Statement: Some of my research since July 2003

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My research area is geometric group theory and geometric structures on manifolds, with the main emphasis on the interaction between algebraic, topological and geometric properties of groups and applying geometric techniques in order to solve problems in the algebraic group theory and algebraic geometry.

1 Interaction of metric geometry and the theory of algebraic groups

In the series of papers [13], [39], [54], [43], [44], [45], [14], I and my collaborators (Tom Haines, Shrawan Kumar, Bernhard Leeb and John Millson) applied ideas and methods of Riemannian geometry to tackle some fundamental problems of Lie theory and representation theory. Surveys of this work are published in [18] and [30]. Some of these problems again originate in the 19-th century linear algebra problems, e.g.

"Given two symmetric matrices with given sets of eigenvalues, what can be said about the eigenvalues of their sum?"

Another basic problem we are addressing is how to decompose the tensor product of two irreducible representations of a Lie group into irreducible summands. The most significant result we obtained here is a generalization of the celebrated *Saturation Theorem* in representation theory (due to Knutson and Tao), to groups other than $GL(n)$, see [54].

Theorem 1 *Given any root system R and positive Weyl chamber Δ, one can define a* saturation factor $k = k_R$ *(currently, ranging between 1 and 60), and a convex cone $C \subset \Delta^3$ (given by the vector-valued triangle inequalities) so that the following holds. Let G be a complex semisimple group with the root system R. Then for every triple (λ, μ, ν) of dominant weights of G, whose sum belongs to the root lattice of G, we have:*

1. If

$$(V_\lambda \otimes V_\mu \otimes V_\nu)^G \neq \{0\}$$

then $(\lambda, \mu, \nu) \in C$.

2. If $(\lambda, \mu, \nu) \in C$ then

$$(V_{k^2\lambda} \otimes V_{k^2\mu} \otimes V_{k^2\nu})^G \neq \{0\}.$$

This theorem contains the saturation theorem of Knutson and Tao [58] as a special case, since for the type A root systems, we have $k = 1$.

A spin-off of this work is my paper with Arkady Berenstein [2]. There we construct thick affine buildings associated with dihedral Coxeter groups $I_2(m)$ of order $2m$, $m \neq 1, 2, 3, 6$. These groups do not correspond to any root systems and any Lie groups. While spherical buildings were constructed by J.Tits in 1977, the existence of Euclidean buildings was unknown prior to our work, except for a construction of buildings for the dihedral group $I_2(8)$ (due to Kramer, Hitzelberger and Weiss [15]: Unlike [2], their proof was based on existence of certain finite groups, called Suzuki groups). The paper [2] completes the classification (initiated by J. Tits) of finite Coxeter groups for which there exist thick affine building.

2 Geometric structures on manifolds

I proved some existence theorems for conformally-flat Riemannian metrics and convex real-projective structures:

[17]. The main result of this paper is:

Theorem 2 *Let M be a smooth closed 4-dimensional spin-manifold. Then there exists a closed 4-dimensional smooth manifold N such that the connected sum $M\#N$ admits a conformally-flat Riemannian metric. The point is that the manifold M need not admit a conformally flat metric, for instance, because it has nonzero signature or has simple infinite fundamental group.*

[20]. In this paper I construct strictly convex real-projective structures on Gromov-Thurston manifolds M^n of dimension $n \geq 4$. These are manifolds which admit negatively curved metrics with small pinching, while admitting no locally symmetric metrics of negative curvature. As the result, I obtain strictly convex domains $\Omega \subset \mathbb{R}P^n$ (for each $n \geq 4$) and properly discontinuous cocompact groups of projective transformations $\Gamma \curvearrowright \Omega$, so that Γ is Gromov-hyperbolic but not isometric to the fundamental group of a closed hyperbolic n-manifold. Prior to this work, such examples existed only for $n = 4$, they were constructed by Y. Benoist in [1] using reflection groups; in the same paper he posed the problem of existence of higher-dimensional examples.

[33]. In this paper I answered a question asked by Gregory Margulis in his ICM-1974 paper: I constructed a finitely generated discrete subgroup Γ of isometries of a 4-dimensional Hadamard manifold of pinched negative curvature, such that the orders of torsion elements in Γ are unbounded.

3 Kleinian groups and discrete subgroups in semisimple Lie groups

[21]. This is an extensive survey (which I was writing on and off since 1992) on topological and geometric aspects of Kleinian groups acting on higher-dimensional

hyperbolic spaces. The main challenge was to select and organize the diverse material so that it takes a coherent shape and connects to various developments in other areas of mathematics. The paper contains some new results as well. For instance, it is a standard fact of the theory of Kleinian groups acting on the hyperbolic 3-space $\mathbf{H}^3$ that if two such groups are quasiconformally conjugate, then one can continuously deform one group to the other. It is shown in [21] that the corresponding assertion is false for torsion-free Kleinian groups acting on $\mathbf{H}^n, n \geq 5$.

[22]. This paper deals with the interplay between homological dimension of Kleinian groups and their critical exponents. The main result is a linkage between two basis invariants of Kleinian groups: An algebraic invariant (the homological dimension) and a geometric invariant (the critical exponent).

Theorem 3 *Let Γ be a discrete subgroup of the group of isometries of hyperbolic n-space, $Isom(\mathbf{H}^n)$. Let $\delta(\Gamma)$ denote the critical exponent of Γ and $d(\Gamma)$ denote the rational homological dimension of Γ relative to its rank ≥ 2 parabolic subgroups. Then*

$$d(\Gamma) \leq \delta(\Gamma) + 1.$$

Corollary 4 *Let Γ be a nonelementary geometrically finite subgroup of $Isom(\mathbf{H}^n)$ so that the topological dimension of its limit set equals the Hausdorff dimension of the limit set. Then the limit set is a round sphere.*

Results of this kind have a long history, going back to H. Poincaré who (about 100 years ago) had observed that the limit set of a quasifuchsian group is either a round circle or is "fractal" (nowhere differentiable curve).

[24]. The main goal of this paper is to prove two results on sequences of discrete representations of a finitely generated group Γ to a rank 1 Lie group G (such theorems are, by now, standard, in the case of discrete and faithful representations). The work was motivated by questions asked (nearly simultaneously) by several people (in particular, Michel Boileau and Fritz Grunewald) about sequences of discrete but non-faithful representations. The main result is

Theorem 5 *1. Let $\rho_i : \Gamma \to G$ be a sequence of Zariski dense representations with discrete images converging to a representation $\rho : \Gamma \to G$. Then $\rho(\Gamma)$ is again discrete.*

2. Suppose above that Γ is finitely-presented and the sequence of representations ρ_i diverges to an action $\Gamma \curvearrowright T$ of Γ on a tree T. Then, as in the Rips theory, Γ admits a nontrivial splitting over a virtually nilpotent subgroup.

The novelty of Part 2 is that the action $\Gamma \curvearrowright T$ need not be *stable* in the sense of Rips. Nevertheless, one can generalize Rips notion of stability, so that the Rips Machine still works.

[25]. In this paper I consider self-similar groups in the sense Nekrashevych. Self-similar groups appear in various constructions of groups of intermediate growth and most of them are rather exotic. The main result of this paper is that such groups appear quite naturally in the context of discrete subgroups of Lie groups:

Theorem 6 *Let Γ be an irreducible lattice in a semisimple algebraic group over the real numbers which is not locally isomorphic to $SL(2, \mathbb{R})$. Then Γ is arithmetic if and only if it admits a self-similar action on a rooted simplicial tree (of finite valence). Moreover, all arithmetic groups (including the surface groups) admit self-similar actions on rooted simplicial trees.*

[27]. A group Γ is called *coherent* if every finitely-generated subgroup of Γ is also finitely-presented. This paper is centered around the following conjecture of mine (for $G = SO(n, 1)$ this conjecture is due to Dani Wise):

Conjecture 7 *Let G be a semisimple algebraic group without compact factors, $\Gamma < G$ be a lattice. Then Γ is* coherent *if and only if G is locally isomorphic to $SL_2(F)$, $F = \mathbb{R}$ or $F = \mathbb{C}$.*

Note that one direction of this conjecture is well-known: Every discrete subgroup of $PSL_2(\mathbb{C})$ is isomorphic to the fundamental group of a 3-dimensional hyperbolic orbifold and, hence, is coherent by the theorem of P. Scott. In the case of lattices in higher rank (i.e., rank ≥ 2) Lie groups this conjecture is currently out of reach, it is not even known if $SL(3, \mathbb{Z})$ is coherent, which is a problem posed by J.-P. Serre in 1979. In [27], building upon my earlier work with Potyagailo and Vinberg [55, 56, 57], I established the following results supporting my conjecture by proving it for "most" arithmetic lattices in rank 1 Lie groups:

- Conjecture 7 holds for all arithmetic subgroups in $SO(n, 1)$ which are associated with hermitian forms for division rings over totally–real number fields. This leaves (among arithmetic subgroups of groups $SO(n, 1)$) the conjecture open only the exceptional class of arithmetic lattices in $SO(7, 1)$.

- Suppose that Γ is a cocompact lattice (arithmetic or not) in $PU(2, 1)$ with positive 1st Betti number. Then Conjecture 7 holds for Γ. In particular, Conjecture 7 holds for all arithmetic lattices in $PU(n, 1)$ of type 1 (i.e., the ones defined by hermitian quadratic forms over number fields).

- Conjecture 7 holds for all lattices acting on quaternionic hyperbolic spaces $\mathbf{HH}^n$ ($n \geq 2$) and the octonionic hyperbolic plane $\mathbf{OH}^2$.

Proofs of these results are based on a combination of ideas of algebraic, geometric, topological and analytical nature. One of the key ingredients was Agol's solution of the virtual fibration conjecture for 3-dimensional hyperbolic manifolds. (When I first wrote [27], I was working modulo the conjecture; luckily for me, Agol proved the conjecture while the paper was being refereed.)

[31]. **Undecidability of the discreteness problem for subgroups of $SL(2, \mathbb{C})$.** This paper is motivated by the following basic question:

Question 8 *Let G be a connected Lie group and let $\mathcal{A} = (A_1, \ldots, A_k)$ be a finite ordered subset of G. Is the discreteness problem for the subgroup $\Gamma_{\mathcal{A}} := \langle A_1, \ldots, A_k \rangle < G$ decidable?*

This question, in the case of $G = PSL(2, \mathbb{C})$, was raised, most recently, in the paper [9] by J. Gilman and L. Keen, who noted that:

"*it is a challenging problem that has been investigated for more than a century and is still open.*"

The decidability problem was solved in the case $G = PSL(2, \mathbb{R})$ by R. Riley [60] and, more efficiently, in the case of 2-generated subgroups, by J. Gilman and B. Maskit [10] and J. Gilman [7].

In [31] I prove that the problem of discreteness for 2-generated nonelementary subgroups of $PSL(2, \mathbb{C})$ is undecidable in the Blum-Schub-Smale computability model (for computations over the real numbers).

Geometric finiteness in Lie groups of higher rank. I wrote a number of papers in collaboration with Bernhard Leeb and Joan Porti and my current student, Subhadip Dey, developing an analogue of the theory of geometrically finite Kleinian groups in the setting of Lie groups of higher rank, like $SL(n, \mathbb{R})$, see [46], [47], [48], [40], [41], [50], [3], [4]. Some of this work is inspired by Mumford's Geometric Invariant Theory (GIT); in some sense we develop GIT for discrete subgroups of Lie groups, defining stable and unstable points, etc. One of the novelties of our approach is that our analogue of the Hilbert-Mumford numerical stability function takes values not in real numbers but in a poset, specifically a finite Weyl group equipped with the strong Bruhat order. These papers by now amount to about 700 pages. The paper [50] is a survey of our work and [41] is an introduction to the subject. Spin-offs of this work are my papers [51, 52] and [34] written with my former students Beibei Liu and Jaejeong Lee, as well as Sungwoon Kim.

4 Metric geometry and Geometric Group Theory.

[19]. In this paper I solved a problem raised by Gromov in [12, page 18]:

Suppose that X is a geodesic metric space which is not quasi-isometric to a point, $\mathbb{R}_+$ and $\mathbb{R}$. Then the necessary and sufficient conditions for existence of a triangle in X with the given side-lengths a, b, c are the usual triangle inequalities on a, b, c.

The same is shown to be true for path-metric spaces, provided that the triples (a, b, c) are "nondegenerate". On the other hand, I constructed examples of complete non-geodesic path metric spaces X, for which no "degenerate" triples $a = b + c, b \neq 0, c \neq 0$, correspond to triangles in X.

[35]. In this joint paper with Bruce Kleiner we develop a homological machinery for geometric group theory, primarily, for studying *duality groups*. An example of application of this machinery is a *coarse analogue* of the Scott compact core theorem for groups of arbitrarily large dimension (the Scott compact core theorem is a fundamental result in 3-dimensional topology).

[37]. In this appendix to the paper by Olshansky, Osin and Sapir "Lacunary hyperbolic groups," I and Bruce Kleiner prove

Theorem 9 *Let* Γ *be a finitely-presented group. Then* Γ *is Gromov-hyperbolic if and only if one of its asymptotic cones is a tree.*

This theorem is well-known to be false for groups which are merely finitely–generated. The main subject of the paper by Olshansky, Osin and Sapir "Lacunary hyperbolic groups" is to develop a structural theory of groups admitting one asymptotic cone which is a tree.

[36]. This joint paper with Bruce Kleiner is motivated by the 3-dimensional *Wall conjecture*, due to C.T.C.Wall, namely that 3-dimensional Poincare duality groups are fundamental groups of 3-dimensional manifolds. One approach to this conjecture is along the lines of the *weak hyperbolization conjecture*, which is the coarse analogue of Thurston's hyperbolization conjecture (now Perelman's theorem) for 3-dimensional manifolds:

Conjecture 10 *If Γ is a 3-dimensional Poincare duality group then either Γ contains $\mathbb{Z}^2$ or Γ is Gromov-hyperbolic.*

In [36] we prove this conjecture assuming, in addition, that Γ is a CAT(0) group.

[29]. In his essay [11], Gromov gave a brief sketch of the proof of Stallings' theorem on ends of groups via harmonic functions. In [29], I gave a complete proof for Gromov's argument; the key technical result is a new compactness for harmonic functions on Riemannian manifolds:

Theorem 11 *Let M be a Riemannian manifold with infinitely many ends which admits cocompact isometric group action $G \curvearrowright M$. Consider the space H of nonconstant harmonic functions on M which take only values 0 and 1 on the ends of M. Then the energy function on H is proper modulo the action of G.*

[26]. This work deals with finitely-generated subgroups of the group $Ham(M,\omega)$, the group of Hamiltonian symplectomorphisms of a symplectic manifold (M,ω). It was proven by Polterovich and Franks & Handel that non-uniform lattices in higher rank Lie groups do not embed in $Ham(M,\omega)$, where M is a surface (Franks and Handel established this even for some non-uniform lattices in rank 1 Lie groups). These results provide evidence to Zimmer's conjecture that groups with Property T cannot embed in the groups of area-preserving diffeomorphisms of surfaces. These results left open the case of lattices in $SO(n,1)$, which was posed as an open problem at the Oberwolfach workshop "Group Theory, Symplectic Geometry and Dynamics" in Oberwolfach (Germany), 2006. In the paper [26] (solicited for publication by GAFA) I proved the following:

Theorem 12 *Let G be any RAAG (a Right–Angled Artin Group) and (M,ω) be any symplectic manifold. Then G embeds in $Ham(M,\omega)$. In particular, as a corollary it follows that for every arithmetic lattice Γ of simplest type in $SO(n,1)$, there exists a finite-index subgroup $\Gamma' < \Gamma$, so that Γ' embeds in $Ham(M,\omega)$, where (M,ω) is an arbitrary symplectic manifold (e.g., the 2-sphere).*

Results along these lines were known earlier (in the context of area-preserving diffeomorphisms) provided that M has a surface of high genus (depending on G).

Passing from the higher-genus surfaces to the 2-sphere in (which is the key to the proof of Theorem 12) required some substantially new ideas, both of geometrical and analytical nature.

[6]. This joint paper with Koji Fujiwara paper deals with quasi–homomorphisms $f : \Gamma \to (G, d)$ from arbitrary groups Γ into discrete groups equipped with proper left-invariant metrics. A map f is called a *quasi-homomorphism* if

$$d(f(xy), f(x)f(y)) \leq Const$$

for all $x, y \in \Gamma$. Quasi–homomorphisms were introduced by Stanislav Ulam in 1960's [62, Chapter 6], who regarded stability of homomorphisms as a part of the general stability theory and asked the following general question: "When is it true that the solution of an equation differing slightly from a given one, must of necessity be close to the solution of the given equation?" It was known for a long time that homomorphisms of many groups are unstable in Ulam's sense, when the target is commutative (i.e., there are many quasihomomorphisms which are very far from homomorphisms): This was first discovered by Robert Brooks when the domain Γ is a free group and extended by Epstein and Fujiwara to the case of Gromov-hyperbolic groups Γ. Very little was know prior to [6] about quasi-homomorphisms with noncommutative targets. Bill Thurston in an email to Danny Calegari noted that

"About quasi-morphisms to non-abelian groups: they may be hard to construct in general, but it looks like the Heisenberg group will be one interesting case."

In the same email Thurston outlined a construction of "interesting" quasihomomorphisms from fundamental groups of closed hyperbolic 3-manifold groups into the 3-dimensional Heisenberg group. Among other things, the paper [6] justifies Thurston's claim about "interesting" (exotic) quasi-homomorphisms to Heisenberg groups and shows that, in general, homomorphisms can be lifted to exotic quasi-homomorphisms, when the pull-back of a certain 2nd cohomology class is bounded. The main result of [6] is that, no matter what the target group G is, all quasi-homomorphisms to G essentially come from such a lifting construction. In other words, [6] shows that *instability* in Ulam's problem (when the domain and the range groups are *arbitrary discrete groups*) essentially reduces to quasihomomorphisms with commutative target groups. In the special case, when the target group G is torsion-free and hyperbolic, the main theorem of [6] is easy to state:

Theorem 13 *Every quasihomomorphism $f : \Gamma \to G$ is either bounded, or is a homomorphism, or is a quasi-homomorphism whose image is contained in an infinite cyclic subgroup of G.*

While quasihomomorphisms with noncommutative targets were studied before, this rigidity phenomenon was overlooked. In addition to the rigidity results, we also introduce several generalizations of Ulam-quasihomomorphisms and show that for some of them an analogue of the Brooks' construction does work (at least for self-maps of free groups).

Books. My first book "Hyperbolic Manifolds and Discrete Groups", originally published by Birkhauser in 2001, had second edition published in 2009 in the series "Modern Birkhauser Classics."

My second book, "Geometric Group Theory", [5], written in collaboration with Cornelia Drutu, was published by the AMS in 2018.

5 Interactions of geometric topology and algebraic geometry.

[38], [28]. The first paper is my joint work with J.Kollár. It is well-known that there are many restrictions on the fundamental groups of smooth projective varieties. To the contrary, in [61], Carlos Simpson proved that every finitely-presented group appears as the fundamental group of an irreducible singular complex-projective variety. In the same paper he also asked (Question 12.2, [61]):

Question 14 *Is it true that every finitely-presented group appears as the fundamental group of an irreducible singular complex-projective variety where all singularities are "normal crossings"?*[1]

A partial affirmative answer to this question is given in [38], except that one only gets a reducible variety. Using this result, it was proven in [38] that every finitely-presented group appears as the fundamental group of the link of an isolated complex singularity. It was further shown in [38] that a finitely-presented group G is superperfect over $\mathbb{Q}$ (i.e., $H_i(G, \mathbb{Q}) = 0$ for $i = 1, 2$) if and only if G is the fundamental group of the link of a rational complex singularity. All this was quite unexpected.

In the follow-up paper [28] (which was actually published before [38], because the refereeing process of [38] took too long), I used an interesting connection between hyperbolic and algebraic geometry to prove several theorems partially answering Simpson's question for irreducible varieties:

Theorem 15 *1. Suppose that G is the fundamental group of a compact hyperbolic 3-dimensional manifold M^3. Then there exists an irreducible 2-dimensional projective variety Z whose singularities are normal crossings only, such that $\pi_1(Z) \cong G$.*

2. Suppose that G is an arbitrary finitely-presented group. Then there exists an irreducible 2-dimensional projective variety Z whose singularities are normal crossings and Whitney umbrellas, such that $\pi_1(Z) \cong G$.

Note that Whitney umbrellas (given by the equation $v^2 = wu^2$) are the "next simplest singularities" after normal crossings.

The work [38] and [28] required several new ideas both on algebro–geometric side (converting Euclidean and hyperbolic polyhedral complexes into complex-projective varieties) and on the hyperbolic geometry side. One of the key ingredients in the

[1] Normal crossings are the simplest singularities.

proof is a recent theorem of Panov and Petrunin [59] establishing a certain universality phenomenon for 3-dimensional hyperbolic orbifolds. The most difficult part on the hyperbolic geometry side was to find a (partial) correction of a faulty paper by Jorgensen and Marden [16] claiming that "generic" Dirichlet fundamental domains for Kleinian groups are "simple" (residues of faces are isomorphic to posets of simplices). In fact, examples constructed in [28] show some of the claimed genericity results to be false as stated. Presence of Whitney umbrellas in Part 2 of Theorem 15 comes from 2-torsion in Kleinian groups uniformizing certain 3-dimensional hyperbolic orbifolds.

6 Commutative algebra and complex analysis

My paper [32] addresses the question

"*How many nonconstant holomorphic functions can a connected complex manifold have?*"

The precise way to quantify the size of the ring of holomorphic functions $\mathcal{O}_M$ of a complex manifold M is the *Krull dimension* of this ring. In my paper I solve a problem raised by Georges Elencwajg about the possible range of dimensions of $\mathcal{O}_M$:

Theorem 16 *For every connected complex manifold M, the Krull dimension of $\mathcal{O}_M$ is either zero (only constant holomorphic functions) or is infinite (more precisely, has cardinality at least continuum).*

The proof of this theorem is an application of nonstandard real numbers and ultralimits.

References

[1] Y. Benoist, *Convexes hyperboliques et quasiisométries*, Geom. Dedicata, vol. 122 (2006), p. 109–134.

[2] A. Berenstein, M. Kapovich, *Affine buildings for dihedral groups*, Geometria Dedicata, Vol. 156 (2012) p. 171–207.

[3] S. Dey, M. Kapovich, *Patterson-Sullivan theory for Anosov subgroups.* Preprint, 2019.

[4] S. Dey, M. Kapovich, B. Leeb, *A combination theorem for Anosov subgroups*, Math. Zeitschrift, 293 (2019) 1-2, 551-578.

[5] C. Drutu, M. Kapovich, "Geometric group theory". AMS series "Colloquium Publications," 2018.

[6] K. Fujiwara, M. Kapovich, *On quasihomomorphisms with noncommutative targets*, Geometric Analysis and Functional Analysis, 2016. DOI: 10.1007/s00039-016-0364-9.

[7] J. Gilman, *Two-generator discrete subgroups of* $PSL(2,\mathbb{R})$, Mem. Amer. Math. Soc., 117 (1995) no. 561

[8] J. Gilman, *Algorithms, complexity and discreteness criteria in* $PSL(2,\mathbb{C})$, J. Anal. Math. 73 (1997), p. 91–114.

[9] J. Gilman, L. Keen, *Canonical hexagons and the* $PSL(2,\mathbb{C})$ *discreteness problem*, Preprint, August 2015, arxiv.org/pdf/1508.00257v2.

[10] J. Gilman, B. Maskit, *An algorithm for 2-generator Fuchsian groups*, Michigan Math. J., 38 (1991), no. 1, p. 13–32.

[11] M. Gromov, *Hyperbolic groups*, In: "Essays in Group Theory," Math. Sci. Res. Inst. Publ. 8, Springer, New York (1987), p. 75-263.

[12] M. Gromov, "Metric structures for Riemannian and non-Riemannian spaces," Progress in Mathematics 152, Birkhauser, Boston (1999).

[13] T. Haines, M. Kapovich and J. Millson, *Appendix A to Equidimensionality of convolution morphisms and applications to saturation problems* (by T. Haines), Advances in Mathematics, Vol. 207, (2006) Issue 1, p. 321-327.

[14] T. Haines, M. Kapovich and J. Millson, *Ideal triangles in Euclidean buildings and branching to Levi subgroups*, Journal of Algebra, Vol. 361 (2012), p. 41-78.

[15] P. Hitzelberger, L. Kramer and R. Weiss, *Non-discrete Euclidean buildings for the Ree and Suzuki groups,* Amer. J. Math. 132 (2010), no. 4, p. 1113–1152.

[16] T. Jorgensen, A. Marden, *Generic fundamental polyhedra for Kleinian groups.* In: "Holomorphic functions and moduli," Vol. II p. 69–85. Math. Sci. Res. Inst. Publ., 11, Springer, New York, 1988.

[17] M. Kapovich, *Conformally flat metrics on 4-manifolds*, Journal of Differential Geometry, Vol. 66 (2004) p. 1-13.

[18] M. Kapovich, *Generalized triangle inequalities and their applications*, Proceedings of the International Congress of Mathematicians - Madrid, August 22-30, 2006. Eds. Marta Sanz-Sola, Javier Soria, Juan L. Varona, Joan Verdera. Vol. 2, p. 719-742.

[19] M. Kapovich, *Triangle inequalities in path metric spaces*, Geometry and Topology, Vol. 11 (2007) p. 1653-1680.

[20] M. Kapovich, *Convex projective structures on Gromov-Thurston manifolds*, Geometry and Topology, Vol. 11 (2007) p. 1777-1830.

[21] M. Kapovich, *Kleinian groups in higher dimensions*. In "Geometry and Dynamics of Groups and Spaces. In memory of Alexander Reznikov", M.Kapranov et al (eds). **Birkhauser, Progress in Mathematics**, Vol. 265, 2007, p. 485-562.

[22] M. Kapovich, *Homological dimension and critical exponent of Kleinian groups*, Geometric Analysis and Functional Analysis, Vol. 18 (2009), 6, 2017-2054.

[23] M. Kapovich, "Hyperbolic Manifolds and Discrete Groups", 2nd edition, Birkhauser's Modern Classics, 2009.

[24] M. Kapovich, *On sequences of finitely generated discrete groups*. "In the tradition of Ahlfors-Bers. V," **Contemporary Mathematics**, Vol. 510 (2010), p. 165-184.

[25] M. Kapovich, *Arithmetic aspects of self-similar groups*, Groups, Geometry and Dynamics, Vol. 6 (2012) p. 737-754.

[26] M. Kapovich, *RAAGs in Ham*, Geometric Analysis and Functional Analysis, Vol. 22 (2012) p. 733–755.

[27] M. Kapovich, *Non-coherence of arithmetic hyperbolic lattices*, Geometry and Topology, Vol. 17 (2013) p. 39–71.

[28] M. Kapovich, *Dirichlet domains and topology of projective varieties*, Inventiones Mathematicae, Vol. 194, (2013) 3, p. 631-672.

[29] M. Kapovich, *Energy of harmonic functions and Gromov's proof of Stallings' theorem*, Georgian Math. J., Vol. 21 (2014) 3, p. 281-296.

[30] M. Kapovich, *Buildings and tensor product multiplicities*, pages 1122-1142. Appendix to the survey "Additive Eigenvalue Problem" by S. Kumar, **Transformation Groups**, Vol. 19 (2014) 1051-1148.

[31] M. Kapovich, *Discreteness is undecidable*, International Journal of Algebra and Computation, Vol. 26 (2016) p. 467-472.

[32] M. Kapovich, *Krull dimensions of rings of holomorphic functions*. Proceedings of Complex Analysis and Dynamical Systems, VII. Contemporary Mathematics, vol. 699, Amer. Math. Soc., Providence, RI, 2017, pp. 167-173.

[33] M. Kapovich, *A note on Selberg's lemma and negatively pinched Hadamard manifolds*. Preprint, 2018.

[34] M. Kapovich, S. Kim, J. Lee, *Sullivan's structural stability of expanding group actions*. Preprint, 2019.

[35] M. Kapovich, B. Kleiner, *Coarse Alexander duality and duality groups*, Journal of Differential Geometry, Vol. 69, (2005) 2, p. 279-352.

[36] M. Kapovich, B. Kleiner, *Weak hyperbolization conjecture for 3-dimensional* CAT(0) *groups*, Geometry, Groups and Dynamics, Vol. 1 (2007), p. 67-79.

[37] M. Kapovich, B. Kleiner, *Appendix to "Lacunary hyperbolic groups"*, by A. Olshanskii, D.Osin, M.Sapir, Geometry and Topology, Vol. 13 (2009) p. 2132-2137.

[38] M. Kapovich, J. Kollár, *Fundamental groups of links of isolated singularities*, Journal of American Mathematical Society, Vol. 27, (2014) 4, p. 929-952.

[39] M. Kapovich, S. Kumar and J. Millson, *The eigencone and saturation for Spin(8)*, Pure and Applied Mathematics Quarterly, Vol. 5, N2 (Hirzebruch Special Issue, Part 1), 2009, p. 755-780.

[40] M. Kapovich and B. Leeb, *Finsler bordifications of symmetric and certain locally symmetric spaces*, Geometry and Topology, 22 (2018) 2533-2646.

[41] M. Kapovich, B. Leeb, *Discrete isometry groups of symmetric spaces*, Spring 2015 MSRI Lecture Notes. Volume IV of Handbook of Group Actions. The ALM series, International Press, Eds. L.Ji, A.Papadopoulos, S-T.Yau. (2018) Chapter 5, p. 191-290.

[42] M. Kapovich, B. Leeb, *Relativizing characterizations of Anosov subgroups, I.* Preprint, 2018.

[43] M. Kapovich, B. Leeb and J. Millson, "The generalized triangle inequalities in symmetric spaces and buildings with applications to algebra", **Memoirs of AMS**, Vol. 192, 2008.

[44] M. Kapovich, B. Leeb and J. Millson, *Convex functions on symmetric spaces, side lengths of polygons and the stability inequalities for weighted configurations at infinity*, Journal of Differential Geometry, Vol. 81 (2009), p. 297-354.

[45] M. Kapovich, B. Leeb and J. Millson, *Polygons in buildings and their refined side lengths*, Geometric Analysis and Functional Analysis, Vol. 19 (2009), no. 4, p. 1081-1100.

[46] M. Kapovich, B. Leeb and J. Porti, *Morse actions of discrete groups on symmetric spaces*, Preprint, 2014.

[47] M. Kapovich, B. Leeb and J. Porti, *A Morse Lemma for quasigeodesics in symmetric spaces and euclidean buildings*, Geometry and Topology, 22 (2018) 3827-3923.

[48] M. Kapovich, B. Leeb and J. Porti, *Dynamics on flag manifolds: domains of proper discontinuity and cocompactness*, Geometry and Topology, Vol. 22 (2017) 157-234.

[49] M. Kapovich, B. Leeb and J. Porti, *Some recent results on Anosov representations*. Transformation Groups, Vol. 21 (2016) 4, 1105-1121.

[50] M. Kapovich, B. Leeb and J. Porti, *Lectures on Anosov representations: Dynamical and geometric characterizations*. European Mathematical Journal. Vol. 3 (2017) 808-898.

[51] M. Kapovich, B.Liu, Geometric finiteness in negatively pinched Hadamard manifolds. Ann. Acad. Sci. Fennicae, 44 (2019) 2, 841-875.

[52] M. Kapovich, B. Liu, *Hausdorff dimension of nonconical limit sets*, Preprint, 2019.

[53] M. Kapovich, J. Millson, *Structure of the tensor product semigroup*, Asian Math. Journal (S.S.Chern memorial issue) Vol. 10, N. 3 (2006) p. 493-540.

[54] M. Kapovich, J. Millson, *A path model for geodesics in Euclidean buildings and its applications to representation theory*, Groups, Geometry and Dynamics, Vol 2 (2008), p. 405-480.

[55] M. Kapovich, L. Potyagailo, *On absence of Ahlfors' finiteness theorem for Kleinian groups in dimension 3*, Topology and its Applications, 40 (1991), p. 83– 91.

[56] M. Kapovich, L. Potyagailo, *On the absence of finiteness theorems of Ahlfors and Sullivan for Kleinian groups in higher dimensions,* Siberian Math. J. 32 (1991), no. 2, p. 227–237.

[57] M. Kapovich, L. Potyagailo and E. Vinberg, *Non-coherence of some non-uniform lattices in* $\mathrm{Isom}(\mathbf{H}^n)$, Geometry and Topology Monographs (Zieschang's memorial volume), Vol. 14, 2008, p. 335-351.

[58] A. Knutson and T. Tao, *The honeycomb model of* $\mathrm{GL}_n(\mathbb{C})$ *tensor products. I. Proof of the saturation conjecture,* J. Amer. Math. Soc., vol. **12** (1999), no. 4, p. 1055–1090.

[59] D. Panov, A. Petrunin, *Telescopic actions*, Geom. Funct. Anal. 22 (2012), no. 6, p. 1814–1831.

[60] R. Riley, *Applications of a Computer Implementation of Poincaré's Theorem on Fundamental Polyhedra*, Math of Computation, 40 (1983) p. 607–632.

[61] C. Simpson, *Local systems on proper algebraic V-manifolds*, Pure and Applied Math. Quarterly, 7 (2011), p. 1675–1760.

[62] S. Ulam, "A collection of mathematical problems." Interscience Tracts in Pure and Applied Mathematics, no. 8, Interscience Publishers, New York–London, 1960.