MIRZAKHANI'S RECURSION RELATIONS, VIRASORO CONSTRAINTS AND THE KDV HIERARCHY

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Abstract. We present in this paper a differential version of Mirzakhani's recursion relation for the Weil-Petersson volumes of the moduli spaces of bordered Riemann surfaces. We discover that the differential relation, which is equivalent to the original integral formula of Mirzakhani, is a Virasoro constraint condition on a generating function for these volumes. We also show that the generating function for ψ and κ_1 intersections on $\overline{\mathcal{M}}_{g,n}$ is a 1-parameter solution to the KdV hierarchy. It recovers the Witten-Kontsevich generating function when the parameter is set to be 0.

1. Introduction

In her striking series of papers [18, 19], Mirzakhani obtained a beautiful recursion formula for the Weil-Petersson volume of the moduli spaces of bordered Riemann surfaces. Her recursion relation is an integral formula involving a kernel function that appears in the work of McShane [17] on hyperbolic geometry of surfaces. We have discovered that the differential version of the Mirzakhani recursion formula, which is *equivalent* to the original integral form, is indeed a Virasoro constraint condition imposed on a generating function of these volumes.

Mirzakhani proves in [19] that her recursion relation reduces to the Virasoro constraint condition as the length parameters of the boundary components of Riemann surfaces go to infinity, and moreover, it recovers the celebrated Witten-Kontsevich theorem of intersection numbers of tautological classes on the moduli spaces of stable algebraic curves. Our result reveals that the Virasoro structure exists essentially in the Mirzakhani theory, and that it is not the consequence of the large boundary limit.

The Virasoro constraint formulas for the generating functions of Gromov-Witten invariants of various target manifolds have been extensively studied in recent years [4, 8, 9, 22]. Although Mirzakhani's hyperbolic method does not immediately apply to these cases with higher dimensional target spaces, the Virasoro structure we identify in this paper strongly suggests that the Virasoro constraint conjecture of [5, 6] is a reflection of the combinatorial structure of building the domain Riemann surface from simpler objects such as pairs of pants or three punctured spheres.

Although it is more than 15 years old, the Witten-Kontsevich theory [23, 15] has never lost its place as one of the most beautiful and prime theories in the study of algebraic curves and their moduli spaces. The theory provides a complete computational method for all intersection numbers of the tautological cotangent

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classes (the ψ-classes) defined on the moduli space $\overline{\mathcal{M}}_{g,n}$ of stable algebraic curves of genus g with n marked points. Recently several new proofs have appeared [21, 19, 14, 13]. We note that all these new proofs are based on very different ideas and techniques, including random matrix theory, random graphs, Hurwitz theory, representation theory of symmetric groups, symplectic geometry and hyperbolic geometry.

The mystery of the Witten-Kontsevich theory has been the following question: where does the KdV equation, and also the Virasoro constraint condition, come from? Once we accept the Kontsevich matrix model expression of the generating function of all cotangent class intersections, then both the KdV and the Virasoro are an easy corollary of the analysis of matrix integrals. Thus the real question is: where do these structures appear in the geometry of moduli spaces of algebraic curves?

The insight of some of the the new proofs [14, 13], which do not rely on matrix integrals but rather use the counting of ramified overings of $\mathbb{P}^1$, is that the KdV equation is a direct consequence of the *cut and join* mechanism of [10].

The proof [19] due to Mirzakhani utilizes hyperbolic geometry and has a markedly different nature from the others, whose origins are rooted in algebraic geometry. Mirzakhani's work concerns the Weil-Petersson volume of the moduli space of *bordered* Riemann surfaces $\mathcal{M}_{g,n}(\boldsymbol{L})$, where $\boldsymbol{L} = (L_1, \ldots, L_n)$ specifies the geodesic lengths of the boundaries of Riemann surfaces. Here the moduli space is equipped with the structure of a differentiable orbifold realized as the quotient of the Teichmüller space by the action of a mapping class group. Mirzakhani shows that these volumes satisfy a recursion relation, and that in the limit $\boldsymbol{L} \to \infty$ her recursion formula recovers the Virasoro constraint condition for the generating function of ψ-class intersection numbers of $\overline{\mathcal{M}}_{g,n}$, or equivalently, the generating function of the Gromov-Witten invariants of a point. A striking theorem of [19] relates, via the method of symplectic reduction, the Weil-Petersson volume of $\mathcal{M}_{g,n}(\boldsymbol{L})$ and the intersection numbers involving both the first Mumford class κ_1 and the ψ-classes on $\overline{\mathcal{M}}_{g,n}$. As a consequence, she proves that the volume $\mathrm{Vol}\big(\mathcal{M}_{g,n}(\boldsymbol{L})\big)$, after an appropriate normalization with powers of π, is a polynomial in $\boldsymbol{L}$ with rational coefficients.

Since there is no particular reason to believe that there should be a direct relation between the Weil-Petersson volume of the moduli spaces of bordered Riemann surfaces and Virasoro constraint condition, our discovery suggests the existence of another, more algebraic, point of view in the Mirzakhani theory.

Our Virasoro structure also bears an interesting consequence: it leads to the natural normalization of the Weil-Petersson volume of the moduli spaces of bordered (or unbordered) Riemann surfaces. Although the geometric orbifold picture and the algebraic stack picture give the same moduli space for most of the cases, there is one exception: the moduli space of one-pointed stable elliptic curves $\overline{\mathcal{M}}_{1,1}$. If we define this space as an orbifold, then its canonical Weil-Petersson volume is $\zeta(2) = \pi^2/6$. On the other hand, the Virasoro constraint condition dictates that we need to have

$$\mathrm{Vol}(\overline{\mathcal{M}}_{1,1}) = \frac{\zeta(2)}{2}$$

as its canonical symplectic volume. This makes sense if we consider $\overline{\mathcal{M}}_{1,1}$ as an algebraic stack. The factor 2 difference is due to the fact that every elliptic curve with one marked point possesses a $\mathbb{Z}/2\mathbb{Z}$ automorphism. It is remarkable that even a purely hyperbolic geometry argument leads us to this stack picture.

To summarize our main results, let us consider the rational volume of $\mathcal{M}_{g,n}(\boldsymbol{L})$ defined by

$$v_{g,n}(\boldsymbol{L}) \stackrel{\text{def}}{=} \frac{\mathrm{Vol}\left(\mathcal{M}_{g,n}(\boldsymbol{L})\right)}{2^d \pi^{2d}},$$

where $d = 3g-3+n$ and

$$\mathrm{Vol}\left(\mathcal{M}_{g,n}(\boldsymbol{L})\right) \stackrel{\text{def}}{=} \int_{\mathcal{M}_{g,n}(\boldsymbol{L})} \frac{\omega_{WP}^d}{d!}$$

is the Weil-Petersson volume of $\mathcal{M}_{g,n}(\boldsymbol{L})$. Then the Mirzakhani recursion formula reads

$$\begin{aligned} v_{g,n}(\boldsymbol{L}) = &\frac{2}{L_1}\int_0^{L_1}\int_0^\infty\int_0^\infty xyK(x+y,t)v_{g-1,n+1}(x,y,\boldsymbol{L}_{\hat{1}})dxdydt \\ &+ \frac{2}{L_1}\sum_{\substack{g_1+g_2=g\\ \mathcal{I}\coprod\mathcal{J}=\{2,\dots,n\}}}\int_0^{L_1}\int_0^\infty\int_0^\infty xyK(x+y,t)v_{g_1,n_1}(x,\boldsymbol{L}_{\mathcal{I}}) \\ &\qquad\qquad\qquad\qquad \times v_{g_2,n_2}(y,\boldsymbol{L}_{\mathcal{J}})dxdydt \\ &+ \frac{1}{L_1}\sum_{j=2}^n\int_0^{L_1}\int_0^\infty x\left(K(x,t+L_j)+K(x,t-L_j)\right) \\ &\qquad\qquad\qquad \times v_{g,n-1}(x,\boldsymbol{L}_{\widehat{\{1,j\}}})dxdt, \end{aligned}$$

where the kernel function of the integral transform is given by

$$K(x,t) = \frac{1}{1+e^{\pi(x+t)}} + \frac{1}{1+e^{\pi(x-t)}},$$

and the symbol $\widehat{\ }$ indicates the complement of the indices. Recall that our normalized Weil-Petersson volume is a polynomial in $\boldsymbol{L}$ with coefficients given by intersection numbers of κ_1 and ψ-classes:

$$v_{g,n}(\boldsymbol{L}) = \sum_{\substack{d_0+\cdots+d_n\\ =d}}\prod_{i=0}^n \frac{1}{d_i!}\left\langle \kappa_1^{d_0}\prod \tau_{d_i}\right\rangle_{g,n}\prod_{i=1}^\infty L_i^{2d_i}.$$

Instead of defining our generating function directly from these rational volumes, let us consider the generating function of the mixed κ_1 and ψ-class intersections

$$\begin{aligned} G(s,t_0,t_1,t_2,\dots) &\stackrel{\text{def}}{=} \sum_g \left\langle e^{s\kappa_1+\sum t_i\tau_i}\right\rangle_g \\ &= \sum_g\sum_{m,\{n_i\}}\left\langle \kappa_1^m\tau_0^{n_0}\tau_1^{n_1}\cdots\right\rangle_g \frac{s^m}{m!}\prod_{i=0}^\infty \frac{t_i^{n_i}}{n_i!}. \end{aligned}$$

The main results of the present paper are the following *differential* version of the integral recursion formula.

Theorem 1.1. *For every $k \geq -1$, let us define*

$$V_k = -\frac{1}{2}\sum_{i=0}^\infty (2(i+k)+3)!!\alpha_i s^i \frac{\partial}{\partial t_{i+k+1}} + \frac{1}{2}\sum_{j=0}^\infty \frac{(2(j+k)+1)!!}{(2j-1)!!} t_j \frac{\partial}{\partial t_{j+k}}$$

$$+\frac{1}{4}\sum_{\substack{d_1+d_2=k-1\\ d_1,d_2\geq 0}}(2d_1+1)!!(2d_2+1)!!\frac{\partial^2}{\partial t_{d_1}\partial t_{d_2}}+\frac{\delta_{k,-1}t_0^2}{4}+\frac{\delta_{k,0}}{48},$$

where $\alpha_i = \frac{(-2)^i}{(2i+1)!}$. *Then we have:*

(1) *The operators* V_k *satisfy Virasoro relations*

$$[V_n, V_m] = (n-m)V_{n+m}.$$

(2) *The function* $\exp(G)$ *satisfies the Virasoro constraint condition*

$$V_k \exp(G) = 0 \quad \text{for } k \geq -1.$$

Moreover, these properties uniquely determine G *and enable one to calculate all coefficients of the expansion. Since* G *contains all information of the rational volumes* $v_{g,n}(\boldsymbol{L})$, *we conclude that the Virasoro constraint condition is indeed equivalent to the Mirzakhani recursion relation.*

Since the generating function for ψ-class intersections

$$F(t_0, t_1, \ldots) = \sum_g \langle e^{\sum \tau_i t_i}\rangle_g$$
$$= \sum_g \sum_{\{n_*\}} \langle \prod \tau_i^{n_i}\rangle_g \prod \frac{t_i^{n_i}}{n_i!}$$

is a solution of the KdV hierarchy, it is natural to ask if G satisfies any integrable equations. Indeed, we prove the following.

Theorem 1.2. *The function* $\exp(G)$ *is a* τ*-function for the KdV hierarchy for any fixed value of* s. *In fact, we have an explicit relation*

$$G(s, t_0, t_1, \ldots) = F(t_0, t_1, t_2 + \gamma_2, t_3 + \gamma_3, \ldots), \tag{1.1}$$

where $\gamma_i = \frac{(-1)^i}{(2i+1)i!}s^{i-1}$.

We remark that it is well known to algebraic geometers that generating functions F and G contain the same information [2, 3, 7, 12, 16, 26].

An important consequence of Theorem 1.2 is that G is also completely determined by the property of being a τ-function, together with the string equation $V_{-1}\exp(G) = 0$. It is fruitful to think of the string equation as being the initial condition for the KdV flow. Since G is determined, we note that Theorem 1.2 is again equivalent to Mirzakhani's recursion formula.

Here we recall that in the theory of integrable systems, every variable has a weighted degree so that all natural operators have homogenous weights. Coming from the KdV equations, we assign $\deg t_j = 2j+1$. The quantity γ_j has the same degree, which defines that $\deg s^j = 2j+3$. The Virasoro operator V_k then has homogenous degree $-2k$ for every $k \geq -1$. Another way to view the degree of s^i comes from the generalized Kontsevich integral

$$\log \int_{\mathcal{H}_N} e^{i\sum_{j=0}^{\infty}(-\frac{1}{2})^j s_j \frac{\operatorname{tr} X^{2j+1}}{2j+1}} e^{-\frac{\operatorname{tr}(X^2\Lambda)}{2}}\, dX.$$

As indicated in the work of Mondello [20], there should be a substitution $s_j = c_j s^{j-1}$ which transforms the asymptotic expansion of the integral into the generating function G. Since s_j has degree $2j+1$, we confirm that s^j must have degree $2j+3$. As well, we should point out that it is quite natural for (1.1) to leave variables t_0 and

t_1 unchanged. The reason is that in any expression relating intersections involving κ classes to those involving τ terms alone, τ_0 and τ_1 never make an appearance.

A few of the natural questions that crop up from this work are:

(1) Is there a matrix integral expression for the function G?
(2) Is it possible to prove that G is a solution to the KdV hierarchy without appealing to the Witten-Kontsevich theorem?
(3) What is the direct geometric connection between the cut and join mechanism and the Mirzakhani recursion?

We note that the essence of the original Virasoro constraint conjecture is that the generating function of Gromov-Witten invariants *should* have a matrix integral expression. The analysis of matrix integrals [1] indicates that once a matrix integral formula is established, the Virasoro constraints and integrable systems of KdV type are obvious consequences. Since the ribbon graph expansion method provides a powerful tool to matrix integrals, the very existence of both the KdV equations and the Virasoro constraint for the Weil-Petersson volume of the moduli spaces of bordered Riemann surfaces points to a matrix model expression and ribbon graph interpretation of the Mirzakhani formulas. These questions, however, are beyond the scope of our present work.

This paper is organized as follows. In section 2 we review the work of Mirzakhani [18, 19]. Since the Virasoro structure very delicately depends on all the subtle points of the theory, we provide a detailed discussion on some of the key ingredients of the work, including the case of genus one with one boundary, precise combinatorial description of cutting a surface along geodesics, and the choice of a canonical orientation of the circle bundle when the Duistermaat-Heckman formula is applied to the extended moduli spaces. Section 3 gives a proof of Theorem 1.1. Finally, in section 4 we prove Theorem 1.2.

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2. Mirzakhani's Recursion Relation

2.1. **Notations.** Since the orbifold picture and the stack structure are the same for the moduli spaces of algebraic curves except for genus 1 with one marked point, we employ the orbifold view point throughout the paper. As mentioned above, however, when we interpret the canonical volume of the moduli spaces, we need to use the stack picture.

Let $\mathcal{M}_{g,n}$ denote the moduli space of smooth algebraic curves, or equivalently, the moduli orbifold consisting of finite area hyperbolic metrics on a surface, of topological type (g, n). A surface of type (g, n) is a surface with genus g and n punctures. Since we are interested in the stable, noncompact case, we impose $2g - 2 + n > 0$ and $n > 0$ throughout this paper. When referring to the underlying topological type of the surface we will consistently employ the notation $S_{g,n}$. We will also use the notation $\mathcal{M}_S$ for the moduli space of surfaces of topological type S.

Mirzakhani's breathtaking theory is about the moduli space $\mathcal{M}_{g,n}(\boldsymbol{L})$ of genus g hyperbolic surfaces with n geodesic boundary components of specified length $\boldsymbol{L} = (L_1, \ldots, L_n)$. This space relates to the algebro-geometric moduli space via the equality $\mathcal{M}_{g,n} = \mathcal{M}_{g,n}(0)$. The moduli space of bordered Riemann surfaces is defined as an orbifold

$$\mathcal{M}_{g,n}(\boldsymbol{L}) = \mathcal{T}_{g,n}(\boldsymbol{L}) / \operatorname{Mod}_{g,n},$$

where $\operatorname{Mod}_{g,n}$ is the mapping class group of the surface of type (g,n), i.e., the set of isotopy classes of diffeomorphisms which preserve the boundaries setwise, and $\mathcal{T}_{g,n}(\boldsymbol{L})$ is the Teichmüller space. The Deligne-Munford type compactification of this moduli space is obtained by pinching non-trivial cycles.

The tautological classes we consider in this paper are the κ_1 and ψ classes. Let

$$\pi : \overline{\mathcal{M}}_{g,n+1} (= \mathcal{C}_{g,n}) \longrightarrow \overline{\mathcal{M}}_{g,n}$$

be the forgetful morphism which forgets the $n+1$-st marked point, and

$$\sigma_i(C, x_1, \ldots, x_n) = x_i \in C, \qquad i = 1, 2, \ldots, n$$

its canonical sections. We denote by $\omega_{\mathcal{C}/\mathcal{M}}$ the relative dualizing sheaf, and let $\mathcal{D}_i = \sigma_i(\overline{\mathcal{M}}_{g,n})$, which is a divisor in $\overline{\mathcal{M}}_{g,n+1}$. The tautological classes are defined by

$$\begin{aligned}
\mathcal{L}_i &= \sigma_i^*(\omega_{\mathcal{C}/\mathcal{M}}), \\
\psi_i &= c_1(\mathcal{L}_i), \\
\kappa_1 &= \pi_*\left(c_1\big(\omega_{\mathcal{C}/\mathcal{M}}(\textstyle\sum \mathcal{O}(\mathcal{D}_i)\big)^2 \right).
\end{aligned}$$

We are interested in the intersection numbers

$$\left\langle \kappa_1^m \tau_{d_1} \cdots \tau_{d_n} \right\rangle_g = \int_{\overline{\mathcal{M}}_{g,n}} \kappa_1^m \psi_1^{d_1} \cdots \psi_n^{d_n}.$$

The class κ_1 has a nice geometric interpretation, coming from the symplectic structure of $\mathcal{M}_{g,n}$. The Fenchel-Nielsen coordinates are associated with a pair of pants decompostion of the surface $S_{g,n}$, which is a disjoint set of simple closed curves $\Gamma = \{\gamma_1, \ldots, \gamma_d\}$ such that $S_{g,n} \setminus \Gamma$ is a disjoint union of pairs of pants (triply punctured spheres). Since pairs of pants have no moduli (they are uniquely fixed after specifying the boundary lengths), all that remains to recover the original hyperbolic structure is to specify how the matching geodesics are glued together. Hence for every curve γ_i in the pair of pants decomposition, one has the freedom of two parameters (l_i, τ_i) where l_i is the length of the curve and τ_i is the twist parameter. These coordinates give an isomorphism $\mathcal{T}_{g,n} = \mathbb{R}_+^d \times \mathbb{R}^d$. At the moduli space level we must quotient out by the mapping class group action. Since one full twist around a curve is a Dehn twist (element of $\operatorname{Mod}_{g,n}$), we get local coordinates of the form $\mathbb{R}_+^d \times (S^1)^d$.

In Fenchel-Nielsen coordinates, the Weil-Petersson form is in Darboux coordinates [24]:

$$\omega_{WP} = \sum dl_i \wedge d\tau_i.$$

This is a closed, nondegenerate 2-form on $\mathcal{T}_{g,n}$ which is invariant under the action of the mapping class group, hence gives a well defined symplectic form on $\mathcal{M}_{g,n}$. Note that by Wolpert [25] the Weil-Petersson form extends as a closed current on $\overline{\mathcal{M}}_{g,n}$. In particular, the Weil-Petersson volume

$$\operatorname{Vol}_{g,n}(\boldsymbol{L}) \stackrel{\text{def}}{=} \int_{\mathcal{M}_{g,n}(\boldsymbol{L})} \frac{\omega_{WP}^d}{d!}$$

is a finite quantity. (Note that Wolpert defines the Weil-Petersson form to be half of the above expression. Our convention is adopted from the algebraic geometry community [2, 12, 26].) The relation to tautological classes is provided by the well-known formula

$$\omega_{WP} = 2\pi^2 \kappa_1.$$

For use in the sequel, we note that the vector field generated by a Fenchel-Nielsen twist about a simple closed geodesic is symplectically dual to the length of the geodesic. A Fenchel-Nielsen twist is defined by cutting the surface along the curve, twisting one component with respect to the other and than regluing. As a formula, we have

$$\omega_{WP}(\,\cdot\,, \frac{\partial}{\partial \tau_i}) = dl_i.$$

2.2. **McShane's identity.** A crucial step in Mirzakhani's program [18, 19] is to use McShane's identity to write a constant function on the moduli space as a sum over mapping class group orbits of simple closed curves. To state Mirzakhani's generalization of McShane's identity, we introduce the following notation for an arbitrary hyperbolic surface X with boundaries $(\beta_1, \dots, \beta_n)$ of length $(L_1, \dots, L_n)$: $\mathcal{I}_j$ denotes the set of simple closed geodesics γ such that $(\beta_1, \beta_j, \gamma)$ bound a pair of pants; and $\mathcal{J}$ the set of pairs of simple closed geodescis (α_1, α_2) such that $(\beta_1, \alpha_1, \alpha_2)$ bound a pair of pants. Using the functions

$$\mathcal{D}(x, y, z) = 2 \log \left(\frac{e^{x/2} + e^{(y+z)/2}}{e^{-x/2} + e^{(y+z)/2}} \right) \text{ and}$$
$$\mathcal{R}(x, y, z) = x - \log \left(\frac{\cosh \frac{y}{2} + \cosh \frac{x+z}{2}}{\cosh \frac{y}{2} + \cosh \frac{x-z}{2}} \right),$$

Mirzakhani proves

Theorem 2.1 (Mirzakhani [18])**.** *For X as above, we have*

$$L_1 = \sum_{(\alpha_1, \alpha_2) \in \mathcal{J}} \mathcal{D}\big(L_1, l(\alpha_1), l(\alpha_2)\big) + \sum_{j=2}^{n} \sum_{\gamma \in \mathcal{I}_j} \mathcal{R}\big(L_1, L_j, l(\gamma)\big).$$

2.3. **Integration over the moduli spaces.** The idea is to find a fundamental domain for a particular cover of $\mathcal{M}_{g,n}$, enabling one to integrate functions over this covering space. Then a specific class of functions defined on the moduli space (such as those arising from McShane's identity) can be lifted to this cover and integrated. To that end, let $\Gamma = \{\gamma_1, \dots, \gamma_n\}$ be a collection of disjoint simple closed curves on the surface $S_{g,n}$, where $S_{g,n}$ is the underlying topology of the hyperbolic surface $X \in \mathcal{M}_{g,n}$. We define

$$\operatorname{Stab}\Gamma = \cap \operatorname{Stab}\gamma_i = \{f \in \operatorname{Mod}_{g,n} \mid f(\gamma_i) = \gamma_i, \text{ for } i = 1, \dots, n\},$$

and set

$$\begin{aligned}\mathcal{M}_{g,n}^{\Gamma} &= \mathcal{T}_{g,n}/\operatorname{Stab}\Gamma \\ &= \{(X, \eta_1, \dots, \eta_n) \,|\, X \in \mathcal{M}_{g,n},\ \eta_i \text{ is a simple closed geodesic in } \operatorname{Mod}\gamma_i\ \}.\end{aligned}$$

Note that as a quotient of Teichmüller space, $\mathcal{M}_{g,n}^{\Gamma}$ inherits the Weil-Petersson symplectic form. Hence we can talk about integration over $\mathcal{M}_{g,n}^{\Gamma}$ with respect to the symplectic volume form. The advantage of integration on $\mathcal{M}_{g,n}^{\Gamma}$ as opposed to the usual moduli space is that we can exploit the existence of a hamiltonian torus

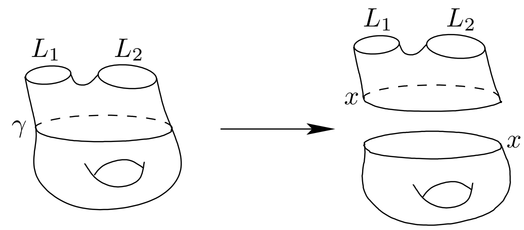

FIGURE 1. Decomposing a surface

action. In fact, by a result of Wolpert, the vector field generated by a Fenchel-Nielsen twist along a geodesic is symplectically dual to the length function of the geodesic (as a function on Teichmüller space). The space $\mathcal{M}_{g,n}^{\Gamma}$ is the intermediate covering space on which the circle actions on $\{\gamma_1, \cdots, \gamma_n\}$ descend. The problem with attempting to construct such a circle action on $\mathcal{M}_{g,n}$ is that there is no well defined notion of a geodesic curve on an element of moduli space; the best one can obtain is a mapping class group orbit of curves.

Thus we have the moment map for the torus action:

$$\begin{aligned} \boldsymbol{l} : \mathcal{M}_{g,n}^{\Gamma} &\to \mathbb{R}_+^n \\ (X, \boldsymbol{\eta}) &\mapsto \big(l(\eta_1), \ldots, l(\eta_n)\big). \end{aligned}$$

Hence we see that $\boldsymbol{l}^{-1}(\boldsymbol{x})/T^n$ is symplectomorphic to $\mathcal{M}_{S_{g,n}\setminus\Gamma}(\boldsymbol{L}, \boldsymbol{x}, \boldsymbol{x})$. Recall that $\mathcal{M}_{S_{g,n}\setminus\Gamma}$ is the moduli space with underlying topological type the (possibly disconnected) surface $S_{g,n}\setminus\Gamma$ and with boundary lengths following the rule outlined in Figure 1.

The most straightforward way to prove the above assertion is to take a pair of pants decomposition for the surface $S_{g,n}$ which contains the curves Γ. Then $\boldsymbol{l}^{-1}(\boldsymbol{x})$ fixes the lengths of the geodesics $\eta_1, \ldots, \eta_n$, while quotienting by the torus action removes the twist variable from these curves. This gives the diffeomorphism. The symplectic equivalence follows immediately from the Fenchel-Nielsen coordinate expression for the Weil-Petersson form. What emerges is an exceptionally clear local picture for the space $\mathcal{M}_{g,n}^{\Gamma}$. In fact, it is a fibre bundle over $\mathbb{R}_+^n$ where the fibres are (locally) equal to the product of a torus and $\mathcal{M}_{S_{g,n}\setminus\Gamma}(\boldsymbol{L}, \boldsymbol{x}, \boldsymbol{x})$.

Consider a map

$$f : \mathcal{M}_{g,n}^{\Gamma} \to \mathbb{R},$$

which is a function of the lengths of the marked geodesics $l(\eta_i)$. In other words, $f(X, \eta_1, \ldots, \eta_n) = f\big(l(\eta_1), \ldots, l(\eta_n)\big)$. By the previously discussed decomposition of $\mathcal{M}_{g,n}^{\Gamma}$ we can write

$$\int_{\mathcal{M}_{g,n}^{\Gamma}} f\big(\boldsymbol{l}(\boldsymbol{\eta})\big) e^{\omega_{WP}(\boldsymbol{L})} = \int_{\mathbb{R}_+^n} f(\boldsymbol{x}) \operatorname{Vol}_{S_{g,n}\setminus\Gamma}(\boldsymbol{L}, \boldsymbol{x}, \boldsymbol{x}) \boldsymbol{x} \cdot d\boldsymbol{x}. \tag{2.1}$$

Here $e^{\omega_{WP}}$ means we are integrating over the maximal power of the Weil-Petersson form $\frac{\omega_{WP}^d}{d!}$, $d = 3g-3+n$. Note that if $S_{g,n}\setminus\Gamma$ is disconnected then $\mathcal{M}_{S_{g,n}\setminus\Gamma}$ is the direct product of the component moduli spaces, with the volume being the product of each.

2.4. **Volume calculation.** To relate the above discussion to integration on the moduli space, Mirzakhani uses her generalized McShane identity. As a lead in to the main results, consider the following simplified situation. Suppose γ is a simple closed curve on $S_{g,n}$, with $\operatorname{Mod}\gamma$ its mapping class group orbit. Then given any hyperbolic structure X on $S_{g,n}$, every $\alpha \in \operatorname{Mod}\gamma$ has a unique geodesic in its isotopy class. Denote $l_X(\alpha)$ the corresponding geodesic length. Hence for any function $f : \mathbb{R}_+ \to \mathbb{R}$ and thinking of $X \in [X]$ as a representative of an element of $\mathcal{M}_{g,n}$ we have the following well defined function on $\mathcal{M}_{g,n}$:

$$\begin{aligned} f^\gamma : \mathcal{M}_{g,n} &\to \mathbb{R} \\ [X] &\mapsto \sum_{\alpha \in \operatorname{Mod}\gamma} l_X(\alpha). \end{aligned}$$

One can easily check that this function does not depend on the choice of representative $X \in [X]$. However, it is not a priori clear whether or not f^γ will be a convergent sum. At minimum one requires $\lim_{x\to\infty} f(x) = 0$. We can similarly define a function

$$\tilde{f}^\gamma : \mathcal{M}^\gamma_{g,n} \to \mathbb{R}$$

by the rule

$$\tilde{f}^\gamma(X, \eta) = f(l_X(\eta)),$$

which gives the relation

$$f^\gamma(X) = \sum_{(X,\eta)\in\pi^{-1}(X)} \tilde{f}^\gamma(X, \eta).$$

In particular, since the pullback of the Weil-Petersson form is the Weil-Petersson form on the cover, we have

$$\int_{\mathcal{M}_{g,n}} f^\gamma e^{\omega_{WP}} = \int_{\mathcal{M}^\gamma_{g,n}} \tilde{f}^\gamma e^{\omega_{WP}}.$$

Note that for any curve $\gamma \in \mathcal{I}_j$, we have $\mathcal{I}_j = \operatorname{Mod}\gamma$. This follows because two curves are in the same orbit of the mapping class group if and only if the surfaces obtained by cutting along the curves are homeomorphic, with a homeomorphism preserving the boundary components setwise. The homeomorphism will extend continuously to the curves to give the map of the entire surface. This tells us that the set $\mathcal{J}$ is not the orbit of a single pair of curves $(\alpha_1, \alpha_2) \in \mathcal{J}$. In fact, we further refine this set of curves as follows. For any $(\alpha_1, \alpha_2) \in \mathcal{J}$ set $P(\beta_1, \alpha_1, \alpha_2) \subset S_{g,n}$ to be the pair of pants bounded by the curves $\beta_1, \alpha_1, \alpha_2$. Now we define (see Figure 2)

$$\begin{aligned} \mathcal{J}_{\text{conn}} &= \{(\alpha_1, \alpha_2) \in \mathcal{J} \mid S_{g,n} \setminus P(\beta_1, \alpha_1, \alpha_2) \text{ is connected } \} \\ \mathcal{J}_{g_1,\{i_1,\ldots,i_{n_1}\}} &= \{(\alpha_1, \alpha_2) \in \mathcal{J} \mid S_{g,n} \setminus P(\beta_1, \alpha_1, \alpha_2) \text{ breaks into 2 pieces} \\ &\qquad \text{one of which is a surface of type } (g_1, n_1 + 1) \text{ with} \\ &\qquad \text{boundary } (\alpha_i, \beta_{i_1}, \ldots, \beta_{i_{n_1}}) \ \}. \end{aligned}$$

Other than the obvious identification

$$\mathcal{J}_{g_1,\{i_1,\ldots,i_{n_1}\}} = \mathcal{J}_{g-g_1,\{1,\ldots,n\}\setminus\{i_1,\ldots,i_{n_1}\}},$$

these subsets form disjoint orbits under the mapping class group. Moreover $\operatorname{Mod}_{g,n}$ acts transitively on each set.

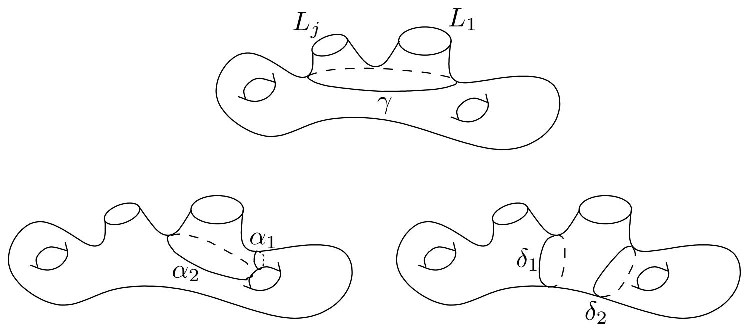

FIGURE 2. Removing a pair of pants from a surface

Hence we can write the Mirzakhani-McShane identity in the following form

$$\begin{aligned} L_1 = \frac{1}{2} &\sum_{\substack{g_1+g_2=g \\ \mathcal{A}\coprod\mathcal{B}= \\ \{2,\dots,n\}}} \sum_{(\alpha_1,\alpha_2)\in\mathcal{J}_{g_1,\mathcal{A}}} \mathcal{D}\big(L_1, l(\alpha_1), l(\alpha_2)\big) \\ &+ \sum_{(\delta_1,\delta_2)\in\mathcal{J}_{\mathrm{conn}}} \mathcal{D}\big(L_1, l(\delta_1), l(\delta_2)\big) \\ &+ \sum_{j=2}^{n} \sum_{\gamma\in\mathcal{I}_j} \mathcal{R}\big(L_1, L_j, l(\gamma)\big). \end{aligned}$$

There is a slight inaccuracy - we undercount by half for terms with $n = 1$ and $g_1 = g_2$. However, we will see in a moment that this makes further calculations somewhat simpler.

Each of the terms in the above sum can be lifted to a function on an appropriate cover $\mathcal{M}^{\Gamma}_{g,n}$. We see that

$$\begin{aligned} \int_{\mathcal{M}_{g,n}(\boldsymbol{L})} L_1 e^{\omega_{WP}} = \frac{1}{2} &\sum_{\substack{g_1+g_2=g \\ \mathcal{A}\coprod\mathcal{B}= \\ \{2,\dots,n\}}} \sum_{(\alpha_1,\alpha_2)\in\mathcal{J}_{g_1,\mathcal{A}}} \int_{\mathcal{M}_{g,n}(\boldsymbol{L})} \mathcal{D}(L_1, l(\alpha_1), l(\alpha_2)) e^{\omega_{WP}} \\ &+ \sum_{(\delta_1,\delta_2)\in\mathcal{J}_{\mathrm{conn}}} \int_{\mathcal{M}_{g,n}(\boldsymbol{L})} \mathcal{D}(L_1, l(\delta_1), l(\delta_2)) e^{\omega_{WP}} \\ &+ \sum_{j=2}^{n} \sum_{\gamma\in\mathcal{I}_j} \int_{\mathcal{M}_{g,n}(\boldsymbol{L})} \mathcal{R}(L_1, L_j, l(\gamma)) e^{\omega_{WP}}. \end{aligned}$$

Hence

$$\begin{aligned} L_1 \operatorname{Vol}_{g,n}(\boldsymbol{L}) = \frac{1}{2} &\sum_{\substack{g_1+g_2=g \\ \mathcal{A}\coprod\mathcal{B}=\{2,\dots,n\}}} \int_{\mathcal{M}^{\{\alpha_1,\alpha_2\}}_{g,n}} \mathcal{D}(L_1, l(\eta_1), l(\eta_2)) e^{\omega_{WP}} \\ &+ \frac{1}{2} \int_{\mathcal{M}^{\{\delta_1,\delta_2\}}_{g,n}} \mathcal{D}(L_1, l(\eta_1), l(\eta_2)) e^{\omega_{WP}} \\ &+ \sum_{j=2}^{n} \int_{\mathcal{M}^{\gamma}_{g,n}} \mathcal{R}(L_1, L_j, l(\gamma)) e^{\omega_{WP}}. \end{aligned}$$

Note that the factor $\frac{1}{2}$ that appears in front of the second term in the sum is needed because the mapping class group orbit double counts the set of curves (δ_1, δ_2). In other words, there is a diffeomorphism exchanging δ_1 and δ_2. Similarly, the factor $\frac{1}{2}$ discrepancy for the term in the first sum with $g_1 = g_2$ and $n = 1$ has now disappeared and the presented sum is unambiguously correct. Applying the results of the previous section we see that

$$\begin{aligned} L_1 \operatorname{Vol}_{g,n}(\boldsymbol{L}) &= \frac{1}{2} \sum_{\substack{g_1+g_2=g \\ \mathcal{A} \coprod \mathcal{B}}} \int_{\mathbb{R}_+^2} xy\mathcal{D}(L_1, x, y) \operatorname{Vol}_{g_1,n_1}(x, \boldsymbol{L}_{\mathcal{A}}) \operatorname{Vol}_{g_2,n_2}(y, \boldsymbol{L}_{\mathcal{B}}) dxdy \\ &+ \frac{1}{2} \int_{\mathbb{R}_+^2} xy\mathcal{D}(L_1, x, y) \operatorname{Vol}_{g-1,n+1}(x, y, \boldsymbol{L}_{\hat{1}}) dxdy \\ &+ \sum_{j=2}^{n} \int_{\mathbb{R}_+} x\mathcal{R}(L_1, L_j, x) \operatorname{Vol}_{g,n-1}(x, \boldsymbol{L}_{\widehat{1,j}}) dx. \end{aligned}$$

For any subset $\mathcal{A} = \{i_1, \ldots, l_k\} \subset \{1, \ldots, n\}$ the notation $\boldsymbol{L}_{\mathcal{A}}$ means the vector $(L_{i_1}, \ldots, L_{i_k})$ while $\boldsymbol{L}_{\hat{\mathcal{A}}} = \boldsymbol{L}_{\{1,\ldots,n\}\setminus\mathcal{A}}$.

There is one additional subtlety that crops up at this point. Note that for the case of $\mathcal{M}_{1,1}(L)$, there is an order two automorphism obtained by rotating around the boundary by half a turn. There are two ways to deal with this issue. The approach taken in [18] is to divide the appropriate integrals by 2 every time such a term appears in the above integral. Our approach, which is computationally equivalent, is to *define* the volume of $\mathcal{M}_{1,1}(L)$ to be half the value obtained by calculations using the above techniques. In other words, we have initial conditions

$$\begin{aligned} \operatorname{Vol}_{0,3}(\boldsymbol{L}) &= 1 \\ \operatorname{Vol}_{1,1}(L) &= \frac{1}{48}(L^2 + 4\pi^2). \end{aligned}$$

We will see that this viewpoint simplifies further calculations; as well, it agrees with known results from algebraic geometry.

The final step is to differentiate both sides with respect to L_1 and then integrate, which has the effect of simplifying the integrands on the right side of the equation. The result is

$$\begin{aligned} \operatorname{Vol}_{g,n}(\boldsymbol{L}) &= \frac{1}{2L_1} \sum_{\substack{g_1+g_2=g \\ \mathcal{A} \coprod \mathcal{B}}} \int_0^{L_1} \int_0^{\infty} \int_0^{\infty} xyH(t, x+y) \\ &\qquad \times \operatorname{Vol}_{g_1,n_1}(x, \boldsymbol{L}_{\mathcal{A}}) \operatorname{Vol}_{g_2,n_2}(y, \boldsymbol{L}_{\mathcal{B}}) dxdydt \\ &+ \frac{1}{2L_1} \int_0^{L_1} \int_0^{\infty} \int_0^{\infty} xyH(t, x+y) \\ &\qquad \times \operatorname{Vol}_{g-1,n+1}(x, y, \boldsymbol{L}_{\hat{1}}) dxdydt \\ &+ \frac{1}{2L_1} \sum_{j=2}^{n} \int_0^{L_1} \int_0^{\infty} x\big(H(x, L_1 + L_j) + H(x, L_1 - L_j)\big) \\ &\qquad \times \operatorname{Vol}_{g,n-1}(x, \boldsymbol{L}_{\widehat{1,j}}) dxdt, \end{aligned}$$

where

$$H(x, y) = \frac{1}{1 + e^{(x+y)/2}} + \frac{1}{1 + e^{(x-y)/2}}.$$

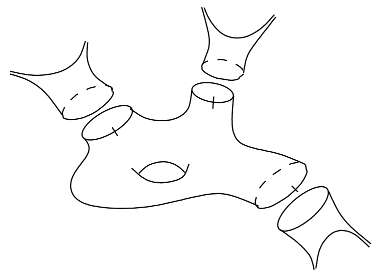

FIGURE 3. Capping off a bordered surface

2.5. **Relation to intersection numbers.** In this subsection we review the idea of Mirzakhani to write the integral over $\mathcal{M}_{g,n}(\boldsymbol{L})$ as an integral of an appropriately modified volume form over $\mathcal{M}_{g,n}$. This will relate the Weil-Petersson volumes to intersection numbers of tautological classes. Following [19], let

$$\widehat{\mathcal{M}}_{g,n} = \{(X, p_1, \dots, p_n) \,|\, X \in \overline{\mathcal{M}}_{g,n}(\boldsymbol{L}), \boldsymbol{L} \in \mathbb{R}^n_{\geq 0}, p_i \in \beta_i\}$$

be the moduli space of bordered hyperbolic surfaces of arbitrary boundary length, with the additional information of a marked point on each boundary component. If $L_i = 0$, then we can think of p_i as a point on a horocycle about the cusp.

The marked point can be used as a twist parameter, so by gluing on pairs of pants with two cusps and the third boundary having length matching the surface's boundary, we obtain a map $\widehat{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,2n}$. In fact, we have $\widehat{\mathcal{M}}_{g,n} = \overline{\mathcal{M}}^{\Gamma}_{g,2n}$ where $\Gamma = \{\gamma_1, \dots, \gamma_n\}$ is a collection of curves which group the cusps into pairs. We refer to Figure 3 for a descriptive picture of this construction.

This tells us that $\widehat{\mathcal{M}}_{g,n}$ has a symplectic structure from the Weil-Petersson form on $\mathcal{M}^{\Gamma}_{g,2n}$. Moreover it has a hamiltonian torus action given by rotating the marked points on the boundary. However, we need to take some care here. We are interested in studying the symplectic action in a neighborhood of surfaces $X \in \widehat{\mathcal{M}}_{g,n}$ with $l(\beta_i) = 0$. Moreover, we want to construct the action in such a way that these points are not fixed by the torus action. In other words, we need to non-trivially extend the action to the cusped surfaces.

It is a simple matter of defining the twists to be proportional to the lengths of the boundaries. In other words, we scale the action so that a twist parameter of 1 is always the identity. The model is the change from the cartesian (x, y) coordinates in the plane to the polar coordinate (r, θ). In the first case rotation around the origin leaves it fixed, but $(0, \theta)$ is not fixed by $\theta \mapsto \theta + \epsilon$. From the point of view of $\widehat{\mathcal{M}}_{g,n}$, the marking on the boundary degenerates to a marking on a horocycle of the cusp. The result, after a change of coordinates to the reparametrized twist coordinate $\theta_i = \tau_i/l_i$, is

$$\omega_{WP} = \sum l_i dl_i \wedge d\theta_i,$$

and the moment map corresponding to the twist vector field $\frac{\partial}{\partial \theta_i}$ is $\frac{1}{2}l_i^2$.

Given the map $\boldsymbol{L} : \widehat{\mathcal{M}}_{g,n} \to \mathbb{R}^n_{\geq 0}$ determined by mapping the marked surface $X \in \widehat{\mathcal{M}}_{g,n}$ to the lengths of its boundary components, we see that $\boldsymbol{L}^{-1}(\boldsymbol{x})$ is a principal torus bundle over $\overline{\mathcal{M}}_{g,n}(\boldsymbol{x})$. In fact, over $\overline{\mathcal{M}}_{g,n}(0)$ it is the principal bundle associated to the vector bundle $\mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_n$. At first glance this is a rather counter-intuitive statement, as marked points on the boundary map naturally to the tangent bundle, rather than the cotangent bundle. However, the principal torus bundle in question is naturally oriented from the induced orientations on the boundaries coming from the orientation of the surface. This orientation is opposite to the natural complex orientation on the tangent bundle. This is most easily seen by studying the clockwise orientation induced on the unit circle from the standard orientation of the plane.

Using symplectic reduction, we see that the reduced space $\boldsymbol{L}^{-1}(\boldsymbol{x})/T^n$ is symplectomorphic to $\overline{\mathcal{M}}_{g,n}(\boldsymbol{x})$ with the Weil-Petersson form. We may use the techniques of the Duistermaat–Heckman theorem to compare $\omega_{WP}(\boldsymbol{L})$ to $\omega_{WP}(0)$. The result is

$$\omega_{WP}(\boldsymbol{L}) = \omega_{WP}(0) - \frac{1}{2}\sum L_i^2 \operatorname{Curv}(\mathcal{L}_i),$$

where $\operatorname{Curv}(\mathcal{L}_i)$ is the curvature of the bundle. Since $c_1(\mathcal{L}_i) = -\operatorname{Curv}(\mathcal{L}_i)$ we get

$$\omega_{WP}(\boldsymbol{L}) = \omega_{WP}(0) + \frac{1}{2}\sum L_i^2 \psi_i.$$

2.6. **A rational recursion relation.** Using Wolpert's equivalence $\kappa_1 = \frac{\omega_{WP}}{2\pi^2}$ we define the rational volume of $\mathcal{M}_{g,n}(\boldsymbol{L})$ to be

$$\begin{aligned} v_{g,n}(\boldsymbol{L}) &\stackrel{\text{def}}{=} \frac{\operatorname{Vol}_{g,n}(2\pi \boldsymbol{L})}{2^d \pi^{2d}} \qquad d = 3g-3+n \\ &= \frac{1}{d!}\int_{\mathcal{M}_{g,n}} (\kappa_1 + \sum L_i^2 \psi_i)^d \\ &= \sum_{\substack{d_0+\cdots+d_n \\ =d}} \prod_{i=0}^{n} \frac{1}{d_i!} \langle \kappa_1^{d_0} \prod \tau_{d_i} \rangle_{g,n} \prod_{i=1}^{\infty} L_i^{2d_i}. \end{aligned}$$

We reformulate Mirzakhani's recursion relation for $\operatorname{Vol}_{g,n}$ into a recursion relation for $v_{g,n}$. Making the above change of variables to the recursion relation gives

$$\begin{aligned} v_{g,n}(\boldsymbol{L}) = &\frac{2}{L_1}\int_0^{L_1}\int_0^\infty\int_0^\infty xyK(x+y,t) v_{g-1,n+1}(x,y,\boldsymbol{L}_{\hat{1}})dxdydt \\ &+ \frac{2}{L_1} \sum_{\substack{g_1+g_2=g \\ \mathcal{I}\coprod\mathcal{J}=\{2,\ldots,n\}}} \int_0^{L_1}\int_0^\infty\int_0^\infty xyK(x+y,t) \\ &\qquad\qquad \times v_{g_1,n_1}(x,\boldsymbol{L}_{\mathcal{I}}) v_{g_2,n_2}(y,\boldsymbol{L}_{\mathcal{J}})dxdydt \\ &+ \frac{1}{L_1}\sum_{j=2}^{n}\int_0^{L_1}\int_0^\infty x\left(K(x,t+L_j)+K(x,t-L_j)\right) \\ &\qquad\qquad \times v_{g,n-1}(x,\boldsymbol{L}_{\{\widehat{1,j}\}})dxdt, \end{aligned}$$

with normalizations $v_{0,3}(L) = 1$ and $v_{1,1}(L) = \frac{1}{24}(1+L^2)$.

The integral kernel $K(x,t)$ is defined as

$$K(x,t) = \frac{1}{1+e^{\pi(x+t)}} + \frac{1}{1+e^{\pi(x-t)}},$$

which gives the following integral identities:

$$\mathfrak{h}_{2k+1}(t) \stackrel{\text{def}}{=} \int_0^\infty \frac{x^{2k+1}}{(2k+1)!}K(x,t)dx \tag{2.2}$$

$$= \sum_{m=0}^{k+1} (-1)^{m-1}(2^{2m}-2)\frac{B_{2m}}{(2m)!}\frac{t^{2k+2-2m}}{(2k+2-2m)!},$$

$$\mathfrak{h}_{2i+2j+3}(t) = \int_0^\infty \int_0^\infty \frac{x^{2i+1}y^{2j+1}}{(2i+1)!(2j+1)!} K(x+y,t)dxdy. \tag{2.3}$$

Note that B_{2m} is the $2m$-th Bernoulli number.

3. From Mirzakhani's Recursion Relation to the Virasoro Algebra

Our aim is to show that the Mirzakhani recursion relations are equivalent to an algebraic constraint on the generating function for κ_1 and ψ class intersections. We introduce the formal generating function for all κ_1 and ψ class intersections

$$G(s, t_0, t_1, t_2, \ldots) \overset{\text{def}}{=} \sum_g \big\langle e^{s\kappa_1 + \sum t_i \tau_i}\big\rangle_g$$

$$= \sum_g \sum_{m, \{n_i\}} \big\langle \kappa_1^m \tau_0^{n_0} \tau_1^{n_1} \cdots \big\rangle_g \frac{s^m}{m!} \prod_{i=0}^\infty \frac{t_i^{n_i}}{n_i!}.$$

The main result of the paper is the following.

Theorem 3.1. *There exist a sequence of differential operators $V_{-1}, V_0, V_1, \ldots$ satisfying Virasoro relations*

$$[V_n, V_m] = (n-m)V_{n+m}$$

and annihilating $\exp(G)$*:*

$$V_k \exp(G) = 0 \quad \textit{for } k = -1, 0, 1, \ldots$$

This property uniquely fixes G and enables one to calculate all coefficients of the expansion.

The proof is obtained by differentiating Mirzakhani's recursion relation. For reference, we note that

$$\frac{\partial^{2k_1}}{\partial L_1^{2k_1}} \cdots \frac{\partial^{2k_n}}{\partial L_n^{2k_n}} v_{g,n}(\boldsymbol{L}) \tag{3.1}$$

$$= \sum_{\substack{d_0+\cdots+d_n=d \\ d_i \ge k_i}} \frac{1}{d_0!} \prod_{i=1}^n \left(\frac{(2d_i)!}{d_i!(2(d_i-k_i))!} L_i^{2(d_i-k_i)}\right) \big\langle \kappa_1^{d_0} \tau_{d_1} \cdots \tau_{d_n} \big\rangle_g,$$

and

$$\frac{\partial^{2k_1}}{\partial L_1^{2k_1}} \cdots \frac{\partial^{2k_n}}{\partial L_n^{2k_n}} v_{g,n}(0) = \frac{1}{k_0!} \prod_{i=1}^n \left(\frac{(2k_i)!}{k_i!}\right) \big\langle \kappa_1^{k_0} \tau_{k_1} \cdots \tau_{k_n} \big\rangle, \tag{3.2}$$

where $k_0 = 3g-3+n-\sum_{i=1}^n k_i$.

The recursion relation gives the following identity for $(g,n) \neq (0,3), (1,1)$.

$$\frac{\partial^{2k_1}}{\partial L_1^{2k_1}} \cdots \frac{\partial^{2k_n}}{\partial L_n^{2k_n}} v_{g,n}(0)$$

$$= \frac{\partial^{2k_1}}{\partial L_1^{2k_1}} \frac{2}{L_1} \int_0^{L_1} \int_0^\infty \int_0^\infty xyK(x+y,t)$$

$$
\begin{aligned}
&\times \frac{\partial^{2k_2}}{\partial L_2^{2k_2}} \cdots \frac{\partial^{2k_n}}{\partial L_n^{2k_n}} v_{g-1,n+1}(x, y, \boldsymbol{L}_{\hat{1}}) dx dy dt \Big|_{\boldsymbol{L}=0} \\
&+ \frac{\partial^{2k_1}}{\partial L_1^{2k_1}} \frac{2}{L_1} \sum_{\substack{g_1+g_2=g \\ \mathcal{I} \coprod \mathcal{J}}} \int_0^{L_1} \int_0^\infty \int_0^\infty xyK(x+y,t) \\
&\times \frac{\partial^{2k(\mathcal{I})}}{\partial \boldsymbol{L}_{\mathcal{I}}^{2k(\mathcal{I})}} v_{g_1,n_1}(x, \boldsymbol{L}_{\mathcal{I}}) \frac{\partial^{2k(\mathcal{J})}}{\partial \boldsymbol{L}_{\mathcal{J}}^{2k(\mathcal{J})}} v_{g_2,n_2}(x, \boldsymbol{L}_{\mathcal{J}}) dx dy dt \Big|_{\boldsymbol{L}=0} \\
&+ \sum_{j=2}^n \frac{\partial^{2(k_1+k_j)}}{\partial L_1^{2k_1} \partial L_j^{2k_j}} \frac{1}{L_1} \int_0^{L_1} \int_0^\infty x(K(x, t+L_j) + K(x, t-L_j)) \\
&\times \frac{\partial^{2k(\widehat{1,j})}}{\partial \boldsymbol{L}_{\widehat{1,j}}^{2k(\widehat{1,j})}} v_{g,n-1}(x, \boldsymbol{L}_{\widehat{1,j}}) dx dt \Big|_{\boldsymbol{L}=0}.
\end{aligned}
$$

Plugging in the expressions for derivatives of volume functions (3.1), (3.2) and integrating against the kernel function using (2.2) and (2.3) gives

$$
\begin{aligned}
&\frac{1}{k_0!} \prod_{i=1}^n \frac{(2k_i)!}{k_i!} \left\langle \kappa_1^{k_0} \tau_{k_1} \cdots \tau_{k_n} \right\rangle \\
&= \sum_{\substack{d_0+d_1+d_2 \\ =k_0+k_1-2}} \frac{(2d_1+1)!(2d_2+1)!}{d_0! d_1! d_2!} \prod_{i=2}^n \frac{(2k_i)!}{k_i!} \left\langle \kappa_1^{d_0} \tau_{d_1} \tau_{d_2} \boldsymbol{\tau}_{k(\hat{1})} \right\rangle_{g-1,n+1} \\
&\times \frac{\partial^{2k_1}}{\partial L_1^{2k_1}} \frac{2}{L_1} \int_0^{L_1} \mathfrak{h}_{2(d_1+d_2)+3}(t) dt \Big|_{L_1=0} \\
&+ \sum_{\substack{g_1+g_2=g \\ \mathcal{I} \coprod \mathcal{J}}} \sum_{\substack{d_0+d_1=3g_1-3 \\ +n_1-k(\mathcal{I}) \\ d_0'+d_1'=3g_2-3 \\ +n_2-k(\mathcal{J})}} \frac{(2d_1+1)!(2d_1'+1)!}{d_0! d_1! d_0'! d_1'!} \prod_{i=2}^n \frac{(2k_i)!}{k_i!} \left\langle \kappa_1^{d_0} \tau_{d_1} \boldsymbol{\tau}_{k(\mathcal{I})} \right\rangle_{g_1,n_1} \\
&\times \left\langle \kappa_1^{d_0'} \tau_{d_1'} \boldsymbol{\tau}_{k(\mathcal{J})} \right\rangle_{g_2,n_2} \frac{\partial^{2k_1}}{\partial L_1^{2k_1}} \frac{2}{L_1} \int_0^{L_1} \mathfrak{h}_{2(d_1+d_1')+3}(t) dt \Big|_{L_1=0} \\
&+ \sum_{j=2}^n \sum_{\substack{d_0+d_1= \\ k_0+k_1+k_j-1}} \frac{(2d_1+1)!}{d_0! d_1!} \prod_{i \neq 1,j} \frac{(2k_i)!}{k_i!} \left\langle \kappa_1^{d_0} \tau_{d_1} \boldsymbol{\tau}_{k(\widehat{1,j})} \right\rangle_{g,n-1} \\
&\times \frac{\partial^{2k_1}}{\partial L_1^{2k_1}} \frac{2}{L_1} \int_0^{L_1} \mathfrak{h}_{2d_1+1}^{(2k_j)}(t) dt \Big|_{L_1=0}.
\end{aligned}
$$

We rewrite this sum by introducing the sequence of nonnegative integers $\{n_0, n_1, n_2, \ldots\}$ such that $n_j = \#|\{k_i \,|\, i \neq 1, k_i = j\}|$, and relabel $k_1 = k$. In other words, we have

$$
\left\langle \kappa_1^{k_0} \tau_{k_1} \cdots \tau_{k_n} \right\rangle_g = \left\langle \kappa_1^{k_0} \tau_k \tau_0^{n_0} \tau_1^{n_1} \cdots \right\rangle_g.
$$

We further define

$$\beta_i = (-1)^{i-1} 2^i (2^{2i} - 2) \frac{B_{2i}}{(2i)!}, \tag{3.3}$$

which results in the equation

$$\begin{aligned}
(3.4)\quad & (2k+1)!! \big\langle \kappa_1^{k_0} \tau_k \prod_{i=0}^{\infty} \tau_i^{n_i} \big\rangle_g \\
&= \frac{1}{2} \sum_{\substack{d_0+d_1+d_2= \\ k_0+k-2}} \frac{k_0!}{d_0!} \beta_{(k_0-d_0)} (2d_1+1)!(2d_2+1)! \big\langle \kappa_1^{d_0} \tau_{d_1} \tau_{d_2} \prod_{i=0}^{\infty} \tau_i^{n_i} \big\rangle_{g-1,n+1} \\
&\quad + \frac{1}{2} \sum_{\substack{g_1+g_2=g \\ \{l_i\}+\{m_i\}=\{n_i\}}} \sum_{\substack{d_0+d_1=3g_1-3 \\ +n_1-k(\mathcal{I}) \\ d_0'+d_1'=3g_2-3 \\ +n_2-k(\mathcal{J})}} \frac{k_0!}{d_0! d_0'!} \beta_{(k_0-d_0-d_0')} (2d_1+1)!(2d_1'+1)! \\
&\qquad \times \prod_{i=0}^{\infty} \frac{n_i!}{l_i! m_i!} \big\langle \kappa_1^{d_0} \tau_{d_1} \prod \tau_i^{l_i} \big\rangle_{g_1} \big\langle \kappa_1^{d_0'} \tau_{d_1'} \prod \tau_i^{m_i} \big\rangle_{g_2} \\
&\quad + \sum_{j=0}^{\infty} \sum_{\substack{d_0+d_1= \\ k_0+k+j-1}} \frac{k_0!}{d_0!} \beta_{(k_0-d_0)} \frac{(2d_1+1)!}{(2j-1)!} n_j \big\langle \kappa_1^{d_0} \tau_{d_1} \tau_j^{-1} \prod \tau_i^{n_i} \big\rangle_g .
\end{aligned}$$

By looking at expressions of the form

$$\begin{aligned}
\frac{\partial G}{\partial t_i} &= \sum_g \sum_{m,\{n_i\}} \big\langle \kappa_1^m \tau_i \tau_0^{n_0} \tau_1^{n_1} \cdots \big\rangle_g \frac{s^m}{m!} \prod_{i=0}^{\infty} \frac{t_i^{n_i}}{n_i!}, \\
s^i t_j \frac{\partial G}{\partial t_k} &= \sum_g \sum_{m,\{n_i\}} \frac{m!}{(m-i)!} n_j \big\langle \kappa_1^{m-i} \tau_j^{-1} \tau_k \tau_0^{n_0} \tau_1^{n_1} \cdots \big\rangle_g \frac{s^m}{m!} \prod_{i=0}^{\infty} \frac{t_i^{n_i}}{n_i!}, \\
s^i \frac{\partial^2 G}{\partial t_j \partial t_k} &= \sum_g \sum_{m,\{n_i\}} \frac{m!}{(m-i)!} \big\langle \kappa_1^{m-i} \tau_j \tau_k \tau_0^{n_0} \tau_1^{n_1} \cdots \big\rangle_g \frac{s^m}{m!} \prod_{i=0}^{\infty} \frac{t_i^{n_i}}{n_i!}, \\
s^i \frac{\partial G}{\partial t_j} \frac{\partial G}{\partial t_k} &= \sum_{\substack{g \\ m,\{n_i\}}} \sum_{\substack{g_1+g_2=g \\ d_1+d_2=m-i \\ \{k_i\}+\{l_i\}=\{n_i\}}} \frac{m!}{d_1! d_2!} \big\langle \kappa_1^{d_1} \tau_j \tau_0^{k_0} \cdots \big\rangle_{g_1} \big\langle \kappa_1^{d_2} \tau_k \tau_0^{l_0} \cdots \big\rangle_{g_2} \frac{s^m}{m!} \prod_{i=0}^{\infty} \frac{t_i^{n_i}}{n_i!},
\end{aligned}$$

we see that (3.4) leads to the following expression for all $k > 0$:

$$\begin{aligned}
(2k+3)!! \frac{\partial}{\partial t_{k+1}} G &= \sum_{i,j=0}^{\infty} \frac{(2(i+j+k)+1)!!}{(2j-1)!!} \beta_i s^i t_j \frac{\partial}{\partial t_{i+j+k}} G \\
&\quad + \frac{1}{2} \sum_{i=0}^{\infty} \sum_{\substack{d_1+d_2= \\ i+k-1}} (2d_1+1)!!(2d_2+1)!! \beta_i s^i \left(\frac{\partial^2 G}{\partial t_{d_1} \partial t_{d_2}} + \frac{\partial G}{\partial t_{d_1}} \frac{\partial G}{\partial t_{d_2}} \right).
\end{aligned}$$

Note that similar expressions are possible for $k = -1, 0$ by taking special care of the base cases $(g, n) = (0, 3), (1, 1)$. We introduce the family of differential operators for $k \geq -1$

$$\begin{aligned}\hat{V}_k = &-\frac{(2k+3)!!}{2}\frac{\partial}{\partial t_{k+1}} + \delta_{k,-1}(\frac{t_0^2}{4}+\frac{s}{48}) + \frac{\delta_{k,0}}{48}\\ &+\frac{1}{2}\sum_{i,j=0}^{\infty}\frac{(2(i+j+k)+1)!!}{(2j-1)!!}\beta_i s^i t_j \frac{\partial}{\partial t_{i+j+k}}\\ &+\frac{1}{4}\sum_{i=0}^{\infty}\sum_{\substack{d_1+d_2=\\ i+k-1}}(2d_1+1)!!(2d_2+1)!!\beta_i s^i \frac{\partial^2}{\partial t_{d_1}\partial t_{d_2}}.\end{aligned}$$

We have proven the following statement.

Theorem 3.2. *For $k \geq -1$*

$$\hat{V}_k \exp(G) = 0.$$

A reasonable question is: what is the algebra spanned by the operators $\hat{V}_k$? The answer is that they span a subalgebra of the Virasoro algebra. One can check directly that the operators satisfy the relations

$$[\hat{V}_n, \hat{V}_m] = (n-m)\sum_{i=0}^{\infty}\beta_i s^i \hat{V}_{n+m+i}.$$

On the surface, this looks to be a deformation of the Virasoro relations (setting $s = 0$ recovers Virasoro). However, this is, in fact, a simple reparametrization of the representation. These statements can all be proved by direct calculations. Here we make some simplifications. Let us introduce new variables $\{T_{2j+1}\}_{j=0,1,\ldots}$ defined by

$$T_{2i+1} = \frac{t_i}{(2i+1)!!},$$

which transform the operators $\hat{V}_k$ into

$$\begin{aligned}\hat{V}_k = &-\frac{1}{2}\frac{\partial}{\partial T_{2k+3}} + \delta_{k,-1}(\frac{t_0^2}{4}+\frac{s}{48}) + \frac{\delta_{k,0}}{16}\\ &+\frac{1}{2}\sum_{i,j=0}^{\infty}(2j+1)\beta_i s^i T_{2j+1}\frac{\partial}{\partial T_{2(i+j+k)+1}}\\ &+\frac{1}{4}\sum_{i=0}^{\infty}\sum_{\substack{d_1+d_2=\\ i+k-1}}\beta_i s^i \frac{\partial^2}{\partial T_{2d_1+1}\partial T_{2d_2+1}}.\end{aligned}$$

This admits the following 'boson' representation, similar to that used by Kac and Schwarz [11]. Define operators J_p for $p \in \mathbb{Z}$ by

$$J_p = \begin{cases} (-p)T_{-p} & \text{if } p < 0,\\ \frac{\partial}{\partial T_p} & \text{if } p > 0.\end{cases}$$

Then

$$\hat{V}_k = -\frac{1}{2}J_{2k+3} + \sum_{i=0}^{\infty}\beta_i s^i E_{k+i},$$

where

$$E_k = \frac{1}{4}\sum_{p\in\mathbb{Z}} J_{2p+1}J_{2(k-p)-1} + \frac{\delta_{k,0}}{16}.$$

To recover operators satisfying the Virasoro constraint we need a better handle on the constants β_i, as defined by (3.3). Starting from the defining formula for the Bernoulli numbers

$$\sum_{n=0}^{\infty} \frac{B_{2n}}{(2n)!} z^{2n} = \frac{z}{2}\frac{e^{z/2}+e^{-z/2}}{e^{z/2}-e^{-z/2}},$$

we see that

$$\begin{aligned}\sum_{i=0}^{\infty} \beta_i s^i &= \sqrt{2s}(\cot\sqrt{s/2} - \cot\sqrt{2s}) \\ &= \frac{\sqrt{2s}}{\sin\sqrt{2s}}.\end{aligned}$$

This motivates the definition of the constants α_i by the series

$$\sum_{i=0}^{\infty} \alpha_i s^i = \frac{\sin\sqrt{2s}}{\sqrt{2s}},$$

from which we obtain the operators

$$\begin{aligned}V_k &\stackrel{\text{def}}{=} \sum_{i=0}^{\infty} \alpha_i s^i \hat{V}_{k+i} \\ &= -\frac{1}{2}\sum_{i=0}^{\infty} \alpha_i s^i J_{2k+3} + E_k. \qquad (3.5)\end{aligned}$$

We are now ready to prove the following.

Proposition 3.3. *The operators V_k, $k \geq -1$ satisfy the Virasoro relations*

$$[V_n, V_m] = (n-m)V_{n+m}.$$

Proof. The first step is to verify that operators E_k satisfy the Virasoro relations, which is a straightforward calculation. Since $[J_{2k+3}, E_m] = (2k+3)J_{2(k+m)+3}$ we see that

$$\begin{aligned}[V_n, V_m] &= \left[-\frac{1}{2}\sum_{i=0}^{\infty}\alpha_i s^i J_{2(n+i)+3} + E_n, -\frac{1}{2}\sum_{j=0}^{\infty}\alpha_i s^i J_{2(m+j)+3} + E_m\right] \\ &= -\frac{1}{2}\sum_{i=0}^{\infty}\alpha_i s^i\Big(\big[J_{2(n+i)+3}, E_m\big] + \big[E_n, J_{2(m+i)+3}\big]\Big) \\ &= (n-m)V_{n+m}.\end{aligned}$$

□

4. Relationship to KdV hierarchy

The Witten-Kontsevich theorem [23, 15] states that the generating function for ψ class intersections

$$\begin{aligned} F(t_0, t_1, \ldots) &= \sum_g \langle e^{\sum \tau_i t_i} \rangle_g \\ &= \sum_g \sum_{\{n_*\}} \langle \prod \tau_i^{n_i} \rangle_g \prod \frac{t_i^{n_i}}{n_i!} \end{aligned}$$

is a τ-function for the KdV hierarchy. The property of being a τ function, combined with the string equation

$$\langle \tau_0 \prod_{i=1}^n \tau_{d_i} \rangle_g = \sum_{j=1}^n \langle \prod_{i=1}^n \tau_{d_i - \delta_{ij}} \rangle_g,$$

completely determines the function F. Another way of determining F is the Virasoro constraint condition. Let us define the sequence of operators L_k for $k \geq -1$:

$$\begin{aligned} L_k = &-\frac{(2k+3)!!}{2} \frac{\partial}{\partial t_{k+1}} + \frac{1}{2} \sum_{j=0}^{\infty} \frac{(2(j+k)+1)!!}{(2j-1)!!} t_j \frac{\partial}{\partial t_{j+k}} \\ &+ \frac{1}{4} \sum_{\substack{d_1+d_2=k-1 \\ d_1, d_2 \geq 0}} (2d_1+1)!!(2d_2+1)!! \frac{\partial^2}{\partial t_{d_1} \partial t_{d_2}} + \frac{\delta_{k,-1} t_0^2}{4} + \frac{\delta_{k,0}}{48}. \end{aligned} \tag{4.1}$$

The Witten-Kontesevich theorem, together with the string equation, implies

$$L_k(\exp F) = 0$$

for $k \geq -1$. This property is also sufficient to uniquely fix F. Note that $L_{-1} e^F = 0$ is equivalent to the string equation. The consistency of the infinite set of differential equations follows from the fact that operators L_n satisfy the Virasoro relations:

$$[L_n, L_m] = (n-m) L_{n+m}.$$

Recall the operators V_k defined in equation 3.5 (rewritten in terms of the variables t_i)

$$\begin{aligned} V_k = &-\frac{1}{2} \sum_{i=0}^{\infty} (2(i+k)+3)!! \alpha_i s^i \frac{\partial}{\partial t_{i+k+1}} + \frac{1}{2} \sum_{j=0}^{\infty} \frac{(2(j+k)+1)!!}{(2j-1)!!} t_j \frac{\partial}{\partial t_{j+k}} \\ &+ \frac{1}{4} \sum_{\substack{d_1+d_2=k-1 \\ d_1, d_2 \geq 0}} (2d_1+1)!!(2d_2+1)!! \frac{\partial^2}{\partial t_{d_1} \partial t_{d_2}} + \frac{\delta_{k,-1} t_0^2}{4} + \frac{\delta_{k,0}}{48}, \end{aligned}$$

where $\alpha_i = \frac{(-2)^i}{(2i+1)!}$. The change of variables

$$\tilde{t}_i = \begin{cases} t_i & \text{for } i = 0, 1 \ , \\ t_i - (2i-1)!! \alpha_{i-1} s^{i-1} & \text{otherwise,} \end{cases}$$

transforms the operators V_k into

$$V_k = -\frac{1}{2} (2k+3)!! \frac{\partial}{\partial \tilde{t}_{k+1}} + \frac{1}{2} \sum_{j=0}^{\infty} \frac{(2(j+k)+1)!!}{(2j-1)!!} \tilde{t}_j \frac{\partial}{\partial \tilde{t}_{j+k}}$$

$$+\frac{1}{4}\sum_{\substack{d_1+d_2=k-1\\ d_1,d_2\geq 0}}(2d_1+1)!!(2d_2+1)!!\frac{\partial^2}{\partial\tilde{t}_{d_1}\partial\tilde{t}_{d_2}}+\frac{\delta_{k,-1}\tilde{t}_0^2}{4}+\frac{\delta_{k,0}}{48}.$$

But these are precisely the operators L_k (4.1). We have thus proven the following.

Theorem 4.1.

$$G(s,t_0,t_1,\dots)=F(t_0,t_1,t_2+\gamma_2,t_3+\gamma_3,\dots),$$

where $\gamma_i=\frac{(-1)^i}{(2i+1)i!}s^{i-1}$*. In particular, for any fixed value of* s*,* G *is a* τ *function for the KdV hierarchy.*

That the more general generating function G is expressible in terms of F is not a surprise. It has been known since at least the work of Witten [23] that intersections involving κ classes are expressible in terms of ψ classes. Moreover, Faber's formula [7] for this correspondence gives an explicit proof of the above theorem. In fact, one has

$$\kappa_1^n=\sum_{\substack{\sigma\in S_n\\ (\sigma=\gamma_1\cdots\gamma_k\\ \text{is cycle decomp})}}\frac{(-1)^{n-k}}{\prod_{i=1}^k(|\gamma_i|-1)!}\pi_{\{q_1,\dots q_k\}*}(\psi_{q_1}^{|\gamma_1|+1}\cdots\psi_{q_k}^{|\gamma_k|+1}),$$

which gives a short, direct proof of Theorem 4.1. This is essentially the approach taken by Zograf [26] for his calculation of the Weil-Petersson volumes of $\mathcal{M}_{g,n}$.

References

[1] M. Adler and P. van Moerbeke, *Hermitian, symmetric and symplectic random ensembles: PDEs for the distribution of the spectrum*, Ann. of Math. (2) **153** (2001), no. 1, 149–189.

[2] E. Arbarello and M. Cornalba, *Combinatorial and algebro-geometric cohomology classes on the moduli spaces of curves*, J. Algebraic Geom. **5** (1996), no. 4, 705–749, arXiv:alg-geom/9406008.

[3] ———, *Calculating cohomology groups of moduli spaces of curves via algebraic geometry*, Inst. Hautes Études Sci. Publ. Math. (1998), no. 88, 97–127, arXiv:math.AG/9803001.

[4] B. Dubrovin and Y. Zhang, *Frobenius manifolds and Virasoro constraints*, Selecta Math. (N.S.) **5** (1999), no. 4, 423–466, arXiv:math.AG/9808048.

[5] T. Eguchi, K. Hori, and C.-S. Xiong, *Quantum cohomology and Virasoro algebra*, Phys. Lett. B **402** (1997), no. 1-2, 71–80.

[6] T. Eguchi, M. Jinzenji, and C.-S. Xiong, *Quantum cohomology and free-field representation*, Nuclear Phys. B **510** (1998), no. 3, 608–622.

[7] C. Faber, *A conjectural description of the tautological ring of the moduli space of curves*, Moduli of curves and abelian varieties, Aspects Math., vol. E33, 1999, arXiv:math.AG/9711218.

[8] E. Getzler, *The Virasoro conjecture for Gromov-Witten invariants*, Algebraic geometry: Hirzebruch 70 (Warsaw, 1998), Contemp. Math., vol. 241, Amer. Math. Soc., Providence, RI, 1999, arXiv:math.AG/9812026, pp. 147–176.

[9] A. B. Givental, *Gromov-Witten invariants and quantization of quadratic Hamiltonians*, Mosc. Math. J. **1** (2001), no. 4, 551–568, 645, arXiv:math.AG/0108100.

[10] I. P. Goulden, D. M. Jackson, and A. Vainshtein, *The number of ramified coverings of the sphere by the torus and surfaces of higher genera*, Ann. Comb. **4** (2000), no. 1, 27–46.

[11] V. Kac and A. Schwarz, *Geometric interpretation of the partition function of 2d gravity*, Phys. Lett. B **257** (1991), no. 3-4, 329–334.

[12] R. Kaufmann, Yu. Manin, and D. Zagier, *Higher Weil-Petersson volumes of moduli spaces of stable n-pointed curves*, Comm. Math. Physics **181** (1996), no. 3, 763–787, arXiv:alg-geom/9604001.

[13] M. E. Kazarian and S. K. Lando, *An algebro-geometric proof of Witten's conjecture*, arXiv:math.AG/0601760.

[14] Y.S. Kim and K. Liu, *A simple proof of Witten conjecture through localization*, arXiv:math.AG/0508384.
[15] M. Kontsevich, *Intersection theory on the moduli space of curves and the matrix Airy function*, Comm. Math. Physics **147** (1992), no. 1, 1–23.
[16] Yu. Manin and P. Zograf, *Invertible cohomological field theories and Weil-Petersson volumes*, Ann. Inst. Fourier **50** (2000), no. 2, 519–535, arXiv:math.AG/9902051.
[17] G. McShane, *Simple geodesics and a series constant over Teichmuller space*, Invent. Math. **132** (1998), no. 3, 607–632.
[18] M. Mirzakhani, *Simple geodesics and Weil-Petersson volumes of moduli spaces of bordered riemann surfaces*, preprint.
[19] ———, *Weil-Petersson volumes and intersection theory on the moduli space of curves*, preprint.
[20] G. Mondello, *Combinatorial classes on the moduli space of curves are tautological*, arXiv:math.AT/0303207.
[21] A. Okounkov and R. Pandharipande, *Gromov–Witten theory, Hurwitz numbers, and matrix models, I*, arXiv:math.AG/0101147.
[22] ———, *Virasoro constraints for target curves*, arXiv:math.AG/0308097.
[23] E. Witten, *Two-dimensional gravity and intersection theory on moduli space*, Surveys in Differential Geometry **1** (1990), 243–310.
[24] S. Wolpert, *On the symplectic geometry of deformations of a hyperbolic surface*, Ann. of Math. **117** (1983), no. 2, 207–234.
[25] ———, *On the Weil-Petersson geometry of the moduli space of curves*, Amer. J. Math. **107** (1985), no. 4, 969–997.
[26] P. Zograf, *The Weil-Petersson volumes of moduli spaces of curves and the genus expansion in two dimensional gravity*, arXiv:math.AG/9811026.

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