On the Horizon of a Surface

Speech Given to Young Students Starting College

Motohico Mulase

University of California, Davis

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Abstract

Euler (1707-1783), Gauss (1777-1855), and Riemann (1826-1866) are counted among the greatest of all mathematicians. Although they worked on many areas of mathematics, the mystery of prime numbers was a recurrent theme of their research throughout their lives. The most fundamental question of prime numbers, i.e., how they are distributed among all numbers, is still unsolved today, and carries a $1,000,000 prize to whoever solves it. The particular function that has all the information of prime numbers is called the Riemann zeta function, although it was first introduced by Euler and studied by Gauss. The prime number distribution question is equivalent to a specific property of the zeta function called the Riemann hypothesis.

Another fascination these three giants shared is topology and analysis of surfaces. A Platonic solid may have a different number of faces, but the number of vertices minus the number of edges plus the number of faces is always 2. We call this value the Euler characteristic. Gauss noticed that each surface carries different geometries, and Riemann proposed to study the collection of all geometries on a given surface. This is an example of the moduli spaces which form a central theme of the 21st century mathematics. We wonder what Euler, Gauss, and Riemann would say if they learned that the Euler characteristic of Riemann's moduli space is again given by their zeta function!

In this talk I will invite you to stand on the boundary of a surface. We view where we have come, and look into the horizon of mathematics yet to come. Prime numbers and topology play an infinite fugue.

On the Horizon of a Surface

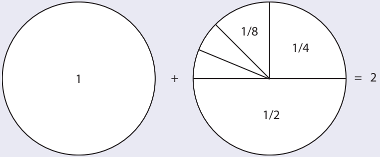
1. Mathematics = Science of Infinity
   - Adding an infinite number of numbers
   - Euler
   - Prime Numbers Come In Here
2. Fascination on the Distribution of Prime Numbers
   - Gauss
   - The Number of Primes
   - Riemann
3. Surfaces
   - Euler
   - Gauss
   - Riemann
   - Laplace
4. Beyond the Horizon

Mathematics = Science of Infinity

Adding an infinite number of numbers

What is

$$1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \cdots \quad ?$$



How about

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \frac{1}{5\cdot 6} + \cdots \quad ?$$

We calculate

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\frac{1}{5\cdot 6}+\cdots$$

$$=\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)+\cdots$$

$$= 1+\left(-\frac{1}{2}+\frac{1}{2}\right)+\left(-\frac{1}{3}+\frac{1}{3}\right)+\left(-\frac{1}{4}+\frac{1}{4}\right)+\left(-\frac{1}{5}+\frac{1}{5}\right)+\cdots$$

$$= 1.$$

Leonhard Euler (1707 - 1783)

What is

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$= 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots \quad ?$$

How about

$$\sum_{n=1}^{\infty} \frac{1}{n^3} = 1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \cdots \quad ?$$

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots \quad ?$$

We know:

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots = \infty.$$

Why?

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \cdots$$

$$> 1 + \frac{1}{2} + \overbrace{\frac{1}{4} + \frac{1}{4}}^{2} + \overbrace{\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}}^{4} + \overbrace{\frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \cdots +}^{8}$$

$$= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots$$

$$= \infty.$$

What Euler did:

By around 1735 (Euler was about 28 years old), he calculated

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1.64493406684822643647\ldots \qquad \text{(20 decimal places)}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^3} = 1.202056903159594\ldots \qquad \text{(15 decimal places)}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = 1.08232323371113819\ldots \qquad \text{(18 decimal places)}$$

And

$$\pi = 3.141592653589793\ldots \qquad \text{(15 decimal places)}$$

Then a Miracle Happened in 1735

$$\frac{3.141592653589793 \times 3.141592653589793}{6}$$
$$= 1.644934066848226\ldots$$

Euler's Formula

Euler noticed, "quite unexpectedly" as he said, that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{6}.$$

The Miracle Continues

Euler's Formula for the Zeta Function (1739)

Let us define the Zeta Function by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \cdots .$$

Then

$$\zeta(2) = \frac{\pi^2}{6} \qquad \zeta(4) = \frac{\pi^4}{90} \qquad \zeta(6) = \frac{\pi^6}{945}$$

$$\zeta(8) = \frac{\pi^8}{9450} \qquad \zeta(10) = \frac{\pi^{10}}{93555}$$

$$\zeta(12) = \frac{691\pi^{12}}{638512875} \qquad \textit{etc.}$$

Prime Numbers Come In Here

Why was Euler interested in the Zeta Function?

Theorem of Euclid

Every positive integer is factorized into a product of prime numbers in exactly one way.

Examples:

$$
\begin{aligned}
2010 &= 2 \times 3 \times 5 \times 67. \\
20102707 &= 20102707 \quad \text{(it is a prime number)}. \\
638512875 &= 3^6 \times 5^3 \times 7^2 \times 11 \times 13.
\end{aligned}
$$

A Consequence of Euclid's Theorem

$$
\begin{aligned}
1 &+ \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \frac{1}{6^s} + \frac{1}{7^s} + \frac{1}{8^s} + \frac{1}{9^s} + \frac{1}{10^s} + \frac{1}{11^s} + \cdots \\
&= \left(1 + \frac{1}{2^s} + \frac{1}{(2^2)^s} + \frac{1}{(2^3)^s} + \frac{1}{(2^4)^s} + \frac{1}{(2^5)^s} + \cdots\right) \\
&\times \left(1 + \frac{1}{3^s} + \frac{1}{(3^2)^s} + \frac{1}{(3^3)^s} + \frac{1}{(3^4)^s} + \frac{1}{(3^5)^s} + \cdots\right) \\
&\times \left(1 + \frac{1}{5^s} + \frac{1}{(5^2)^s} + \frac{1}{(5^3)^s} + \frac{1}{(5^4)^s} + \frac{1}{(5^5)^s} + \cdots\right) \\
&\times \left(1 + \frac{1}{7^s} + \frac{1}{(7^2)^s} + \frac{1}{(7^3)^s} + \frac{1}{(7^4)^s} + \frac{1}{(7^5)^s} + \cdots\right) \\
&\times \cdots
\end{aligned}
$$

Let us now apply the summation formula of powers.

A Summation Formula

If $|x| < 1$, then we have

$$1 + x + x^2 + x^3 + x^4 + x^5 + \cdots = \frac{1}{1-x}.$$

How do we know it?

$$\begin{aligned}
\text{Let us put} \quad S &= 1 + x + x^2 + x^3 + x^4 + x^5 + \cdots \\
\text{Then we have} \quad Sx &= \phantom{1 + {}} x + x^2 + x^3 + x^4 + x^5 + x^6 + \cdots \\
\text{Therefore} \quad S - Sx &= 1. \\
\text{Hence} \quad S &= \frac{1}{1-x}.
\end{aligned}$$

In particular,

$$1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \cdots = \frac{1}{1-\frac{1}{2}} = 2.$$

The Product Formula of Euler for The Zeta Function

Euler's Product Formula

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p=\text{ prime number}} \frac{1}{1 - \frac{1}{p^s}}.$$

An Immediate Corollary

There are infinitely many prime numbers.

If there were only finitely many primes, then the right-hand side of the product formula would be a finite product. Therefore, it has a concrete value for the input $s = 1$. But we know the left-hand side at $s = 1$ gives an infinity. Contradiction!

Euler: $\zeta(s)$ has something to do with the distribution of primes!

Garl Friedrich Gauss (1777 - 1855) Asked:

Take an arbitrary positive integer *n*.

What is the **probability** that *n* is actually a prime number?

Of course whether *n* is prime or not is determined. It is not random. But if *n* is a large number something $\sim 10^{400}$, then even the fastest computer of our time has hard time factoring it.

Experiments

Gauss calculated a large table of prime numbers. Then he found that experimentally

The Experimental Formula of Gauss

The probability for n to be a prime number is about

$$\frac{1}{\log n}.$$

What is this $\log n$? Of course we can give a mathematical definition. But in the Age of Information Technology...

Logarithm: A Common Function in the Age of I.T.

Logarithm = Entropy = Measure of the Amount of Information

Suppose there are n different choices of making a thing happen. Then the amount of information contained in this collection of "things" is $\log n$.

Example.

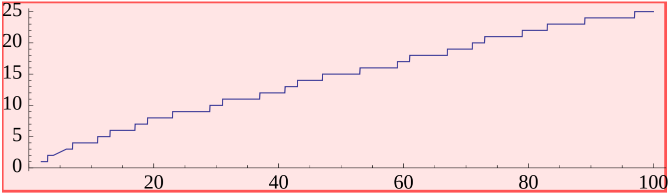
We use the unit of "bit" for a measurement of information. A collection of 8-bit data means we have a choice of $2^8 = 256$ objects. The collection of all alphabets, both lower case and UPPER CASE, Arabic numbers from 0 to 9, punctuation characters and all the special symbols @, $\#, \$, \%, \wedge, \&, \ast ...$, form a part of the 8-bit information. Here $8 = \log_2 256$.

The Number of Primes

$\pi(N)$ = the number of primes $\leq N$.

Let $\pi(N)$ denote the number of primes less than or equal to N. We have

$$\pi(2) = 1, \quad \pi(3) = 2, \quad \pi(5) = 3,$$
$$\pi(1000) = 168, \quad \pi(1000000) = 78498, \ldots$$



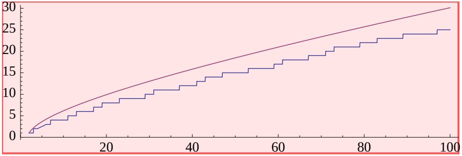
Approximation of $\pi(N)$

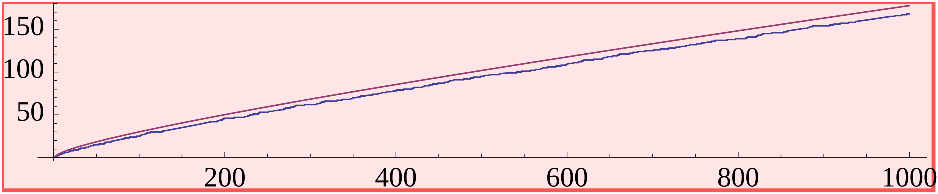
Gauss thought, since n would be a prime with probability $\frac{1}{\log n}$, the total number $\pi(N)$ of primes $\leq N$ should be given by

$$Li(N) = \int_2^N \frac{dx}{\log x} \sim \sum_{n=2}^{N} \frac{1}{\log n}.$$

There is NO formula for the function $\pi(x)$. Gauss is the first person to ask what is the best formula that gives a good approximation of $\pi(x)$, and has come up with the *smooth* analytic function $Li(x)$.

Gauss $Li(x)$-Function and the Prime Distribution





Bernhard Riemann's (1826 - 1866) Hypothesis

$\pi(x)$ is a function given to us by Nature.

$Li(x)$ is a smooth function living in an ideal, harmonious, world.

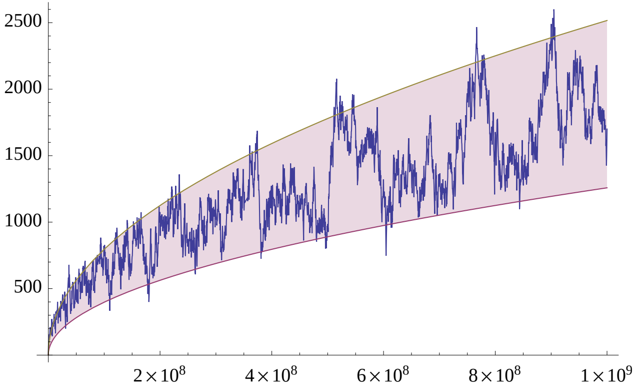
Then what is the *error* of the approximation of $\pi(x)$ by $Li(x)$?

The difference $\pi(x) - Li(x)$ must be a function obeying the law of large random numbers:

$$|\pi(x) - Li(x)| < \frac{1}{8\pi}\sqrt{x}\log x.$$

Nature $=$ Harmony $+$ Randomness

The difference between $\pi(x)$ and $Li(x)$



Plot of $Li(x) - \pi(x)$, $\sqrt{x}/4\pi$, and $\sqrt{x}/8\pi$. Riemann calculated many values of $Li(x) - \pi(x)$, and noticed its random behavior.

Zeta Function and the Riemann Hypothesis

The Zeta function makes sense for a *complex* number $s = x + \sqrt{-1}y$.

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \sum_{n=1}^{\infty} e^{-s \log n}$$

Nature $=$ Harmony $+$ Randomness

More traditional formulation of the Riemann Hypothesis: If

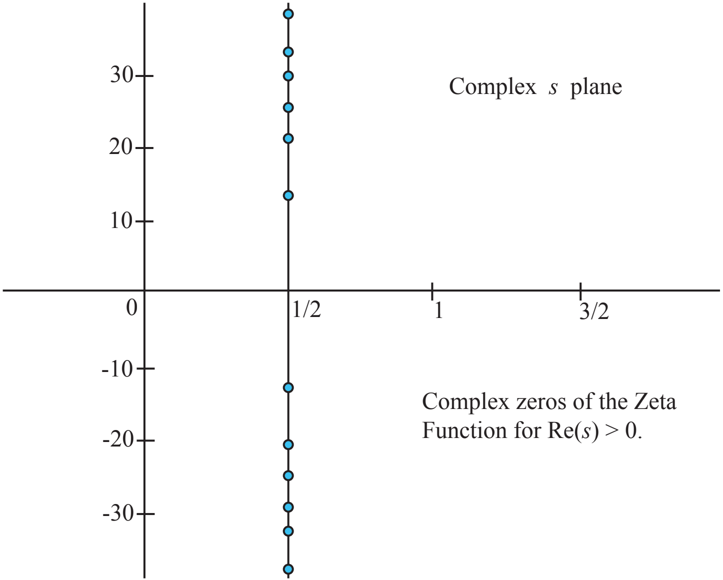
$$\zeta(s) = 0$$

and $Re(s) = x > 0$, then $x = \frac{1}{2}$.

This statement is *equivalent* to the error estimate formula

$$|\pi(x) - Li(x)| < \frac{1}{8\pi}\sqrt{x}\log x, \qquad x >> 0.$$

Riemann Hypothesis



Fascination on Prime Numbers

Centuries before the invention of computers, Euler, Gauss, and Riemann spent years on calculating things just *for fun*,

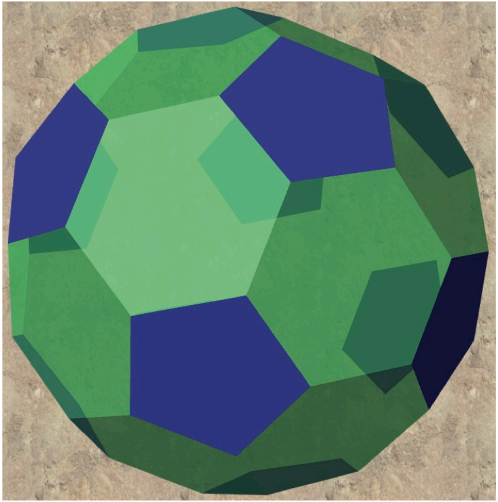
- To discover completely unexpected formulas;
- And to open the door to the future.

Their fascination on prime numbers are still leading us to more mysteries in different fronts of Mathematics today.

Now let's change out attention to *Surfaces*, on which the three giants also shared their fascination.

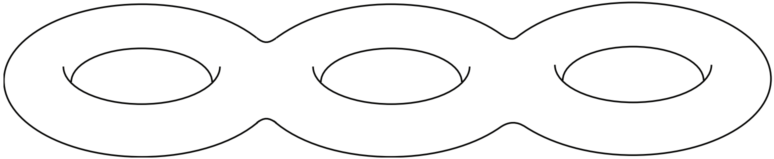
The Euler Characteristic of the Fullerene C_{60}

The number of vertices $-$ the number of edges $+$ the number of faces $=$?



$$\begin{cases} \textit{vertices} = 60 \\ \textit{edges} = ?? \\ \textit{faces} = 32. \end{cases}$$

A Surface with g Handles



If we make a tiling of a surface S_g with g handles, then the number of vertices $-$ the number of edges $+$ the number of faces is always $2 - 2g$.

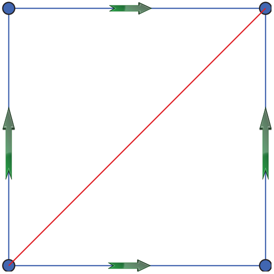
The Euler characteristic of a surface of *genus* g is defined to be

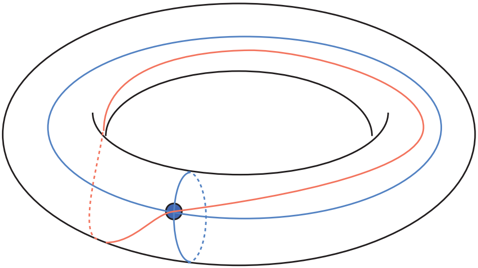
$$\chi(S_g) = 2 - 2g.$$

A sphere has no handles. Therefore, its Euler characteristic is $2 - 0 = 2$.

Torus

A triangulation of a torus with 1 vertex, 3 edges, and 2 triangles.



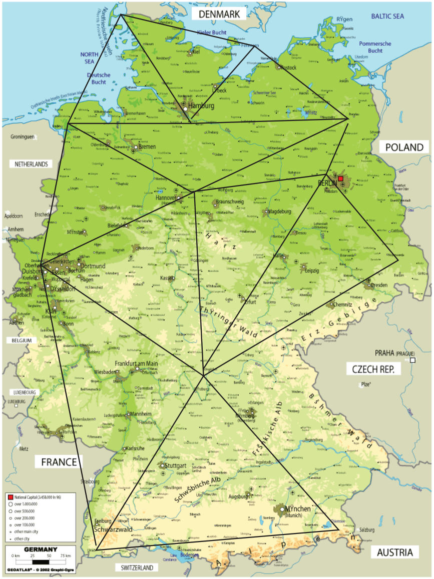


The Euler characteristic is $1 - 3 + 2 = 0 = 2 - 2 \times 1$.

How is Germany Curved?

Gauss was ordered by a German king to measure his territory. That gave Gauss a motivation to develop a new area of mathematics, now we call *Differential Geometry*.

A triangle is determined by the three edges. So Gauss *triangulated* Germany to measure its area.



Geometry of Surfaces

The main concern of Gauss: how to measure the way a surface is curved ($=$ the *curvature* of the surface). A plane is *flat*. A sphere is *curved*. How do you measure the difference?

Gaussian Curvature: Theorema Egregium

The discovery of Gauss: If we give the notion of distance on a surface locally at everywhere, then the curvature of a surface is completely determined.

If we make the mesh of a triangulation small enough, then the curvature of the surface at every point is measured by how the triangles are assembled at each vertex.

KaleidoTile Demo

`http://www.geometrygames.org/KaleidoTile/`

Gauss = Euler!

The curvature of a surface may vary from point to point. So let us consider the *average* curvature of a surface, and call it K. A surface has the total area, which we denote by A.

Gauss-Bonnet Formula

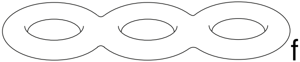
Let S_g be a surface with g handles. Then the average curvature K, the total area A of S_g, and its Euler characteristic satisfy the following equation:

$$K \cdot A = 2\pi \cdot \chi(S_g) = 2\pi(2 - 2g).$$

Corollary. The Average Curvature is

$$\begin{cases} K > 0 & g = 0 \\ K = 0 & g = 1 \\ K < 0 & g > 1. \end{cases}$$

Riemann's Idea = Space of All Geometries

From the consideration of complex analysis, Riemann was led to dealing with the space of all geometries of a given surface. We call this space the *moduli space*. Since we do not distinguish a *large* surface  from a *small* surface , the moduli space is a space of all *shapes*.

Riemann's Moduli Problem

Study the space $\mathcal{M}_g$ of all geometries one can put on a given surface of g handles.

Riemann's Formula for the Dimension of the Moduli Spaces

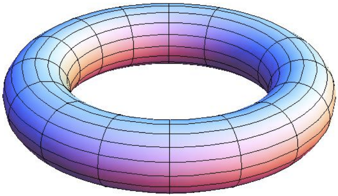
$$\dim \mathcal{M}_0 = 0, \qquad \dim \mathcal{M}_1 = 2$$
$$\dim \mathcal{M}_g = 6g - 6 \qquad g > 1.$$

Mathematical Definition of Moduli Spaces

$\mathcal{M}_g$ is difficult to grasp. It is easier to define:

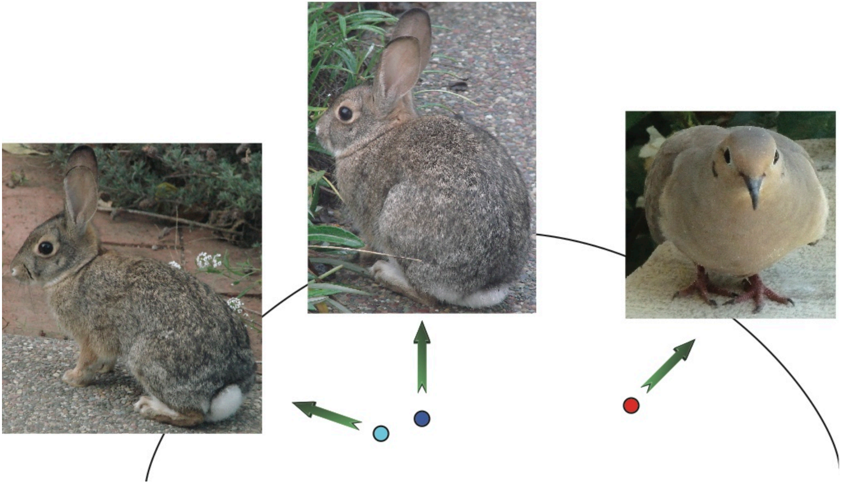
$\mathcal{M}_{g,n} =$ Collection of all different kinds of *Tiling* of a surface S_g with n tiles.

A tiling determines a geometry on the surface. Thus the collection of all tiling is indeed the collection of all geometries. It is a space of dimension $6g - 6 + 2n$.



A rectangular tiling of a torus with 225 tiles

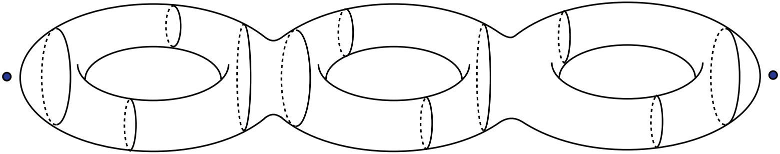
Where do we encounter Moduli Spaces?



The moduli space is a space of shapes in our brain. By measuring the distance between two points in that space, our brain can tell if the two shapes are the same or not.

Another Place = String Theory

A tiny string is created, expands itself, splits into two pieces, and flies in our universe. The pieces may re-unite together. In this way the string sweeps out a surface.



Considering the space of all geometries of a surface corresponds to considering quantum mechanics of strings.

A Consequence of String Theory

Riemann = Euler!

Although the moduli space $\mathcal{M}_g$ is a complicated space of $6g-6$ dimensions, it has a geometric structure, and we can consider its Euler characteristic. Using the idea of string theory, mathematicians have calculated:

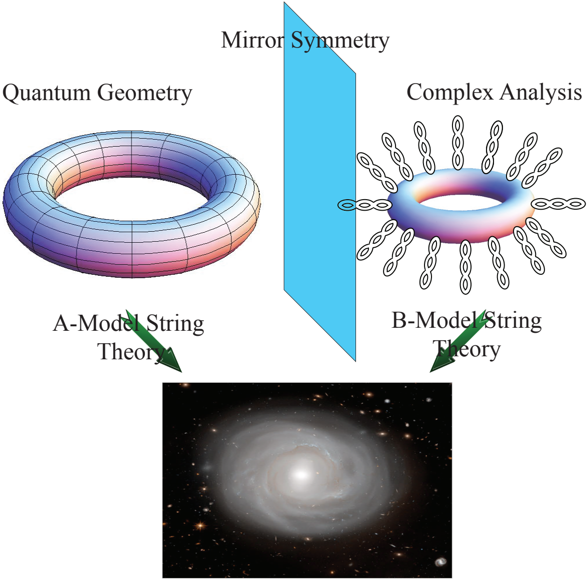
$$\chi(\mathcal{M}_g) = \frac{1}{2-2g}\cdot\zeta(1-2g) \qquad g>1,$$

$$\chi(\mathcal{M}_{g,n}) = (-1)^{n-1}\frac{(2g-3+n)!}{(2g-2)!}\cdot\zeta(1-2g).$$

Here $\zeta(s)$ is the Zeta Function of Euler!

String Theory used to be Physics. It is a branch of Mathematics now. How does it work in Mathematics?

Mirror Symmetry of String Theories in Physics



Mirror Symmetry

Quantum Geometry, or Mathematics of Counting

- Discrete Mathematics
- Enumerative Algebraic Geometry
- Number Theory
- Quantum Knot Invariants
- Topology of the Moduli Space $\mathcal{M}_{g,n}$
- Gromov-Witten Invariants

Complex Analysis

- Zeta Functions
- Complex Analytic Geometry
- Nonlinear Integrable Partial Differential Equations; KdV and KP Equations
- Analysis of Matrix Integrals
- Donaldson-Thomas Invariants

Demo of Solitons

`http://www.math.ucdavis.edu/~mulase/solitons.html`

Pierre-Simon Laplace (1749 - 1827)

Mirror Symmetry is a mysteriously powerful tool in Mathematics.

So what is Mirror Symmetry?

In Terms of Mathematics

Mirror Symmetry
|| ???
Laplace Transform

A mathematical understanding of the Mirror Symmetry is currently sought.

Laplace Transform Changes Mathematics of Counting to Mathematical Analysis

Example

The Zeta Function
$=$ The Laplace Transform of the Prime Number Distribution

Prime numbers are fundamental objects in Mathematics. Then so is the Zeta Function. This explains why the Zeta Function appears everywhere in geometric counting problems.

The idea of Mirror Symmetry, and more general Laplace Transform, has been used to solve many problems in number theory and modern geometry.

Yet there are still interesting problems unexplored on the Horison. Furthermore....

Unsolved Problems are Waiting for YOU!

Euler calculated in 1735 that

$$\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3} = 1.202056903159594\ldots$$

- **We still don’t know what this number is!**
- Roger Apéry (1916–1994) proved that it was an *irrational* number.
- In Physics, $\zeta(3)$ appears in the calculation of the *anomalous magnetic moment* of an electron.
- **Nothing is known about $\zeta(5), \zeta(7), \zeta(9), \ldots$ and their relations.**
- In Mathematics these numbers appear when we deal with *non-orientable* surfaces, such as a Möbius strip and a Klein bottle. This fact does not seem to help so far...

Unsolved Problems are *Still* Waiting for YOU!

The Riemann Hypothesis is about the error of the ideal approximation $Li(x)$ for the prime number counting function $\pi(x)$ given by Nature. Riemann thought the difference is a random process:

$$|\pi(x) - Li(x)| < \frac{1}{8\pi}\sqrt{x}\log x \quad ??$$

The exponent of $\sqrt{x} = x^{\frac{1}{2}}$ is the $\frac{1}{2}$ that appears in the Zeta Function version of the Riemann Hypothesis.

Nobody knows if it is true, or why it should be true.

Joy of Mathematics

To discover two things that look completely different are actually equal.

A $=$ B

The larger the distance is, the greater your joy becomes.

The Riemann Hypothesis Says:

$$\textit{Nature} = \textit{Cosmos} + \textit{Chaos}.$$

While enrolled in a university, you are in a harmonious world, or a Cosmos. After Cosmos, you will be going back to the real world, or Chaos. This is the Nature, as Riemann says.

Enjoy your journey that is about to begin, and

Bon Voyage!