21C: Midterm 1

2014-07-10

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1. (24 points.) For each series below, if the series converges then find its sum; otherwise, state that it diverges.

a) $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$

$$\frac{1/2}{1-1/2} = 1$$

b) $\sum_{n=1}^{\infty} \left(\frac{3}{2}\right)^n$

diverges since $\frac{3}{2} > 1$

c) $\sum_{n=1}^{\infty} \frac{2^n + 3^n}{4^n}$ $= \sum_{n=1}^{\infty} \left(\frac{2}{4}\right)^n + \sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n$

$$= \frac{\frac{1}{2}}{1-\frac{1}{2}} + \frac{\frac{3}{4}}{1-\frac{3}{4}}$$

$$= 4$$

2. (24 points.) For each series below, state whether it converges or diverges.

Integral test:

a) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ Diverges by comparison

with $\displaystyle\int_2^{\infty} \frac{1}{x \ln x}\,dx =$

$\displaystyle\lim_{b\to\infty} \int_2^{b} \frac{1}{x \ln x}\,dx =$

$\displaystyle\lim_{b\to\infty} \left[\ln(\ln x)\right]_2^b = \infty$

b) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ Converges by

comparison with

$\displaystyle\int_2^{\infty} \frac{1}{x(\ln x)^2}\,dx < \infty$

3. (24 points.) For each series below, state whether it converges or diverges.

Limit comparison test:

a) $\sum_{n=1}^{\infty} \frac{n}{n^2+n+1}$ diverges by comparison with $\Sigma \frac{1}{n}$

b) $\sum_{n=1}^{\infty} \frac{n^2}{n^2+n+1}$ diverges by n^{th} term test

c) $\sum_{n=1}^{\infty} \frac{1}{n^2 \ln n}$ converges by comparison with $\Sigma \frac{1}{n^2}$

d) $\sum_{n=1}^{\infty} \frac{n^3+n}{n^5}$ converges by comparison with $\Sigma \frac{1}{n^2}$

4. (24 points.) For each series below, state whether it converges absolutely, converges conditionally, or diverges

a) $\sum_{n=1}^{\infty}(-1)^n \frac{1}{\sqrt{n}+1}$ converges conditionally

- converges by alt. series test
- not absolutely convergent by comparison with $\Sigma \frac{1}{\sqrt{n}}$

b) $\sum_{n=1}^{\infty}(-1)^n \frac{n}{n^2+1}$ converges conditionally

- converges by alt. series test
- not absolutely convergent by comp. with $\Sigma \frac{1}{n}$

c) $\sum_{n=1}^{\infty}(-1)^n \frac{n}{n+1}$

diverges by n^{th} term test

d) $\sum_{n=2}^{\infty}(-1)^n \frac{n}{n^3-1}$ absolutely convergent

by comparison with $\Sigma \frac{1}{n^2}$

5. (24 points.) Does the following series converge absolutely, converge conditionally, or diverge?

$$\sum_{n=1}^{\infty} (-1)^n \left(\ln(n+1) - \ln(n)\right)$$

Converges conditionally.

Let $u_n = \ln(n+1) - \ln(n)$

Then $u_n = \ln\left(\frac{n+1}{n}\right)$.

Since $\frac{n+1}{n} \to 1$, we have

$u_n \to \ln(1) = 0$

Furthermore,

$u_n = \ln\left(1+\frac{1}{n}\right)$ is decreasing in n.

It follows that

$\sum (-1)^n u_n$

converges by alt. series test.

But $\sum u_n$ is a telescoping series whose n^{th} partial sum can be computed as $\ln(n+1) - \ln 1 = \ln(n+1)$

Thus the series diverges since $\ln(n+1) \to \infty$

6. (24 points.) Consider the power series

$$\sum_{n=1}^{\infty} \frac{(x-1)^n}{n2^n}. \quad — a_n$$

a) Find the interval of convergence.

$$|a_n|^{\frac{1}{n}} = \frac{1}{n^{1/n}} \left|\frac{x-1}{2}\right| \to \left|\frac{x-1}{2}\right|.$$

$$\left|\frac{x-1}{2}\right| < 1 \quad \text{if} \quad -1 < x < 3$$

$x = -1$: $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$ converges

$x = 3$: $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

Answer: $[-1, 3)$

b) Find the radius of convergence.

$R = 2$

c) Find all values of x such that the series is conditionally convergent.

$x = -1$