ESP Kouba
Worksheet 14 ½
Solutions

1.) $f(x) = 7x^3 + x - 5 \rightarrow f'(x) = 21x^2 + 1 \rightarrow$

Newton: $x_{n+1} = x_n - \dfrac{7x_n^3 + x_n - 5}{21x_n^2 + 1} \rightarrow$

$x_{n+1} = \dfrac{14x_n^3 + 5}{21x_n^2 + 1}$ ; since $f(0) = -5 < 0$

and $f(1) = 3 > 0$ let $x_1 = 0$. Then

$$x_2 = \frac{14x_1^3 + 5}{21x_1^2 + 1} = \frac{14(0)^3 + 5}{21(0)^2 + 1} = 5,$$

$$x_3 = \frac{14x_2^3 + 5}{21x_2^2 + 1} = \frac{14(5)^3 + 5}{21(5)^2 + 1} \approx 3.33650,$$

$$x_4 = \frac{14x_3^3 + 5}{21x_3^2 + 1} \approx 2.23615,$$

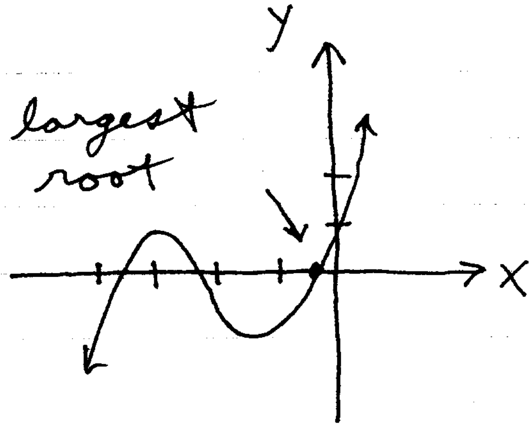
$$x_5 = \frac{14x_4^3 + 5}{21x_4^2 + 1} \approx 1.52387,$$

$x_6 \approx 1.09597$, $x_7 \approx 0.89344$,
$x_8 \approx 0.84358$, $x_9 \approx 0.84070$,
$x_{10} \approx 0.84069$, $x_{11} \approx 0.84069$
so root $r$ to $f(x) = 0$ is $r \approx 0.8406$.

2.) $f(x) = x^3 + 6x^2 + 9x + 1 \rightarrow f'(x) = 3x^2 + 12x + 9 \rightarrow$

Newton: $x_{n+1} = x_n - \dfrac{x_n^3 + 6x_n^2 + 9x_n + 1}{3x_n^2 + 12x_n + 9} \rightarrow$

$x_{n+1} = \dfrac{2x_n^3 + 6x_n^2 - 1}{3x_n^2 + 12x_n + 9}$ ; sketch graph




Let $x_1 = 0$ ; then

$$x_2 = \frac{2x_1^3 + 6x_1^2 - 1}{3x_1^2 + 12x_1 + 9} = \frac{-1}{9} \approx -0.11111,$$

$$x_3 = \frac{2x_2^3 + 6x_2^2 - 1}{3x_2^2 + 12x_2 + 9} \approx -0.08902,$$

$$x_4 = \frac{2x_3^3 + 6x_3^2 - 1}{3x_3^2 + 12x_3 + 9} \approx -0.11989, \quad x_5 \approx -0.12061,$$

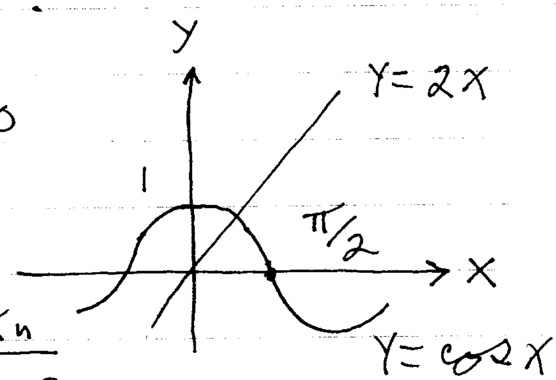
$x_6 \approx -0.12061$ , so largest root $r$ to $f(x) = 0$ is $r \approx -0.1206$ .

3.) $\cos x = 2x \rightarrow \cos x - 2x = 0$

$\rightarrow f(x) = \cos x - 2x$

$\rightarrow f'(x) = -\sin x - 2$

$\rightarrow$ newton: $x_{n+1} = x_n - \dfrac{\cos x_n - 2x_n}{-\sin x_n - 2}$




$\rightarrow$ $x_{n+1} = \dfrac{x_n \sin x_n + \cos x_n}{\sin x_n + 2}$ ; let $x_1 = 1$ then

$$x_2 = \frac{x_1 \sin x_1 + \cos x_1}{\sin x_1 + 2} \approx 0.48628,$$

$x_3 \approx 0.36365$ , $x_4 \approx 0.45165$ ,
$x_5 \approx 0.45018$ , $x_6 \approx 0.45018$ so root $r$ to $f(x) = 0$ is

$r \approx 0.4501$ .

4.) $f(x) = x^3 - 10 = 0 \longrightarrow f'(x) = 3x^2 \longrightarrow$

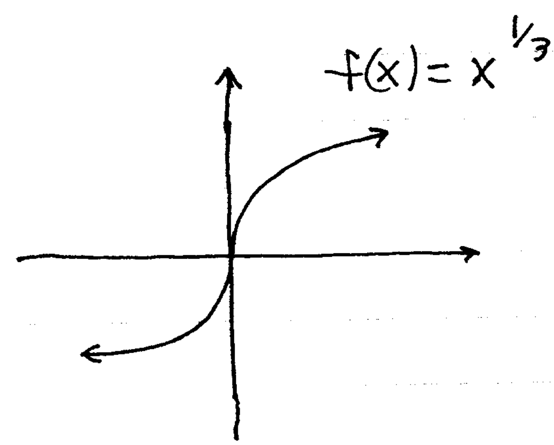
Newton: $x_{n+1} = x_n - \dfrac{x_n^3 - 10}{3x_n^2} \longrightarrow$

$$x_{n+1} = \frac{2x_n^3 + 10}{3x_n^2}$$ ; let $x_1 = 2$ then

$x_2 = \dfrac{2x_1^3 + 10}{3x_1^2} \approx 2.16666$, $x_3 \approx 2.15450$,

$x_4 \approx 2.15443$, $x_5 \approx 2.15443$ so

$10^{1/3} \approx 2.1544$.




5.) $f'(x) = \frac{1}{3}x^{-2/3}$ then

Newton: $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

$\rightarrow x_{n+1} = x_n - \dfrac{x_n^{1/3}}{\frac{1}{3}x_n^{-2/3}}$

$= x_n - 3 \cdot x_n \longrightarrow$ $\boxed{x_{n+1} = -2x_n}$ ;

$x_1 = 1 \longrightarrow x_2 = -2x_1 = -2 \longrightarrow$

$x_3 = -2x_2 = 4 \longrightarrow x_4 = -2x_3 = -8 \longrightarrow$

$x_5 = -2x_4 = 16$ ; <u>conjecture</u>: Newton's method fails on this example because there is a vertical tangent line at the root.

6.) $f(x) = \frac{1+\ln x}{x} \rightarrow$

$$f'(x) = \frac{x \cdot \frac{1}{x} - (1+\ln x)\cdot 1}{x^2} = \frac{-\ln x}{x^2} \rightarrow$$

Newton : $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \rightarrow$

$$x_{n+1} = x_n - \frac{\frac{1+\ln x_n}{x_n}}{\frac{-\ln x_n}{x_n^2}} = x_n + \frac{1+\ln x_n}{\cancel{x_n}} \cdot \frac{x_n^{\cancel{2}}}{\ln x_n} \rightarrow$$

$$= \frac{x_n}{1} + \frac{x_n + x_n \ln x_n}{\ln x_n} = \frac{x_n \ln x_n + x_n + x_n \ln x_n}{\ln x_n} \rightarrow$$

$$x_{n+1} = \frac{2x_n \ln x_n + x_n}{\ln x_n} \rightarrow x_{n+1} = 2x_n + \frac{x_n}{\ln x_n} \ ;$$

a.) $x_1 = 1.2 \rightarrow x_2 = 2x_1 + \frac{x_1}{\ln x_1} \approx 8.9817 \rightarrow$

$x_3 = 2x_2 + \frac{x_2}{\ln x_2} \approx 22.0551 \rightarrow$

$x_4 = 2x_3 + \frac{x_3}{\ln x_3} \approx 51.2396$ ;

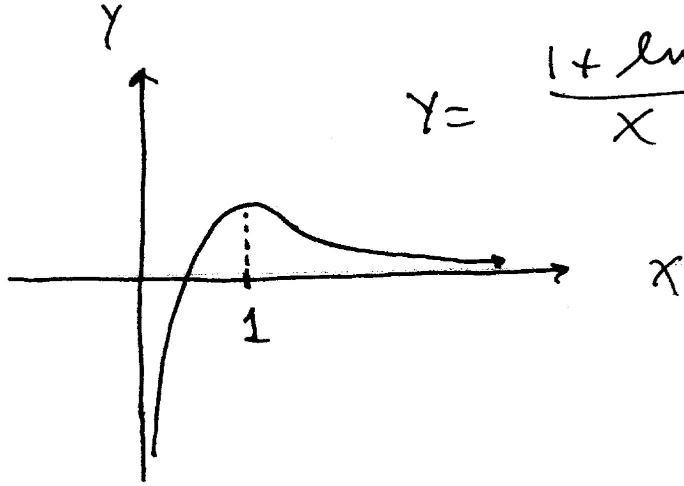
b.) $x_1 = 0.5 \rightarrow x_2 = 2x_1 + \frac{x_1}{\ln x_1} \approx 0.2786 \rightarrow$

$x_3 \approx 0.3392 \rightarrow x_4 \approx 0.3646$ (continue)

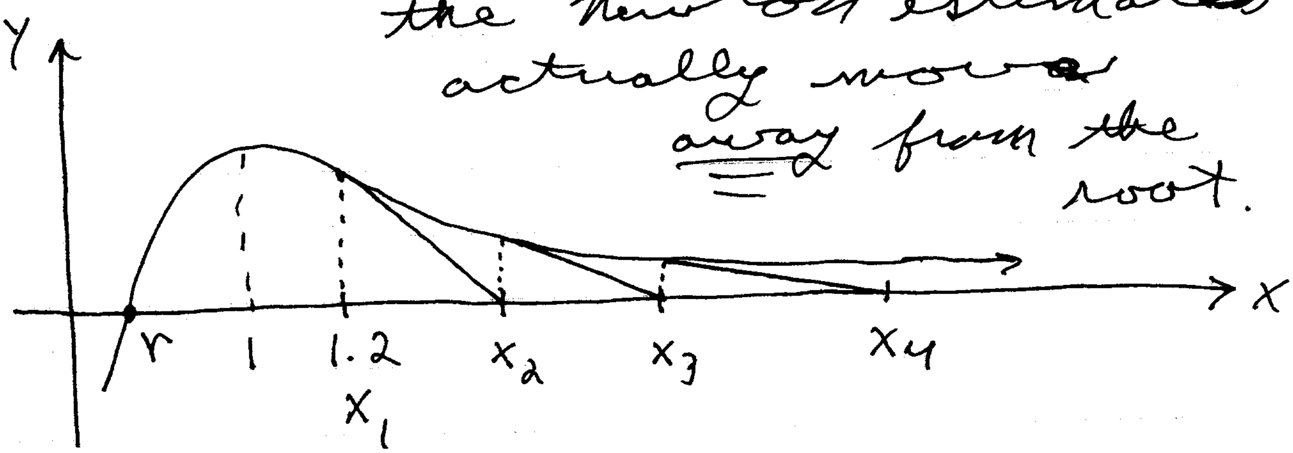
$x_5 \approx 0.3628 \rightarrow x_6 \approx 0.3628$

c.)

$$Y = \frac{1 + \ln x}{x}$$




<u>Conjecture</u>: In part a.) the initial guess of $x_1 = 1.2$ was to the right of the maximum value of the function. Thus, the Newton estimates actually move <u>away</u> from the root.




In part b.) the tangent line at $x_1 = 0.5$ slopes toward the root, so Newton's method works.