ESP Kouba
Worksheet $13\frac{1}{2}$
Solutions

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1.) a.) $x^2 - y^2 = 36 \xrightarrow{D} 2x - 2yy' = 0 \rightarrow$
$2yy' = 2x \rightarrow y' = x/y$.

b.) $\cos(xy^2) = y^3 + x \xrightarrow{D}$
$-\sin(xy^2) \cdot [x \cdot 2yy' + y^2 \cdot 1] = 3y^2y' + 1 \rightarrow$
$-2xyy'\sin(xy^2) - y^2\sin(xy^2) = 3y^2y' + 1 \rightarrow$
$-2xyy'\sin(xy^2) - 3y^2y' = 1 + y^2\sin(xy^2) \rightarrow$
$y'[-2xy\sin(xy^2) - 3y^2] = 1 + y^2\sin(xy^2) \rightarrow$
$$y' = \frac{1 + y^2\sin(xy^2)}{-2xy\sin(xy^2) - 3y^2}$$

c.) $(x-y)^4 = \tan(xy) \xrightarrow{D}$
$4(x-y)^3 \cdot (1-y') = \sec^2(xy) \cdot (xy' + 1 \cdot y) \rightarrow$
$4(x-y)^3 - 4(x-y)^3y' = x\sec^2(xy) \cdot y' + y\sec^2(xy) \rightarrow$
$4(x-y)^3 - y\sec^2(xy) = 4(x-y)^3y' + x\sec^2(xy)y' \rightarrow$
$$y' = \frac{4(x-y)^3 - y\sec^2(xy)}{4(x-y)^3 + x\sec^2(xy)}$$

d.) $\dfrac{x^2}{y^3+1} = \dfrac{x-1}{y+1} \longrightarrow$
$x^2y + x^2 = xy^3 - y^3 + x - 1 \xrightarrow{D}$
$x^2y' + 2x \cdot y + 2x = x \cdot 3y^2y' + y^3 - 3y^2y' + 1 \rightarrow$
$x^2y' - 3xy^2y' + 3y^2y' = y^3 + 1 - 2xy - 2x \rightarrow$
$$y' = \frac{y^3 + 1 - 2xy - 2x}{x^2 - 3xy^2 + 3y^2} .$$

e.) $x \sin y + y \sin x = x + y \xrightarrow{D}$

$x \cos y \cdot y' + 1 \cdot \sin y + y \cos x + y' \sin x = 1 + y' \rightarrow$

$x \cos y \cdot y' + \sin x \cdot y' - y' = 1 - \sin y - y \cos x \rightarrow$

$$y' = \frac{1 - \sin y - y \cos x}{x \cos y + \sin x - 1} .$$

f.) $\sqrt{1+\sqrt{1+\sqrt{1+y}}} = x^2 + \sec(3y) \xrightarrow{D}$

$\frac{1}{2}\left[1+\sqrt{1+\sqrt{1+y}}\right]^{-1/2} \cdot \frac{1}{2}\left[1+\sqrt{1+y}\right]^{-1/2} \cdot \frac{1}{2}(1+y)^{-1/2} \cdot y'$

$= 2x + \sec(3y)\tan(3y) \cdot 3y' \rightarrow$

$\frac{1}{8}\left[1+\sqrt{1+\sqrt{1+y}}\right]^{-1/2} \cdot \left[1+\sqrt{1+y}\right]^{-1/2} \cdot [1+y]^{-1/2} y'$

$- \sec(3y)\tan(3y) \cdot 3y' = 2x \rightarrow$

$$y' = \frac{2x}{\frac{1}{8}\left[1+\sqrt{1+\sqrt{1+y}}\right]^{-1/2} \cdot \left[1+\sqrt{1+y}\right]^{-1/2} \cdot [1+y]^{-1/2} - 3\sec(3y)\tan(3y)} .$$

2.) $(xy^3 + y)^3 = x^2 + 27$ at $x=0 \rightarrow y=3 \xrightarrow{D}$

$3(xy^3+y)^2 \cdot [x \cdot 3y^2 y' + 1 \cdot y^3 + y'] = 2x \rightarrow$

(let $x=0$, $y=3$) $27 \cdot [27 + y'] = 0 \rightarrow$

slope of tangent line is $y' = -27$ so slope of $\perp$ line is $m = 1/27$ and line is

$$y - 3 = 1/27 (x - 0) \quad \text{or} \quad y = \frac{1}{27}x + 3 .$$

3.) $y^3 + y^2 = xy + 2$ at $y=1 \rightarrow x=0 \xrightarrow{D}$

$3y^2 y' + 2y y' = x y' + 1 \cdot y \rightarrow$

$3Y^2Y' - XY' + 2YY' = Y \rightarrow$

$Y' = \dfrac{Y}{3Y^2 - X + 2Y}$; let $X=0, Y=1 \rightarrow$

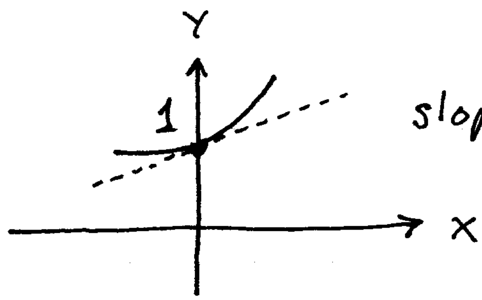
$Y' = \frac{1}{5}$; $\xrightarrow{D}$

$$Y'' = \frac{(3Y^2 - X + 2Y)Y' - Y(6YY' - 1 + 2Y')}{(3Y^2 - X + 2Y)^2} \; ;$$

let $X=0, Y=1, Y' = 1/5 \longrightarrow$

$$Y'' = \frac{(5)(1/5) - (6/5 - 1 + 2/5)}{(5)^2}$$

$$= \frac{1 - 3/5}{25} = \frac{\frac{2}{5}}{25} = \frac{2}{125} \rightarrow Y'' = \frac{2}{125}$$



slope $= 1/5$ and concave up

4.) $Y' = \dfrac{1 - 2XY^2}{2X^2Y - 1} \rightarrow 2X^2YY' - Y' = 1 - 2XY^2$

$\rightarrow (X^2 \cdot 2YY' + 2X \cdot Y^2) - Y' = 1$

$\rightarrow D\{X^2Y^2 - Y\} = 1$

$\rightarrow X^2Y^2 - Y = X + C$; let $X=0, Y=1 \rightarrow$

$0 - 1 = 0 + C \rightarrow C = -1 \rightarrow$

$X^2Y^2 - Y = X - 1$.

5.) $\underline{2X^2 - XY + Y^2 = 42} \xrightarrow{D}$

$4X - (XY' + 1\cdot Y) + 2YY' = 0 \rightarrow$

$(2Y - X)Y' = Y - 4X \rightarrow$

(*) $Y' = \dfrac{Y-4X}{2Y-X}$;

at the maximum and minimum y-values $Y' = 0 \rightarrow Y - 4X = 0 \rightarrow$

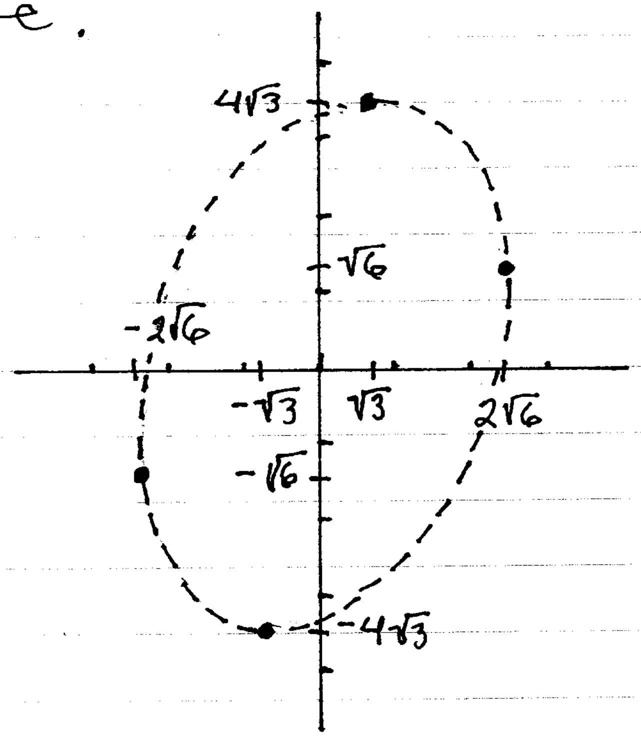
$\boxed{Y = 4X}$;

then $2X^2 - X(4X) + (4X)^2 = 42 \rightarrow$

$2X^2 - 4X^2 + 16X^2 = 42 \rightarrow 14X^2 = 42 \rightarrow$

$X^2 = 3 \rightarrow X = \pm\sqrt{3}$. If $X = +\sqrt{3}$, then $Y = 4\sqrt{3}$ is the <u>maximum</u> Y-value. If $X = -\sqrt{3}$, then $Y = -4\sqrt{3}$ is the <u>minimum</u> Y-value.

at the maximum and minimum X-values, the tangent lines are vertical, i.e., the derivative in (*) is undefined



$\rightarrow 2Y - X = 0 \rightarrow$

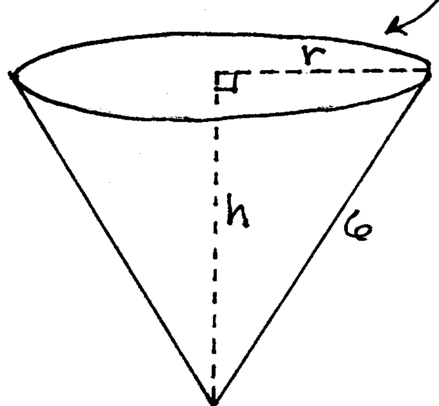
$\boxed{2Y = X}$; then

$2(2Y)^2 - (2Y)Y + Y^2 = 42 \rightarrow$

$8Y^2 - 2Y^2 + Y^2 = 42 \rightarrow 7Y^2 = 42 \rightarrow$

$Y^2 = 6 \rightarrow Y = \pm\sqrt{6}$. If $Y = +\sqrt{6}$, then $X = 2\sqrt{6}$ is the maximum x-value. If $Y = -\sqrt{6}$, then $X = -2\sqrt{6}$ is the minimum x-value

6.)



$$\text{length} = 2\pi(6) - (6)\theta$$
$$= 12\pi - 6\theta \rightarrow$$
$$2\pi r = 12\pi - 6\theta \rightarrow$$
$$r = \frac{12\pi - 6\theta}{2\pi} \rightarrow$$
$$r = 6 - \frac{3}{\pi}\theta$$
; then

$$h^2 + r^2 = 6^2 \rightarrow$$
$$h^2 = 36 - \left(6 - \frac{3}{\pi}\theta\right)^2 = \cancel{36} - \left(\cancel{36} - \frac{36}{\pi}\theta + \frac{9}{\pi^2}\theta^2\right) \rightarrow$$
$$h^2 = \frac{36}{\pi}\theta - \frac{9}{\pi^2}\theta^2 = \frac{9}{\pi^2}\left(4\pi\theta - \theta^2\right) \rightarrow$$
$$h = \frac{3}{\pi}\sqrt{4\pi\theta - \theta^2}$$
; maximize volume of cone.

$$V = \frac{1}{3}\pi r^2 h \rightarrow$$
$$V = \frac{1}{3}\pi\left(6 - \frac{3}{\pi}\theta\right)^2 \cdot \frac{3}{\pi}\sqrt{4\pi\theta - \theta^2} \rightarrow$$
$$\boxed{V = \left(6 - \frac{3}{\pi}\theta\right)^2\sqrt{4\pi\theta - \theta^2}}$$
; then

$$V' = \left(6 - \frac{3}{\pi}\theta\right)^2 \cdot \frac{1}{2}\left(4\pi\theta - \theta^2\right)^{-1/2} \cdot (4\pi - 2\theta)$$
$$+ 2\left(6 - \frac{3}{\pi}\theta\right)\left(\frac{-3}{\pi}\right) \cdot \sqrt{4\pi\theta - \theta^2}$$
$$= 2\left(6 - \frac{3}{\pi}\theta\right)\left\{\frac{\left(6 - \frac{3}{\pi}\theta\right)(2\pi - \theta)}{2\sqrt{4\pi\theta - \theta^2}} - \frac{3\sqrt{4\pi\theta - \theta^2}}{\pi}\right\}$$

$$= 2\left(6-\frac{3}{\pi}\theta\right) \cdot \frac{\left(6-\frac{3}{\pi}\theta\right)(2\pi-\theta)\pi - 6\left(4\pi\theta-\theta^2\right)}{2\pi\sqrt{4\pi\theta-\theta^2}} = 0$$

$\rightarrow \theta = 2\pi$ (impossible) or

$$\left(6-\frac{3}{\pi}\theta\right)(2\pi-\theta)\pi - 6\left(4\pi\theta-\theta^2\right) = 0 \rightarrow$$

$$\left(12\pi - 6\theta - 6\theta + \frac{3}{\pi}\theta^2\right)\pi - 24\pi\theta + 6\theta^2 = 0 \rightarrow$$

$$12\pi^2 - 12\pi\theta + 3\theta^2 - 24\pi\theta + 6\theta^2 = 0 \rightarrow$$

$$9\theta^2 - 36\pi\theta + 12\pi^2 = 0 \rightarrow$$

$$\boxed{3\theta^2 - 12\pi\theta + 4\pi^2 = 0} \rightarrow$$

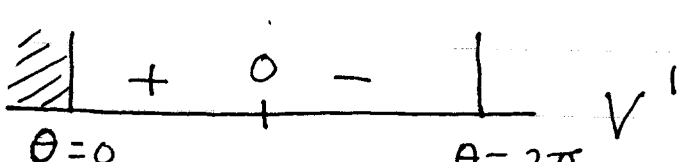
$$\theta = \frac{12\pi \pm \sqrt{144\pi^2 - 48\pi^2}}{6}$$

$$= \frac{12\pi \pm \sqrt{96\pi^2}}{6}$$

$$= \frac{12\pi \pm 4\sqrt{6}\,\pi}{6}$$

$$= \frac{6-2\sqrt{6}}{3}\pi \approx 0.367\pi \text{ radians}$$

or 66.1°



$\theta = 0.367\pi$, $r \approx 4.9$ in, $h \approx 3.5$ in.^{2}, and max. $V \approx 87$ in^3