ESP
Kouba
Worksheet 7 Solutions

1.) a.) $f(x) = \sin x$ is continuous for all values of x.

b.) $f(x) = \dfrac{1}{\sin x}$ is continuous for all values of x except where $\sin x = 0$, i.e., except $x = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \cdots$.

c.) $f(x) = \dfrac{x^4-1}{(x-1)(x+1)}$ is continuous everywhere except $x=1$ and $x=-1$.

d.) $\displaystyle\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{(x^2+1)\cancel{(x^2-1)}}{\cancel{(x^2-1)}} = 2 = f(1)$

and $\displaystyle\lim_{x\to -1} f(x) = \lim_{x\to -1} \frac{(x^2+1)\cancel{(x^2-1)}}{\cancel{(x^2-1)}} = 2 \neq f(-1)$

so f is continuous everywhere except at $x=-1$.

e.) $\displaystyle\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (x^2+x) = 0$ and

$\displaystyle\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{\sin x}{\sqrt{x}} = \lim_{x\to 0^+} \sqrt{x}\cdot\frac{\sin x}{\sqrt{x}\sqrt{x}}$

$\displaystyle= \lim_{x\to 0^+} \sqrt{x}\cdot\frac{\sin x}{x} = 0\cdot 1 = 0$; f is not defined at $x = 2\pi$, so f is continuous for all x-values except $x = 2\pi$.

2.) a.) i.) $\text{Ave} = \dfrac{T(4)-T(1)}{4-1} = \dfrac{48-3}{3} = 15$ mph

ii) $\text{Ave} = \dfrac{T(2)-T(1)}{2-1} = \dfrac{12-3}{1} = 9$ mph

iii) $\text{Ave} = \dfrac{T(1.1)-T(1)}{1.1-1} = \dfrac{3.63-3}{.1} = 6.3$ mph

iv) $\text{Ave} = \dfrac{T(1.01)-T(1)}{1.01-1} = \dfrac{3.0603-3}{1.01-1} = 6.03$ mph

b.) $T' = \lim\limits_{h\to 0} \dfrac{T(t+h)-T(t)}{h} = \lim\limits_{h\to 0} \dfrac{3(t+h)^2 - 3t^2}{h}$

$= \lim\limits_{h\to 0} \dfrac{3(t^2+2ht+h^2)-3t^2}{h} = \lim\limits_{h\to 0} \dfrac{\cancel{3t^2}+6ht+3h^2-\cancel{3t^2}}{h}$

$= \lim\limits_{h\to 0} \dfrac{\cancel{h}(6t+3h)}{\cancel{h}} = 6t$ so velocity at

$t=1$ is $T'(1) = 6(1) = 6$ mph.

3.) a.) i.) $\text{Ave} = \dfrac{\sqrt{3}-\sqrt{2}}{3-2} \approx .3178$ gm./cm.

ii) $\text{Ave} = \dfrac{\sqrt{2.5}-\sqrt{2}}{2.5-2} \approx .3338$ gm./cm.

iii.) $\text{Ave} = \dfrac{\sqrt{2.1}-\sqrt{2}}{2.1-2} \approx .3492$ gm./cm.

iv.) $\text{Ave} = \dfrac{\sqrt{2.01}-\sqrt{2}}{2.01-2} \approx .3531$ gm./cm.

b.) $\text{Density} = \lim\limits_{h\to 0} \dfrac{\sqrt{2+h}-\sqrt{2}}{(2+h)-2} = \lim\limits_{h\to 0} \dfrac{\sqrt{2+h}-\sqrt{2}}{h}\cdot\dfrac{\sqrt{2+h}+\sqrt{2}}{\sqrt{2+h}+\sqrt{2}}$

$$= \lim_{h\to 0} \frac{(\cancel{2}+h) - \cancel{2}}{h(\sqrt{2+h}+\sqrt{2})} = \lim_{h\to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{2+h}+\sqrt{2})} = \frac{1}{2\sqrt{2}} \approx .3535 \text{ gm/cm}$$

4.) a.) i.)

$$m = \frac{\frac{1}{3-1} - \frac{1}{\frac{3}{2}-1}}{3 - \frac{3}{2}} = -1$$

ii)

$$m = \frac{\frac{1}{2-1} - \frac{1}{\frac{3}{2}-1}}{2 - \frac{3}{2}} = -2$$

iii)

$$m = \frac{\frac{1}{\frac{7}{4}-1} - \frac{1}{\frac{3}{2}-1}}{\frac{7}{4} - \frac{3}{2}} = \frac{-8}{3} = -2.6666$$

iv.)

$$m = \frac{\frac{1}{\frac{25}{16}-1} - \frac{1}{\frac{3}{2}-1}}{\frac{25}{16} - \frac{3}{2}} = \frac{-32}{9} = -3.5555$$

b.)

$$\text{Slope} = \lim_{h\to 0} \frac{f(\frac{3}{2}+h) - f(\frac{3}{2})}{h} = \lim_{h\to 0} \frac{\frac{1}{(\frac{3}{2}+h)-1} - \frac{1}{\frac{3}{2}-1}}{h}$$

$$= \lim_{h\to 0} \left(\frac{1}{\frac{1}{2}+h} - 2\right)\cdot\frac{1}{h} = \lim_{h\to 0} \frac{-2\cancel{h}}{(\frac{1}{2}+h)\cancel{h}} = -4$$

5.) a.)

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\{(x+h) - (x+h)^2\} - \{x - x^2\}}{h} = \lim_{h\to 0} \frac{h - 2xh - h^2}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(1 - 2x - h)}{\cancel{h}} = 1 - 2x$$

b.) $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

work!

$$= \lim_{h \to 0} \frac{\dfrac{(x+h)+1}{2(x+h)-3} - \dfrac{x+1}{2x-3}}{h} = \lim_{h \to 0} \frac{-5\cancel{h}}{(2x+2h-3)(2x-3)} \cdot \frac{1}{\cancel{h}}$$

$$= \frac{-5}{(2x-3)^2}$$

c.) $g'(x) = \lim_{h \to 0} \dfrac{g(x+h) - g(x)}{h}$

$$= \lim_{h \to 0} \frac{\sqrt{x+h-5} - \sqrt{x-5}}{h} = \lim_{h \to 0} \frac{\sqrt{x+h-5} - \sqrt{x-5}}{h} \cdot \frac{\sqrt{x+h-5} + \sqrt{x-5}}{\sqrt{x+h-5} + \sqrt{x-5}}$$

$$= \lim_{h \to 0} \frac{(\cancel{x}+h-\cancel{5}) - (\cancel{x}-\cancel{5})}{h\left(\sqrt{x+h-5} + \sqrt{x-5}\right)} = \lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}\left(\sqrt{x+h-5} + \sqrt{x-5}\right)} = \frac{1}{2\sqrt{x-5}}$$

d.) $H'(x) = \lim_{h \to 0} \dfrac{H(x+h) - H(x)}{h}$

$$= \lim_{h \to 0} \frac{\left\{(\cancel{x}+h) - \frac{1}{(x+h)^2}\right\} - \left\{\cancel{x} - \frac{1}{x^2}\right\}}{h}$$

$$= \lim_{h \to 0} \left\{h + \frac{1}{x^2} - \frac{1}{(x+h)^2}\right\} \cdot \frac{1}{h}$$

$$= \lim_{h \to 0} \left\{h + \frac{2xh+h^2}{x^2(x+h)^2}\right\} \cdot \frac{1}{h} = \lim_{h \to 0} \cancel{h}\left\{1 + \frac{2x+h}{x^2(x+h)^2}\right\} \cdot \frac{1}{\cancel{h}}$$

$$= 1 + \frac{2x}{x^4} = 1 + \frac{2}{x^3}.$$

e.) $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

$$= \lim_{h\to 0} \frac{\cos 4(x+h) - \cos 4x}{h} = \lim_{h\to 0} \frac{\cos(4x+4h) - \cos 4x}{h}$$

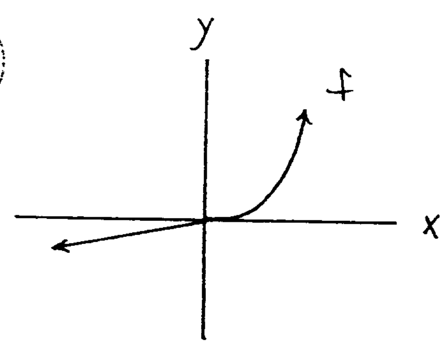
$$= \lim_{h\to 0} \frac{\cos 4x \cdot \cos 4h - \sin 4x \cdot \sin 4h - \cos 4x}{h}$$

$$= \lim_{h\to 0} \left\{ \cos 4x \cdot \frac{\cos 4h - 1}{4h} \cdot 4 - \sin 4x \cdot \frac{\sin 4h}{4h} \cdot 4 \right\}$$

$$= \cos 4x \cdot (0) \cdot 4 - \sin 4x \cdot (1) \cdot 4$$

$$= -4 \sin 4x$$

f.)



<u>For $x > 0$</u> $\quad f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{x^2}}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}} = 2x \;;$$

<u>for $x < 0$</u>

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{2}(x+h) - \frac{1}{2}x}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{\tfrac{1}{2}x} + \frac{1}{2}h - \cancel{\tfrac{1}{2}x}}{h} = \lim_{h\to 0} \frac{1}{2} = \frac{1}{2} \;;$$

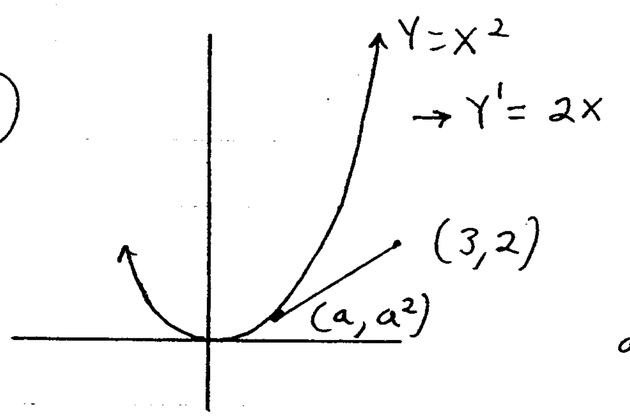
<u>for $x = 0$</u> $\quad f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h}$

$$= \lim_{h\to 0} \frac{f(h) - 0}{h} = \lim_{h\to 0} \frac{f(h)}{h} \Rightarrow$$

$$\begin{cases} \lim_{h\to 0^+} \frac{f(h)}{h} = \lim_{h\to 0^+} \frac{h^2}{h} = \lim_{h\to 0^+} h = 0 \\ \lim_{h\to 0^-} \frac{f(h)}{h} = \lim_{h\to 0^-} \frac{\frac{1}{2}\cancel{h}}{\cancel{h}} = \frac{1}{2} \end{cases}, \text{ so}$$

$f'(0)$ does not exist, i.e.,

$$f'(x) = \begin{cases} 2x & \text{for } x > 0 \\ \text{does not exist} & \text{for } x = 0 \\ 1/2 & \text{for } x < 0 \end{cases}.$$

6.) 

Slope of tangent line at $x = a$ is

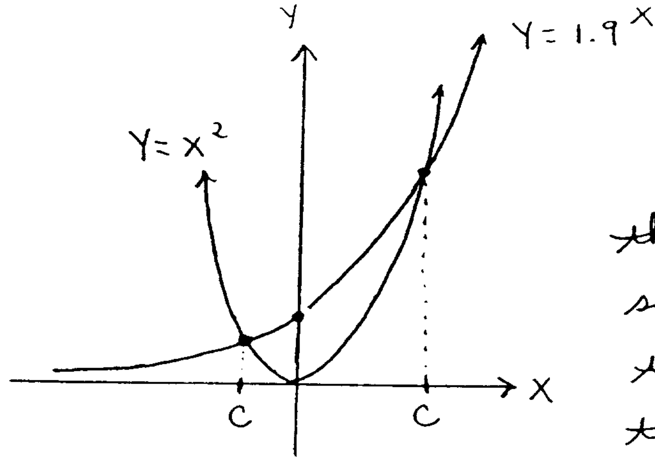
i.) $\frac{a^2 - 2}{a - 3}$

and ii) $2a$, thus

$$\frac{a^2-2}{a-3} = 2a \Rightarrow a^2 - 2 = 2a^2 - 6a \Rightarrow 0 = a^2 - 6a + 2 \Rightarrow$$

$$a = \frac{6 \pm \sqrt{36-8}}{2} = 3 \pm \sqrt{7}$$

7.)



This graph suggests that there are only 2 solutions. However, the IMVT will prove that there are at least 3 solutions:

Let $f(x) = x^2 - 1.9^x$, which is continuous for all values of x since $Y = x^2$ and $Y = 1.9^x$ are continuous for all values of x. Consider f on the interval $[-1, 0]$:

$$f(-1) = 1 - \frac{1}{1.9} = \frac{9}{19} > 0 \quad \text{and}$$
$$f(0) = 0 - 1 = -1 < 0 .$$

By the IMVT there is at least one number c in $[-1, 0]$ satisfying $f(c) = 0$, i.e., solving the equation $x^2 = 1.9^x$. Consider f on the interval $[0, 2]$:

$$f(0) = 0 - 1 = -1 < 0 \quad \text{and}$$
$$f(2) = 4 - 1.9^2 = 0.39 > 0 .$$

By the IMVT there is at least one number c in $[0, 2]$ satisfying $f(c) = 0$, i.e., solving the equation $x^2 = 1.9^x$. Consider f on the interval $[2, 6]$:

$$f(2) = 4 - 1.9^2 = 0.39 > 0 \quad \text{and}$$
$$f(6) = 36 - 1.9^6 = -11.045881 < 0 .$$

By the IMVT there is at least one number c in $[2, 6]$ satisfying $f(c) = 0$, i.e., solving the equation $x^2 = 1.9^x$.

This completes the problem.

8.) a.) Let $f(x) = 5x^3 - x + 7$ on $[-2, 0]$. Then f is continuous (since it's a polynomial) and $m = 0$ is between $f(-2) = -31$ and $f(0) = 7$. So by the IMVT there is a number c satisfying $f(c) = 0$, i.e., $5c^3 - c + 7 = 0$, where $-2 \le c \le 0$.

b.) Since n is odd and $a_n > 0$, $\lim_{x \to +\infty} P(x) = +\infty$ and $\lim_{x \to -\infty} P(x) = -\infty$. Thus, there are numbers a and b satisfying $P(a) > 0$ and $P(b) < 0$. Since P is continuous (since it's a polynomial) and $m = 0$ is between $P(a)$ and $P(b)$, it follows from the IMVT that there is a number r between a and b so that $P(r) = 0$.

9.) Let $f(x) = \frac{1}{x+3} - e^x$, which is continuous on the interval $[-1, 0]$. Constant $m = 0$ is between $f(-1) = \frac{1}{2} - \frac{1}{e} > 0$ and $f(0) = \frac{-2}{3}$. Thus, by the IMVT there is at least one number c, $-1 \le c \le 0$, satisfying $f(c) = 0$, i.e.,

$$\frac{1}{c+3} - e^c = 0 .$$

10.) Let $f(x) = x^3 - 2^x$, which is continuous on the interval $[0, 2]$. Constant $m = 0$ is between $f(0) = -1$ and $f(2) = 4$. Thus, by the IMVT there is at least one number c, $0 \le c \le 2$, satisfying $f(c) = 0$, i.e., $c^3 - 2^c = 0$ or

$$c^3 = 2^c .$$

11.) Let $h(x) = f(x) - g(x)$ on the interval $[a, b]$, which is continuous since f and g are continuous. Let $m = 0$ and note that $h(a) = f(a) - g(a) < 0$ and $h(b) = f(b) - g(b) > 0$. Since m is between $h(a)$ and $h(b)$, the IMVT guarantees that there is a number c, $a \le c \le b$, satisfying $h(c) = 0$, i.e., $f(c) - g(c) = 0$, i.e.,

$$f(c) = g(c) .$$

12) $h(x) = \dfrac{f(x)}{g(x)}$ is continuous on $[a,b]$ since f and g are continuous on $[a,b]$ and $g(x) > 0$ on $[a,b]$; $h(a) = \dfrac{f(a)}{g(a)} = 1$ and $h(b) = \dfrac{f(b)}{g(b)} < 0$ so by IMVT there is c, $a \le c \le b$, satisfying $h(c) = \frac{1}{2}$, i.e., $\frac{f(c)}{g(c)} = \frac{1}{2}$, i.e., $2f(c) = g(c)$.

13.) 1, 5, 9, 17, ..., 12045 is generated by the formula $4n-3$ for $n \ge 1$.
If $4n-3 = 12045 \Rightarrow 4n = 12048 \Rightarrow$ $\boxed{n = 3012}$ numbers are in the list.

The total sum of these numbers is

$$\sum_{n=1}^{3012} (4n-3) = 4\sum_{n=1}^{3012} n - \sum_{n=1}^{3012} 3$$

$$= 4\,\frac{(3012)(3012+1)}{2} - 3\,(3012)$$

$$= 18{,}141{,}276.$$