Introduction to Kleinian Groups

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August 20, 2007

Basics of Hyperbolic Geometry-

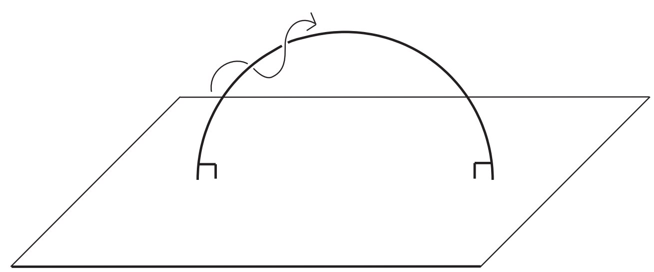
Hyperbolic 3-space, $\mathbb{H}^3$, may be identified with the upper half space $\{(z,t) \mid z \in \mathbb{C},\ t > 0\}$ equipped with the metric

$$\frac{dz^2 + dt^2}{t^2}$$

The isometry group of hyperbolic space $\mathrm{Isom}(\mathbb{H}^3)$ can be identified with the group of Mobius transformations, and the group of orientation preserving isometries $\mathrm{Isom}^+(\mathbb{H}^3)$ can be identified with $\mathrm{PSL}_2(\mathbb{C})$. $\mathrm{PSL}_2(\mathbb{C})$ acts on the boundary of the upper half space by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}$$

and this action can be extended in a natural way to the interior. Non-identity elements of $\mathrm{PSL}_2(\mathbb{C})$ fall into one of the following three categories:
-elliptic isometries (having fixed points in $\mathbb{H}^3$)
-parabolic elements (having one fixed point on $\widehat{\mathbb{C}}$, the sphere at infinity)
-loxodromic elements (having two fixed points at infinity and an axis fixed set-wise):



A Kleinian group Γ is discrete a subgroup of $\mathrm{Isom}(\mathbb{H}^3)$. We will usually assume that it is orientation preserving and without elliptic elements. A discrete group is a group in which the identity element is isolated. Discreteness also implies that the orbit of any point in $\mathbb{H}^3$ is discrete, i.e. for any $x \in \mathbb{H}^3$ there exists a ball B containing x such that $gB \cap B = \varnothing \iff gx = x$. Assuming the group has no elliptic elements, this implies that there exists a B such that $gB \cap B = \varnothing$ for any nonidentity element. In this case $M_\Gamma = \mathbb{H}^3/\Gamma$ is a manifold , and it inherits a complete hyperbolic metric.

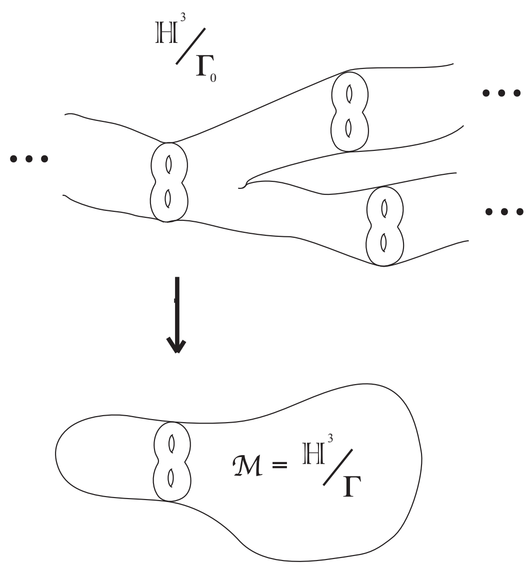
Conversely, given a complete hyperbolic manifold M, there exists an isometry ϕ from the universal cover $\widetilde{M} \to \mathbb{H}^3$ with $\phi^*(\pi_1(M))$ a Kleinian group, so the study of hyperbolic 3-manifolds can be reduced to the study of Kleinian groups.

Surface groups-

A Kleinian group Γ is a called a surface group if $\Gamma \cong \pi_1(S)$ for a closed surface S. In some cases we allow compact surfaces but impose a parabolicity condition on ∂S.

The simplest example of a surface group sitting in $\mathbb{H}^3$ is given by considering the set $\{(z,t) \mid z \in \mathbb{R}\}$, an isometric copy of $\mathbb{H}^2$ sitting in $\mathbb{H}^3$. The subgroup of the isometry group of $\mathbb{H}^3$ perseving this plane is $\mathrm{PSL}_2(\mathbb{R}) \subset \mathrm{PSL}_2(\mathbb{C})$, so we can see hyperbolic three space as a complexification of two dimensional hyperbolic space.

Other examples of surface groups are given by considering the images of topological maps $S \to M$ whose induced map $\pi_1(S) \to \pi_1(M)$ is injective. Then $\Gamma_0 \cong \pi_1(S)$ sits as a subgroup in Γ, and we get a covering map $\mathbb{H}^3/\Gamma_0 \to \mathbb{H}^3/\Gamma \approx M$.



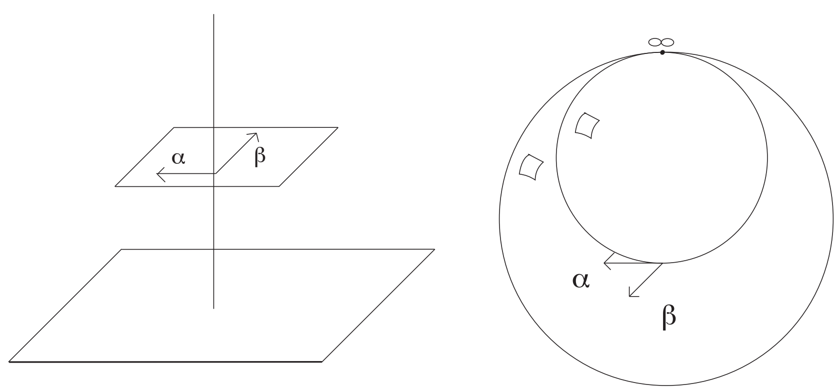
Manifolds foliated by surfaces and fibrations give important examples of such situations.

As another example of how surface groups come up in 3-manifolds, we can consider manifolds M which are homeomorphic to the interior of a compact manifold $\widehat{M}$. In this case surface groups are associated the the ends of the manifold. Considering the ends of a manifold allows us to get a grasp on the deformation theory of hyperbolic structures on manifolds, as we can set up a correspondence between structures on the ends of the manifold and the space of hyperbolic structures on M.

Limit sets-

Let Γ be a general Kleinian group. For any $x \in \mathbb{H}^3$, Γx is a discrete set in $\mathbb{H}^3$, but by compactness it must accumulate on the boundary of $\mathbb{H}^3 \cup \widehat{\mathbb{C}} = \overline{\mathbb{H}^3}$. The compactness of the closure of $\mathbb{H}^3$ is easier to see in the Poincaré ball model of hyperbolic space, which is the unit ball in $\mathbb{R}^3$ with the metric $\frac{4(dx^2+dy^2+dz^2)}{(1-r^2)^2}$. In this model the sphere at infinity $\widehat{\mathbb{C}}$ can be identified with the unit sphere. $\Lambda_\Gamma = \overline{\Gamma x} \cap \widehat{\mathbb{C}}$ is called the limit set of Γ. Notice that the limit set does not depend on the particular orbit we look at. If λ is a limit point of Γx, $\gamma_i x \to \lambda$, and y is any other point in $\mathbb{H}^3$, $\gamma_i x$ and $\gamma_i y$ are the same distance apart for all i as γ_i is a hyperbolic isometry. Given any segment of fixed hyperbolic length in $\mathbb{H}^3$, its Euclidean length goes to zero as it approaches the boundary, so $\gamma_i x$ and $\gamma_i y$ converge as $\gamma_i x$ approaches $\partial\mathbb{H}^3$.

If Γ is a finite group then $\Lambda_\Gamma = \varnothing$. If Γ is infinite cyclic, the limit set consists of two points p_+ and p_- in the case of a group generated by a loxodromic element, or a single point in the case of a group generated by a parabolic element. The limit set can also be a single point if $\Gamma \cong \mathbb{Z} \times \mathbb{Z}$ is generated by two parabolics with a common fixed point.

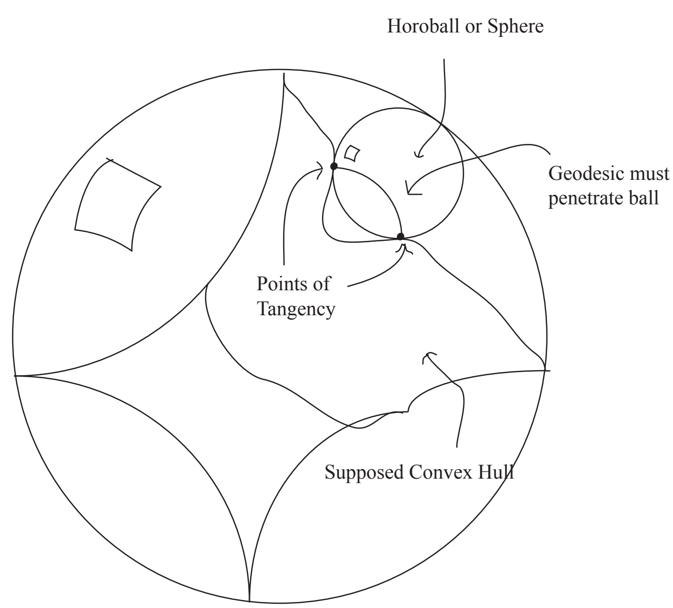


The groups listed above are all called elementary groups. If Γ is not elementary, then its limit set is uncountable. The limit set Λ can be proved to be the smallest nonempty closed Γ-invariant set in $\widehat{\mathbb{C}}$, and is also the closure of the set of fixed points of the parabolic and loxodromic elements of the group.

As Λ is Γ-invariant, $\Omega = \widehat{\mathbb{C}} \backslash \Lambda$ is Γ-invariant. Orbits do not accumulate in Ω, and in fact Γ acts properly discontinuously on Ω, i.e. if $K \subset \Omega$ is compact then only finitely many of the sets γK for $\gamma \in \Gamma$ intersect K. This implies that Ω/Γ is Hausdoff, and it will be a manifold assuming the group does not contain elliptics.

To prove that Γ acts properly discontinuously on Ω, we introduce C_λ, the convex hull of Λ, which is the smallest convex set in $\mathbb{H}^3$ whose closure in the ball contains Λ. Recall that a convex set is a set X such that for $x, y \in X$ the geodesic segment containing x and y also lies in X. Note that the Γ-invariance of the limit set implies the Γ invariance of C_Λ. C_Λ may also be characterized as the intersection of all closed half-spaces whose extension to the sphere at infinity contains λ.

Note that as C_Λ lies in $\mathbb{H}^3$, Γ acts properly discontinuously on C_Λ. We can define a Γ-equivariant retraction $r : \mathbb{H}^3 \cup \partial\mathbb{H}^3 \to C_\Lambda \cup \Lambda_\Gamma$, by sending $r(x)$ to the "closest point" on C_Λ, where closest point is understood literally for points in the interior, and is understood to mean the point of intersection of a horoball through x for a point x on the boundary. To see that this is well defined, note that balls in $\mathbb{H}^3$ are strictly convex, so the following picture can't happen:



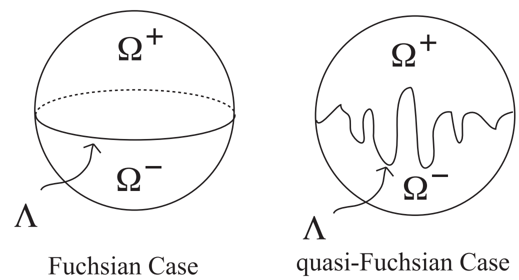
The Γ-equivariance of this retraction is easy to see, as hitting the whole picture by an element of Γ doesn't change the convex hull of the limit set, and if y in the closest point in C_Λ to x then γy is the closest point in C_Λ to γx. If $K \subset \Omega$ is compact, then $r(K)$ is compact, and lies in the interior of $\mathbb{H}^3$. If $\gamma K \cap K \neq \varnothing$ for infinitely many γ's, then $\gamma(r(K)) \cap r(K) = r(\gamma(K)) \cap r(K) \neq \varnothing$ for infinitely many γ, which is a contradiction.

We now have that Γ acts properly discontinuously on $\mathbb{H}^3 \cup \Omega$, and $\mathbb{H}^3 \cup \Omega/\Gamma = \overline{M}$, a manifold with boundary. $\Omega/\Gamma = \partial\overline{M}$ is called the conformal boundary at infinity: as $\mathrm{PSL}_2(\mathbb{C})$ acts conformally on the boundary sphere $\widehat{\mathbb{C}}$, Ω/Γ comes with the structure of a Riemann surface. Note that $\overline{M}$ is not necessarily compact.

Limit sets of Surface Groups-

Let Γ be a surface group. If $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$ preserves a copy of $\mathbb{H}^2$ inside $\mathbb{H}^3$, then in non-elementary cases the limit set is a circle, and the group is a Fuchsian group. The domain of discontinuity is a disjoint union of two disks, Ω_+ and Ω_-, and $\mathbb{H}^2/\Gamma$ is conformally equivalent to both Ω_+/Γ and Ω_-/Γ (the retraction r is in fact conformal in the case when C_Λ is a totally geodesic disc).

As a more interesting case, we can consider the case when the limit set is a general Jordan curve. This is the called the quasi-Fuchsian case. A Jordan curve still separates the sphere into two discs, and in fact Ω_+/Γ and Ω_-/Γ will still be homeomorphic to a single surface S, and $\overline{M} \simeq S \times [0,1] \simeq C_\Lambda/\Gamma$.



One can show that a group that is bi-Lipschitz equivalent to a Fuchsian group is quasi-Fuchsian. Furthermore, Ber's simultaneous uniformization theorem gives that there is a one-to-one correspondence between quasi-Fuchsian groups considered up to an appropriate equivalence and pairs of points (X, Y) in Teichmüller space, the space of hyperbolic structures on X and Y. A central question that will be addressed in these lectures is how information about the pair (X, Y) gives information about the quasi-Fuchsian groups and its quotient manifold.