ESP
Kouba
Worksheet 16 Solutions

1.)

a.) $\lim\limits_{t \to 0} (1+t)^{1/t} = e$

b.) $\lim\limits_{n \to +\infty} \left(1+\frac{1}{n}\right)^{n} = e$

c.) $\lim\limits_{n \to -\infty} \left(1+\frac{1}{n}\right)^{n} = e$

d.) $\lim\limits_{n \to +\infty} \left(1-\frac{1}{n}\right)^{n} = \lim\limits_{n \to +\infty} \left[\left(1+\frac{1}{(-n)}\right)^{(-n)}\right]^{-1} = e^{-1}$

e.) $\lim\limits_{n \to +\infty} \left(1+\frac{3}{n}\right)^{4n} = \lim\limits_{n \to +\infty} \left[\left(1+\frac{1}{\left(\frac{n}{3}\right)}\right)^{\left(\frac{n}{3}\right)}\right]^{12} = e^{12}$

f.) $\lim\limits_{n \to +\infty} \left[\left(1+\frac{1}{(n^2)}\right)^{(n^2)}\right]^{\frac{1}{n}} = e^{0} = 1$

g.) $\lim\limits_{n \to +\infty} \left(1-\frac{2}{n}\right)^{n^2} = \lim\limits_{n \to +\infty} \left[\left(1+\frac{1}{\left(\frac{-n}{2}\right)}\right)^{\left(\frac{-n}{2}\right)}\right]^{-2n}$

$= \lim\limits_{n \to +\infty} \dfrac{1}{\left[\left(1+\frac{1}{\left(\frac{-n}{2}\right)}\right)^{\left(\frac{-n}{2}\right)}\right]^{2n}} = \text{"}\frac{1}{e^{\infty}}\text{"} = 0$

h.) $\lim\limits_{t \to 0^+} \left[(1+t)^{\frac{1}{t}}\right]^{\sqrt{t}} = e^{0} = 1$

i.) $\lim\limits_{n \to +\infty} \left(1+\frac{2}{n}\right)^{3n} = \lim\limits_{n \to +\infty} \left[\left(1+\frac{1}{\left(\frac{n}{2}\right)}\right)^{\left(\frac{n}{2}\right)}\right]^{6} = e^{6}$

j.) $\lim\limits_{n \to -\infty} \dfrac{1}{\left(\frac{n-1}{n}\right)^{n^2}} = \lim\limits_{n \to -\infty} \dfrac{1}{\left(1-\frac{1}{n}\right)^{n^2}}$

$= \lim\limits_{n \to -\infty} \dfrac{1}{\left[\left(1+\frac{1}{(-n)}\right)^{(-n)}\right]^{(-n)}} = \text{"}\frac{1}{e^{\infty}}\text{"} = 0$

2.) a.) $3^x = 3^3 \rightarrow x = 3$

b.) $3^x = 20 \rightarrow \ln 3^x = \ln 20 \rightarrow x \ln 3 = \ln 20 \rightarrow$

$x = \dfrac{\ln 20}{\ln 3}$

c.) $\left(1 + \frac{x}{12}\right) = 200^{1/36} \rightarrow \dfrac{x}{12} = 200^{1/36} - 1 \rightarrow$

$x = 12\left(200^{\frac{1}{36}} - 1\right)$

d.) $\ln x = 10 \rightarrow x = e^{10}$

e.) $2x = \sqrt{x^2+1} \rightarrow 4x^2 = x^2+1 \rightarrow$

$3x^2 = 1 \rightarrow x^2 = \frac{1}{3} \rightarrow \cancel{x = -\frac{1}{\sqrt{3}}}$ or $x = +\dfrac{1}{\sqrt{3}}$

No! why not?

f.) $\ln x(x-3) = \ln 4 \rightarrow x^2 - 3x = 4 \rightarrow$

$x^2 - 3x - 4 = 0 \rightarrow (x-4)(x+1) = 0 \rightarrow$

$x = 4$ or $\cancel{x = -1}$

No! why not?

g.) $\ln\left(\dfrac{2x+1}{x-3}\right) = \ln x \rightarrow \dfrac{2x+1}{x-3} = x \rightarrow 2x+1 = x^2 - 3x \rightarrow$

$0 = x^2 - 5x - 1 \rightarrow x = \dfrac{5 \pm \sqrt{25+4}}{2} = \dfrac{5 \pm \sqrt{29}}{2} \rightarrow$

$x = \dfrac{5 + \sqrt{29}}{2}$

3.) a.) $2x + y' = 0 \rightarrow y' = -2x$

b.) $2x + 2yy' = 0 \rightarrow y' = -x/y$

c.) $x \cdot 3y^2 y' + y^3 + 5y' = 1 \rightarrow$

$(3xy^2 + 5)y' = 1 - y^3 \rightarrow y' = \dfrac{1 - y^3}{3xy^2 + 5}$

d.) $5(X+4Y)^4 \cdot (1+4Y') = 0 \rightarrow$

$5(X+4Y)^4 + 20(X+4Y)^4 Y' = 0 \rightarrow \quad Y' = \frac{-1}{4}$

e.) $X \cdot \sec^2 Y \cdot Y' + \tan Y + 3\cos 3X = \cos Y^3 \cdot 3Y^2 \cdot Y' \rightarrow$

$(X\sec^2 Y - 3Y^2 \cos Y^3)Y' = -\tan Y - 3\cos 3X \rightarrow$

$$Y' = \frac{-\tan Y - 3\cos 3X}{X\sec^2 Y - 3Y^2 \cos Y^3}$$

f.) $X^2 \cdot 2\sec Y^2 \cdot \sec Y^2 \cdot \tan Y^2 \cdot 2Y \cdot Y'$

$+ 2X \cdot \sec^2 Y^2 = Y' - 1 \rightarrow$

$(4X^2 Y \sec^2 Y^2 \cdot \tan Y^2 - 1)Y' = -1 - 2X\sec^2 Y^2 \rightarrow$

$$Y' = \frac{-1 - 2X\sec^2 Y^2}{4X^2 Y \sec^2 Y^2 \cdot \tan Y^2 - 1}$$

g.) $-\csc^2(X - \sin\sqrt{Y}) \cdot \left(1 - \cos\sqrt{Y} \cdot \frac{1}{2\sqrt{Y}} \cdot Y'\right) = \frac{XY' - Y}{X^2} \rightarrow$

$-X^2 \csc^2(X - \sin\sqrt{Y}) + \frac{X^2}{2\sqrt{Y}} \csc^2(X - \sin\sqrt{Y}) \cdot \cos\sqrt{Y} \cdot Y' = XY' - Y \rightarrow$

$\left(\frac{X^2}{2\sqrt{Y}} \csc^2(X - \sin\sqrt{Y}) \cdot \cos(\sqrt{Y}) - X\right)Y' = X^2 \csc^2(X - \sin\sqrt{Y}) - Y \rightarrow$

$$Y' = \frac{X^2 \csc^2(X - \sin\sqrt{Y}) - Y}{\frac{X^2}{2\sqrt{Y}} \csc^2(X - \sin\sqrt{Y}) \cdot \cos\sqrt{Y} - X}$$

4.) $X=3 \rightarrow Y=2$ and $3Y^2 Y' - 2X = 0 \rightarrow Y' = \frac{2X}{3Y^2} \rightarrow$

slope $= \frac{6}{3(4)} = \frac{1}{2} \rightarrow$ tangent line is

$$Y - 2 = \frac{1}{2}(X-3).$$

5.) $3\ln Y = Y^2 \ln X \rightarrow \frac{3}{Y}Y' = Y^2 \cdot \frac{1}{X} + 2YY' \ln X \rightarrow$

$3XY' = Y^3 + 2XY^2 Y' \ln X \rightarrow Y' = \frac{Y^3}{3X - 2XY^2 \ln X}$ so

at $(8^{1/4}, 2)$ slope of tangent line is -4.1 so slope of normal line is $.244$ and equation of normal line is $Y - 2 = .244(X - 8^{1/4})$.

6.)

a.) $Y' = 5X^4$

b.) $5Y^4 Y' = 1 \rightarrow \quad Y' = 1/5Y^4$

c.) $-\sin(XY)\cdot(XY' + Y) = \sec^2(X-Y)\cdot \quad (1-Y') \rightarrow$

$-X\sin(XY)\cdot Y' - Y\sin(XY) = \sec^2(X-Y) - \sec^2(X-Y)\cdot Y' \rightarrow$

$$Y' = \frac{\sec^2(X-Y) + Y\sin(XY)}{\sec^2(X-Y) - X\sin(XY)}$$

d.) $X^2Y' + 2XY + X\cdot 2YY' + Y^2 = 0 \rightarrow Y' = \dfrac{-Y^2 - 2XY}{X^2 + 2XY}$

e.) $\frac{1}{Y}Y' = X\cdot\frac{1}{X} + \ln X \rightarrow \quad Y' = Y(1+\ln X)$

f.) $\ln Y = \ln X^X = X\ln X \rightarrow \quad Y' = Y(1+\ln X)$

g.) $Y' = 3^X \ln 3$

h.) $\ln Y = \cos X^3 \cdot \ln X \rightarrow$

$\frac{1}{Y}Y' = \cos X^3\cdot\frac{1}{X} + \ln X\cdot -\sin X^3\cdot 3X^2 \rightarrow$

$Y' = X^{\cos X^3}\left\{\cos X^3\cdot\frac{1}{X} - \ln X\cdot\sin X^3\cdot 3X^2\right\}$

i.) $\ln Y = Y\ln X \rightarrow \frac{1}{Y}Y' = Y\cdot\frac{1}{X} + Y'\cdot\ln X \rightarrow$

$$Y' = \frac{Y/X}{\frac{1}{Y} - \ln X}$$

j.) $\ln\sqrt{5X} = Y^2\ln X \rightarrow \frac{1}{\sqrt{5X}}\cdot\frac{1}{2}(5X)^{-1/2}\cdot 5 = Y^2\cdot\frac{1}{X}$

$+ 2YY'\cdot\ln X \rightarrow \frac{5}{2\cdot 5X} - \frac{Y^2}{X} = 2YY'\cdot\ln X \rightarrow$

$$Y' = \frac{\frac{1}{2X} - \frac{Y^2}{X}}{2Y\ln X}$$

k.) $Y' = 0$

l.) $\ln Y = X^X\cdot\ln X \rightarrow \ln(\ln Y) = \ln\left[X^X\cdot\ln X\right]$

$= X\ln X + \ln(\ln X) \rightarrow$

$$\frac{1}{\ln Y}\cdot\frac{1}{Y}\cdot Y' = X\cdot\frac{1}{X}+\ln X + \frac{1}{\ln X}\cdot\frac{1}{X} \rightarrow$$

$$Y' = \frac{1+\ln X + \frac{1}{X\ln X}}{\frac{1}{Y\ln Y}} = Y\ln Y + Y\ln X\ln Y + \frac{Y\ln Y}{X\ln X}$$

m.) $Y' = 7^{\ln X}\cdot\frac{1}{X}\cdot\ln 7$

n.) $Y' = X\cdot\frac{1}{|X|\sqrt{X^2-1}} + \operatorname{arcsec} X$

o.) $Y' = 2\arcsin\sqrt{X}\cdot\frac{1}{\sqrt{1-X}}\cdot\frac{1}{2\sqrt{X}}$

p.) $Y' = 5\left(\arccos\frac{3}{X}\right)^4\cdot\frac{-1}{\sqrt{1-\left(\frac{3}{X}\right)^2}}\cdot -\frac{3}{X^2}$

q.) $1 = \frac{1}{1+Y^2}\cdot Y' \rightarrow Y' = 1+Y^2$

r.) $Y' = \frac{1}{1+Y^2}Y' \rightarrow \cancel{Y'}+Y^2Y' = \cancel{Y'}\,0 \rightarrow Y'=0$

7.) a.) $f(X) = X^3+X-1$, $f'(X) = 3X^2+1$

i.) $\Delta f = f(4) - f(1) = 66$

$df = f'(1)\cdot\Delta X = (4)(3) = 12$

ii.) $\Delta f = f(2) - f(1) = 8$

$df = f'(1)\cdot\Delta X = (4)(1) = 4$

iii) $\Delta f = f(1.1) - f(1) = .431$

$df = f'(1)\cdot\Delta X = (4)(.1) = .4$

iv.) $\Delta f = f(1.01) - f(1) = .040301$

$df = f'(1)\cdot\Delta X = (4)(.01) = .04$

b.) $f(X) = \ln X$, $f'(X) = \frac{1}{X}$

i.) $\Delta f = f(e+2) - f(e) = .5514447$

$df = f'(e)\cdot\Delta X = \left(\frac{1}{e}\right)(2) = .73575888$

ii) $\Delta f = f(e+.1) - f(e) = .0361274$

$df = f'(e)\cdot\Delta X = \left(\frac{1}{e}\right)(.1) = .0367879$

iii) $\Delta f = f(e+.001) - f(e) = .0003678$

$df = f'(e)\cdot\Delta X = \left(\frac{1}{e}\right)(.001) = .0003678$

c.) $f(x) = e^{\sin x}$, $f'(x) = e^{\sin x} \cdot \cos x$

i.) $\Delta f = f(-1) - f(0) = -.5689240449$

$df = f'(0) \cdot \Delta x = (1)(-1) = -1.0$

ii) $\Delta f = f(-.001) - f(0) = -.0009995$

$df = f'(0) \cdot \Delta x = (1)(-.001) = -.001$

iii) $\Delta f = f(-.00001) - f(0) = -.000099995$

$df = f'(0) \cdot \Delta x = (1)(-.00001) = -.00001$

8.) a.) Let $f(x) = \sqrt{x}$, $x_1 = 225$ and $x_2 = 229$ then

$\Delta f \approx df \rightarrow \sqrt{229} - \sqrt{225} \approx f'(225) \cdot \Delta x \rightarrow$

$\sqrt{229} - 15 \approx \left(\frac{1}{30}\right)(4) \rightarrow \sqrt{229} \approx 15 + \frac{2}{15}$

$= 15.13333333$ and by calculator

$\sqrt{229} = 15.13274595$.

b.) Let $f(x) = x^5$, $f'(x) = 5x^4$, $x_1 = 250$ and $x_2 = 250.1$

then $\Delta f \approx df \rightarrow (250.1)^5 - (250)^5 \approx f'(250) \cdot \Delta x \rightarrow$

$(250.1)^5 \approx (250)^5 + 5(250)^4(.1) = 976{,}531{,}249{,}00$

and by calculator $(250.1)^5 = 978{,}517{,}188{,}000$.

c.) Let $f(x) = \sin x$, $f'(x) = \cos x$, $x_1 = \pi$, $x_2 = \pi - .04$

then $\Delta f \approx df \rightarrow \sin(\pi - .04) - \sin(\pi) \approx f'(\pi) \cdot \Delta x \rightarrow$

$\sin(\pi - .04) \approx (\cos \pi)(-.04) = .04$ and

by calculator $\sin(\pi - .04) = .0399$.

d.) Let $f(x) = \log_{10} x$, $f'(x) = \frac{1}{x} \cdot \log_{10} e$, $x_1 = 10$, $x_2 = 9.9$

then $\Delta f \approx df \rightarrow \log_{10}(9.9) - \log_{10} 10 = f'(10) \cdot \Delta x \rightarrow$

$\log_{10}(9.9) \approx 1 + \frac{1}{10} \log_{10} e \cdot (-.1) = .995657055$ and by calculator $\log_{10}(9.9) = .995635194$

9.) assume $\frac{|\Delta r|}{r} \le 2\%$

a.) $D = 2r \rightarrow \frac{|\Delta D|}{D} \approx \frac{|dD|}{D} = \frac{|D'\Delta r|}{D} = \frac{|2 \cdot \Delta r|}{2r} = \frac{|\Delta r|}{r} \le 2\%$

b.) $V = \frac{4}{3}\pi r^3 \rightarrow \frac{|\Delta V|}{V} \approx \frac{|dV|}{V} = \frac{|V'\Delta r|}{V} = \frac{|4\pi r^2 \Delta r|}{4/3 \pi r^3} = 3 \frac{|\Delta r|}{r} \le 6\%$

c.) $S = 4\pi r^2 \rightarrow \frac{|\Delta S|}{S} \approx \frac{|dS|}{S} = \frac{|S'\Delta r|}{S} = \frac{|8\pi r \cdot \Delta r|}{4\pi r^2} = 2 \frac{|\Delta r|}{r} \le 4\%$

10.) Let $f(x) = \sqrt{x}$ and $x: 4 \rightarrow 4+h$. Then $f'(x) = \frac{1}{2\sqrt{x}}$ and $\Delta x = h$. The exact change in f is $\Delta Y = f(4+h) - f(4) = \sqrt{4+h} - 2$ and the differential of f is $dY = f'(4) \cdot \Delta x = \frac{1}{4}h$. Thus, $\Delta Y \approx dY \rightarrow$ $\sqrt{4+h} - 2 \approx \frac{1}{4}h \rightarrow \sqrt{4+h} \approx 2 + \frac{1}{4}h$.

11.) Let $f(x) = \frac{1}{x^{1/3}} = x^{-1/3}$ and $x: 8 \rightarrow 8+h^3$. Then $f'(x) = \frac{-1}{3}x^{-4/3} = \frac{-1}{3x^{4/3}}$ and $\Delta x = h^3$;

$\Delta Y = f(8+h^3) - f(8) = \frac{1}{(8+h^3)^{1/3}} - \frac{1}{2}$, $dY = f'(8) \cdot \Delta x$

$= \frac{-1}{3 \cdot 8^{4/3}} \cdot h^3 = \frac{-1}{3 \cdot 16} h^3 = \frac{-1}{48} h^3$. Since $\Delta Y \approx dY \rightarrow$

$\frac{1}{(8+h^3)^{1/3}} - \frac{1}{2} \approx \frac{-1}{48} h^3 \rightarrow \frac{1}{(8+h^3)^{1/3}} \approx \frac{1}{2} - \frac{1}{48} h^3$

12.) Let $f(x) = \ln x$ and $x: 1 \rightarrow 1+h^2$. Then $f'(x) = \frac{1}{x}$ and $\Delta x = h^2$; $\Delta Y = f(1+h^2) - f(1)$

$= \ln(1+h^2) - \cancel{\ln 1}^{\,0} = \ln(1+h^2)$, $dY = f'(1)\cdot\Delta X$
$= 1\cdot h^2 = h^2$, and $\Delta Y \approx dY \rightarrow \ln(1+h^2) \approx h^2$.

13.) $V = \frac{4}{3}\pi r^3$ and $r: 5 \rightarrow 6$ so $V' = 4\pi r^2$
and $\Delta r = 1$; ΔV is exact volume of ice
and $\Delta V \approx dV \rightarrow$

a.) $\Delta V \approx dV = V'(1)\cdot\Delta r = 4\pi(5)^2 = 100\pi \approx 314.2 \text{ in.}^3$

b.) Exact volume $\Delta V = V(6) - V(5)$
$= \frac{4}{3}\pi(6)^3 - \frac{4}{3}\pi(5)^3 \approx 381.2 \text{ in.}^3$

c.) $381.2 - 314.2 = 67$ so % error is
$\frac{67}{381.2} = 18\%$.

14.) Let $\log_c X = Z$ then $c^Z = X \rightarrow$

$\log_b c^Z = \log_b X \rightarrow Z\cdot\log_b c = \log_b X \rightarrow$

$Z = \frac{\log_b X}{\log_b c} \rightarrow \log_c X = \frac{\log_b X}{\log_b c}$.