The Eynard-Orantin Theory in Geometry

A Mystery from Physics

Motohico Mulase

University of California, Davis

December 11, 2010
Pucón, Chile

Outline

Abstract

The Mystery

In 2007 physicists Bertrand Eynard and Nicolas Orantin discovered, from their work on statistical mechanics and random matrix theory, a beautiful universal recursion formula based on a plane analytic curve (i.e., a Riemann surface), the Cauchy differentiation kernel, and the residue calculus on it.

The mystery is that it is not clear what this formula is calculating.

The Conjectures

Then string theorists Bouchard, Dijkgraaf, Klemm, Mariño, Pasquetti, Vafa, and others conjectured that the Eynard-Orantin recursion was computing various important geometric quantities, such as the Hurwitz numbers, the Gromov-Witten invariants of toric Calabi-Yau spaces, and knot invariants. It is also speculated that the recursion may have something to do with the hyperbolic volume conjecture of the knot complement.

The Mathematical Work

In this talk I will report that the first cases of the conjectures have been solved very recently by mathematicians.

Instead of starting from describing the general theory, I will start with giving you a simple example of the theory. This example shows the nature of the recursion and where it is coming from.

Reference

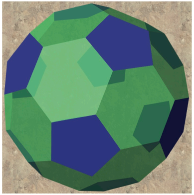
1. M-Penkava: *Topological recursion for the Poincaré polynomial of the combinatorial moduli space of curves*, Preprint arXiv:1009.2135 math.AG (2010).
2. Chapman-M-Safnuk: *The Kontsevich constants for the volume of the moduli of curves and topological recursion*, Preprint arXiv:1009.2055 math.AG (2010).
3. Eynard-M-Safnuk: *The Laplace transform of the cut-and-join equation and the Bouchard-Mariño conjecture on Hurwitz numbers*, Publ. Res. Inst. Math. Sci. (2011).
4. M-Zhang: *Polynomial recursion formula for linear Hodge integrals*, Commun. Number Theory and Physics **4**, (2010).
5. Borot-Eynard-M-Safnuk: *A matrix model for simple Hurwitz numbers, and topological recursion*, J. Geometry and Physics (2011).

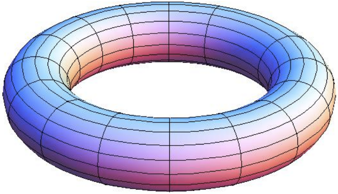
Counting the Number of Tilings of a Topological Surface

Question

How many different ways are there to tile a compact topological surface of genus g by n distinct (i.e., colored) tiles?

$$n = \text{ the number of tiles } = \text{ the number of colors.}$$





A **tiling** Γ of a surface has many different names: Map, Cell-Decomposition, Feynman Diagram, Ribbon Graph, Strebel Differential, Grothendieck's Dessin d'Enfant....

$$\begin{cases} v = & \text{the number of vertices of } \Gamma \\ e = & \text{the number of edges of } \Gamma \end{cases}$$

Euler's Formula

$$v - e + n = 2 - 2g$$

Let

$$Z(g, n, e) = \text{the number of tilings with } e \text{ edges.}$$

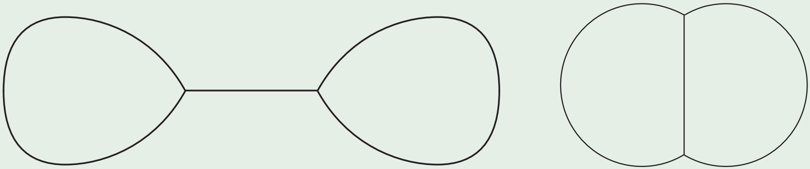
The number of edges is the least when $v = 1$, and the largest when Γ is trivalent ($2e = 3v$).

$$2g - 2 + n + 1 \leq e \leq 2(3g - 3 + n) + n$$

The simplest cases are $g = 0, n = 3$ and $g = n = 1$.

$g = 0, n = 3, e = 3$

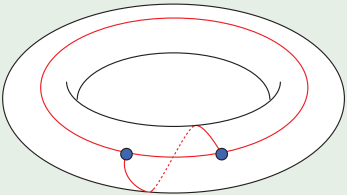
$Z(0,3,3) = 3 + 1 = 4.$

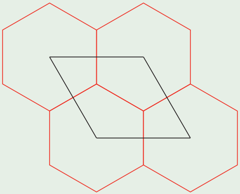


If a homeomorphism of the surface changes one tiling to another, then we identify them.

$g = n = 1, e = 3$

$Z(1,1,3) = \frac{1}{6}$.





If a tiling has an automorphism group G, then this tiling contributes $1/|G|$ to the total number.

Some Calculations

$$Z(0,4,6) = 64$$
$$Z(0,4,5) = 144$$
$$Z(0,4,4) = 99$$
$$Z(0,4,3) = 20$$

$$Z(3,1,15) = \frac{5005}{3}$$
$$Z(3,1,14) = \frac{25025}{2}$$
$$Z(3,1,13) = 41118$$

$$Z(3,1,12) = \frac{929929}{12}$$
$$Z(3,1,11) = \frac{183955}{2}$$
$$Z(3,1,10) = \frac{283767}{4}$$
$$Z(3,1,9) = \frac{317735}{9}$$
$$Z(3,1,8) = 10813$$
$$Z(3,1,7) = \frac{25443}{7}$$
$$Z(3,1,6) = \frac{445}{4}$$

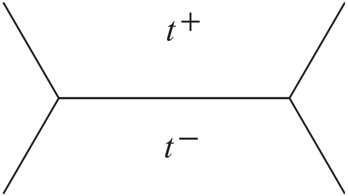
The Counting Formula

Provide a variable t_i for each tile i and define

$$Z(t_i, t_j) = \frac{(t_i + 1)(t_j + 1)}{2(t_i + t_j)},$$

and the **Free Energy**

$$F_{g,n}(t_1, \ldots, t_n) = \sum_{\Gamma \text{ tiling}} \frac{(-1)^{e(\Gamma)}}{|\mathrm{Aut}(\Gamma)|} \prod_{\eta \in \Gamma} z(t_\eta^+, t_\eta^-).$$



Remark-Why the Free Energy important?

One reason is because it is related to the Euler characteristic of the moduli space of pointed algebraic curves. Define

$$z = z(t,t) = \frac{(t+1)^2}{4t}.$$

Then

$$F_{g,n}(t,t,\ldots,t) = \sum_{\substack{\Gamma \text{ tiling of} \\ \text{type } (g,n)}} \frac{(-1)^{e(\Gamma)}}{|\mathrm{Aut}(\Gamma)|} \, z^{e(\Gamma)}.$$

In particular, a special value of the Free Energy is

$$\begin{aligned} F_{g,n}(1,1,\ldots,1) &= (-1)^n \chi\left(\mathcal{M}_{g,n}\right) \\ &= -\frac{(2g-2+n-1)!}{(2g-2)!} \cdot \zeta(1-2g), \end{aligned}$$

Where $\zeta(s)$ is the Riemann zeta function.

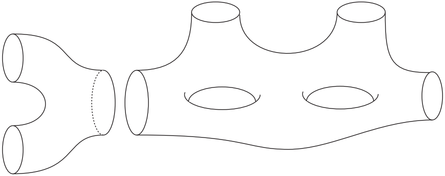
Theorem 1. Topological Recursion Formula (M-Penkava)

There is an effective recursion in terms of $2g-2+n$.

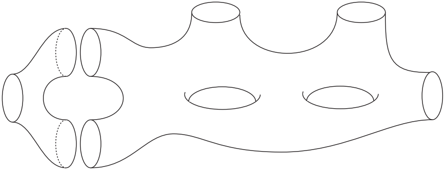
$$\begin{aligned}
F_{g,n}(t_N) = -\frac{1}{16}\int_{-1}^{t_1}\Bigg[\sum_{j=2}^{n}\frac{t_j}{t^2-t_j^2}\Bigg(&\frac{(t^2-1)^3}{t^2}\frac{\partial}{\partial t}F_{g,n-1}(t,t_{N\setminus\{1,j\}})\\
&-\frac{(t_j^2-1)^3}{t_j^2}\frac{\partial}{\partial t_j}F_{g,n-1}(t_{N\setminus\{1\}})\Bigg)\\
&+\sum_{j=2}^{n}\frac{(t^2-1)^2}{t^2}\frac{\partial}{\partial t}F_{g,n-1}(t,t_{N\setminus\{1,j\}})\\
&+\frac{1}{2}\,\frac{(t^2-1)^3}{t^2}\frac{\partial^2}{\partial u_1\partial u_2}\Bigg(F_{g-1,n+1}(u_1,u_2,t_{N\setminus\{1\}})\\
&+\sum_{\substack{g_1+g_2=g\\ I\sqcup J=N\setminus\{1\}}}^{\text{stable}} F_{g_1,|I|+1}(u_1,t_I)F_{g_2,|J|+1}(u_2,t_J)\Bigg)\Bigg|_{u_1=u_2=t}\Bigg]\,dt.
\end{aligned}$$

Topological Recursion = Removing a Pair of Pants

$$(g, n) \Longrightarrow (g, n-1)$$

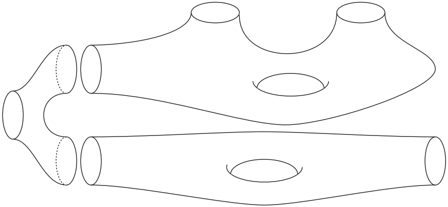


$$(g, n) \Longrightarrow (g-1, n+1)$$



$$(g, n) \Longrightarrow (g_1, n_1) + (g_2, n_2)$$

$$\begin{cases} g = g_1 + g_2 \\ n = n_1 + n_2 - 1 \end{cases}$$



Theorem 2. Laurent Polynomial – Unexpected!

$F_{g,n}(t_1, \dots, t_n)$ is a Laurent polynomial, and its leading terms are given by

$$F^{\mathrm{top}}_{g,n}(t_N) = \sum_{\substack{\Gamma \text{ trivalent tiling} \\ \text{of type } (g,n)}} \frac{(-1)^{e(\Gamma)}}{|\mathrm{Aut}(\Gamma)|} \prod_{\eta \in \Gamma} \frac{t_\eta^+ t_\eta^-}{2\left(t_\eta^+ + t_\eta^-\right)}$$

$$= \frac{(-1)^n}{2^{5g-5+2n}} \sum_{\substack{d_1+\cdots+d_n \\ =3g-3+n}} \langle \tau_{d_1} \cdots \tau_{d_n} \rangle_{g,n} \prod_{j=1}^{n} \frac{(2d_j)!}{d_j!} \left(\frac{t_j}{2}\right)^{2d_j+1} .$$

This formula is identical to the Kontsevich's "Boxed Formula" of his paper on the Witten Conjecture. Indeed, our recursion formula restricts to the top degree terms and recovers the Witten-Kontsevich theory, i.e., the Virasoro Constraint Condition on $\overline{\mathcal{M}}_{g,n}$.

Theorem 3. The Laplace Transform

- $F_{g,n}(t_1, \ldots, t_n)$ is the Laplace transform of the number of lattice points in the canonical orbi-polytope decomposition of the moduli space

$$\mathcal{M}_{g,n} \times \mathbb{R}_+^n \cong \coprod_{\substack{\Gamma \text{ Tiling of} \\ \text{type } (g,n)}} \frac{\mathbb{R}_+^{e(\Gamma)}}{\mathrm{Aut}(\Gamma)}$$

  due to Harer, Mumford, Strebel, and Thurston.
- The variable t_j is the Laplace dual coordinate of the perimeter length of the j-th tile.
- Our recursion formula is the Laplace transform of the combinatorial formula that corresponds to the **edge-removal** operation of a tiling.

The Combinatorial Part of the Theory

Let Γ be a tiling of a compact Riemann surface of genus g with n colored tiles indexed by the set $N = \{1, 2, \ldots, n\}$, and $e = e(\Gamma)$ labeled edges indexed by $E = \{1, 2, \ldots, e\}$. Define

$$a_{i\eta} = \text{the number of times Edge } \eta \text{ appears in Face } i,$$

and the $N \times E$ **incidence matrix** of the tiling

$$A_\Gamma = [a_{i\eta}]_{i\in N, \eta\in E}.$$

The number of lattice points in question is

$$N_{g,n}(\mathbf{p}) = \sum_{\Gamma \text{ tiling of type } (g,n)} \frac{\left|\{\mathbf{x} \in \mathbb{Z}_+^{e(\Gamma)} \mid A_\Gamma \mathbf{x} = \mathbf{p}\}\right|}{|\mathrm{Aut}(\Gamma)|}$$

for each collection $\mathbf{p} \in \mathbb{Z}_+^n$ of the perimeter length of tiles.

Theorem: The Edge-Removal Formula (C-M-S)

$$
\begin{aligned}
p_1 N_{g,n}(p_N) = \frac{1}{2}\sum_{j=2}^{n} \Bigg[&\sum_{q=0}^{p_1+p_j} q(p_1+p_j-q) N_{g,n-1}(q, p_{N\setminus\{1,j\}}) \\
&+ H(p_1-p_j) \sum_{q=0}^{p_1-p_j} q(p_1-p_j-q) N_{g,n-1}(q, p_{N\setminus\{1,j\}}) \\
&- H(p_j-p_1) \sum_{q=0}^{p_j-p_1} q(p_j-p_1-q) N_{g,n-1}(q, p_{N\setminus\{1,j\}}) \Bigg] \\
+ \frac{1}{2} \sum_{0\le q_1+q_2\le p_1} q_1 q_2 (p_1-q_1-q_2) \Bigg[&N_{g-1,n+1}(q_1, q_2, p_{N\setminus\{1\}}) \\
&+ \sum_{\substack{g_1+g_2=g \\ I\sqcup J = N\setminus\{1\}}}^{\text{stable}} N_{g_1,|I|+1}(q_1, p_I) N_{g_2,|J|+1}(q_2, p_J) \Bigg].
\end{aligned}
$$

Here

$$H(x) = \begin{cases} 1 & x > 0 \\ 0 & x \leq 0 \end{cases}$$

is the Heaviside function.

Remark

The counting function $N_{g,n}(\mathbf{p})$ is a complicated piece-wise polynomial function. For example, if $p_1 + p_2 + \cdots + p_n$ is odd, then $N_{g,n}(\mathbf{p}) = 0$.

The Free Energy

Compared to $N_{g,n}(\mathbf{p})$, the Free Energy $F_{g,n}(t_N)$ is a much nicer, and easy to deal-with, function, which is indeed a Laurent polynomial.

Combinatorics vs. the Free Energy

The interesting situation we have here, however, is that it is easy to prove the combinatorial recursion formula using the edge-removal operation, but there is no direct proof of the topological recursion for the Free Energy. We are able to prove it only by using the **Laplace transform** of the combinatorial formula.

The Mystery

This is just the tip of the iceberg.

Our recursion formula is an example of the universal recursion formula discovered by physicists Eynard and Orantin, based on a plane curve

$$x = y + \frac{1}{y}.$$

The Theme

The Laplace transform changes a combinatorial problem to an equation in complex analysis. The Free Energy $F_{g,n}$ appears there as a holomorphic function, defined on the n-symmetric product of a particular Riemann surface. The Free Energies satisfy a universal recursion relation on $2g - 2 + n$ in terms of the residue calculus.

Topological Recursion of Eynard and Orantin, 2007

The Input

- A plane analytic curve

$$C = \{(x, y) \in \mathbb{C}^2 \mid f(x, y) = 0\}$$

  (erroneously???) called the **Spectral Curve** of the theory, due to the influence of Krichever.
- Fay's Fundamental Form of the Second Kind, or simply the Cauchy Differentiation Kernel on C^2

$$W_{0,2}(t_1, t_2) = \frac{dt_1 \otimes dt_2}{(t_1 - t_2)^2} + \text{holomorphic.}$$

The Output

An infinite sequence of meromorphic symmetric differentials of degree n

$$W_{g,n}(t_1,\ldots,t_n) \quad \text{defined on} \quad C^n,$$

$g = 0,1,2,3\ldots$, and $n = 0,1,2,3,\ldots$.

and

the Free Energies $F_{g,n}(t_1,\ldots,t_n)$ that are defined by

$$d^{\otimes n}F_{g,n}(t_1,\ldots,t_n) = W_{g,n}(t_1,\ldots,t_n).$$

These are the *Primitives* in the sense of Kyoji Saito, and are conjectured to be related to a nonlinear integrable system of the KdV/KP type.

- $\Lambda^1(C)$ = sheaf of meromorphic 1-forms on C
- $H^n = H^0\big(C^n, \mathrm{Sym}^n(\Lambda^1(C))\big)$ = the space of meromorphic symmetric differentials of degree n.

A bilinear operator

$$K : H \otimes H \longrightarrow H$$

naturally extends to

$$K : H^{n_1+1} \otimes H^{n_2+1} \ni V(t_0, t_1, \dots, t_{n_1}) \otimes W(s_0, s_1, \dots, s_{n_2})$$
$$\longmapsto K_{(t_0,s_0)} V(t_0, t_1, \dots, t_{n_1}) \otimes W(s_0, s_1, \dots, s_{n_2}) \in H^{n_1+n_2+1}$$

$$K : H^{n+1} \ni W(t_0, t_1, \dots, t_n) \longmapsto K_{(t_0,t_1)} W(t_0, t_1, \dots, t_n) \in H^n.$$

The Eynard Kernel $K : H \otimes H \longrightarrow H$ for C

$$
\begin{aligned}
&K\big(f_1(t_1)dt_1, f_2(t_2)dt_2\big) \\
&\quad = \frac{1}{2\pi i} \sum_{\lambda=1}^{r} \oint_{|t-a_\lambda|<\epsilon} K_\lambda(t,t_1)\Big(f_1(t)dt \otimes f_2\big(s_\lambda(t)\big)ds_\lambda(t) \\
&\qquad\qquad + f_2(t)dt \otimes f_1\big(s_\lambda(t)\big)ds_\lambda(t)\Big),
\end{aligned}
$$

where

$$
K_\lambda(t,t_1) = \frac{1}{2}\left(\int_t^{s_\lambda(t)} W_{0,2}(t,t_1)\ dt\right) \otimes dt_1 \cdot \frac{1}{\Big(y(t) - y\big(s_\lambda(t)\big)\Big)dx(t)},
$$

$a_1, \ldots, a_r =$ the simple ramification points of the x-projection,
$s_\lambda =$ local deck transformation around a_λ.

The Eynared-Orantin Topological Recursion

The Topological Recursion

$$W_{g,n} = K(W_{g,n-1}, W_{0,2}) + K(W_{g-1,n+1}) + \sum_{\substack{g_1+g_2=g \\ n_1+n_2=n-1}}^{\text{stabel}} K(W_{g_1,n_1+1}, W_{g_2,n_2+1})$$

The topological Recursion determines $W_{g,n}(z_1, \ldots, z_n)$ recursively w.r.t. $2g-2+n$ from $W_{g,n-1}$, $W_{g-1,n+1}$, and pairs $(W_{g_1,n_1}, W_{g_2,n_2})$ such that $g = g_1 + g_2$, $n = n_1 + n_2 - 1$, by an integral transform using the kernel function defined on C.

Where does it come from?

The topological recursion comes from statistical mechanics and Random Matrix Theory / Matrix Models.

Define the partition function by

$$Z = \int_{\mathcal{H}_{N\times N}} \exp\big(-N \,\mathrm{trace}\, V(M)\big)\, dM,$$

where $\mathcal{H}_{N\times N}$ is a real N^2-dimensional space of $N \times N$ matrices with a measure dM (such as the space of Hermitian matrices), and $V(x)$ is a function in one variable that defines the potential function of the theory. Let us denote

$$\langle F(M)\rangle = \int_{\mathcal{H}_{N\times N}} F(M) \exp\big(-N \,\mathrm{trace}\, V(M)\big)\, dM.$$

Then the correlation function of resolvents

$$W_n(x_1, \cdots, x_n) = \left\langle \mathrm{tr}\left(\frac{1}{x_1 - M}\right) \cdots \mathrm{tr}\left(\frac{1}{x_n - M}\right) \right\rangle_{\text{cumulant}}$$

has a large N expansion

$$W_n(x_1, \ldots, x_n) = \sum_{g \geq 0} N^{2-2g-n} W_{g,n}(x_1, \ldots, x_n).$$

The coefficients $W_{g,n}$ satisfy the topological recursion with the *resolvent curve* C as the input, that is defined as the Riemann surface of the analytic function

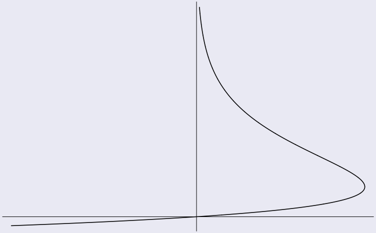
$$W_1(x) = \left\langle \mathrm{tr}\left(\frac{1}{x - M}\right) \right\rangle.$$

Thus if we know C, then we can calculate $Z = W_0$ and all W_n's.

Mystery #1

Bouchard-Mariño Conjecture, September 2007

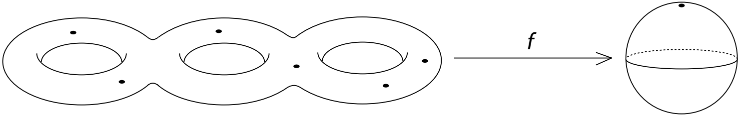
- Take the plane curve $x = ye^{-y}$ as an input.



- Then $W_{g,n}(t_1, \ldots, t_n)$ is the generating function of *Hurwitz* numbers of genus g and an arbitrary *profile* μ given by a partition of length n.

A Big Surprise

A *Hurwitz* cover is a meromorphic function $f : \Sigma \longrightarrow \mathbb{P}^1$ on a Riemann surface Σ of genus g with a prescribed order μ_i at each pole x_i. The Hurwitz number counts such covers.



The Topological Recursion for $x = ye^{-y}$ gives a *Previously Unknown* effective method of counting Hurwitz numbers

$$F_{g,n}(t_1, \ldots, t_n) = \sum_{n_1,\ldots,n_\ell} \langle \tau_{n_1} \cdots \tau_{n_\ell} \Lambda_g^\vee(1) \rangle_{g,\ell} \prod_{i=1}^{\ell} \hat{\xi}_{n_i}(t_i),$$

where $\Lambda_g^\vee(1) = 1 - \lambda_1 + \cdots + (-1)^g \lambda_g$, and $\hat{\xi}_n(t)$ are certain holomorphic functions.

Mystery Continues

Geometric Setting

- (X, ω) = Toric Calabi-Yau 3-fold as a Kähler manifold.
- $\widetilde{X}$ = Mirror dual of X.
- $\widetilde{X}$ as a complex manifold is a conic fibration on $\mathbb{C}^2$

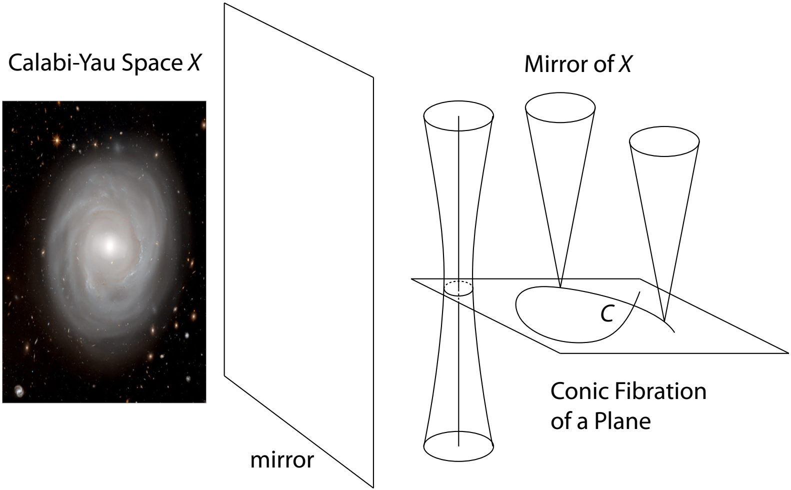
$$\widetilde{X} = \left\{(x, y, u, v) \in \mathbb{C}^4 \;\middle|\; uv = f(x, y)\right\},$$

which degenerates on a plane curve $C \subset \mathbb{C}^2$

$$C = \left\{(x, y) \in \mathbb{C}^2 \;\middle|\; 0 = f(x, y)\right\}$$

called the *Mirror Curve*.

Mirror Symmetry of Toric Calabi-Yau Geometry



Mystery #2

Remodeling Conjecture, Bouchard-Klemm-Mariño-Pasquetti, September 2007

- Take a toric Calabi-Yau 3-fold (X, ω) as an input.
- Find its mirror dual $\widetilde{X}$, and identify the mirror curve C.

Then

The Open and Closed Gromov-Witten Invariants of X

$$||$$

The Eynard-Orantin Invariants $W_{g,n}$ and the Free Energies $F_{g,n}$ defined on the Mirror Curve C

The Discovery

The Bouchard-Mariño Conjecture was Solved in Fall 2009!

1. Borot-Eynard-M-Safnuk: Found a matrix integral expression for Hurwitz numbers.
2. Eynard-M-Safnuk: The topological recursion is the Laplace transform of a combinatorial equation known as the cut-and-join equation in representation theory.
3. M-Zhang: Found that $F_{g,n}(t_1, \dots, t_n)$ are the Laplace transform of Hurwitz numbers, and are indeed *polynomials* in the coordinate of the spectral curve $x = ye^{-y}$. As corollaries, we obtained an extremely simple proof of the Witten conjecture and the λ_g-conjecture in one stroke.

Further Developments in 2010

1. Zhou and Chen (independently): Using the method of Eynard-M-Safnuk, proved the BKMP Remodeling Conjecture for $X = \mathbb{C}^3$.
2. Zhu: Based on M-Zhang, discovered that $F_{g,n}$ for $X = \mathbb{C}^3$ is again a polynomial in the coordinate of the mirror curve C, and obtained new results on the tautological intersection numbers on $\overline{\mathcal{M}}_{g,n}$.
3. Ooguri-Sulkowski-Yamazaki: Discovered a new matrix model for the Donaldson-Thomas invariants (counting BPS states) of Calabi-Yau spaces, and gave a physical evidence for the *general* case of the BKMP Remodeling Conjecture.

Mystery Continues! Dijkgraaf-Fuji-Manabe

Topological Recursion in Knot Theory and the Volume Conjecture

- The Eynard-Orantin recursion computs knot invariants that have the same information of the colored Jones polynomials $J_N(K, q)$ for all values of the dimension N of irreducible representations of $G = SL(2, \mathbb{C})$.
- The spectral curve is the subvariety of the character variety $\mathrm{Hom}(\pi_1(T^2), G)/G = \mathbb{C}^* \times \mathbb{C}^*$ consisting of the flat connections that extend to flat connections on the 3-fold obtained by removing a tubular neighborhood of the knot K from S^3.
- $F_{0,1}$ is essentially the hyperbolic volume of the knot complement $\mathrm{vol}(S^3 \setminus K)$.
- $F_{0,2}$ gives the Reidemeister torsion.

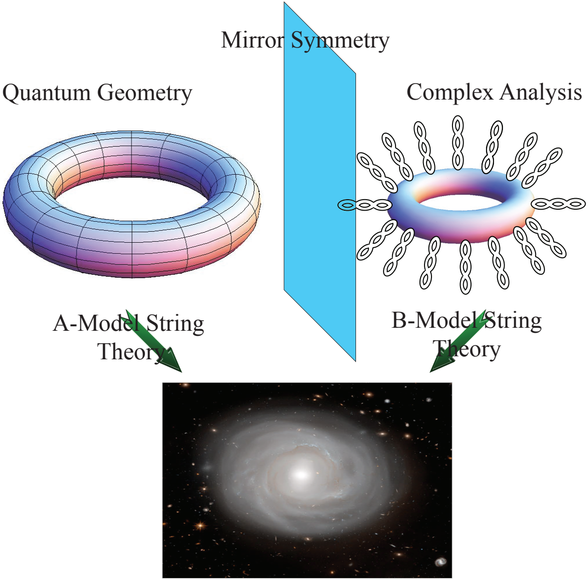
Dijkgraaf-Fuji-Manabe related the Eynard-Orantin theory and the Kashaev-Murakami-Murakami Volume Conjecture.

The Volume Conjecture

$$\frac{1}{\pi}\lim_{N\to\infty}\frac{\left|\log J_N(K, e^{\frac{2\pi i}{N}})\right|}{N} = \mathrm{vol}(S^3 \setminus K).$$

A similar theory exists for torus knots. The Eynard-M-Safnuk theorem appears as the special case of unknot!

Mirror Symmetry of String Theories



Mirror Symmetry

Quantum Geometry

- Mirzakhani recursion for the Weil-Petersson volume of the moduli of bordered hyperbolic surfaces
- Lattice point counting in $\mathcal{M}_{g,n} \times \mathbb{R}_+^n$
- Simple Hurwitz numbers and the cut-and-join equation
- Gromov-Witten Invariants of toric Calabi-Yau 3-folds
- Quantum knot invariants

Complex Analysis

- Intersection theory on $\overline{\mathcal{M}}_{g,n}$
- Residue calculus on the "Spectral Curve" and the Eynard-Orantin recursion
- Nonlinear Integrable PDEs of the KdV and KP type, and Frobenius manifolds
- Donaldson-Thomas Invariants
- Volume of the knot complements, Reidemeister torsion, ...

Pierre-Simon Laplace (1749 - 1827)

What is Mirror Symmetry?

In Terms of Mathematics

Mirror Symmetry
|| ???
Laplace Transform

The Laplace transform changes mathematics of counting to Complex Analysis.

Laplace Transform: Gauss $\longrightarrow$ Riemann

Thank you very much for your attention!