

# Some Geometric Aspects of the Work of Lars Ahlfors

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Let me begin with a personal reminiscence of Lars Ahlfors. My first contact with him came when he was my undergraduate advisor at Harvard. It was the advisor's job to review and approve the courses that the student had selected. When I entered his office and handed him my study card, he looked it over quickly, said "You seem to know what you are doing," and signed the card. This was the only advice I ever got from him. Knowing myself at the time, this was probably the only kind of advice I was likely to accept, and perhaps Ahlfors sensed that. At any rate, it struck me as a good example of how Ahlfors combined a great deal of insight with a minimum of actual work.

Years later, at the conference in honor of his 75th birthday, I reminded him of this story. He then asked me, "What was your name again?" When I told him, he responded, "Well, I guess I was right."

Ahlfors' attitude towards Riemannian geometry *per se* seems to have been somewhat mixed. In lecture notes he wrote on quasiconformal mappings in

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several variables [Ah4], he refers to Riemannian geometry as “simply a set of rules for changing coordinates.” Nonetheless, a strong geometric feeling seems to have permeated Ahlfors’ work, in a way that may seem hard to quantify.

Perhaps his greatest contribution to geometry was his book *Complex Analysis* [Ah2]. I have yet to meet a geometer who prefers any other introductory book in complex analysis to this masterpiece. When read with sufficient tenacity, it makes the subject of complex analysis not only accessible but actually appealing to the geometer.

Some kind of explicit geometric feeling appears to have been in Ahlfors’ conscious thought from the very beginning. For instance, it appears that a primary motivation for his work in Nevanlinna Theory was a certain dissatisfaction with the state of the art as he learned it from Nevanlinna. The subject was not really properly understood, according to his view, unless it was understood geometrically. This pedagogical insistence on geometry appears to have been a primary motivation in his early work, as one sees from the commentary on his papers that he wrote for his collected works [Ah3].

If I were to try to quantify Ahlfors’ attitude towards geometry, I would put it this way: the proper way to do a calculation in complex analysis is to be guided primarily by geometric intuition. Indeed, a calculation is not properly understood, and probably not carried out properly, if it is not primarily guided by geometric intuition.

Anyone who has spent time carefully going through the exercises in Ahlfors’ book will certainly be convinced that Ahlfors believed this. Indeed, the book itself is a compelling argument for the truth of the proposition.

In the remainder of this paper, I would like to focus on the paper, “An Extension of Schwarz’s Lemma,” Trans. AMS 43 (1938), pp. 359-364 [Ah1], where Ahlfors proves the now-famous Ahlfors-Schwarz Lemma. In addition to being a minor masterpiece on its own, this paper provides a great deal of insight into how Ahlfors thought of and used geometry as a tool in complex analysis.

In his commentary on his collected works [Ah3], Ahlfors wrote of this paper that it “had more substance than I was aware of.” He also writes that “Without applications, my lemma would have been too lightweight for publication.” It is my belief that the Ahlfors-Schwarz Lemma continues to

have more substance than people are aware of, and that there remain many opportunities for it to be put to good use. I would therefore like to put this paper in both a historical and a contemporary setting.

I should add that most of what I learned about how to use the Ahlfors-Schwarz Lemma comes from Scott Wolpert, who was using it to great effect at the time we first met.

To recall the statement of the lemma, recall the classical Schwarz Lemma:

**Lemma 1** *Let  $D$  be the unit disk*

$$D = \{z : |z| < 1\}.$$

*If  $f : D \rightarrow D$  is any holomorphic map with  $f(0) = 0$ , then*

$$|f(z)| \leq |z| \quad \text{and} \quad |f'(0)| \leq 1,$$

*with equality if and only if  $f$  is a rotation.*

Ahlfors observes in his book [Ah2] that one should regard the assumptions of the lemma as normalizations. A more geometric version of the lemma was given by Pick as:

**Lemma 2** *Any holomorphic map  $f : D \rightarrow D$  is distance decreasing in the hyperbolic metric, and strictly decreasing unless  $f$  is an isometry.*

It was Ahlfors' insight to see that the essential point here is that the curvature of the target metric be  $\leq -1$ , rather than just constant curvature:

**Theorem 1 (Ahlfors-Schwarz [Ah1])** *Let  $W$  be a Riemann surface carrying a conformal metric of curvature  $\leq -1$ , and*

$$f : D \rightarrow W$$

*a holomorphic map. If we denote by  $d\sigma$  the hyperbolic metric on  $D$  and  $ds$  the “pulled-back metric” from  $W$  to  $D$ , then*

$$ds \leq d\sigma.$$

**Proof:** We first observe that the “pulled-back metric” needn’t be a metric at all, because at the branch points of the map it is 0. But at these points,  $0 = ds < d\sigma$ , so we may restrict attention to the points at which  $f$  is regular.

We now choose local holomorphic coordinates at a regular point, where we can write any conformal metric as

$$\lambda|dz|$$

for some positive function  $\lambda$ . Our first observation is that the Gaussian curvature is given by

$$\kappa = -\frac{\Delta(\log(\lambda))}{\lambda^2},$$

where

$$\Delta(f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

is the “analyst’s Laplacian”—that is to say, the Laplacian in the flat metric in these coordinates, and not the geometric Laplacian.

It follows that the right-hand side is independent of the choice of local holomorphic coordinates, even though the top and bottom terms are highly dependent. Also, given two metrics, it is evident that the ratio  $\frac{\lambda_1}{\lambda_2}$  is also independent of the local coordinates.

We now pick a local coordinate  $z$ , and write

$$d\sigma = \lambda_1|dz| \quad ds = \lambda_2|dz|.$$

Setting  $u_i = \log(\lambda_i)$ , we then have that  $u_1 - u_2$  is independent of the local coordinate, and

$$\Delta(u_1 - u_2) = -\kappa_1 e^{2u_1} + \kappa_2 e^{2u_2}.$$

Hence, at a minimum, where  $\Delta(u_1 - u_2) \geq 0$ , we have

$$-\kappa_1 e^{2u_1} \geq -\kappa_2 e^{2u_2}.$$

Assuming that  $\kappa_2$  is negative, this gives us that

$$\frac{\kappa_1}{\kappa_2} \geq e^{2u_2 - 2u_1},$$

so that if  $\kappa_2 \leq \kappa_1$ , and hence  $\frac{\kappa_1}{\kappa_2} \leq 1$ , we must have that  $u_1 \geq u_2$ . Thus, at an interior minimum of  $u_1 - u_2$ , we must have  $u_1 \geq u_2$ .

We now must worry about boundary points. To that end, we will apply the above argument to  $d\sigma = ds_1$  equal to the hyperbolic metric on the disk of radius  $R$ , for  $R < 1$ , and  $ds_2 = ds$ . Here there is no worry about boundary points, since  $ds_1$  is infinite on  $|z| = R$ , while  $ds$  is of course finite. Hence an interior minimum must exist, and we conclude  $|ds_2| \leq |ds_1|$  everywhere. Letting  $R$  tend to 1 gives us the theorem.

A particularly useful version of this was given by Scott Wolpert, as follows:

**Theorem 2 ([W])** *Let  $d\sigma_1$  and  $d\sigma_2$  be conformally equivalent metrics on a compact Riemann surface  $S$ , with negative curvature  $\kappa_1$  and  $\kappa_2$  respectively. If  $\kappa_1 \leq \kappa_2$ , then  $d\sigma_1 \leq d\sigma_2$ .*

The proof is identical with the above, except that the compactness is used to exclude worrying about the behavior at infinity.

We remark that Ahlfors' original argument allows us to drop the compactness assumption in Wolpert's Theorem, at the expense of two additional assumptions: first, that the metric  $d\sigma_2$  be complete (which is automatic in the compactness assertion), and secondly that the pointwise curvature estimate be strengthened to a sup-norm estimate.

Here is one possible formulation, which would probably satisfy most geometers:

**Theorem 3** *Let  $ds_1$  and  $ds_2$  be conformal metrics on the disk, with  $ds_2$  complete.*

*If  $\sup(\kappa_1) \leq \inf(\kappa_2) < 0$ , then  $ds_1 \leq ds_2$ .*

As a corollary, we have:

**Corollary 1** *Suppose that  $ds_1$  and  $ds_2$  are conformal metrics on the disk which are complete, and suppose there is a constant  $C > 1$  such that*

$$C \sup(\kappa_1) \leq \kappa_2 \leq (1/C) \inf(\kappa_1) < 0.$$

*Then*

$$(1/C)ds_1^2 \leq ds_2^2 \leq Cds_1^2.$$

The idea of the proof of the theorem is to pick  $r \rightarrow 1$ , and let  $ds_{r,2}$  be the pullback to the disk of radius  $r$  under the map  $z \rightarrow (1/r)z$ . Completeness then guarantees that this metric blows up on the circle of radius  $r$ , so that an interior minimum must exist. We messed up the pointwise bounds in the course of this argument, but the rest of the argument is intact.

To prove the corollary from the theorem, we merely multiply the metrics by scalars until they are in the form of the theorem, and then switch the roles of  $ds_1$  and  $ds_2$ .

Wolpert's result was employed in [PS] to deal with the following question: suppose we have a Riemann surface  $S$  with a finite number of punctures  $p_1, \dots, p_k$ . Then we may compactify  $S$  in a canonical way by filling in the punctures to get a surface  $\bar{S}$ .

While this compactification process is easily described conformally, it is hard in general to describe this process geometrically. Indeed, any Riemann surface of negative Euler characteristic carries a hyperbolic metric in its conformal class, that is to say a complete metric of constant curvature  $-1$ . So we can ask the following question: what is the relationship between the hyperbolic metric on  $S$  and the geometry of  $\bar{S}$ ?

A moment's reflection will convince one that the relationship need not be close. For instance,  $\bar{S}$  need not carry a hyperbolic metric at all, since filling in points raises the Euler characteristic. So one would expect that in general the process of conformal compactification does great damage to the hyperbolic metric.

A key observation of [PS] is that this need not be the case when the cusps of  $S$  are large, in a sense which we will now define.

If  $p_i$  is a cusp, then a neighborhood of  $p_i$  is isometric to

$$\{z : 0 \leq \Re z \leq 1, \Im(z) \geq y_0\} / (iy \sim 1 + iy)$$

in the upper-half plane metric

$$ds^2 = \frac{1}{y^2}(dx^2 + dy^2),$$

for some  $y_0$ . The line

$$x + iy_0 : 0 \leq x \leq 1$$

is then a closed horocycle of length  $1/y_0$ . We will say that the cusp neighborhood is large if we may choose  $y_0$  small, so that the length of the closed

horocycle is large. We will further say that the cusp neighborhoods of  $S$  are large if there are cusp neighborhoods of each of the  $p_i$  such that these neighborhoods are large and mutually disjoint.

Wolpert's version of the Ahlfors-Schwarz Lemma then gives us:

**Theorem 4 ([PS])** *Suppose that the cusp neighborhoods of  $S$  are sufficiently large. Then for each  $i$  there is a small neighborhood  $U_i$  of  $p_i$  such that outside these neighborhoods, the hyperbolic metrics on  $S$  and  $\bar{S}$  are uniformly quasi-isometric, with the measure of quasi-isometry depending only on the size of the cusp neighborhood and the size of  $U_i$ .*

*Furthermore, the constant of quasi-isometry tends to 1 as the size of the cusp neighborhood tends to infinity.*

We are being a bit vague here, but refer the reader to [PS] for details.

The theorem has as a consequence that, for instance, if the cusp neighborhoods are sufficiently large, then the Cheeger constants and hence the first eigenvalue of the Laplacian of  $S$  and  $\bar{S}$  are bounded uniformly in terms of one another.

The idea of the proof is to use the largeness of the cusp neighborhood to find a metric on  $\bar{S}$  which has negative curvature, and arbitrarily close to -1 as the size of the cusp increases, but which agrees with the hyperbolic metric on  $S$  outside the cusp neighborhood. To see this, we may consider two metrics  $ds_D^2$  and  $ds_C^2$  on the punctured unit disk  $D - \{0\}$ : the metric  $ds_D^2$  is the usual hyperbolic metric on the disk, and the metric  $ds_C^2$  is the hyperbolic metric on the punctured disk. We may write

$$ds_C^2 = f^2(r)ds_D^2,$$

where  $f(r)$  is a function of the distance to 0.

The usual Schwarz Lemma tells us that  $f > 1$ , but it is not difficult to see from direct computation that  $f \rightarrow 1$  and its derivatives  $\rightarrow 0$  as  $r \rightarrow 1$ . Using this, one can show that, given  $\varepsilon$ , it is possible to find numbers  $r_0, r_1$  and a function  $g$  such that:

$$(i) \ g \equiv 1 \text{ for } r < r_0.$$

$$(ii) \ g \equiv f \text{ for } r > r_1.$$

- (iii) The metric  $g^2 ds_D^2$  has curvature between  $-(1 - \varepsilon)$  and  $-(1 + \varepsilon)$  everywhere.

If the cusps are sufficiently large, we may paste this metric in to each of the cusp neighborhoods to get two metrics on  $\overline{S}$ , one the usual hyperbolic metric on  $\overline{S}$ , and the other a metric which has curvature everywhere between  $-(1 + \varepsilon)$  and  $-(1 - \varepsilon)$ , which furthermore agrees with the hyperbolic metric on  $S$  outside the cusp neighborhoods.

Wolpert's result then tells us that if the curvatures are quasi-equivalent, then the metrics themselves are quasi-isometric, qed.

If we replace Theorem refscott by Theorem 3, then we may reverse the roles of  $S$  and  $\overline{S}$  in the above argument. To be precise, let  $S$  be a closed Riemann surface, and let  $\{p_1, \dots, p_k\}$  be  $k$  points on  $S$ . Set  $S^O = S - \{p_1, \dots, p_k\}$ . We then have:

**Theorem 5** *Suppose that for some  $r_0$  the injectivity radius at  $p_i$  is greater than  $r_0$  for all  $i$ , and the balls of radius  $r_0$  about the  $p_i$  are disjoint.*

*If  $r_0$  is sufficiently large, then outside the balls of radius  $r_0$ , the hyperbolic metrics on  $S$  and  $S^O$  are uniformly quasi-isometric. Furthermore, the measure of quasi-isometry  $\rightarrow 1$  as  $r_0 \rightarrow \infty$ .*

*In particular, the cusps of  $S^O$  are large.*

The idea is identical to the idea in Theorem 4, except that one constructs two metrics on  $S^O$  rather than on  $\overline{S}$ . One must therefore use Theorem 3 rather than Theorem 2.

In this manner, one may introduce cusps into a Riemann surface with little alteration of geometric properties such as the Cheeger constant and the first eigenvalue.

I regard the argument used in Theorems 4 and 5 as something of a prototype, of which there should be many possible useful variations.

Another prototype of applications of the Ahlfors-Schwarz Lemma is given by the work of Phil Griffiths and his school on Nevanlinna Theory in several variables. See Griffiths' Weyl lectures [Gr] for an excellent account of this. Griffiths characterizes the Ahlfors-Schwarz Lemma as "ubiquitous" (that is, present everywhere) in the subject. Basically, its unique role in the subject is



given by the fact that it generalizes easily to higher dimensions. The complex structure allows one to regard the Ricci tensor as a 2-form  $\text{Ricc}$ . If one is in complex dimension  $n$ , one may then compare  $\text{Ricc}^n$  with the volume form. One may also put the hyperbolic metric on the polydisk. Once these two moves are done, Ahlfors' original arguments may be translated directly, with very little change. In this way, one obtains estimates of key expressions which one might think were only available by normal families arguments.

The prototype that Ahlfors had in mind was quite different. The application he gave computing an explicit lower bound for the Bloch constant is a very illustrative example:

**Theorem 6 ([Ahl])** *Let  $f$  be a holomorphic map from  $D$  to  $\mathbb{C}$  satisfying  $|f'(0)| = 1$ . Then there is a ball of radius  $B$  about some point  $z$  in the image of  $f$  on which  $f$  is schlicht, i.e. there is a component of  $f^{-1}(B_B(z))$  on which  $f$  is 1-to-1, with*

$$B = \frac{\sqrt{3}}{4} \sim .433.$$

The idea of the proof is to consider the function  $\rho(w)$  = the radius of the largest ball about  $w$  on which  $f$  is schlicht. Now  $\rho$  is easily seen to be continuous, but it is not smooth. Indeed,  $\rho$  is the Euclidean distance from  $w$  to either a branch point or a boundary point of  $f$ . If  $w$  lies equidistant to two such points, then  $\rho$  will not be smooth at  $w$ .

Let us set  $B_0 = \sup(\rho)$ . The content of the theorem is then to say that  $B_0 \geq B$ .

We now consider the metric given by  $g|dw|$ , with

$$g = \frac{A}{\rho^{1/2}(A^2 - \rho)},$$

where  $A$  will be chosen later. In order for this to be a metric, we must have  $A > \sqrt{B_0}$ .

This metric may be described geometrically as the orbifold hyperbolic metric on the disk of radius  $A^2$  with an orbifold point of order 2 at 0, as one can see by comparing with the usual hyperbolic metric on the disk of radius  $A$  given by

$$|ds| = \left(\frac{2A}{A^2 - r^2}\right)|dz|.$$

From this it is clear that the metric has constant curvature  $-1$ , at least at those points at which it is smooth. In order for this to be a metric, we must have  $A > \sqrt{B_0}$ .

We now consider the pullback of this metric via  $f$  on the disk  $D$ , with the aim of comparing it to the hyperbolic metric on  $D$ . If  $w$  is a point that is closest to a unique branch point or boundary point, then it is easily seen that the pullback of this metric has curvature  $-1$ , and that the pullback is finite everywhere, and vanishes precisely at the branch points of order 3 or more. Furthermore, if we choose  $A > \sqrt{3B_0}$ , then the denominator is an increasing function of  $\rho$  in the range between 0 and  $B_0$ .

We would like to apply the lemma at  $z = 0$ . If it applied, then, using that  $|f'(0)| = 1$ , we would get that

$$\frac{A}{B_0^{1/2}(A^2 - B_0)} \leq \frac{A}{(\rho(f(0)))^{1/2}(A^2 - \rho(f(0)))} \leq 2,$$

where we have used the fact that the denominator is increasing to replace  $\rho(f(0))$  by  $B_0$ .

We may then rewrite this as

$$A \leq 2B_0^{1/2}(A^2 - B_0),$$

and letting  $A$  tend to  $\sqrt{3B_0}$  would then yield  $B_0 \geq \frac{\sqrt{3}}{4}$ , as desired.

The only problem is how to deal with the non-smoothness of  $\rho$ . So say that  $w$  is equidistant to two (or more) branch points. We may consider the corresponding functions  $\rho_1$  and  $\rho_2$ , which are the distance functions to these points. Replacing  $\rho$  with either of these functions now gives a smooth function, such that the metric computed with respect to this function is  $\leq$  to the original metric everywhere, but agrees with the original metric at  $w$ . If  $w$  is a local minimum for the expression occurring in the proof of the Ahlfors-Schwarz Lemma, then it remains so for the new metric, which is now smooth, and hence the Laplacian argument still applies.

In this way, the analytic difficulties caused by the non-smoothness of  $\rho$  disappear, almost without effort.

The main point here is that the metric Ahlfors considers is not really a metric of variable negative curvature. Rather, it is a metric of constant curvature, which is however not smooth. The non-smoothness of the metric indeed captures the geometric difficulties of the problem. The resolution of the difficulty is then that the same minimum argument which was available in the smooth case can be made to work in the non-smooth case as well. Thus, for practical purposes, these non-smooth constant curvature metrics may be thought of in the same way as metrics of variable curvature.

All of the applications Ahlfors gives of his lemma in [Ah1] are of this type. Rather than dealing with metrics of *variable* curvature, they deal with metrics of constant curvature  $-1$ , which however are not smooth. The heart of the argument is then to say that the non-smoothness does not really affect the conclusion.

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