countable union of real semialgebraic subsets. Due to the work of Y. Minsky [15], the topological boundary of $\mathcal{DF}$ is a topological arc α properly embedded in $\mathbb{C}$. The complement to $\mathcal{DF}$ in $\mathbb{C}$ is also diffeomorphic to $\mathbb{R}^2$.

Before proving the next lemma, we recall that an *accidental parabolic element* of a representation

$$\rho : F_2 = \pi_1(T^2 - \text{point}) \to SL(2, \mathbb{C})$$

is an element of F_2 represented by a (necessarily simple) nonperipheral loop γ (not representing the conjugacy class of the generator A of F_2) on the punctured torus T^2 − point, such that $\pi_1(\gamma)$ is a parabolic element of $SL(2, \mathbb{C})$. The equivalence class $[\rho] \in \mathcal{DF}$ of a representation ρ is called a *cusp* if ρ has an accidental parabolic element. It again follows from Minsky's work (Theorem B in [15]) that *cusps are dense* in the boundary of $\mathcal{DF}$ (cf. the earlier work of C. McMullen [14]).

Lemma 2.2. *The arc α contains no smooth subarcs (which are not singletons).*

Proof. H. Miyachi proved [17] the arc α is not smooth at each cusp, which are dense in α. Therefore, α does not contain nondegenerate smooth subarcs. □

We now can conclude the proof of Theorem 1.6.

Proof. Suppose that $\mathcal{DF}$ is a countable union

$$\bigcup_{j \in J} E_j$$

of real algebraic subsets E_j of $\mathbb{C}$. Each E_j is either finite or its topological frontier ∂E_j in $\mathbb{C}$ is a finite union of real-algebraic arcs. Since, as noted above, the arc α does not contain real-algebraic subarcs, each E_j intersects α in a nowhere dense (in α) subset. By the Baire Theorem, the union

$$\bigcup_{j \in J} \partial E_j \cap \alpha$$

has empty interior in α. Therefore, the union of subsets E_j cannot be equal to $\mathcal{DF}$. This contradiction concludes the proof of Theorem 1.6. □

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