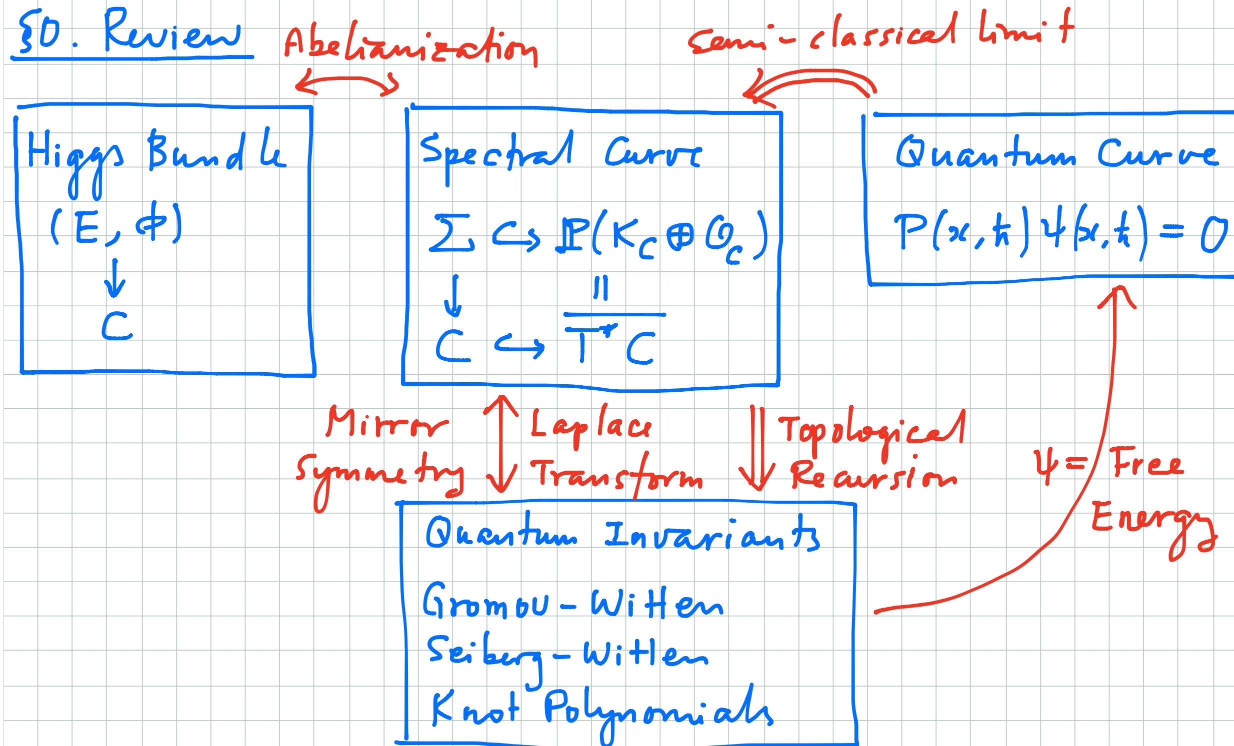
Lecture II : Quantum Curves for Catalan Numbers

§0. Review



Example : Higgs Bundle $E = K_{\mathbb{P}^1}^{-1} \oplus \mathcal{O}_{\mathbb{P}^1}$

$$\begin{array}{c} E \\ \downarrow \\ \mathbb{P}^1 \end{array}$$

$$\phi = \begin{bmatrix} & 1 \\ x(dx)^2 & \end{bmatrix} : E \longrightarrow E \otimes K_{\mathbb{P}^1}(4)$$

Spectral Curve $\Sigma = \{ (x,y) \in \mathbb{F}_2 \mid y^2 - x = 0 \}$

$$\subset \mathbb{F}_2 = \mathbb{P}(K_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}).$$

Quantum Curve $\left[\left(\hbar \frac{d}{dx} \right)^2 - x \right] \psi(x,\hbar) = 0.$

Exact Asymptotic Expansion (valid $|\mathrm{Arg}\, x| < \pi - \delta$)

$$\psi(x,\hbar) = \exp\left(\sum_{m=0}^{\infty} \hbar^{m-1} S_m(x) \right)$$

$$S_m(x) = \sum_{2g-2+n=m-1} \frac{1}{n!} F_{g,n}(x)$$

$$F_{g,n}(x) = F_{g,n}(x_1, x_2, \cdots, x_n)\Big|_{x_1=x_2=\cdots=x_n=x}$$

Principal Specialization

$$F_{g,n}(x_1, x_2, \cdots, x_n)$$

$$= \frac{(-1)^n}{2^{2g-2+n}} \sum_{\substack{d_1+\cdots+d_n\\=3g-3+n}} \langle \tau_{d_1} \cdots \tau_{d_n} \rangle_{g,n} \prod_{i=1}^{n} (2d_i-1)!! \left(\frac{1}{\sqrt{x_i}}\right)^{2d_i+1}.$$

Remark : For $\hbar \neq 0$, the solution $\psi(x, \hbar)$ is an entire function in x.

Thus the above expansion, based in $\frac{1}{\sqrt{x}}$ and $x \to \infty$, is an expansion at the

essential singularity !

- We usually think that

  We need a convergent power series to determine a holomorphic function.

- A convergent power series determines a holomorphic function. But the converse is not true.

- The entire function $\psi(x, \hbar)$ is indeed uniquely determined by the above asymptotic formula.

Wall Crossing

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- At a different wedge domain, we need a different formula.

§1. What is the mirror dual of the Catalan Numbers?

Let us start with a Higgs bundle on $C = \mathbb{P}^1$:

$$E = K_{\mathbb{P}^1}^{-1} \oplus \mathcal{O}_{\mathbb{P}^1}$$

$$\phi = \begin{bmatrix} 0 & 1 \\ -(dx)^2 & -x\,dx \end{bmatrix} : K^{-1} \oplus \mathcal{O} \longrightarrow (\mathcal{O} \oplus K) \otimes \mathcal{O}(4)$$

- Since ϕ has a pole of order 4 at $x = \infty$, we consider $\overline{T^*\mathbb{P}^1} = \mathbb{P}(K_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}) = \mathbb{F}_2$.
- $\eta = y\,dx = v\,du$, $u = \frac{1}{x}$, $w = \frac{1}{v}$

  affine coordinate expression of the tautological 1-form on $T^*\mathbb{P}^1$
- Spectral curve $\boxed{\det(\eta\, \mathrm{Id} - \phi) = 0}$

In affine coordinates,

$$\det(\eta - \phi) = \eta^2 - \eta\, \mathrm{tr}\,\phi + \det\phi$$
$$= (y^2 + xy + 1)(dx)^2 = 0.$$

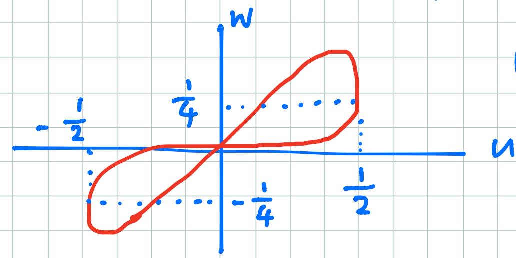
$$\boxed{y^2 + xy + 1 = 0}$$

$$y\,dx = v\,du = y\, d\left(\frac{1}{u}\right) = -y \frac{1}{u^2} du.$$

$$\therefore \quad -u^2 v = y.$$

$$w = \frac{1}{v}$$

$$u^4 v^2 - uv + 1 = 0 \quad \therefore \quad \boxed{u^4 - uw + w^2 = 0}$$



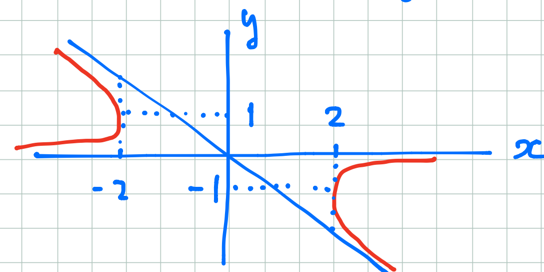
$$(w - \tfrac{1}{2}u)^2 + u^4 - \tfrac{1}{4}u^2 = 0$$

$$w = \tfrac{1}{2}u$$

$$\Rightarrow \quad u^2 = \tfrac{1}{4}$$

- Let's solve $y^2 + xy + 1 = 0$

Answer : $x = -\left(y + \frac{1}{y}\right).$



- Do you see Catalan numbers here ?

$$x = -\left(y + \frac{1}{y}\right).$$

- The inverse function $y = y(x)$, with the boundary condition $y(\infty) = 0$, is given by

$$\boxed{y(x) = -\sum_{m=0}^{\infty} C_m \frac{1}{x^{2m+1}}},$$

Mirror dual of Catalan Numbers !

$$C_m = \frac{1}{m+1}\binom{2m}{m}.$$

~~proof~~ $\quad C_m = \sum_{a+b=m} C_a C_b \Longrightarrow y^2 + xy + 1 = 0.$ $\square$

- Our general theory proves the following :

<u>Quantum Curve</u> : $\left[\left(\hbar \frac{d}{dx}\right)^2 + x\left(\hbar\frac{d}{dx}\right) + 1\right]\psi(x,\hbar) = 0.$

Remark: The result is non-trivial, because the quantization of xy can be either

$$x\left(\hbar\frac{d}{dx}\right) \quad \text{or} \quad \left(\hbar\frac{d}{dx}\right)\cdot x = x\left(\hbar\frac{d}{dx}\right)+\hbar .$$

(Actually, it can be $x^{1-p}\left(\hbar\frac{d}{dx}\right)x^{p}$ for any p.)

- Exact asymptotic expansion:

$$\psi(x,\hbar)=\exp\left(\sum_{m=0}^{\infty}\hbar^{m-1}S_m(x)\right) \quad \text{valid on} \begin{Bmatrix} |x|>2, \ \hbar>0 \\ |\operatorname{Arg} x|<\frac{\pi}{4} \end{Bmatrix}$$

$$S_0(x)=-\frac{1}{2}y(x)^2+\log(-y(x))$$

$$S_1(x)=-\frac{1}{2}\log(1-y(x)^2)$$

$$m\geq 2 \qquad S_m(x)=\sum_{2g-2+n=m-1}\frac{1}{n!}F_{g,n}^{Cat}(x,x,\cdots,x),$$

$$F_{g,n}^{Cat}(x_1,\cdots,x_n)\underset{\text{def}}{=}\sum_{\mu_1,\cdots,\mu_n>0}\frac{C_{g,n}(\vec{\mu})}{\mu_1\mu_2\cdots\mu_n}\prod_{i=1}^{n}x_i^{-\mu_i}.$$

Here, $C_{g,n}(\vec{\mu})$ = generalized Catalan number of genus g and n vertices of degrees $\mu_1,\cdots,\mu_n$.

Before explaining the generalized Catalan numbers, let us remark why $F_{g,n}^{Cat}$ is of any interest.

- $x=-\left(y+\frac{1}{y}\right)$, $\quad y=-\frac{t+1}{t-1}$. Consider $F_{g,n}^{Cat}$ as a function in $t_1,\cdots,t_n\in\mathbb{P}^1$.

$$F_{g,n}^{Cat}=F_{g,n}^{Cat}(t_1,\cdots,t_n).$$

Theorem (Dumitrescu – M – Safnuk – Sorkin 2013)

1) $F^{Cat}_{g,n}(t_1,\cdots,t_n)$ for $2g-2+n>0$ is a Laurent polynomial of degree $6g-6+3n$ with

$$F^{Cat}_{g,n}(t_1,\cdots,t_n) = F^{Cat}_{g,n}\left(\frac{1}{t_1},\cdots,\frac{1}{t_n}\right).$$

2) $F^{Cat}_{g,n}(1,\cdots,1) = (-1)^n \chi(\mathcal{M}_{g,n})$,

here $\mathcal{M}_{g,n}$ = moduli space of smooth pointed curves

2) $$\lim_{\lambda\to\infty} \frac{F^{Cat}_{g,n}(\lambda t_1,\cdots,\lambda t_n)}{\lambda^{6g-6+3n}}$$

$$= \frac{(-1)^n}{2^{2g-2+n}} \sum_{\substack{d_1+\cdots+d_n \\ =3g-3+n}} \langle \tau_{d_1}\cdots\tau_{d_n}\rangle_{g,n} \prod_{i=1}^n (2d_i-1)!! \left(\frac{t_i}{2}\right)^{2d_i+1}.$$

Thus $F^{Cat}_{g,n}(t_1,\cdots,t_n)$ knows both intersection numbers of $\overline{\mathcal{M}}_{g,n}$ and $\chi(\mathcal{M}_{g,n})$!

§2. Generalized Catalan Numbers.

- Generalizations depend on which definition of Catalan numbers you choose.

Definition C_m = # of "legal" ways of placing m pairs of parentheses.

$m=1$ ( ) $C_1 = 1$. ( )(= illegal !)

$m=2$ ( )( ), (( )) $C_2 = 2$

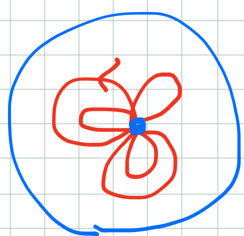
$m=3$ ( )( )( ), ( )(( )), (( ))( ), (( )( )), ((( ))) $C_3 = 5$

This definition is equivalent to the following:

C_m = # of <u>cellular</u> graphs on S^2 with 1 vertex of degree $2m$, on which one of the half-edges has an outgoing arrow (to kill the automorphism).

(Cellular graph = 1-skeleton of a cell-decomposition)

S^2

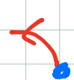


$m = 5$

$\longleftrightarrow \quad (())(())()$

1) Start with the arrowed edge.

2) Follow the orientation of S^2 ⟲ around the vertex.

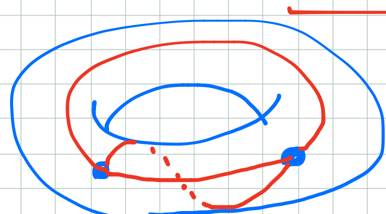
3)  gives (open. Then  gives ( ).

etc.

<u>Definition</u>: $C_{g,n}(\mu_1, \cdots, \mu_n)$ = # of cellular graphs on an oriented genus g surface with n labeled vertices of degrees $\mu_1, \cdots, \mu_n$, on which one of the half-edges at each vertex has an outgoing arrow.

$\boxed{\text{It is an integer !}}$

Ex:



$g = 1,\ n = 2,\ \mu_1 = \mu_2 = 3.$

$C_{g,1}(2m) =$

$g=1$	1	$m=2$
	10	3
	70	4
	420	5
	2310	6
	12012	7

<u>Theorem</u> (DMSS 2013)

The generalized Catalan numbers satisfy

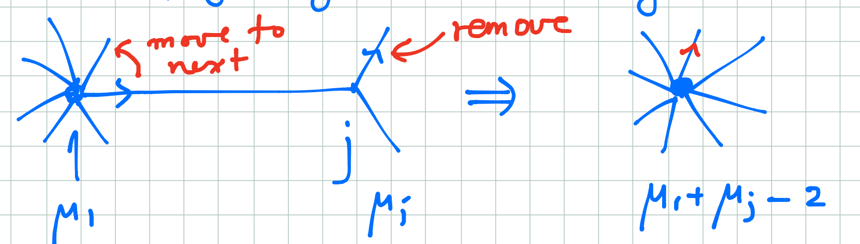
$$C_{g,n}(\mu_1,\cdots,\mu_n) = \sum_{j=2}^{n} \mu_j \, C_{g,n-1}(\mu_1+\mu_j-2, \mu_2, \cdots, \widehat{\mu_j}, \cdots, \mu_n)$$

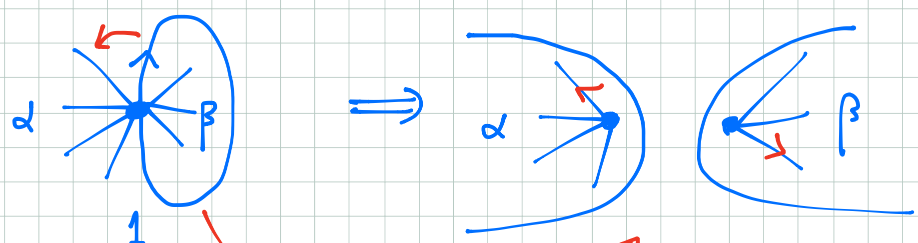
$$+ \sum_{\alpha+\beta=\mu_1-2} \Big[C_{g-1,n+1}(\alpha,\beta,\mu_2,\cdots,\mu_n) + \sum_{\substack{g_1+g_2=g \\ I \sqcup J = \{2,\cdots,n\}}} C_{g_1,|I|+1}(\alpha,\mu_I)\, C_{g_2,|J|+1}(\beta,\mu_J) \Big].$$

<u>Remark</u>. $C_{0,1}(2m) = C_m$.

$$C_{0,1}(2m) = \sum_{a+b=m-1} C_{0,1}(\underset{\alpha}{\underbrace{2a}})\, C_{0,1}(\underset{\beta}{\underbrace{2b}}) \quad !!$$

<u>Proof</u> Apply edge-shrinking operation.

1) 

2) 

pinch a cycle $\Rightarrow$ separate it out

$$\begin{cases} g \longmapsto g-1 & \text{connected} \\ g \longmapsto g_1+g_2 & \text{disconnected} \end{cases}$$

$\square$

<u>Remark</u>. Note that $C_{g,n}$ appears on the right-hand side as well. So this formula is only good for computers to find numbers. We do not know any closed formula for $C_{g,n}(\vec{\mu})$ as a function in $\vec{\mu}$.

§3. Mirror symmetry = Laplace transform

Now define $F_{g,n}(x_1,\cdots,x_n) = \sum_{\mu_1,\cdots,\mu_n>0} \frac{C_{g,n}(\vec{\mu})}{\mu_1\cdots\mu_n} \prod_{i=1}^{n} x_i^{-\mu_i}$.

Laplace transform $(x = e^w)$

Theorem (DM(TBP))

① $W_{g,n} = d_1\cdots d_n F_{g,n}$ is a meromorphic differential form on $\hat{\Sigma}^n$, where $\hat{\Sigma}$ is the normalization of Σ.

② $W_{g,n}$'s satisfy the Topological Recursion:

$$W_{g,n}(y_1,\cdots,y_n) = \frac{1}{2\pi i}\sum_{p\in R}\oint_{\gamma_p} \frac{\int_y^{\tilde{y}} W_{0,2}(\cdot, y_1)}{W_{0,1}(\tilde{y}) - W_{0,1}(y)}$$
$$\times\Big[W_{g,n-1}(y,\tilde{y},y_2,\cdots,y_n) + \sum^{\text{no }(0,1)}_{\substack{g_1+g_2=g\\ I\sqcup J=\{2,\cdots,n\}}} W_{g_1,|I|+1}(y,y_I)\,W_{g_2,|J|+1}(\tilde{y},y_J)\Big],$$

where

① R is the divisor of zeros of $W_{0,1}$.

②
$$\begin{tikzcd} \hat{\Sigma} \arrow[r] \arrow[drr, "\hat{\pi}"'] & \Sigma \arrow[r, phantom, "\subset"] \arrow[dr, "\pi"] & T^*\mathbb{P}^1 \arrow[d, "\pi"] \\ & & \mathbb{P}^1 \end{tikzcd}$$

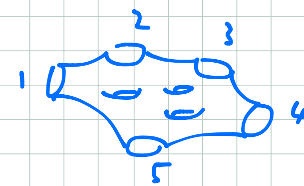
$y \mapsto \tilde{y}$ is the Galois conjugation of $\hat{\pi}: \hat{\Sigma} \to \mathbb{P}^1$

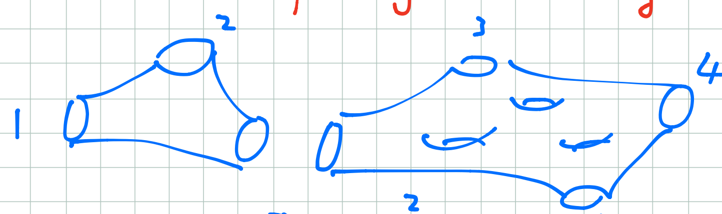
③ No $(0,1)$ means $(g_1=0, I=\phi)$ and $(g_2=0, J=\emptyset)$ are excluded.

Remark ① $F_{g,n}$'s satisfy a similar PDE recursion. Its principal specialization is the quantum curve.

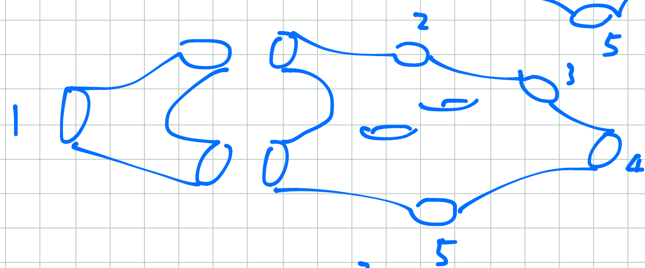
② The topological recursion is a genuine recursion, meaning that the right-hand side does <u>not</u> contain $W_{g,n}$.

Reduction of $2g-2+n$ by 1.

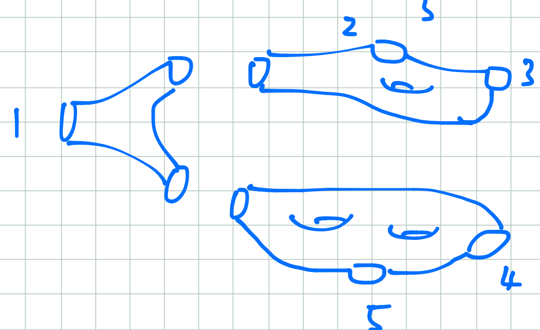




$(g,n) \mapsto (g,n-1)$



$(g,n) \mapsto (g-1,n+1)$



$$(g,n) \mapsto \begin{cases} (g_1, n_1+1) \\ (g_2, n_2+1) \end{cases}$$

$$\begin{cases} n_1+n_2 = n-1 \\ g_1+g_2 = g \end{cases}$$

$$\begin{array}{rl} & 2g_1 - 2 + (n_1+1) \\ +) & 2g_2 - 2 + (n_2+1) \\ \hline & 2g - 4 + (n-1) + 2 = (2g-2+n) - 1 \end{array}$$

③ Thus the topological recursion is the pair-of-pants decomposition of punctured surfaces.

④ The nature of the recursion is the edge contraction operation of generalized Catalan numbers.

§4. Quantum Curve at $\hbar = 1$

Hermite differential eq.

$$\left[\left(\frac{d}{dx}\right)^2 + x\frac{d}{dx} + 1\right]\psi(x) = 0, \qquad \psi(+\infty) = 0$$

Solution : $\psi(x) = e^{-\frac{1}{2}x^2}$.

$$\begin{cases} \psi' = -x e^{-\frac{1}{2}x^2} \\ \psi'' = -e^{-\frac{1}{2}x^2} + x^2 e^{-\frac{1}{2}x^2} \end{cases}$$

- Our asymptotic expansion is in x^{-1}. Thus we are expanding this entire function at the essential singularity.
- The Hermite differential equation has an irregular singularity at $x = \infty$.

  This is why the solution has an essential singularity at $x = \infty$.