Quantization of Spectral Curves of Higgs Bundles via a B-model Topological Recursion

Lectures based on joint work with
Olivia Dumitsescu – arriving next week
(Leibniz Universität Hannover)

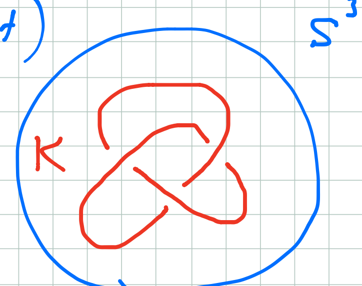
Lecture I: A Story of Quantum Curves

- Let me begin by thanking organizers Graeme and Richard for giving us the opportunity to speak in the Summer School.
- The purpose of the series of lectures is to present a mathematical theory of "Quantum Curves" (Dijkgraaf – Hollands – Sułkowski 2009) in particular, an algebro-geometric construction of quantum curves.

§1. Motivation – Quantum Knot Invariants

2003 Aganagic – Dijkgraaf – Klemm – Mariño – Vafa
2004 Garoufalidis (topologist)
2011 Gukov – Sułkowski

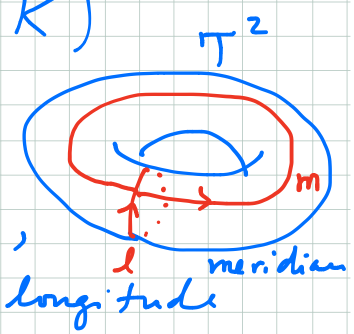
- Speculation / Conjectures



Consider a knot K in S^3:

$M = S^3 \setminus$ (tubular neighborhood of K)

$\partial M = T^2 =$ torus $\subset M$



Since $\pi_1(T^2) \longrightarrow \pi_1(S^3 \setminus K)$,

$$\pi_1(T^2) = \mathbb{Z}^2$$

We have $\mathrm{Hom}(\pi_1(S^3\setminus K), SL_2(\mathbb{C}))/\!/SL_2(\mathbb{C}) = X$

$$\big\downarrow$$

$$\mathrm{Hom}(\pi_1(T^2), SL_2(\mathbb{C}))/\!/SL_2(\mathbb{C})$$

$$\parallel \qquad \text{pair of commuting matrices up to conjugation}$$

$$(\mathbb{C}^*)^2 \longrightarrow (\mathbb{C}^*)^2/(\mathbb{Z}/2\mathbb{Z})$$

$$\begin{array}{ccc} & \cap & \qquad\qquad \cup\!\!\shortmid \\ (x,y) \in & \mathbb{C}^2 & \qquad \left(\begin{bmatrix}\lambda & \\ & \lambda^{-1}\end{bmatrix}, \begin{bmatrix}\mu & \\ & \mu^{-1}\end{bmatrix}\right), \ \lambda,\mu \in \mathbb{C}^* \\ & \cap & \qquad \lambda \leftrightarrow \lambda^{-1}, \ \mu \leftrightarrow \mu^{-1} \ \ \mathbb{Z}/2\mathbb{Z}\text{-action} \\ & \mathbb{P}^2 & \end{array}$$

- $\Sigma \subset \mathbb{C}^2$ closure of all 1-dimensional components of the character variety X in $\mathbb{C}^2$.

<u>Fact</u>: ① $\Sigma = \{(x,y) \in \mathbb{C}^2 \mid \exists A_K(x,y) = 0\}$,

Cooper
Culler
Gillet
Long
Shalen

$A_K(x,y) \in \mathbb{Z}[x,y]$. (A-polynomial)

② $\mathbb{C}(\bar{\Sigma})$ = function field of $\Sigma \subset \bar{\Sigma} \subset \mathbb{P}^2$.

$\Rightarrow K_2(\mathbb{C}(\bar{\Sigma}))$ is a torsion group.

<u>Conjecture</u> : There is a <u>quantization</u> $\hat{A}_K(x, y)$ such that

$$\hat{A}_K\left(e^u, e^{\hbar \frac{d}{du}}\right) J_K(q; N) = 0.$$

Here, $J_K(q; N)$ is the colored Jones polynomial associated with the irreducible representation of $SL_2(\mathbb{C})$ of dimension N, and

$$\begin{cases} \hbar = \frac{1}{N} \\ q = e^{\hbar u}. \end{cases}$$

related to the fact that $K_2(\mathbb{C}(\bar{\Sigma}))$ is torsion.

- The difference operator $\hat{A}_K$ is an example of a "quantum curve." Its existence is essentially
- Although for some simple knots the conjecture is verified, our understanding is far from complete.
- <u>Question</u> : Can't we construct a mathematical theory, for which all features of the speculations are rigorously proven?

It turns out that the Hitchin formulation is most suitable for the mathematical theory! [DM 2014]

§2. The Simplest Example

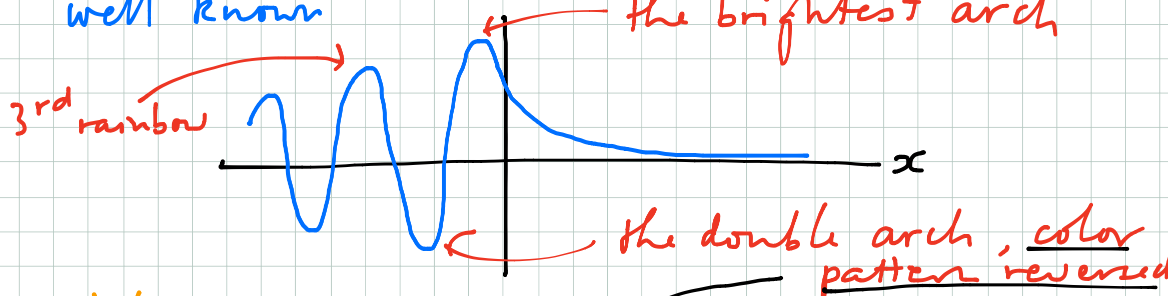
- Mathematicians often keep their childhood dreams for a long time.
- When you saw a perfect rainbow in your childhood, you wondered what awaits you

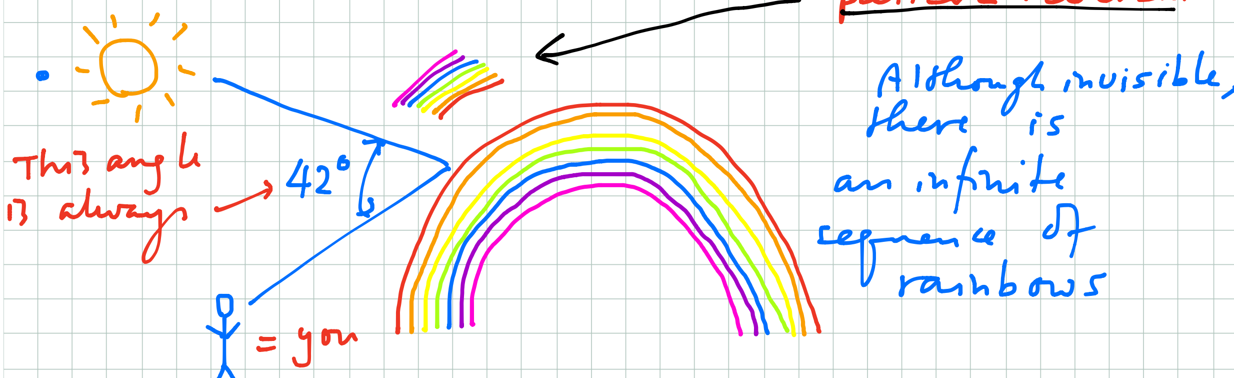
On the Other Side of the Rainbow

- Sir George Biddel Airy deviced the rainbow integral, in his attempt to explain the diffraction mechanism of a rainbow.

$$Ai(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ipx}\, e^{i\frac{p^3}{3}}\, dp .$$

It is a real valued function, and its shape is well known





- For $x > 0$, the Airy function exponentially decays.

$$Ai(x) \sim \frac{1}{2\sqrt{\pi}}\,\frac{1}{x^{1/4}}\, e^{-\frac{2}{3}x^{\frac{3}{2}}} .$$

Therefore, you see no rainbows below the brightest arch.

- Is there really nothing on the other side of the rainbow?
- A better asymptotics:

What are these rational numbers?

$$Ai(x) = \frac{1}{2\sqrt{\pi}} \exp\left(-\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{4}\log x - \frac{5}{48}x^{-\frac{3}{2}} + \frac{5}{64}x^{-3} - \cdots\right)$$

- Actually, there is a closed formula for the exact asymptotics:

$$Ai(x) = \frac{1}{2\sqrt{\pi}} \exp\left(\sum_{m=0}^{\infty} S_m(x)\right),$$

$$\begin{cases} S_0(x) = -\frac{2}{3} x^{\frac{3}{2}} \\ S_1(x) = -\frac{1}{4}\log x \end{cases}$$

$m \geq 2 \Rightarrow$

$$S_m(x) = \sum_{2g-2+n=m-1} \frac{1}{n!} F_{g,n}(x),$$

$$F_{g,n}(x) = \frac{(-1)^n}{2^{2g-2+n}} x^{-\frac{6g-6+3n}{2}} \sum_{\substack{d_1+\cdots+d_n \\ =3g-3+n}} \prod_{i=1}^{n} (2d_i-1)!!$$

$$\times \langle \tau_{d_1}\tau_{d_2}\cdots\tau_{d_n}\rangle_{g,n},$$

where $\langle \tau_{d_1}\cdots\tau_{d_n}\rangle_{g,n} = \int_{\overline{\mathcal{M}}_{g,n}} c_1(\mathbb{L}_1)^{d_1}\cdots c_1(\mathbb{L}_n)^{d_n},$

$$\begin{array}{ccc} \mathbb{L}_i & \supset & T^*_{p_i}C \\ \downarrow & & \\ \overline{\mathcal{M}}_{g,n} & \ni & (C,(p_1,\cdots,p_n)). \end{array}$$

- The Airy function, i.e., the rainbow, knows the intersection numbers of the moduli space of stable curves! (Actually, $S_m(x)$ does NOT recover $F_{g,n}$. But still we know $F_{g,n}$ from the Airy!)
- Where is the Higgs bundle? This summer school is all about Higgs bundles and Hitchin fibrations. (I'm one of the few who are NOT experts! :)) explained later!
- The base curve $C = \mathbb{P}^1$.

  The vector bundle $E = K_{\mathbb{P}^1}^{-1} \oplus \mathcal{O}_{\mathbb{P}^1}$

  The Higgs field $\phi = \begin{bmatrix} & 1 \\ x(dx)^2 & \end{bmatrix}$.

$$\phi : K^{-1} \oplus \mathcal{O} \longrightarrow \mathcal{O} \oplus K \otimes \mathcal{O}(4)$$

with $1 : \mathcal{O} \to \mathcal{O}$ and $x(dx)^2 : K^{-1} \to K$,

$$x(dx)^2 \in H^0(\mathbb{P}^1, K^2 \otimes \underbrace{\mathcal{O}(-1\cdot \bar{0}) \otimes \mathcal{O}(5 \cdot \bar{\infty})}_{= \mathcal{O}_{\mathbb{P}^1}}) \cong \mathbb{C}.$$

The unique object (Natural)

- Since ϕ is singular at $x = \infty$, we need to consider the compactified cotangent bundle

$$\overline{T^*\mathbb{P}^1} := \mathbb{P}(K_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}) = \mathbb{F}_2$$

  - Kontsevich–Soibelman 2013
  - Dumitrescu–M

  explained by David on Monday → Hirzebruch surface

- The spectral curve Σ is defined by

$$\det(y\,dx - \phi) = (y^2 - x)(dx)^2 = 0,$$

$$\boxed{y^2 - x = 0} \quad \text{in } \mathbb{F}_2 .$$

- Here $y\,dx$ = tautological 1-form on $T^*\mathbb{P}^1$, so that $-d(y\,dx) = dx \wedge dy$ is the symplectic form on $T^*\mathbb{P}^1$.
- $y\,dx$ extends as a meromorphic 1-form on $\mathbb{F}_2$.
- Quantum curve :

$$\boxed{\left[\left(\hbar \frac{d}{dx}\right)^2 - x\right] \psi(x,\hbar) = 0 .}$$

$$\psi(x,\hbar) = \hbar^{-1/6} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i \frac{p}{\hbar^{2/3}} x} e^{i \frac{p^3}{3}} dp$$

Solution that decays exponentially for $x \to \infty$, $x > 0$, $\hbar > 0$.

- <u>Semi-classical limit</u>

We wish to solve the equation $\left[\left(\hbar \frac{d}{dx}\right)^2 - x\right] \psi(x,\hbar) = 0$ by the singular perturbation method, or the Wentzel - Kramers - Brillouin (WKB) method.

$$\psi(x,\hbar) = \exp\left(\sum_{m=0}^{\infty} \hbar^{m-1} S_m(x) \right) .$$

We can inductively solve for S_m one by one.

The "classical limit" $\hbar \to 0$ does not make sense.

- But the following equation makes sense :

$$0 = e^{-\frac{1}{\hbar}S_0(x)}\left[\left(\hbar\frac{d}{dx}\right)^2 - x\right]e^{\frac{1}{\hbar}S_0(x)}\exp\Big(\sum_{m=1}^{\infty}\hbar^{m-1}S_m(x)\Big)$$

only for $m>0$

$$= \left[\left(\hbar\frac{d}{dx}\right)^2 - x + 2\hbar S_0'(x)\frac{d}{dx} + \hbar S_0''(x) + (S_0'(x))^2\right]$$

$$\cdot \exp\Big(\sum_{m=1}^{\infty}\hbar^{m-1}S_m(x)\Big)$$

$$\xrightarrow{\hbar\to 0} \left[(S_0'(x))^2 - x\right]\cdot 1 .$$

$$\therefore\ (S_0'(x))^2 - x = 0 \Leftarrow S_0(x) = \pm\frac{2}{3}x^{\frac{3}{2}} \quad !$$

- For the solution that exponentially decays, we choose $S_0(x) = -\frac{2}{3}x^{\frac{3}{2}}$.
- $dS_0(x) = S_0'(x)\,dx$ is a section of $T^*\mathbb{P}^1$. Thus $y = S_0'(x)$ is the fiber coordinate of $T_x^*\mathbb{P}^1$.

$\therefore$ We recover $\boxed{y^2 - x = 0}$, The spectral curve!

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§3. What do we learn from this example?

Higgs Bundle	Spectral Curve	Quantum Curve
$E = K_{\mathbb{P}^1}^{-1}\oplus\mathcal{O}_{\mathbb{P}^1}$	$y^2 - x = 0$	$\left[\left(\hbar\frac{d}{dx}\right)^2 - x\right]\psi(x,\hbar) = 0$
$\downarrow$	$\cap$	① A particular holomorphic solution is constructed by Topological Recursion.
$C = \mathbb{P}^1$	$T^*\mathbb{P}^1$	
$\phi = \begin{bmatrix} & 1 \\ x(dx)^2 & \end{bmatrix}$	$\parallel$ $\mathbb{P}(K_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1})$	

③ The QC equation knows everything about the ψ-class intersection #s on $\overline{\mathcal{M}}_{g,n}$!

② At $\hbar = 1$, the Q.C. gives the non-Abelian Hodge correspondence of ϕ!

④ At $\hbar = 1$, $L = \left(\frac{d}{dx}\right)^2 - x$ is known as the <u>Lax operator</u> for the Korteweg-de Vries (KdV) equation. There is <u>the</u> unique solution to the "KdV hierarchy" $L(p) = \left(\frac{d}{dx}\right)^2 - u(x;p)$, $p = (p_1, p_3, p_5, \cdots)$, $u(x;0) = x$, such that (Witten 1991, Kontsevich 1992)

$$F(x;p) = \left\langle \exp\left(\sum_{j=0}^{\infty} t_j \tau_j \right) \right\rangle, \quad x = t_0$$

$$u(x;p) = \frac{\partial^2 F}{\partial x^2}$$

$$p_{2n+1} = \frac{t_n}{(2n-1)!!}.$$

KdV evolution variables, or the power-sum symmetric functions

- We want to construct a general mathematical theory!

- Warning: <u>There is NO general mathematical theory, yet</u> !!!

You are the ones to find it :)

- What are we expecting?

① From the physics side, the Seiberg-Witten theory with $SU(2)$ gauge group is directly coming into the picture.

Spectral curve = Seiberg-Witten curve

Base curve C = "Gaiotto curve"

② On the mathematics side,

(i) Gromov-Witten invariants of some "spaces," and

(ii) Topology of 4-dimensional instanton moduli spaces.

are behind the scene.

③ The Topological Recursion connects the Higgs/Hitchin side and the GW/SW side.

- Topological Recursion
  - Eynard - Orantin 2007 (Eynard is an ICM 14 speaker.)
  - A mathematical framework, using Higgs bundles and Hitchin spectral curves, is discovered in Dumitrescu - M. 2014 (LMP 2014).

    This also gives the first mathematical construction of quantum curves for $SL_2(\mathbb{C})$ Higgs bundles.

④ In our previous work (DM 2014), we assumed that the spectral curve Σ is non-singular.

- To include most interesting examples, however, we need to generalize the theory to singular spectral curves.
- This is what Olivia and I have recently accomplished, for the case of GL_2-meromorphic Higgs bundles.
- Now we can present our theory, using attractive examples!

§4. Plan of the Series of Lectures

Lecture II : A mathematical theory of Topological Recursion and Quantum Curves for $\boxed{\text{Catalan Numbers}}$ Joint w/ Olivia and Piotr Sułkowski

Next week (Mostly by Olivia Dumitrescu)

Lecture III : A general theory for quantization of a GL_2 - Higgs bundle

1) defined on a smooth projective algebraic curve of an arbitrary genus, and
2) with an arbitrary meromorphic Higgs field. (in particular, spectral curve is singular.)

Lecture IV : Birational geometry of the compactified cotangent bundle

$$\overline{T^*C} = \mathbb{P}(K_C \oplus \mathcal{O}_C)$$

and desingularization of spectral curves

Lecture V : The construction of Quantum Curves in terms of Topological Recursion