LECTURES ON THE TOPOLOGICAL RECURSION FOR HIGGS BUNDLES AND QUANTUM CURVES

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ABSTRACT. The paper aims at giving an introduction to the notion of *quantum curves*. The main purpose is to describe the new discovery of the relation between the following two disparate subjects: one is the topological recursion, that has its origin in random matrix theory and has been effectively applied to many enumerative geometry problems; and the other is the quantization of Hitchin spectral curves associated with Higgs bundles. Our emphasis is on explaining the motivation and examples. Concrete examples of the direct relation between Hitchin spectral curves and enumeration problems are given. A general geometric framework of quantum curves is also discussed.

CONTENTS

> We're not going to tell you the story
> the way it happened.
> We're going to tell it
> the way we remember it.

1. Introduction

Mathematicians often keep their childhood dream for a long time. When you saw a perfect rainbow as a child, you might have wondered what awaited you when you went over the arch. In a lucky situation, you might have seen the double, or even triple, rainbow arches spanning above the brightest one, with the reversing color patterns on the higher arches. Yet we see nothing underneath the brightest arch.

One of the purposes of these lectures is to offer you a vision: **on the other side of the rainbow, you see quantum invariants.** This statement describes only the tip of the iceberg. We believe something like the following is happening: Let C be a smooth projective curve over $\mathbb{C}$, and

$$(1.1) \qquad \Sigma \xrightarrow{\ i\ } \overline{T^*C}$$
$$\pi \searrow \quad \downarrow \pi$$
$$C$$

be an arbitrary Hitchin *spectral curve* associated with a particular *meromorphic* Higgs bundle (E, ϕ) on C. Then the asymptotic expansion at an essential singularity of a solution (flat section) of the $\hbar$-connection on C, that is the image of the **quantization** applied to Σ, carries the information of quantum invariants of a totally different geometric structure, which should be considered as the **mirror** to the geometric context (1.1).

In this introduction, we are going to tell you a story of an example to this mirror correspondence using the rainbow integral of Airy. The Hitchin spectral curve is a singular compactification of a parabola $x = y^2$ in a Hirzebruch surface. The corresponding quantum invariants, the ones hiden underneath the rainbow, are the cotangent class intersection numbers of the moduli space $\overline{\mathcal{M}}_{g,n}$. These numbers then determine the coefficients of the tautological relations among the generators of the tautological rings of $\mathcal{M}_g$ and $\overline{\mathcal{M}}_{g,n}$. The uniqueness of the asymptotic expansion relates the WKB analysis of the quantization of the parabola at infinity to the intersection numbers on $\overline{\mathcal{M}}_{g,n}$, through a combinatorial estimate of the *volume* of the moduli space $\mathcal{M}_{g,n}$.
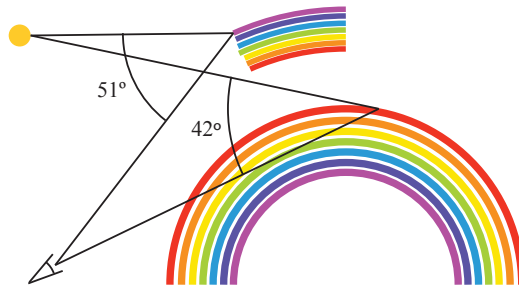
The story begins in 1838.

FIGURE 1.1. Rainbow archs

1.1. **On the other side of the rainbow.** Sir George Biddel Airy devised a simple formula, which he called the *rainbow integral*

$$(1.2) \qquad Ai(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ipx} e^{i\frac{p^3}{3}} \, dp$$

and now carries his name, in his attempt of explaining the rainbow phenomena in terms of wave optics [2]. The angle between the sun and the observer measured at the brightest arch is always about $42°$. The higher arches also have definite angles, independent of the rainbow. Airy tried to explain these angles and the brightness of the rainbow arches by the peak points of the rainbow integral.
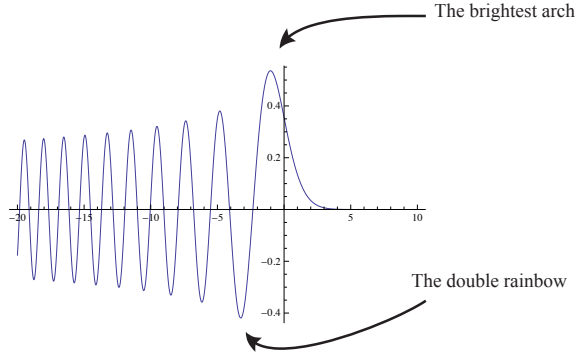
FIGURE 1.2. The Airy function

We note that (1.2) is an *oscillatory integral*, and determines a real analytic function in $x \in \mathbb{R}$. It is easy to see, by integration by parts and taking care of the boundary contributions in oscillatory integral, that

$$\frac{d^2}{dx^2} Ai(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (-p^2) e^{ipx} e^{i\frac{p^3}{3}} \, dp = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ipx} \left(i\frac{d}{dp} e^{i\frac{p^3}{3}} \right) dp$$

$$= -\frac{1}{2\pi} \int_{-\infty}^{\infty} \left(i\frac{d}{dp} e^{ipx} \right) e^{i\frac{p^3}{3}} \, dp = \frac{1}{2\pi} \, x \int_{-\infty}^{\infty} e^{ipx} e^{i\frac{p^3}{3}} \, dp.$$

Thus the Airy function satisfies a second order differential equation (known as the Airy differential equation)

$$(1.3) \qquad \left(\frac{d^2}{dx^2} - x \right) Ai(x) = 0.$$

Now we consider $x \in \mathbb{C}$ as a complex variable. Since the coefficients of (1.3) are entire functions (they are just 1 and x), any solution of this differential equation is automatically entire, and has a convergent power series expansion

$$(1.4) \qquad Ai(x) = \sum_{n=0}^{\infty} a_n x^n$$

at the origin with the radius of convergence ∞. Plugging (1.4) into (1.3), we obtain a recursion formula

$$(1.5) \qquad a_{n+2} = \frac{1}{(n+2)(n+1)} \, a_{n-1}, \qquad n \geq 0,$$

with the initial condition $a_{-1} = 0$. Thus we find

$$a_{3n} = a_0 \cdot \frac{\prod_{j=1}^n (3j-2)}{(3n)!}, \qquad a_{3n+1} = a_1 \cdot \frac{\prod_{j=1}^n (3j-1)}{(3n+1)!}, \qquad a_{3n+2} = 0.$$

Here, a_0 and a_1 are arbitrary constants, so that the Airy differential equation has a two-dimensional space of solutions. These coefficients do not seem to be particularly interesting. The oscillatory integral (1.2) tells us that as $x \to +\infty$ on the real axis, the Airy function defined by the rainbow integral vanishes, because $e^{ipx+ip^3/3}$ oscillates so much that the integral cancels. More precisely, $Ai(x)$ satisfies a limiting formula

$$(1.6) \qquad \lim_{x \to +\infty} \frac{Ai(x)}{\frac{1}{2\sqrt{\pi}} \cdot \frac{1}{\sqrt[4]{x}} \exp\left(-\frac{2}{3} x^{\frac{3}{2}}\right)} = 1.$$

Hence it exponentially decays, as $x \to +\infty$, $x \in \mathbb{R}$. Among the Taylor series solutions (1.4), there is *only one* solution that satisfies this exponential decay property, which is given by the following initial condition for (1.5):

$$a_0 = \frac{1}{3^{\frac{2}{3}} \Gamma(\frac{2}{3})}, \qquad a_1 = -\frac{1}{3^{\frac{1}{3}} \Gamma(\frac{1}{3})}.$$

The exponential decay on the positive real axis explains why we do not see any rainbows under the brightest arch. Then what do we really see underneath the rainbow? Or on the *other side* of the rainbow?

The differential equation (1.3) tells us that the Airy function has an essential singularity at $x = \infty$. Otherwise, the solution would be a polynomial in x, but (1.5) does not terminate at a finite n. How do we analyze the behavior of a holomorphic function at its essential singularity? And what kind of information does it tell us?

Definition 1.1 (Asymptotic expansion). Let $f(z)$ be a holomorphic function defined on an open domain Ω of the complex plane $\mathbb{C}$ having the origin 0 on its boundary. A formal power series

$$\sum_{n=0}^{\infty} a_n z^n$$

is an *asymptotic expansion* of $f(z)$ on Ω at $z = 0$ if

$$(1.7) \qquad \lim_{\substack{z \to 0 \\ z \in \Omega}} \frac{1}{z^{m+1}} \left(f(z) - \sum_{n=0}^m a_n z^n \right) = a_{m+1}$$

holds for every $m \geq 0$.

The asymptotic expansion is a domain specific notion. For example, $f(z) = e^{-1/z}$ is holomorphic on $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, but it does not have any asymptotic expansion on all of $\mathbb{C}^*$. However, it has an asymptotic expansion

$$e^{-1/z} \sim 0$$

on

$$\Omega = \left\{ z \in \mathbb{C}^* \;\middle|\; |\mathrm{Arg}(z)| < \frac{\pi}{2} - \epsilon \right\}, \qquad \epsilon > 0.$$

If there is an asymptotic expansion of f on a domain Ω, then it is unique, and if $\Omega' \subset \Omega$ with $0 \in \partial\Omega'$, then obviously the same asymptotic expansion holds on Ω'.

The Taylor expansion (1.4) shows that $Ai(x)$ is real valued on the real axis, and from (1.6), we see that the value is positive for $x > 0$. Therefore, $\log Ai(x)$ is a holomorphic function on $Re(x) > 0$.

Theorem 1.2 (Asymptotic expansion of the Airy function). *Define*

$$(1.8) \qquad S_0(x) = -\frac{2}{3}x^{\frac{3}{2}}, \qquad S_1(x) = -\frac{1}{4}\log x - \log(2\sqrt{\pi}).$$

Then $\log Ai(x) - S_0(x) - S_1(x)$ has the following asymptotic expansion on $Re(x) > 0$.

$$(1.9) \qquad \log Ai(x) - S_0(x) - S_1(x) = \sum_{m=2}^{\infty} S_m(x),$$

$$(1.10) \quad S_m(x) := x^{-\frac{3}{2}(m-1)} \cdot \frac{1}{2^{m-1}} \sum_{\substack{g \geq 0, n > 0 \\ 2g-2+n=m-1}} \frac{(-1)^n}{n!} \sum_{\substack{d_1+\cdots+d_n \\ =3g-3+n}} \langle \tau_{d_1} \cdots \tau_{d_n} \rangle_{g,n} \prod_{i=1}^{n} |2d_i - 1|!!$$

for $m \geq 2$. The key coefficients are defined by

$$(1.11) \qquad \langle \tau_{d_1} \cdots \tau_{d_n} \rangle_{g,n} := \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n},$$

where, $\overline{\mathcal{M}}_{g,n}$ is the moduli space of stable curves of genus g with n smooth marked points. Let $[C, (p_1, \ldots, p_n)] \in \overline{\mathcal{M}}_{g,n}$ be a point of the moduli space. We can construct a line bundle $\mathbb{L}_i$ on the smooth Deligne-Mumford stack $\overline{\mathcal{M}}_{g,n}$ by attaching the cotangent line $T^*_{p_i}C$ at the point $[C, (p_1, \ldots, p_n)]$ of the moduli space. The symbol

$$\psi_i = c_1(\mathbb{L}_i) \in H^2(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$$

denotes its first Chern class. Since $\overline{\mathcal{M}}_{g,n}$ has dimension $3g - 3 + n$, the integral (1.11) is automatically 0 unless $d_1 + \cdots + d_n = 3g - 3 + n$.

Surprisingly, on the other side of the rainbow, i.e., when $x > 0$, we see the intersection numbers (1.11)!

Remark 1.3. The relation between the Airy function and the intersection numbers was discovered by Kontsevich [55]. He replaces the variables x and p in (1.2) by *Hermitian matrices*. It is a general property that a large class of Hermitian matrix integrals satisfy an integrable system of KdV and KP type (see, for example, [62]). Because of the cubic polynomial in the integrand, the matrix Airy function of [55] counts trivalent ribbon graphs through the Feynman diagram expansion, which represent open dense subsets of $\overline{\mathcal{M}}_{g,n}$. By identifying the intersection numbers and the Euclidean volume of these open subsets, Kontsevich proves the Witten conjecture [86]. Our formulas (1.9) and (1.10) are a consequence of the results reported in [15, 27, 65, 68]. We will explain the relation more concretely in these lectures.

Remark 1.4. The numerical value of the asymptotic expansion (1.9) is given by

$$(1.12) \quad \log Ai(x) = -\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{4}\log x - \log(2\sqrt{\pi})$$

$$- \frac{5}{48}x^{-\frac{3}{2}} + \frac{5}{64}x^{-3} - \frac{1105}{9216}x^{-\frac{9}{2}} + \frac{565}{2048}x^{-6} - \frac{82825}{98304}x^{-\frac{15}{2}} + \frac{19675}{6144}x^{-9}$$

$$- \frac{1282031525}{88080384}x^{-\frac{21}{2}} + \frac{80727925}{1048576}x^{-12} - \frac{1683480621875}{3623878656}x^{-\frac{27}{2}} + \cdots.$$

This follows from the WKB analysis of the Airy differential equation, which will be explained in this introduction.

Remark 1.5. Although the asymptotic expansion is not *equal* to the holomorphic function itself, we use the equality sign in these lectures to avoid further cumbersome notations.

The Airy differential equation appears in many different places, showing the feature of a *universal object* in the WKB analysis. It reflects the fact that the intersection numbers (1.11) are the most fundamental objects in Gromov-Witten theory. In contrast to the Airy differential equation, the gamma function is a universal object in the context of *difference equations*. We recall that

$$\Gamma(z+1) = z\Gamma(z),$$

and its asymptotic expansion for $Re(z) > 0$ is given by

$$(1.13) \qquad \log\Gamma(z) = z\log z - z - \frac{1}{2}\log z + \frac{1}{2}\log(2\pi) + \sum_{m=1}^{\infty} \frac{B_{2m}}{2m(2m-1)} z^{-(2m-1)},$$

where B_{2m} is the $(2m)$-th Bernoulli number defined by the generating function

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n.$$

This is called **Stirling's formula**, and its main part gives the well-known approximation of the factorial:

$$n! \sim \sqrt{2\pi n}\, \frac{n^n}{e^n}.$$

The asymptotic expansion of the gamma function is deeply related to the moduli theory of algebraic curves. For example, from Harer and Zagier [45] we learn that the orbifold Euler characteristic of the moduli space of smooth algebraic curves is given by the formula

$$(1.14) \qquad \chi(\mathcal{M}_{g,n}) = (-1)^{n-1} \frac{(2g-3+n)!}{(2g-2)!n!} \zeta(1-2g).$$

Here, the special value of the Riemann zeta function is the Bernoulli number

$$\zeta(1-2g) = -\frac{B_{2g}}{2g}.$$

The expression (1.14) is valid for $g = 0, n \geq 3$ if we use the gamma function for $(2g-2)!$. Stirling's formula (1.13) is much simpler than the expansion of $\log Ai(x)$. As the prime factorization of one of the coefficients

$$\frac{1683480621875}{3623878656} = \frac{5^5 \cdot 13 \cdot 17 \cdot 2437619}{2^{27} \cdot 3^3}$$

shows, we do not expect any simple closed formula for the coefficients of (1.12), like Bernoulli numbers. Amazingly, still there is a close relation between these two asymptotic expansions (1.12) and (1.13) through the work on *tautological relations* of Chow classes on the moduli space $\mathcal{M}_g$ by Mumford [70], followed by recent exciting developments on the Faber-Zagier conjecture [36, 49, 77]. In Theorem 2.7, we will see yet another close relationship between the Euler characteristic of $\mathcal{M}_{g,n}$ and the intersection numbers on $\overline{\mathcal{M}}_{g,n}$, through two special values of the *same* function.

The asymptotic expansion of the Airy function $Ai(x)$ itself for $Re(x) > 0$ has actually a rather simple expression:

$$
(1.15) \quad
\begin{aligned}
Ai(x) &= \frac{e^{-\frac{2}{3}x^{\frac{3}{2}}}}{2\sqrt{\pi}x^{\frac{1}{4}}} \sum_{m=0}^{\infty} \frac{\left(-\frac{3}{4}\right)^m \Gamma\left(m+\frac{5}{6}\right)\Gamma\left(m+\frac{1}{6}\right)}{2\pi m!} x^{-\frac{3}{2}m} \\
&= \frac{e^{-\frac{2}{3}x^{\frac{3}{2}}}}{2\sqrt{\pi}x^{\frac{1}{4}}} \sum_{m=0}^{\infty} (-1)^m \left(\frac{1}{576}\right)^m \frac{(6m)!}{(2m)!(3m)!} x^{-\frac{3}{2}m}.
\end{aligned}
$$

The expansion in terms of the gamma function values of the first line of (1.15) naturally arises from a hypergeometric function. The first line is equal to the second line because

$$
\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z},
$$

and induction on m. Since the $m = 0$ term in the summation is 1, we can apply the formal logarithm expansion

$$
\log(1-X) = -\sum_{j=1}^{\infty} \frac{1}{j}X^j
$$

to (1.15) with

$$
X = -\sum_{m=1}^{\infty}(-1)^m \left(\frac{1}{576}\right)^m \frac{(6m)!}{(2m)!(3m)!} x^{-\frac{3}{2}m},
$$

and obtain

$$
(1.16) \quad \log Ai(x) = -\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{4}\log x - \log(2\sqrt{\pi})
$$

$$
- \frac{1}{576}\frac{6!}{2!3!}x^{-\frac{3}{2}} + \left(\frac{1}{576}\right)^2 \left(\frac{12!}{4!6!} - \frac{1}{2}\left(\frac{6!}{2!3!}\right)^2\right) x^{-3}
$$

$$
- \left(\frac{1}{576}\right)^3 \left(\frac{18!}{6!9!} - \frac{12!}{4!6!} \cdot \frac{6!}{2!3!} + \frac{1}{3}\left(\frac{6!}{2!3!}\right)^3\right) x^{-\frac{9}{2}} + \cdots.
$$

In general, for $m \geq 1$, we have

$$
(1.17) \quad S_{m+1}(x) = (-1)^m x^{-\frac{3}{2}m} \left(\frac{1}{576}\right)^m \sum_{\lambda \vdash m}(-1)^{\ell(\lambda)-1}\frac{(\ell(\lambda)-1)!}{|\mathrm{Aut}(\lambda)|} \prod_{i=1}^{\ell(\lambda)} \frac{(6\lambda_i)!}{(2\lambda_i)!(3\lambda_i)!},
$$

where λ is a partition of m, $\ell(\lambda)$ its length, and $\mathrm{Aut}(\lambda)$ is the group of permutations of the parts of λ of equal length.

Comparing (1.10) and (1.17), we establish concrete relations among the intersection numbers.

Theorem 1.6 (Rainbow formula). *The cotangent class intersection numbers of $\overline{\mathcal{M}}_{g,n}$ satisfy the following relation for every $m \geq 1$:*

$$
(1.18) \quad
\begin{aligned}
&\sum_{\substack{g \geq 0, n > 0 \\ 2g-2+n=m}} \frac{1}{n!} \sum_{\substack{d_1+\cdots+d_n \\ =3g-3+n}} \langle \tau_{d_1}\cdots\tau_{d_n}\rangle_{g,n} \prod_{i=1}^{n} |2d_i - 1|!! \\
&= \left(\frac{1}{288}\right)^m \sum_{\lambda \vdash m}(-1)^{\ell(\lambda)-1}\frac{(\ell(\lambda)-1)!}{|\mathrm{Aut}(\lambda)|} \prod_{i=1}^{\ell(\lambda)} \frac{(6\lambda_i)!}{(2\lambda_i)!(3\lambda_i)!}.
\end{aligned}
$$

Here, we use the fact that $(-1)^n = (-1)^m$ if $2g - 2 + n = m$. For example, for $m = 2$, we have

$$\frac{1}{6}\langle\tau_0^3\tau_1\rangle_{0,4} + \frac{1}{2}\langle\tau_1^2\rangle_{1,2} + 3\langle\tau_0\tau_2\rangle_{1,2} = \left(\frac{1}{288}\right)^2 \left(\frac{12!}{4!6!} - \frac{1}{2}\left(\frac{6!}{2!3!}\right)^2\right) = \frac{5}{16}.$$

This can be verified by evaluating

$$\langle\tau_0^3\tau_1\rangle_{0,4} = 1, \qquad \langle\tau_1^2\rangle_{1,2} = \langle\tau_0\tau_2\rangle_{1,2} = \frac{1}{24}.$$

Remark 1.7. The main purpose of these lectures is to relate the topological recursion of [34] and quantization of Hitchin spectral curves. The left-hand side of (1.18) represents the topological recursion in this example, since the intersection numbers can be computed through this mechanism, as explained below. Actually, this is an important example that leads to the universal structure of the topological recursion. The right-hand side is the asymptotic expansion of a function that is coming from the geometry of the Hitchin spectral curve of a Higgs bundle.

The structure of the cohomology ring (or more fundamental Chow ring) of the moduli space $\overline{\mathcal{M}}_{g,n}$, and its open part $\mathcal{M}_{g,n}$ consisting of smooth curves, attracted much attention since the publication of the inspiring paper by Mumford [70] mentioned above. Let us focus on a simple situation

$$\pi : \mathcal{M}_{g,1} \longrightarrow \mathcal{M}_g,$$

which *forgets* the marked point on a smooth curve. By gluing the canonical line bundle of the fiber of each point on the base $\mathcal{M}_g$, which is the curve represented by the point on the moduli, we obtain the relative dualizing sheaf ω on $\mathcal{M}_{g,1}$. Its first Chern class, considered as a divisor on $\mathcal{M}_{g,1}$ and an element of the Chow group $A^1(\mathcal{M}_{g,1})$, is denoted by ψ. In the notation of (1.11), this is the same as ψ_1. Mumford defines *tautological* classes

$$\kappa_a := \pi_*(\psi^{a+1}) \in A^a(\mathcal{M}_g).$$

One of the statements of the Faber-Zagier conjecture of [36], now a theorem due to Ionel [49], says the following.

Conjecture 1.8 (A part of the Faber-Zagier Conjecture [36]). *Define rational numbers* $a_j \in \mathbb{Q}$ *by*

(1.19) $$\sum_{j=1}^\infty a_j t^j = -\log\left(\sum_{m=0}^\infty \frac{(6m)!}{(2m)!(3m)!}t^m\right).$$

Then the coefficient of t^ℓ of the expression

$$\exp\left(\sum_{j=1}^\infty a_j\kappa_j t^j\right) \in \big(\mathbb{Q}[\kappa_1,\kappa_2,\dots]\big)[[t]]$$

for each $\ell \geq 1$ gives the unique codimension ℓ tautological relation among the κ-classes on the moduli space $\mathcal{M}_{3\ell-1}$.

We see from (1.18), these coefficients a_j's are given by the intersection numbers (1.11), by a change of the variables

$$t = -\frac{1}{576}\, x^{-\frac{3}{2}}.$$

Indeed, we have

$$(1.20) \qquad a_j = -288^j \sum_{\substack{g \geq 0, n > 0 \\ 2g-2+n=j}} \frac{1}{n!} \sum_{\substack{d_1+\cdots+d_n \\ =3g-3+n}} \langle \tau_{d_1} \cdots \tau_{d_n} \rangle_{g,n} \prod_{i=1}^{n} |2d_i - 1|!! \,.$$

These tautological relations are generalized for the moduli spaces $\overline{\mathcal{M}}_{g,n}$, and are proved in [77]. Amazingly, still the rainbow integral (1.2) plays the essential role in determining the tautological relations in this generalized work.

Remark 1.9. As we have seen above, the asymptotic expansions of the gamma function and the Airy function carry information of quantum invariants, in particular, certain topological information of $\mathcal{M}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}$. We note that these quantum invariants are stored in the **insignificant** part of the asymptotic expansions.

Here come questions.

Question 1.10. *The Airy function is a one-variable function. It cannot be a generating function of all the intersection numbers* (1.11). *Then how do we obtain all intersection numbers from the Airy function, or the Airy differential equation, alone?*

Question 1.11. *The relations between the Airy function, the gamma function, and intersection numbers are all great. But then how does this relation have anything to do with* **Higgs bundles**?

Question 1.12. *As we remarked, the information of the quantum invariants is stored in the insignificant part of the asymptotic expansion. In the Airy example, they correspond to $S_m(x)$ of* (1.10) *for $m \geq 2$. Then what does the main part of the asymptotic behavior of the function, i.e., those functions in* (1.8), *determine?*

As we have remarked earlier, Kontsevich [55] utilized matrix integral techniques to answer Question 1.10. The key idea is to replace the variables in (1.2) by Hermitian matrices, and then use the asymptotic expansion on the result. Through the Feynman diagram expansion, he was able to obtain a generating function of all the intersection numbers.

What we explain in these lectures is the concept of **topological recursion** of [34]. Without going into matrix integrals, we can directly obtain (a totally different set of) generating functions of the intersection numbers from the Airy function. Here, the Airy differential equation is identified as a **quantum curve**, and application of the **semi-classical limit** and the topological recursion enable us to calculate generating functions of the intersection numbers.

But before going into detail, let us briefly answer Question 1.11 below. The point is that the geometry of the Airy function is a special example of Higgs bundles.

For the example of the Airy differential equation, the topological recursion is exactly the same as the *Virasoro constraint conditions* for the intersection numbers (1.11), and the semi-classical limit recovers the **Hitchin spectral curve** of the corresponding Higgs bundle. The information stored in the main part of the asymptotic expansion (1.8), as asked in Question 1.12, actually determines the spectral curve and its geometry. We can turn the story in the other way around: we will see that the functions $S_0(x)$ and $S_1(x)$ corresponding to (1.8) in the general context are indeed determined by the geometry of the Hitchin spectral curve of an appropriate Higgs bundle.

The stage setting is the following. As a base curve, we have $\mathbb{P}^1$. On this curve we have a vector bundle

$$(1.21) \qquad E = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(1) = K_{\mathbb{P}^1}^{\frac{1}{2}} \oplus K_{\mathbb{P}^1}^{-\frac{1}{2}}$$

of rank 2. The main character of the Second Act is a *meromorphic* Higgs field

$$(1.22) \qquad \phi = \begin{bmatrix} & x(dx)^2 \\ 1 & \end{bmatrix} : E \longrightarrow K_{\mathbb{P}^1}(5) \otimes E.$$

Here, x is an affine coordinate of $\mathbb{A}^1 \subset \mathbb{P}^1$, 1 on the $(2,1)$-component of ϕ is the natural morphism

$$1 : K_{\mathbb{P}^1}^{\frac{1}{2}} \xrightarrow{=} K_{\mathbb{P}^1}^{\frac{1}{2}} \longrightarrow K_{\mathbb{P}^1}^{\frac{1}{2}} \otimes \mathcal{O}_{\mathbb{P}^1}(5),$$

and

$$x(dx)^2 \in H^0\big(\mathbb{P}^1, K_{\mathbb{P}^1}^{\otimes 2}(5-1)\big) \cong \mathbb{C}$$

is the unique (up to a constant factor) meromorphic *quadratic differential* on $\mathbb{P}^1$ that has one zero at $x = 0$ and a pole of order 5 at $x = \infty$. We use K_C to denote the canonical sheaf on a projective algebraic curve C. The data (E, ϕ) is called a *Higgs pair*. Although ϕ contains a quadratic differential in its component, because of the shape of the vector bundle E, we see that

$$E = K_{\mathbb{P}^1}^{\frac{1}{2}} \oplus K_{\mathbb{P}^1}^{-\frac{1}{2}} \xrightarrow{\phi} \left(K_{\mathbb{P}^1}^{\frac{3}{2}} \oplus K_{\mathbb{P}^1}^{\frac{1}{2}} \right) \otimes \mathcal{O}_{\mathbb{P}^1}(5) = K_{\mathbb{P}^1}(5) \otimes \left(K_{\mathbb{P}^1}^{\frac{1}{2}} \oplus K_{\mathbb{P}^1}^{-\frac{1}{2}} \right),$$

hence

$$\phi \in H^0\big(\mathbb{P}^1, K_{\mathbb{P}^1}(5) \otimes \mathrm{End}(E)\big)$$

is indeed an $\mathrm{End}(E)$-valued meromorphic 1-form on $\mathbb{P}^1$.

The cotangent bungle

$$\pi : T^*\mathbb{P}^1 \longrightarrow \mathbb{P}^1$$

is the total space of $K_{\mathbb{P}^1}$. Therefore, the pull-back bundle $\pi^* K_{\mathbb{P}^1}$ has a tautological section $\eta \in H^0(T^*\mathbb{P}^1, \pi^* K_{\mathbb{P}^1})$, which is a globally defined holomorphic 1-form on $T^*\mathbb{P}^1$. The global holomorphic 2-form $-d\eta$ gives the holomorphic symplectic structure, hence a hyper-Kähler structure, on $T^*\mathbb{P}^1$. If we trivialize the cotangent bundle on the affine neighborhood of $\mathbb{P}^1$ with a coordinate x, and use a fiber coordinate y, then $\eta = ydx$. We wish to define the spectral curve of this Higgs pair. Due to the fact that ϕ is singular at $x = \infty$, we cannot capture the whole story within the cotangent bundle. We note that the cotangent bundle $T^*\mathbb{P}^1$ has a natural compactification

$$\overline{T^*\mathbb{P}^1} := \mathbb{P}(K_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}) = \mathbb{F}_2 \xrightarrow{\pi} \mathbb{P}^1,$$

which is known as a Hirzebruch surface. The holomorphic 1-form η extends to $\overline{T^*\mathbb{P}^1}$ as a meromorphic 1-form with simple poles along the divisor at infinity.

Now we can consider the characteristic polynomial

$$\det(\eta - \pi^*\phi) \in H^0\left(\overline{T^*\mathbb{P}^1}, \pi^*\big(K_{\mathbb{P}^1}^{\otimes 2}(5)\big) \right)$$

as a meromorphic section of the line bundle $\pi^* K_{\mathbb{P}^1}^{\otimes 2}$ on the compact space $\overline{T^*\mathbb{P}^1}$. It defines the Hitchin spectral curve

$$(1.23) \qquad \Sigma = \big(\det(\eta - \pi^*\phi) \big)_0 \subset \overline{T^*\mathbb{P}^1}$$

as a divisor. Again in terms of the local coordinate (x, y) of $\overline{T^*\mathbb{P}^1}$, the spectral curve Σ is simply given by

$$(1.24) \qquad\qquad\qquad\qquad x = y^2.$$

It is a perfect parabola in the (x, y)-plane. But our Σ is in the Hirzebruch surface, not in the projective plane. Choose the coordinate $(u, w) \in \mathbb{F}_2$ around $(x, y) = (\infty, \infty)$ defined by

$$(1.25) \qquad \begin{cases} u = \frac{1}{x} \\ \frac{1}{w} du = y dx \end{cases} .$$

Then the local expression of Σ around $(u, w) = (0, 0)$ becomes a quintic cusp

$$(1.26) \qquad w^2 = u^5.$$

So the spectral curve Σ is indeed highly singular at infinity!

1.2. **Quantum curves, semi-classical limit, and the WKB analysis.** At this stage we have come to the point to introducing the notion of *quantization*. We wish to quantize the spectral curve Σ of (1.23). In terms of the affine coordinate (x, y), the **quantum curve** of $y^2 - x = 0$ should be the Airy differential equation

$$(1.27) \qquad \left(\left(\hbar \frac{d}{dx} \right)^2 - x \right) \Psi(x, \hbar) = 0.$$

This is the Weyl quantization, in which we change the commutative algebra $\mathbb{C}[x, y]$ to a Weyl algebra $\mathbb{C}[\hbar]\langle x, y \rangle$ defined by the commutation relation

$$(1.28) \qquad [x, y] = -\hbar.$$

We consider $x \in \mathbb{C}[\hbar]\langle x, y \rangle$ as the multiplication operator by the coordinate x, and $y = \hbar \frac{d}{dx}$ as a differential operator.

How do we know that (1.27) is the right quantization of the spectral curve (1.24)? Apparently, the limit $\hbar \to 0$ of the differential operator does not reproduce the spectral curve. Let us now recall the WKB method for analyzing differential equations like (1.27). This is a method that relates classical mechanics and quantum mechanics. As we see below, the WKB method is not for finding a convergent analytic solution to the differential equation. Since the equation we wish to solve is considered to be a quantum equation, the corresponding classical problem, if it exists, should be recovered by taking $\hbar \to 0$. We denote by an unknown function $S_0(x)$ the "solution" to the corresponding classical problem. To emphasize the classical behavior at the $\hbar \to 0$ limit, we expand the solution to the quantum equation as

$$(1.29) \qquad \Psi(x, \hbar) = \exp \left(\sum_{m=0}^{\infty} \hbar^{m-1} S_m(x) \right) := \exp \left(\frac{1}{\hbar} S_0(x) \right) \cdot \exp \left(\sum_{m=1}^{\infty} \hbar^{m-1} S_m(x) \right).$$

The idea is that as $\hbar \to 0$, the effect of $S_0(x)$ is magnified. But as a series in $\hbar$, (1.29) is ill defined because the coefficient of each power of $\hbar$ is an infinite sum. It is also clear that $\hbar \to 0$ does not make sense for $\Psi(x, \hbar)$. Instead of expanding (1.29) immediately in $\hbar$ and take its 0 limit, we use the following standard procedure. First we note that (1.27) is equivalent to

$$\left[\exp \left(-\frac{1}{\hbar} S_0(x) \right) \cdot \left(\left(\hbar \frac{d}{dx} \right)^2 - x \right) \cdot \exp \left(\frac{1}{\hbar} S_0(x) \right) \right] \exp \left(\sum_{m=1}^{\infty} \hbar^{m-1} S_m(x) \right) = 0.$$

Since the conjugate differential operator

$$\exp \left(-\frac{1}{\hbar} S_0(x) \right) \cdot \left(\left(\hbar \frac{d}{dx} \right)^2 - x \right) \cdot \exp \left(\frac{1}{\hbar} S_0(x) \right)$$

$$= \left(\hbar \frac{d}{dx}\right)^2 + 2\hbar S_0'(x)\frac{d}{dx} + \left(S_0'(x)\right)^2 - x + \hbar S_0''(x)$$

is a well-defined differential operator, *and* its limit $\hbar \to 0$ makes sense, we interpret (1.27) as the following differential equation:

$$(1.30) \quad \left[\left(\hbar \frac{d}{dx}\right)^2 + 2\hbar S_0'(x)\frac{d}{dx} + \left(S_0'(x)\right)^2 - x + \hbar S_0''(x)\right] \exp\left(\sum_{m=1}^{\infty} \hbar^{m-1} S_m(x)\right) = 0.$$

Here, $'$ indicates the x-derivative. This equation is equivalent to

$$(1.31) \qquad \left(\sum_{m=0}^{\infty} \hbar^m S_m'(x)\right)^2 + \sum_{m=0}^{\infty} \hbar^{m+1} S_m''(x) - x = 0$$

for every $m \geq 0$. The coefficient of the $\hbar^0$, or the $\hbar \to 0$ limit of (1.31), then gives

$$(1.32) \qquad \left(S_0'(x)\right)^2 - x = 0,$$

and that of $\hbar^1$ gives

$$(1.33) \qquad S_0''(x) + 2S_0'(x)S_1'(x) = 0.$$

The $\hbar^0$ term is what we call the **semi-classical limit** of the differential equation (1.27). From (1.32) we obtain

$$(1.34) \qquad S_0(x) = \pm\frac{2}{3}x^{\frac{3}{2}} + c_0,$$

with a constant of integration c_0. Then plugging $S_0(x)$ into (1.33) we obtain

$$S_1(x) = -\frac{1}{4}\log x - \log(2\sqrt{\pi}) + c_1,$$

again with a constant of integration c_1. Note that these solutions are consistent with (1.8). For $m \geq 1$, the coefficient of $\hbar^{m+1}$ gives

$$(1.35) \qquad S_{m+1}'(x) = -\frac{1}{2S_0'(x)}\left(S_m''(x) + \sum_{a=1}^{m} S_a'(x)S_{m+1-a}'(x)\right),$$

which can be solved recursively, term by term from $S_0(x)$. This mechanism is the method of Wentzel-Kramers-Brillouin (WKB) approximation.

We can ignore the constants of integration when solving (1.35) because it is easy to restore them, if necessary, just by adding c_m to each $S_m(x)$ in (1.29). The solution then simply changes to another one

$$\left(\exp\left(\frac{1}{\hbar}c_0\right)\exp\left(\frac{1}{\hbar}S_0(x)\right)\right) \cdot \exp\left(\sum_{m=1}^{\infty} h^{m-1}c_m\right)\exp\left(\sum_{m=1}^{\infty} h^{m-1}S_m(x)\right).$$

In terms of the main variable x, the above solution is a constant multiple of the original one. The two choices of the sign in (1.34) lead to two linearly independent solutions of (1.27). If we impose

$$(1.36) \qquad \lim_{x \to \infty} S_m(x) = 0, \qquad m \geq 2,$$

then the differential equation (1.35) uniquely determines all terms $S_m(x)$. Thus, with the choice of the negative sign in (1.34) and imposing $c_0 = c_1 = 0$ and (1.36), we obtain the unique exponentially decaying solution for $x \to \infty$ along the real line. This solution agrees with (1.10) and (1.17). Thus we obtain the second line of the Rainbow formula (1.18).

We also see from the semi-classical limit (1.32) that if we put $y = S'_0(x)$, then we recover the Hitchin spectral curve $x = y^2$. The functions $S_m(x)$ actually live on the spectral curve rather than the base $\mathbb{P}^1$, because of the appearance of $\sqrt{x}$ in (1.10).

Remark 1.13. One can ask a question: **Does (1.29) give a convergent solution?** The answer is a flat **No!** Suppose we solve the Airy differential equation with the WKB method explained above, and define a "solution" by (1.29). Expand the second exponential factor as a power series in $\hbar$, and write the solution as

$$\Psi(x, \hbar) = \exp\left(\frac{1}{\hbar} S_0(x)\right) \sum_{n=0}^{\infty} f_n(x) \hbar^n.$$

Then for any compact subset $K \subset \mathbb{C} \setminus \{0\}$, there is a constant C_K such that

$$\sup_{x \in K} |f_n(x)| \leq C_K^n n!.$$

Therefore, unless we are in an extremely special case, the second exponential factor in the expression (1.29) does not converge as a power series in $\hbar$ at all! Since

$$\Psi(x, \hbar) := Ai\left(x/\hbar^{\frac{2}{3}}\right)$$

is a solution that is entire in x and any $\hbar$ for which $1/\hbar^{\frac{2}{3}}$ makes sense, the WKB method around $x \in \mathbb{C} \setminus \{0\}$ is the same as the asymptotic expansion of the Airy function $Ai(x)$ given in (1.15). There we see the factorial growth of the coefficients. Thus the WKB method is **not** for finding a convergent analytic solution.

1.3. **The topological recursion as quantization.** Then what is good about the WKB method and the purely formal solution (1.29)? Let us examine (1.10). We note that $S_m(x)$s are one variable functions, and different values of g and n are summed in its definition. Therefore, knowing the solution $\Psi(x, \hbar)$ of (1.27) that decays exponentially as $x \to \infty$ along the real axis, assuming $\hbar > 0$, does not seem to possibly recover intersection numbers $\langle \tau_{d_1} \cdots \tau_{d_n} \rangle_{g,n}$ for all values of $(d_1, \ldots, d_n)$ and (g, n). Then how much information does the quantum curve (1.27) really have?

Here comes the idea of **topological recursion** of Eynard and Orantin [34]. This mechanism gives a refined expression of each $S_m(x)$, and computes all intersection numbers. The solution Ψ of (1.29) is never holomorphic, and it makes sense only as the asymptotic expansion of a holomorphic solution at its essential singularity. The expansion of the Airy function $Ai(x)$ at a holomorphic point does not carry any interesting information. The function's key information is concentrated in the expansion at the essential singularity. The topological recursion is for obtaining this hidden information when applied at the essential singularity of the solution, by giving an explicit formula for the WKB expansion. And the WKB analysis is indeed a method that determines the relation between the quantum behavior and the classical behavior of a given system, i.e., the process of quantization.

As we have seen above, the quantum curve (1.27) recovers the spectral curve (1.24) by the procedure of semi-classical limit. We recall that the spectral curve lives in the Hirzebruch surface $\mathbb{F}_2$, and it has a quintic cusp singularity (1.26) at $(x, y) = (\infty, \infty)$. It requires two blow-ups of $\mathbb{F}_2$ to resolve the singularity of Σ. Let us denote this minimal resolution by $\widetilde{\Sigma}$. The proper transform is a smooth curve of genus 0, hence it is a $\mathbb{P}^1$. Let $B \cong \mathbb{P}^1$ be the 0-section, and $F \cong \mathbb{P}^1$ a fiber, of $\mathbb{F}_2$. Then after two blow-ups, $\widetilde{\Sigma} \subset Bl(\mathbb{F}_2)$ is identified as a divisor by the equation

$$\widetilde{\Sigma} = 2B + 5F - 4E_2 - 2E_1 \in \mathrm{Pic}\big(Bl(\mathbb{F}_2)\big),$$

where E_i is the exceptional divisor introduced at the i-th blow-up [25, Section 5].
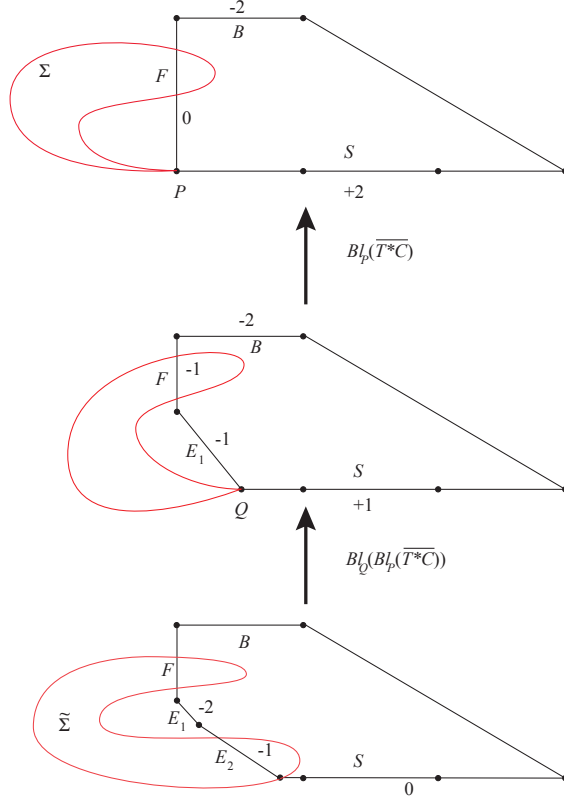

FIGURE 1.3. Blowing up $\mathbb{F}_2$ twice. The parabola in $\mathbb{F}_2$ has a quintic cusp singularity at infinity (top). After the second blow-up, the proper transform becomes non-singular (bottom). Since all fibers of $\mathbb{F}_2 \longrightarrow \mathbb{P}^1$ are equivalent, the toric picture has only one representative of the fiber $F = \mathbb{P}^1$. It is a stretch to place Σ and $\widetilde{\Sigma}$ in this diagram, because the spectral curve is a double cover of the base B, rather than a degree 4 covering that the picture may suggest.

Since the desingularization $\widetilde{\Sigma}$ is just a copy of a $\mathbb{P}^1$, we can choose a **normalization coordinate** t on it so that the map $\tilde{\pi} : \widetilde{\Sigma} \longrightarrow \mathbb{P}^1$ to the base curve $\mathbb{P}^1$ is given by

$$(1.37) \qquad \begin{cases} x = \frac{4}{t^2} \\ y = -\frac{2}{t} \end{cases} , \qquad \begin{cases} u = \frac{t^2}{4} \\ w = \frac{t^5}{32} \end{cases} .$$

With respect to the normalization coordinate, define a homogeneous polynomial of degree $6g - 6 + 3n$ for $2g - 2 + n > 0$ by

$$(1.38) \qquad F_{g,n}^A(t_1, \cdots, t_n) := \frac{(-1)^n}{2^{2g-2+n}} \sum_{\substack{d_1 + \cdots + d_n \\ = 3g-3+n}} \langle \tau_{d_1} \cdots \tau_{d_n} \rangle_{g,n} \prod_{i=1}^n |2d_i - 1|!! \left(\frac{t_i}{2} \right)^{2d_i+1}$$

as a function on $(\widetilde{\Sigma})^n$, and an n-linear differential form

$$(1.39) \qquad W_{g,n}^A(t_1, \ldots, t_n) := d_{t_1} \cdots d_{t_n} F_{g,n}^A(t_1, \ldots, t_n).$$

For *unstable geometries* $(g,n) = (0,1)$ and $(0,2)$, we need to define differential forms separately:

$$(1.40) \qquad W_{0,1}^A(t) := \eta = ydx = \frac{16}{t^4}dt,$$

$$(1.41) \qquad W_{0,2}^A(t_1, t_2) := \frac{dt_1 \cdot dt_2}{(t_1 - t_2)^2}.$$

The definition of $W_{0,1}^A$ encodes the geometry of the singular spectral curve $\Sigma \subset \mathbb{F}_2$ embedded in the Hirzebruch surface, and $W_{0,2}^A$ depends only on the intrinsic geometry of the normalization $\widetilde{\Sigma}$. Then we have

Theorem 1.14 (Topological recursion for the intersection numbers, [27])**.**

$$(1.42) \quad W_{g,n}^A(t_1, \ldots, t_n)$$

$$= -\frac{1}{2\pi i} \int_\gamma \left(\frac{1}{t + t_1} + \frac{1}{t - t_1} \right) \frac{t^4}{64} \frac{1}{dt} \, dt_1 \Bigg[W_{g-1,n+1}^A(t, -t, t_2, \ldots, t_n)$$

$$+ \overset{no \; (0,1) \; terms}{\underset{\substack{g_1+g_2=g \\ I \sqcup J = \{2,\ldots,n\}}}{\sum}} W_{g_1,|I|+1}^A(t, t_I) W_{g_2,|J|+1}^A(-t, t_J) \Bigg],$$

where the integral is taken with respect to the contour of Figure 1.4, and the sum is over all partitions of g and set partitions of $\{2, 3, \ldots, n\}$ without including $g_1 = 0$ and $I = \emptyset$, or $g_2 = 0$ and $J = \emptyset$. The notation $\frac{1}{dt}$ represents the ratio operation, which acts on a differential 1-form to produce a global meromorphic function. When acted on the quadratic differential, $\frac{1}{dt}$ yields a 1-form.
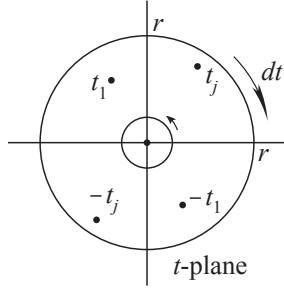

FIGURE 1.4. The integration contour γ. This contour encloses an annulus bounded by two concentric circles centered at the origin. The outer one is a small circle around $\infty \in \widetilde{\Sigma} = \mathbb{P}^1$, and the inner one a small circle around the origin. Both circles are positively oriented on $\widetilde{\Sigma}$. Using the t-plane as the affine chart, the small loop around ∞ becomes a circle of a large radius $r \gg \infty$ with the opposite orientation, as in the figure. Geometrically, $t = 0$ and $t = \infty$ correspond to the two simple ramification points of the Galois covering $\tilde{\pi} : \widetilde{\Sigma} \longrightarrow \mathbb{P}^1$, and the contour γ consists of two small loops around these ramification points.

This is an example of the axiomatic mechanism of [34] called the *topological recursion.* We give a more geometric interpretation of the formula in the later sections. The derivation of the topological recursion (1.42) from (1.38) is the subject of Section 2 where the origin of the recursion formula is identified via graph enumeration. The power of this formula is

that all $W_{g,n}^A$ for $2g - 2 + n > 0$ are calculated from the initial values (1.40) and (1.41). For example,

$$
\begin{aligned}
W_{1,1}^A(t_1) &= -\left[\frac{1}{2\pi i}\int_\gamma \left(\frac{1}{t + t_1} + \frac{1}{t - t_1}\right)\frac{t^4}{64}\frac{1}{dt}W_{0,2}^A(t, -t)\right]dt_1 \\
&= -\left[\frac{1}{2\pi i}\int_\gamma \left(\frac{1}{t + t_1} + \frac{1}{t - t_1}\right)\frac{t^4}{64}\frac{(-dt)}{4t^2}\right]dt_1 \\
&= -\frac{1}{128}t_1^2 dt_1 \\
&= -\frac{3}{16}\langle\tau_1\rangle_{1,1}t_1^2 dt_1.
\end{aligned}
$$

Thus we find $\langle\tau_1\rangle_{1,1} = \frac{1}{24}$.

The functions $F_{g,n}^A$ for $2g - 2 + n > 0$ can be calculated by integration:

$$(1.43) \qquad F_{g,n}^A(t_1,\ldots,t_n) = \int_0^{t_1}\cdots\int_0^{t_n} W_{g,n}^A(t_1,\ldots,t_n).$$

Note that

$$t \to 0 \iff x \to \infty.$$

Therefore, we are considering the expansion of quantities at the essential singularity of $Ai(x)$. It is surprising to see that the topological recursion indeed determines all intersection numbers (1.11)! Now define

$$(1.44) \qquad S_m(x) = \sum_{2g-2+n=m-1}\frac{1}{n!}F_{g,n}^A\big(t(x),\ldots,t(x)\big),$$

where we choose a branch of $\pi : \Sigma \longrightarrow \mathbb{P}^1$, and consider $t = t(x)$ as a function in x. It coincides with (1.10). Since the *moduli* spaces $\overline{\mathcal{M}}_{0,1}$ and $\overline{\mathcal{M}}_{0,2}$ do not exist, we do not have an expression (1.38) for these unstable geometries. So let us formally apply (1.43) to (1.40):

$$(1.45) \qquad F_{0,1}^A(t) := \int_0^t \frac{16}{t^4}dt = -\frac{16}{3}t^{-3} = -\frac{2}{3}x^{\frac{3}{2}} = S_0(x).$$

We do not have this type of integration procedure to produce $S_1(x)$ from $W_{0,2}^A$. So we simply define $S_1(x)$ by solving (1.33). Then the residues in (1.42) can be concretely computed, and produce a system of recursive partial differential equations among the $F_{g,n}^A$s. Their **principal specialization** (1.44) produces (1.35)! Therefore, we obtain (1.18).

In this context, the topological recursion is the process of quantization, because it actually constructs the function $\Psi(x, \hbar)$ by giving a closed formula for the WKB analysis, and hence the differential operator (1.27) that annihilates it, all from the classical curve $x = y^2$.

1.4. **Non-Abelian Hodge correspondence and quantum curves.** Then what is the quantum curve? Since it is a second order differential equation with a deformation parameter $\hbar$, and its semi-classical limit is the spectral curve of a Higgs bundle, which is a rank 2 bundle in our example, we can easily *guess* that it should be the result of the **non-Abelian Hodge correspondence**. Since the quantization procedure is a *holomorphic* correspondence, while the non-Abelian Hodge correspondence is *not* holomorphic as a map, we do not expect that these two are the same.

To have a glimpse of the geometric effect of quantization, let us start with a Higgs bundle (E, ϕ) of (1.21) and (1.22). The transition function of the vector bundle

$$E = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$$

on $\mathbb{P}^1 = U_\infty \cup U_0$ defined on $\mathbb{C}^* = U_\infty \cap U_0$ is given by $\begin{bmatrix} x & \\ & \frac{1}{x} \end{bmatrix}$, where $U_0 = \mathbb{P}^1 \setminus \{\infty\}$ and $U_\infty = \mathbb{A}^1 = \mathbb{P}^1 \setminus \{0\}$. The trivial extension

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-1) \longrightarrow E \longrightarrow \mathcal{O}_{\mathbb{P}^1}(1) \longrightarrow 0$$

has a unique 1-parameter family of deformations as the extension of $\mathcal{O}_{\mathbb{P}^1}(1)$ by $\mathcal{O}_{\mathbb{P}^1}(-1)$:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-1) \longrightarrow E_\hbar \longrightarrow \mathcal{O}_{\mathbb{P}^1}(1) \longrightarrow 0,$$

where $\hbar \in \operatorname{Ext}^1\left(\mathcal{O}_{\mathbb{P}^1}(1), \mathcal{O}_{\mathbb{P}^1}(-1)\right) \cong H^1\left(\mathbb{P}^1, K_{\mathbb{P}^1}\right) \cong \mathbb{C}$, and the transition function of $E_\hbar$ is given by

$$\begin{bmatrix} x & \hbar \\ & \frac{1}{x} \end{bmatrix}.$$

Since

$$\begin{bmatrix} 1 & \\ -\frac{1}{\hbar x} & 1 \end{bmatrix} \begin{bmatrix} x & \hbar \\ & \frac{1}{x} \end{bmatrix} \begin{bmatrix} & -\hbar \\ \frac{1}{\hbar} & x \end{bmatrix} = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix},$$

(1.46)
$$E_\hbar \cong \begin{cases} \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(1) & \hbar = 0 \\ \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} & \hbar \neq 0. \end{cases}$$

The Higgs field (1.22) satisfies the transition relation

$$-\begin{bmatrix} 1 & \frac{1}{u^5} \end{bmatrix} du = \begin{bmatrix} x & \\ & \frac{1}{x} \end{bmatrix} \begin{bmatrix} & x \\ 1 & \end{bmatrix} dx \begin{bmatrix} x & \\ & \frac{1}{x} \end{bmatrix}^{-1},$$

where $u = 1/x$ is a coordinate on U_∞. Because of the relation

$$-\begin{bmatrix} 1 & \frac{1}{u^5} \end{bmatrix} du = \begin{bmatrix} x & \hbar \\ & \frac{1}{x} \end{bmatrix} \begin{bmatrix} & x \\ 1 & \end{bmatrix} dx \begin{bmatrix} x & \hbar \\ & \frac{1}{x} \end{bmatrix}^{-1} - d\begin{bmatrix} x & \hbar \\ & \frac{1}{x} \end{bmatrix} \begin{bmatrix} x & \hbar \\ & \frac{1}{x} \end{bmatrix}^{-1},$$

somewhat miraculously, $\nabla^\hbar = \hbar d + \phi$ with the same Higgs fields as a connection matrix becomes an $\hbar$-connection in $E_\hbar$:

(1.47)
$$\nabla^\hbar = \hbar d + \phi : E_\hbar \longrightarrow K_{\mathbb{P}^1}(5) \otimes E_\hbar.$$

It gives the Higgs field at $\hbar = 0$. A flat section with respect to $\nabla^\hbar$ for $\hbar \neq 0$ can be obtained by solving

(1.48)
$$\left(\hbar \frac{d}{dx} + \begin{bmatrix} & x \\ 1 & \end{bmatrix}\right) \begin{bmatrix} -\hbar\Psi(x, \hbar)' \\ \Psi(x, \hbar) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

where $\Psi(x, \hbar)'$ is the x-derivative. Clearly, (1.48) is equivalent to (1.27).

For $\hbar = 1$, (1.47) is a holomorphic flat connection on $E_1|_{\mathbb{A}^1} = \mathcal{O}_{\mathbb{A}^1}^{\oplus 2}$, the restriction of the vector bundle E_1 on the affine coordinate neighborhood. Therefore, $\left(E_\hbar, \nabla^\hbar\right)\big|_{\hbar=1}$ defines a $\mathcal{D}$-module on $E_1|_{\mathbb{A}^1}$. As a holomorphic $\mathcal{D}$-module over $\mathbb{A}^1$, we have an isomorphism

$$\left(\mathcal{O}_{\mathbb{A}^1}^{\oplus 2}, \nabla^{\hbar=1}\right)\Big|_{\mathbb{A}^1} \cong \mathcal{D}/\left(\mathcal{D} \cdot P(x, 1)\right),$$

where

(1.49)
$$P(x, \hbar) := \left(\hbar \frac{d}{dx}\right)^2 - x,$$

and $\mathcal{D}$ denotes the sheaf of linear differential operators with holomorphic coefficients on $\mathbb{A}^1$. The above consideration indicates how we constructs a quantum curve as a $\mathcal{D}$-module from a given particular Higgs bundle.

Hitchin's original idea [46] of constructing *stable* Higgs bundles is to solve a system of differential equations, now known as *Hitchin's equations.* The stability condition for the Higgs bundles can be translated into a system of nonlinear elliptic partial differential equations defined on a **Hermitian** vector bundle $E \longrightarrow C$ over a compact Riemann surface C. It takes the following form:

$$
F(D) + [\phi, \phi^{\dagger h}] = 0
$$
(1.50)
$$
D^{0,1}\phi = 0.
$$

Here, h is a Hermitian metric in E, D is a unitary connection in E with respect to h, $D^{0,1}$ is the $(0,1)$-component of the covariant differentiation with respect to the complex structure of the curve C, $F(D)$ is the curvature of D, $\dagger_h$ is the Hermitian conjugation with respect h, and ϕ is a differentiable Higgs field on E. Solving Hitchin's equation is equivalent to constructing a 1-parameter family of flat connections of the form

$$
(1.51) \qquad\qquad D(\zeta) = \frac{\phi}{\zeta} + D + \phi^{\dagger h}\zeta, \qquad \zeta \in \mathbb{C}^*
$$

(see [38, 71, 80]). The non-Abelian Hodge correspondence is the association of $\left(\widetilde{E}, D(1)^{1,0}\right)$ to the given stable holomorphic Higgs bundle (E, ϕ), i.e., a solution to Hitchin's equations. Here, $D(1) = \phi + D + \phi^{\dagger h}$, and $\widetilde{E}$ is a holomorphic vector bundle with the complex structure given by the flat connection $D(1)^{0,1}$. The connection $D(1)^{1,0}$ is then a holomorphic connection in $\widetilde{E}$.

A new idea that relates the non-Abelian Hodge correspondence and *opers* is emerging [23]. The role of the topological recursion in this context, and the identification of opers as globally defined quantum curves, are being developed. Since it is beyond our scope of the current lecture notes, it will be discussed elsewhere.

So far, we have considered $\Psi(x, \hbar)$ as a formal "function." What is it indeed? Since our vector bundle E is of the form (1.21), the shape of the equation (1.48) suggests that

$$
(1.52) \qquad\qquad \Psi(x, \hbar) \in \widehat{H}^0\big(\mathbb{P}^1, K_{\mathbb{P}^1}^{-\frac{1}{2}}\big),
$$

where the hat sign indicates that we really do not have any good space to store the formal wave function $\Psi(x, \hbar)$. The reason for the appearance of $K_{\mathbb{P}^1}^{-\frac{1}{2}}$ as its home is understood as follows. Since $W_{g,n}^A$ for $2g - 2 + n$ is an n-linear differential form, (1.43) tells us that $F_{g,n}^A$ is just a number. Therefore, $S_m(x)$ that is determined by (1.44) for $m \geq 2$ is also just a number. Here, by a "number" we mean a genuine function in x. The differential equation (1.32) should be written

$$
(1.53) \qquad\qquad (dS_0(x))^{\otimes 2} = x(dx)^2 = q(x)
$$

as an equation of quadratic differentials. Here,

$$
q(x) \in K_{\mathbb{P}^1}^{\otimes 2}
$$

can be actually any meromorphic quadratic differential on $\mathbb{P}^1$, including $x(dx)^2$, so that

$$
\phi = \begin{bmatrix} & 1 \\ q & \end{bmatrix} \in K_{\mathbb{P}^1} \otimes \mathrm{End}(E)
$$

is a meromorphic Higgs field on the vector bundle E in the same way. Similarly, (1.33) should be interpreted as

$$
(1.54) \qquad S_1(x) = -\frac{1}{2}\int d\log(dS_0) = -\frac{1}{2}\log\sqrt{q(x)} = -\frac{1}{4}\log q(x).
$$

Now recall the conjugate differential equation (1.30). Its solution takes the form

$$(1.55) \qquad \frac{1}{\sqrt[4]{q(x)}} \exp\left(\sum_{m=2}^{\infty} S_m(x)\hbar^{m-1} \right),$$

and as we have noted, the exponential factor is just a number. Therefore, the geometric behavior of this solution is determined by the factor

$$\frac{1}{\sqrt[4]{q(x)}} \in K_{\mathbb{P}^1}^{-\frac{1}{2}},$$

which is a meromorphic section of the negative half-canonical sheaf.

We recall that $\Psi(x,\hbar)$ has another factor $\exp\left(S_0(x)/\hbar\right)$. It should be understood as a "number" defined on the spectral curve Σ, because $dS_0(x) = \eta$ is the tautological 1-form on $T^*\mathbb{P}^1$ restricted to the spectral curve, and $S_0(x)$ is its integral on the spectral curve. Therefore, this factor tells us that the equation (1.27) should be considered on Σ. Yet its local x-dependence is indeed determined by $K_{\mathbb{P}^1}^{-\frac{1}{2}}$.

We have thus a good answer for Question 1.12 now. The main part of the asymptotic expansion (1.8) tells us what geometry we should consider. It tells us what the Hitchin spectral curve should be, and it also includes the information of Higgs bundle (E, ϕ) itself.

Remark 1.15. The Airy example, and another example we consider in the next section, are in many ways special, in the sense that the $S_2(x)$-term of the WKB expansion is given by integrating the solutions $W_{1,1}$ and $W_{0,3}$ of the topological recursion (1.42). In general, the topological recursion mechanism of computing $W_{1,1}$ and $W_{0,3}$ from $W_{0,2}$ does not correspond to the WKB equation for S_2. As discovered in [24], the topological recursion in its PDE form is equivalent to the WKB equations for all $S_m(x)$ in the range of $m \geq 3$. But the PDE recursion, which we discuss in detail in the later sections, does not determine S_2. It requires a new way of viewing the topological recursion in its differential equation formulation: **we consider $F_{1,1}$ and $F_{0,3}$ as the initial condition for topological recursion**, rather than $W_{0,1}$ and $W_{0,2}$, which have been more commonly considered as the starting point for the topological recursion.

1.5. **The Lax operator for Witten-Kontsevich KdV equation.** Surprisingly, the operator $P(x, 1)$ of (1.49) at $\hbar = 1$ is the initial value of the *Lax operator* for the KdV equations that appears in the work of Witten [86] and Kontsevich [55]. Witten considered a different generating function of the intersection numbers (1.11) given by

$$F(s_0, s_1, s_2, \dots) = \left\langle \exp\left(\sum_{d=0}^{\infty} s_d \tau_d \right) \right\rangle$$

$$= \sum_{k_0, k_1, k_2, \dots = 0}^{\infty} \left\langle \tau_0^{k_0} \tau_1^{k_1} \tau_2^{k_2} \cdots \right\rangle \prod_{j=0}^{\infty} \frac{s_j^{k_j}}{k_j!}$$

$$(1.56) \qquad = \sum_{g=0}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\substack{d_1 + \cdots + d_n \\ = 3g-3+n}} \left\langle \tau_{d_1} \cdots \tau_{d_n} \right\rangle_{g,n} s_{d_1} \cdots s_{d_n}$$

$$= \left\langle \tau_0^3 \right\rangle_{0,3} \frac{s_0^3}{3!} + \left\langle \tau_1 \right\rangle_{1,1} s_1 + \left\langle \tau_0^4 \right\rangle_{0,4} \frac{s_0^4}{4!} + \cdots .$$

Define

$$t_{2j+1} := \frac{s_j}{(2j+1)!!},$$

and

$$(1.57) \qquad u(t_1, t_3, t_5, \dots) := \left(\frac{\partial}{\partial s_0}\right)^2 F(s_0, s_1, s_2, \dots).$$

Then $u(t_1, t_3, t_5, \dots)$ satisfies the system of KdV equations, whose first equation is

$$(1.58) \qquad u_{t_3} = \frac{1}{4} u_{t_1 t_1 t_1} + 3u u_{t_1}.$$

The system of KdV equations are the deformation equation for the universal iso-spectral family of second order ordinary differential operators of the form

$$(1.59) \qquad L(X, t) := \left(\frac{d}{dX}\right)^2 + 2u(X + t_1, t_3, t_5, \dots)$$

in X and deformation parameters $t = (t_1, t_3, t_5, \dots)$. The operator is often referred to as the *Lax operator* for the KdV equations. The expression

$$\sqrt{L} = \frac{d}{dX} + u \cdot \left(\frac{d}{dX}\right)^{-1} - \frac{1}{2} u' \cdot \left(\frac{d}{dX}\right)^{-2} + \cdots$$

makes sense in the ring of pseudo-differential operators, where $'$ denotes the X-derivative. The KdV equations are the system of *Lax equations*

$$\frac{\partial L}{\partial t_{2m+1}} = \left[\left(\sqrt{L}^{2m+1}\right)_+, L\right], \qquad m \geq 0,$$

where $_+$ denotes the differential operator part of a pseudo-differential operator. The commutator on the right-hand side explains the invariance of the eigenvalues of the Lax operator L with respect to the deformation parameter t_{2m+1}. The t_1-deformation is the translation $u(X) \longmapsto u(X + t_1)$, and the t_3-deformation is given by the KdV equation (1.58).

For the particular function (1.57) defined by the intersection numbers (1.56), the *initial value* of the Lax operator is

$$\left(\frac{d}{dX}\right)^2 + 2u(X + t_1, t_3, t_5, \dots)\big|_{t_1 = t_3 = t_5 = \cdots = 0} = \left(\frac{d}{dX}\right)^2 + 2X.$$

In terms of yet another variable $x = -\frac{2}{\sqrt[3]{4}} X$, the initial value becomes

$$\frac{1}{\sqrt[3]{4}} \left(\left(\frac{d}{dX}\right)^2 + 2X\right) = \left(\frac{d}{dx}\right)^2 - x = P(x, \hbar)\big|_{\hbar=1},$$

which is the Airy differential operator.

Kontsevich [55] used the matrix Airy function to obtain all intersection numbers. The topological recursion replaces the asymptotic analysis of matrix integrations with a series of residue calculations on the spectral curve $x = y^2$.

1.6. **All things considered.** What we have in front of us is an interesting example of a theory yet to be constructed.

- We have generating functions of quantum invariants, such as Gromov-Witten invariants. They are symmetric functions.
- We take the *principal specialization* of these functions, and form a generating function of the specialized generating functions.
- This function then solves a 1-dimensional stationary Schrödinger equation. The equation is what we call a *quantum curve*.

- From this Schrödinger equation (or a quantum curve), we construct an algebraic curve, a *spectral curve* in the sense of Hitchin, through the process of *semi-classical limit*.
- The differential version of the *topological recursion* [24, 25] applied to the spectral curve then recovers the starting quantum invariants.
- The spectral curve can be also expressed as the Hitchin spectral curve of a particular meromorphic Higgs bundle.
- Then the quantum curve is equivalent to the $\hbar$-connection in the $\hbar$-deformed vector bundle on the base curve, on which the initial Higgs bundle is defined.
- The topological recursion therefore constructs a flat section, although formal, of the $\hbar$-connection from the Hitchin spectral curve. At least locally, the $\hbar$-connection itself is thus constructed by the topological recursion.

We do not have a general theory yet. In particular, we do not have a global definition of quantum curves. As mentioned above, right at this moment, the notion of *opers* is emerging as a mathematical definition of quantum curves, at least for the case of smooth spectral covers in the cotangent bundle T^*C of a smooth curve C of genus greater than 1. We will report our finding in this exciting direction elsewhere. Here, we present what we know as of now.

The idea of topological recursion was devised for a totally different context. In the authors' work [24], for the first time the formalism of Eynard and Orantin was placed in the Higgs bundle context. The formalism depends purely on the geometry of the Hitchin spectral curve. Therefore, quantities that the topological recursion computes should represent the geometric information. Then in [25], we have shown through examples that quantization of singular spectral curves are related to certain enumerative geometry problems, when the quantum curve is analyzed at its singular point and the function $\Psi(x, \hbar)$, which should be actually considered as a formal section of $K_C^{-\frac{1}{2}}$, is expanded at its essential singularity. In the example described above, the corresponding counting problem is computing the intersection numbers of certain cohomology classes on $\overline{\mathcal{M}}_{g,n}$. The original topological recursion of [34] is generalized to singular spectral curves in [25] for this purpose.

The question we still do not know its answer is how to directly connect the Higgs bundle information with the geometric structure whose quantum invariants are captured by the topological recursion.

Since the time of inception of the topological recursion [16, 34], numerous papers have been produced, in both mathematics and physics. It is far beyond the authors' ability to make any meaningful comments on this vast body of literature in the present article. Luckily, interested readers can find useful information in Eynard's talk at the ICM 2014 [32]. Instead of attempting the impossible, we review here a glimpse of **geometric developments** inspired by the topological recursion that have taken place in the last few years.

The geometry community's keen attention was triggered when a concrete **remodeling conjecture** was formulated by string theorists, first by Mariño [59], and then in a more precise and generalized framework by Bouchard, Klemm, Mariño and Pasquetti [11, 12]. The conjecture states that open Gromov-Witten invariants of an arbitrary toric Calabi-Yau orbifold of dimension 3 can be calculated by the topological recursion formulated on the *mirror curve*. A physical argument for the correctness of the conjecture is pointed out in [76]. Bouchard and Mariño [13] then derived a new conjectural formula for simple Hurwitz numbers from the remodeling conjecture. The correctness of the Hurwitz number conjecture can be easily checked by a computer for many concrete examples. At the same

time, it was clear that the conjectural formula was totally different from the combinatorial formula known as the **cut-and-join equation** of [42, 43, 84].

After many computer experiments, one of the authors noticed that the conjectural formula of Bouchard and Mariño was exactly the *Laplace transform* of a particular variant of the cut-and-join equation. Once the precise relation between the knowns and unknowns is identified, often the rest is straightforward, even though a technical difficulty still remains. The conjecture for simple Hurwitz numbers of [13] was solved in [33, 69]. Its generalization to the orbifold Hurwitz numbers is then established in [10]. In each case, the Laplace transform plays the role of the mirror symmetry, changing the combinatorial problem on the A-model side to a complex analysis problem on the B-model side. The first case of the remodeling conjecture for $\mathbb{C}^3$ was solved, using the same idea, shortly afterwords in [90]. The remodeling conjecture in its full generality is recently solved in its final form by Fang, Liu, and Zong (announced in [37]), based on an earlier work of [35].

Independent of these developments, the relation between the topological recursion and combinatorics of enumeration of various graphs drawn on oriented topological surfaces has been studied by the Melbourne group of mathematicians, including Do, Leigh, Manesco, Norbury, and Scott (see, for example, [22, 72, 73, 74]). The authors' earlier papers [15, 27, 65] are inspired by their work. A surprising observation of the Melbourne group, formulated in a conjectural formula, is that the Gromov-Witten invariants of $\mathbb{P}^1$ themselves should satisfy the topological recursion. Since $\mathbb{P}^1$ is not a Calabi-Yau manifold, this conjecture does not follow from the remodeling conjecture.

The $GW(\mathbb{P}^1)$ conjecture of [74] is solved by the Amsterdam group of mathematicians, consisting of Dunin-Barkowski, Shadrin, and Spitz, in collaboration with Orantin [30]. Their discovery, that the topological recursion on a disjoint union of open discs as its spectral curve is equivalent to *cohomological field theory*, has become a key technique of many later works [3, 28, 37].

The technical difficulty of the topological recursion lies in the evaluation of residue calculations involved in the formula. When the spectral curve is an open disc, this difficulty does not occur. But if the global structure of the spectral curve has to be considered, then one needs a totally different idea. The work of [10, 33, 69] has overcome the complex analysis difficulty in dealing with simple and orbifold Hurwitz numbers. There, the key idea is the use of the (piecewise) *polynomiality* of these numbers through the ELSV formula [31] and its orbifold generalization [51]. The paper [28] proves the converse: they first prove the polynomiality of the Hurwitz numbers without assuming the relation to the intersection numbers over $\overline{\mathcal{M}}_{g,n}$, and then establish the ELSV formula from the topological recursion, utilizing a technique of [69].

The topological recursion is a byproduct of the study of random matrix theory/matrix models [16, 34]. A recursion of the same nature appeared earlier in the work of Mirzakhani [60, 61] on the Weil-Petersson volume of the moduli space of bordered hyperbolic surfaces. The Laplace transform of the Mirzakhani recursion is an example of the topological recursion. Its spectral curve, the sine curve, was first identified in [66] as the intertwiner of two representations of the Virasoro algebra.

The notion of quantum curves goes back to [1]. It is further developed in the physics community by Dijkgraaf, Hollands, Sułkowski, and Vafa [18, 19, 48]. The geometry community was piqued by [17, 44], which speculated on the relation between the topological recursion, quantization of the $SL(2,\mathbb{C})$-character variety of the fundamental group of a knot complement in S^3, the AJ-conjecture due to [39, 40], and the K_2-group of algebraic K-theory. Although it is tantalizingly interesting, so far no mathematical results have been

established in this direction for hyperbolic knots, or even it may be impossible (see for example, [9]), mainly due to the reducible nature of the spectral curve in this particular context. For torus knots, see also [14]. A rigorous construction of quantization of spectral curves was established for a few examples in [68] for the first time, but without any relation to knot invariants. One of the examples there will be treated in these lectures below. By now, we have many more mathematical examples of quantum curves [10, 22, 29, 67, 91, 92]. We note that in many of these examples, the spectral curves have a global parameter, even though the curves are not necessarily the rational projective curve. Therefore, the situation is in some sense still the "genus 0 case." The difficulty of quantization lies in dealing with complicated entire functions, and the fact that the quantum curves are **difference equations**, rather than differential equations of finite order.

We are thus led to another question.

Question 1.16. *What can we do when we have a different situation, where the spectral curve of the theory is global, compact, and of a high genus?*

It comes as a surprise that there is a system of recursive partial differential equations, resembling the residue calculation formula for the topological recursion, when the spectral curve of the topological recursion is precisely a Hitchin spectral curve associated with a rank 2 Higgs bundle. The result of the calculation then leads us to a construction of a quantum curve. In this way a connection between quantization of Hitchin spectral curves and the topological recursion is discovered in [24].

If we regard the topological recursion as a method of calculation of quantum invariants, then we need to allow singular spectral curves, as we have seen earlier. The simplest quantization of the singular Hitchin spectral curve is then obtained by the topological recursion again, but this time, it has to be applied to the normalization of the singular spectral curve constructed by a particular way of using blow-ups of the ambient compactified cotangent bundle. This is the content of [25], and we obtain a **Rees $\mathcal{D}$-module** as the result of quantization.

We note here that what people call by quantization is not unique. Depending on the purpose, one needs to use a different guideline for quantization. The result would be a different differential equation, but still having the same semi-classical limit.

For example, it has been rigorously proved in [50] that surprisingly the quantization procedure of [24, 25], including the desingularization of the spectral curve, *automatically* leads to an iso-monodromic deformation family, for the case of the Painlevé I equation. Their global parameter is essentially the normalization coordinate of the singular elliptic curve. In their work [50], they ask what one obtains if the straightforward topological recursion is applied for the quantization of a singular elliptic curve with a prescribed parameter in a particular way. They then find that the quantum curve is a Schrödinger equation whose coefficients have nontrivial dependence on $\hbar$, yet it is an iso-spectral family with respect to the parameter. This work, and also the numerous mathematical examples of quantum curves that have been already constructed, suggest that the idea of using $\mathcal{D}$-modules for the definition of quantum curves ([18, 25]) is not the final word. Differential operators of an infinite order, or difference operators mixed with differential operators, also have to be considered.

For the mirror curves of toric Calabi-Yau orbifolds of dimension 3 appearing in the context of the remodeling conjecture [11, 12, 37, 59], the conjectural quantum curves acquire a very different nature. It has deep connections to number theory and quantum dilogarithm functions [53].

A suggestion of using deformation quantization modules for quantum curves is made by Kontsevich in late 2013 in a private communication to the authors. An interesting work toward this direction is proposed by Petit [78].

The relation of the quantization discussed in these lectures with the **non-Abelian Hodge correspondence** and **opers** is being investigated as of now [23]. A coordinate-free global definition of quantum curves is emerging, and a direct relationship among quantum curves, non-Abelian Hodge correspondence, and opers is being developed.

The story is expanding its horizon. We have come to the other side of the rainbow. And there we find ourselves on Newton's seashore. So far we have found only a few smoother pebbles or prettier shells, whilst...

2. From Catalan numbers to the topological recursion

Let us consider a function $f(X)$ in one variable, where X is an $N \times N$ Hermitian matrix. One of the main problems of matrix integration theory is to calculate the *expectation value*

$$(2.1) \qquad \langle \mathrm{tr} f(X) \rangle := \frac{\int_{H_{N \times N}} \mathrm{tr} f(X) e^{V(X)} dX}{\int_{H_{N \times N}} e^{V(X)} dX},$$

where the *potential function* $V(X)$ is given, such as the Gaussian potential

$$(2.2) \qquad V(X) = -\frac{1}{2} \mathrm{tr}(X^2),$$

so that

$$C_N = \int_{H_{N \times N}} e^{V(X)} dX$$

is finite. The integration measure dX is the standard $U(N)$-invariant Lebesgue measure of the space $H_{N \times N} = \mathbb{R}^{N^2}$ of Hermitian matrices of size N. When a Gaussian potential (2.2) is chosen, $e^{V(X)} dX$ is a probability measure after an appropriate normalization, and $\langle \mathrm{tr}(X^m) \rangle$ is the m-th *moment*. If we know all the moments, then we can calculate the expectation value of any polynomial function $f(X)$. Therefore, the problem changes into calculating a generating function of the moments

$$(2.3) \qquad \frac{1}{C_N} \sum_{m=0}^{\infty} \frac{1}{z^{m+1}} \int_{H_{N \times N}} \mathrm{tr}(X^m) e^{V(X)} dX.$$

For a norm bounded matrix X, we have

$$tr\left(\frac{1}{z-X}\right) = \sum_{m=0}^{\infty} \frac{1}{z^{m+1}} \mathrm{tr}(X^m).$$

Therefore, the *resolvent* of a random matrix X,

$$(2.4) \qquad \left\langle \mathrm{tr}\left(\frac{1}{z-X}\right) \right\rangle = \frac{1}{C_N} \int_{H_{N \times N}} \mathrm{tr}\left(\frac{1}{z-X}\right) e^{V(X)} dX,$$

looks the same as (2.3). But they are not the same. For example, let us consider the $N = 1$ case and write $X = x \in \mathbb{R}$. Then the formula

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{z-x} e^{-\frac{1}{2}x^2} dx \overset{?}{=} \sum_{m=0}^{\infty} \frac{(2m-1)!!}{z^{2m+1}} = \frac{1}{\sqrt{2\pi}} \sum_{m=0}^{\infty} \frac{1}{z^{m+1}} \int_{-\infty}^{\infty} x^m\, e^{-\frac{1}{2}x^2} dx$$

is valid only as the asymptotic expansion of the analytic function

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{z-x} \; e^{-\frac{1}{2}x^2} dx$$

in z for $Im(z) \neq 0$ and $Re(z) \to +\infty$. Still we can see that the information of the generating function of the moment (2.3) can be extracted from the resolvent (2.4) if we apply the technique of asymptotic expansion of a holomorphic function near at an essential singularity. The asymptotic method of matrix integrals leads to many interesting formulas, such as calculating the orbifold Euler characteristic $\chi(\mathcal{M}_{g,n})$ of the moduli space of smooth pointed curves [45]. We refere to [63, 64] for introductory materials of these topics.

More generally, we can consider *multi-resolvent correlation functions*

$$(2.5) \qquad \left\langle \prod_{i=1}^{n} \operatorname{tr}\left(\frac{1}{z_i - X}\right) \right\rangle = \frac{1}{C_N} \int_{H_{N \times N}} \prod_{i=1}^{n} \operatorname{tr}\left(\frac{1}{z_i - X}\right) e^{V(X)} dX.$$

When we say "calculating" the expectation value (2.5), we wish to identify it as a *holomorphic* function in all the parameters, i.e., $(z_1, \ldots, z_n)$, the coefficients of the potential V, and the matrix size N. In particular, the analytic dependence on the parameter N is an important feature we wish to determine.

It is quite an involved problem in analysis, and we do not attempt to follow this route in these lecture notes. The collaborative effort of the random matrix community has devised a recursive method of solving this analysis problem (see, for example, [16, 32, 34]), which is now known as the **topological recursion**. One thing we can easily expect here is that since (2.5) is an analytic function in $(z_1, \ldots, z_n)$, there must be an obvious relation between the topological recursion and algebraic geometry.

What is amazing is that the exact same recursion formula happens to appear in the context of many different enumerative geometry problems, again and again. Even though the counting problems are different, the topological recursion always takes the same general formalism. Therefore, to understand the nature of this formalism, it suffices to give the simplest non-trivial example. This is what we wish to accomplish in this section.

2.1. **Counting graphs on a surface.** Let us start with the following problem:

Problem 2.1. *Find the number of distinct cell-decompositions of a given closed oriented topological surface of genus g, with the specified number of 0-cells, and the number of 1-cells that are incident to each 0-cell.*

Denote by C_g a compact, 2-dimensional, oriented topological manifold of genus g without boundary. Two cell-decompositions of C_g are identified if there is an orientation-preserving homeomorphism of C_g onto itself that brings one to the other. If there is such a map for the same cell-decomposition, then it is an automorphism of the data. The 1-*skeleton* of a cell-decomposition, which we denote by γ, is a *graph* drawn on C_g. We call a 0-cell a *vertex*, a 1-cell an *edge*, and a 2-cell a *face*. The midpoint of an edge separates the edge into two *half-edges* joined together at the midpoint. The *degree* of a vertex is the number of half-edges incident to it. For the purpose of counting, we label all vertices. To be more precise, we give a total ordering to the set of vertices. Most of the time we simply use $[n] = \{1, \ldots, n\}$ to label the set of n vertices.

The 1-skeleton γ is usually called a *ribbon graph*, which is a graph with a cyclic order assigned to incident half-edges at each vertex. The face-labeled ribbon graphs describe an orbifold cell-decomposition of $\mathcal{M}_{g,n} \times \mathbb{R}_+^n$. Since we label vertices of γ, there is a slight

difference as to what the graph represents. It is the *dual* graph of a ribbon graph, and its vertices are labeled. To emphasize the dual nature, we call γ a **cell graph**.

Most cell graphs do not have any non-trivial automorphisms. If there is one, then it induces a cyclic permutation of half-edges at a vertex, since we label all vertices. Therefore, if we pick one of the incident half-edges at each vertex, assign an outgoing arrow to it, and require that an automorphism also fix the arrowed half-edges, then the graph has no non-trivial automorphisms. For a counting problem, no automorphism is a desirable situation because the bijective counting method works better there. Let us call such a graph an **arrowed cell graph**. Now the refined problem:

Problem 2.2. *Let $\vec{\Gamma}_{g,n}(\mu_1, \ldots, \mu_n)$ denote the set of arrowed cell graphs drawn on a closed, connected, oriented surface of genus g with n labeled vertices of degrees $\mu_1, \ldots, \mu_n$. Find its cardinality*

$$(2.6) \qquad C_{g,n}(\mu_1, \ldots, \mu_n) := |\vec{\Gamma}_{g,n}(\mu_1, \ldots, \mu_n)|.$$

When $\gamma \in \vec{\Gamma}_{g,n}(\mu_1, \ldots, \mu_n)$, we say γ has *type* (g, n). Denote by $c_\alpha(\gamma)$ the number of α-cells of the cell-decomposition associated with γ. Then we have

$$2 - 2g = c_0(\gamma) - c_1(\gamma) + c_2(\gamma), \qquad c_0(\gamma) = n, \qquad 2c_1(\gamma) = \mu_1 + \cdots + \mu_n.$$

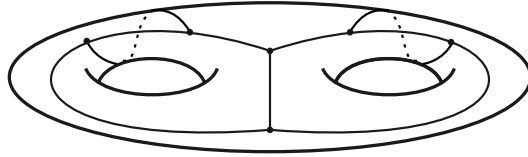Therefore, $\vec{\Gamma}_{g,n}(\mu_1, \ldots, \mu_n)$ is a finite set.

FIGURE 2.1. A cell graph of type $(2, 6)$.

Example 2.1. An arrowed cell graph of type $(0, 1)$ is a collection of loops drawn on a plane as in Figure 2.2. If we assign a pair of parenthesis to each loop, starting from the arrowed one as (, and go around the unique vertex counter-clock wise, then we obtain the parentheses pattern $((( )))$. Therefore,

$$(2.7) \qquad C_{0,1}(2m) = C_m = \frac{1}{m+1}\binom{2m}{m}$$

is the m-th Catalan number. Thus it makes sense to call $C_{g,n}(\mu_1, \cdots, \mu_n)$ **generalized Catalan numbers**. Note that it is a symmetric function in n integer variables.
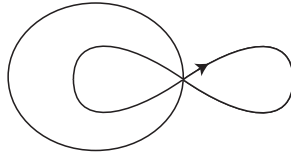

FIGURE 2.2. A cell graph of type $(0, 1)$ with a vertex of degree 6.

Theorem 2.3 (Catalan Recursion, [27, 85]). *The generalized Catalan numbers satisfy the following equation.*

$$(2.8) \quad C_{g,n}(\mu_1, \ldots, \mu_n) = \sum_{j=2}^{n} \mu_j C_{g,n-1}(\mu_1 + \mu_j - 2, \mu_2, \ldots, \widehat{\mu_j}, \ldots, \mu_n)$$

$$+ \sum_{\alpha+\beta=\mu_1-2} \left[C_{g-1,n+1}(\alpha, \beta, \mu_2, \cdots, \mu_n) + \sum_{\substack{g_1+g_2=g \\ I \sqcup J = \{2,\ldots,n\}}} C_{g_1,|I|+1}(\alpha, \mu_I) C_{g_2,|J|+1}(\beta, \mu_J) \right],$$

where $\mu_I = (\mu_i)_{i \in I}$ for an index set $I \subset [n]$, $|I|$ denotes the cardinality of I, and the third sum in the formula is for all partitions of g and set partitions of $\{2, \ldots, n\}$.

Proof. Let γ be an arrowed cell graph counted by the left-hand side of (2.8). Since all vertices of γ are labeled, we write the vertex set by $\{p_1, \ldots, p_n\}$. We take a look at the half-edge incident to p_1 that carries an arrow.

Case 1. The arrowed half-edge extends to an edge E that connects p_1 and p_j for some $j > 1$.

In this case, we contract the edge and join the two vertices p_1 and p_j together. By this process we create a new vertex of degree $\mu_1 + \mu_j - 2$. To make the counting bijective, we need to be able to go back from the contracted graph to the original, provided that we know μ_1 and μ_j. Thus we place an arrow to the half-edge next to E around p_1 with respect to the counter-clockwise cyclic order that comes from the orientation of the surface. In this process we have μ_j different arrowed graphs that produce the same result, because we must remove the arrow placed around the vertex p_j in the original graph. This gives the first line of the right-hand side of (2.8). See Figure 2.3.
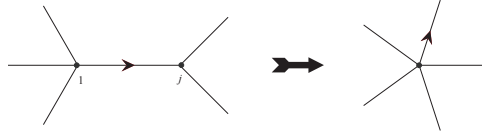

FIGURE 2.3. The process of contracting the arrowed edge E that connects vertices p_1 and p_j, $j > 1$.

Case 2. The arrowed half-edge at p_1 is a loop E that goes out from, and comes back to, p_1.

The process we apply is again contracting the loop E. The loop E separates all other incident half-edges at p_1 into two groups, one consisting of α of them placed on one side of the loop, and the other consisting of β half-edges placed on the other side. It can happen that $\alpha = 0$ or $\beta = 0$. Contracting a loop on a surface causes pinching. Instead of creating a pinched (i.e., singular) surface, we separate the double point into two new vertices of degrees α and β. Here again we need to remember the place of the loop E. Thus we put an arrow to the half-edge next to the loop in each group. See Figure 2.4.

After the pinching and separating the double point, the original surface of genus g with n vertices $\{p_1, \ldots, p_n\}$ may change its topology. It may have genus $g - 1$, or it splits into two pieces of genus g_1 and g_2. The second line of (2.8) records all such cases. Normally we would have a factor $\frac{1}{2}$ in front of the second line of the formula. We do not have it here
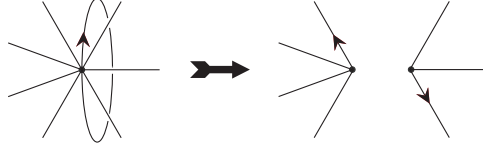
FIGURE 2.4. The process of contracting the arrowed loop E that is attached to p_1.

because the arrow on the loop could be in two different directions. Placing the arrow on the other half-edge of the loop is equivalent to interchanging α and β.

This completes the proof.   □

Remark 2.4. For $(g, n) = (0, 1)$, the above formula reduces to

$$(2.9) \qquad C_{0,1}(\mu_1) = \sum_{\alpha+\beta=\mu_1-2} C_{0,1}(\alpha)C_{0,1}(\beta).$$

Since the degree of the unique vertex is always even for type $(0, 1)$ graphs, by defining $C_{0,1}(0) = 1$, (2.9) gives the Catalan recursion. Only for $(g, n) = (0, 1)$, this irregular case of $\mu_1 = 0$ happens, because a degree 0 single vertex is *connected*, and gives a cell-decomposition of S^2. All other cases, if one of the vertices has degree 0, then the Catalan number $C_{g,n}(\mu_1, \ldots, \mu_n)$ is simply 0 because there is no corresponding *connected* cell decomposition.

Remark 2.5. Eqn. (2.8) is a recursion with respect to

$$2g - 2 + n + \sum_{i=1}^{n} \mu_i.$$

The values are therefore completely determined by the initial value $C_{0,1}(0) = 1$. The formula does not give a recursion of a *function* $C_{g,n}(\mu_1, \ldots, \mu_n)$, because the same type (g, n) appears on the right-hand side.

The classical Catalan recursion (2.9) determines all values of $C_{0,1}(2m)$, but the closed formula (2.7) requires a different strategy. Let us introduce a generating function

$$(2.10) \qquad z = z(x) = \sum_{m=0}^{\infty} C_{0,1}(2m)x^{-2m-1}.$$

From (2.9) we have

$$z^2 = \sum_{m=0}^{\infty} \left(\sum_{a+b=m} C_{0,1}(2a)C_{0,1}(2b) \right) x^{-2m-2} = \sum_{m=0}^{\infty} C_{0,1}(2m+2)x^{-2m-2}.$$

Since

$$xz = \sum_{m=-1}^{\infty} C_{0,1}(2m+2)x^{-2m-2},$$

we obtain an equation $xz = z^2 + 1$, or

$$(2.11) \qquad x = z + \frac{1}{z}.$$

This is the inverse function of the complicated-looking generating function $z(x)$ at the branch $z \to 0$ as $x \to +\infty$! Thus the curve (2.11) knows everything about the Catalan

numbers. For example, we can prove the closed formula (2.7). The solution of (2.11) as a quadratic equation for z that gives the above branch is given by

$$z(x) = \frac{x - \sqrt{x^2 - 4}}{2} = \frac{x}{2}\left(1 - \sqrt{1 - \frac{4}{x^2}}\right).$$

The binomial expansion of the square root

$$\sqrt{1 + X} = \sum_{m=0}^{\infty} \binom{\frac{1}{2}}{m} X^m,$$

$$\binom{\frac{1}{2}}{m} := \frac{\frac{1}{2}(\frac{1}{2} - 1)(\frac{1}{2} - 2)\cdots(\frac{1}{2} - m + 1)}{m!} = (-1)^{m-1}\frac{(2m-3)!!}{2^m m!} = \frac{(-1)^{m-1}}{4^m}\frac{1}{2m-1}\binom{2m}{m},$$

then gives the closed formula (2.7) for the Catalan numbers:

$$z(x) = \sum_{m=0}^{\infty} C_{0,1}(2m)x^{-2m-1}$$

$$= \frac{x}{2}\left(1 - \sqrt{1 - \frac{4}{x^2}}\right)$$

$$= \frac{x}{2}\left(1 - \left(\sum_{m=0}^{\infty}\frac{(-1)^{m-1}}{4^m}\frac{1}{2m-1}\binom{2m}{m}(-1)^m\left(\frac{4}{x^2}\right)^2\right)\right)$$

$$= \frac{1}{2}\sum_{m=1}^{\infty}\frac{1}{2m-1}\binom{2m}{m}x^{-2m+1}$$

$$= \sum_{m=0}^{\infty}\frac{1}{m+1}\binom{2m}{m}x^{-2m-1}.$$

Another piece of information we obtain from (2.11) is the radius of convergence of the infinite series $z(x)$ of (2.10). Since

$$dx = \left(1 - \frac{1}{z^2}\right)dz,$$

the map (2.11) is critical (or ramified) at $z = \pm 1$. The critical values are $x = \pm 2$. On the branch we are considering, (2.10) is the inverse function of (2.11) for all values of $|x| > 2$. This means the series $z(x)$ is absolutely convergent on the same domain.

Remark 2.6. In combinatorics, we often consider a generating function of interesting quantities only as a formal power series. The idea of topological recursion tells us that we should consider the **Riemann surface** of the Catalan number generating function $z = z(x)$. We then recognize that there is a global algebraic curve hidden in the scene, which is the curve of Figure 2.5. The topological recursion mechanism then tells us how to calculate all $C_{g,n}(\vec{\mu})$ from this curve alone, known as the **spectral curve**.

2.2. **The spectral curve of a Higgs bundle and its desingularization.** To consider the quantization of the curve (2.11), we need to place it into a cotangent bundle. Here again, we use the base curve $\mathbb{P}^1$ and the same vector bundle E of (1.21) on it. As a Higgs field, we use

(2.12) $$\phi := \begin{bmatrix} -xdx & -(dx)^2 \\ 1 & \end{bmatrix} : E \longrightarrow K_{\mathbb{P}^1}(4) \otimes E.$$
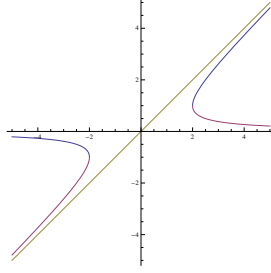
FIGURE 2.5. The **Riemsnn surface** of the Catalan generating function $z = z(x)$.

Here, $(dx)^2 \in H^0\big(\mathbb{P}^1, K_{\mathbb{P}^1}^{\otimes 2}(4)\big)$ is the unique (up to a constant factor) quadratic differential with an order 4 pole at the infinity, and $xdx \in H^0\big(\mathbb{P}^1, K_{\mathbb{P}^1}(2)\big)$ is the unique meromorphic differential with a zero at $x = 0$ and a pole of order 3 at $x = \infty$. In the affine coordinate (x, y) of the Hirzebruch surface $\mathbb{F}^2$ as before, the spectral curve Σ is given by

$$(2.13) \qquad \det\big(\eta - \pi^*(\phi)\big) = (y^2 + xy + 1)(dx)^2 = 0,$$

where $\pi : \mathbb{F}_2 \longrightarrow \mathbb{P}^1$ is the projection. Therefore, the generating function $z(x)$ of (2.10) gives a parametrization of the spectral curve

$$(2.14) \qquad \begin{cases} x = z(x) + \frac{1}{z(x)} \\ y = -z(x). \end{cases}$$

In the other affine coordinate (u, w) of (1.25), the spectral curve is singular at $(u, w) = (0, 0)$:

$$(2.15) \qquad u^4 - uw + w^2 = 0.$$
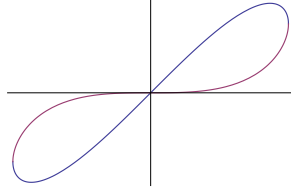
FIGURE 2.6. The spectral curve Σ of (2.13). The horizontal line is the divisor $w = 0$ at infinity, and the vertical line is the fiber class $u = 0$. The spectral curve intersects with $w = 0$ four times. One of the two curve germ components is given by $w = u$, and the other by $w = u^3$.

Blow up $\mathbb{F}_2 = \overline{T^*\mathbb{P}^1}$ once at the nodal singularity $(u, w) = (0, 0)$ of the spectral curve Σ, and let $\widetilde{\Sigma} \longrightarrow \Sigma$ be the proper transform of Σ.

$$(2.16)$$

In terms of the coordinate w_1 defined by $w = w_1 u$, (2.15) becomes

$$(2.17) \qquad u^2 + \left(w_1 - \frac{1}{2} \right)^2 = \frac{1}{4},$$

which is the defining equation for $\widetilde{\Sigma}$. From this equation we see that its geometric genus is 0, hence it is just another $\mathbb{P}^1$. The covering $\tilde{\pi} : \widetilde{\Sigma} \longrightarrow \mathbb{P}^1$ is ramified at two points, corresponding to the original ramification points $(x, y) = (\pm 2, \mp 1)$ of $\pi : \Sigma \longrightarrow \mathbb{P}^1$. The rational parametrization of (2.17) is given by

$$(2.18) \qquad \begin{cases} u = \frac{1}{2} \cdot \frac{t^2 - 1}{t^2 + 1} \\ w_1 = \frac{1}{2} - \frac{t}{t^2 + 1}, \end{cases}$$

where t is the affine coordinate of $\widetilde{\Sigma}$ so that $t = \pm 1$ gives $(u, w) = (0, 0)$. Indeed, the parameter t is a **normalization coordinate** of the spectral curve Σ:

$$(2.19) \qquad \begin{cases} x = 2 + \frac{4}{t^2 - 1} \\ y = -\frac{t+1}{t-1}, \end{cases} \qquad \begin{cases} u = \frac{1}{2} \cdot \frac{t^2 - 1}{t^2 + 1} \\ w = \frac{1}{4} \cdot \frac{(t-1)^3 (t+1)}{(t^2+1)^2}. \end{cases}$$

Although the expression of x and y in terms of the normalization coordinate is more complicated than (2.14), it is important to note that the spectral curve $\widetilde{\Sigma}$ is now non-singular.

2.3. **The generating function, or the Laplace transform.** For all (g, n) except for $(0, 1)$ and $(0, 2)$, let us introduce the generating function of (2.6) as follows:

$$(2.20) \qquad F_{g,n}^C (x_1, \ldots, x_n) := \sum_{\mu_1 \geq 1, \ldots, \mu_n \geq 1} \frac{C_{g,n}(\mu_1, \ldots, \mu_n)}{\mu_1 \cdots \mu_n} x_1^{-\mu_1} \cdots x_n^{-\mu_n}.$$

If we consider $x_i = e^{w_i}$, then the above sum is just the discrete Laplace transform of the function in n integer variables:

$$\frac{C_{g,n}(\mu_1, \ldots, \mu_n)}{\mu_1 \cdots \mu_n}.$$

Our immediate goal is to compute the Laplace transform as a holomorphic function. Since the only information we have now is the generalized Catalan recursion (2.8), how much can we say about this function? Actually, the following theorem is proved, all from using the recursion alone!

Theorem 2.7. *Let us consider the generating function* (2.20) *as a function in the normalization coordinates* $(t_1, \ldots, t_n)$ *satisfying*

$$x_i = 2 + \frac{4}{t_i^2 - 1}, \qquad i = 1, 2, \ldots, n,$$

as identified in (2.19), *and by abuse of notation, we simply write it as* $F_{g,n}^C (t_1, \ldots, t_n)$. *For the range of* (g, n) *with the stability condition* $2g - 2 + n > 0$, *we have the following.*

- *The generating function* $F_{g,n}^C (t_1, \ldots, t_n)$ *is a **Laurent polynomial** in the t_i-variables of the total degree* $3(2g - 2 + n)$.
- *The reciprocity relation holds:*

$$(2.21) \qquad F_{g,n}^C (1/t_1, \ldots, 1/t_n) = F_{g,n}^C (t_1, \ldots, t_n).$$

- *The special values at* $t_i = -1$ *are given by*

$$(2.22) \qquad F_{g,n}^C (t_1, \ldots, t_n) \big|_{t_i = -1} = 0$$

  for each i.

- *The diagonal value at $t_i = 1$ gives the orbifold Euler characteristic of the moduli space $\mathcal{M}_{g,n}$:*

$$(2.23) \qquad F_{g,n}^C(1,\ldots,1) = (-1)^n \chi(\mathcal{M}_{g,n}).$$

- *The restriction of the Laurent polynomial $F_{g,n}^C(t_1,\ldots,t_n)$ to its highest degree terms gives a homogeneous polynomial defined by*

$$(2.24) \quad F_{g,n}^{C,\,highest}(t_1,\ldots,t_n) = \frac{(-1)^n}{2^{2g-2+n}} \sum_{\substack{d_1+\cdots+d_n \\ =3g-3+n}} \langle \tau_{d_1}\cdots\tau_{d_n}\rangle_{g,n} \prod_{i=1}^n |2d_i-1|!! \left(\frac{t_i}{2}\right)^{2d_i+1}.$$

Thus the function $F_{g,n}^C(t_1,\ldots,t_n)$ knows the orbifold Euler characteristic of $\mathcal{M}_{g,n}$, and all the cotangent class intersection numbers (1.11) for all values of (g,n) in the stable range! It is also striking that it is actually a Laurent polynomial, while the definition (2.20) is given only as a formal Laurent series. The reciprocity (2.21) is the reflection of the invariance of the spectral curve Σ under the rotation

$$(x,y) \longmapsto (-x,-y).$$

This surprising theorem is a consequence of the Laplace transform of the Catalan recursion itself.

Theorem 2.8 (Differential recursion, [93]). *The Laplace transform $F_{g,n}^C(t_1,\ldots,t_n)$ satisfies the following differential recursion equation for every (g,n) subject to $2g-2+n \geq 2$.*

$$(2.25) \quad \frac{\partial}{\partial t_1} F_{g,n}^C(t_1,\ldots,t_n)$$

$$= -\frac{1}{16}\sum_{j=2}^n \left[\frac{t_j}{t_1^2-t_j^2} \left(\frac{(t_1^2-1)^3}{t_1^2} \frac{\partial}{\partial t_1} F_{g,n-1}^C(t_1,\ldots,\widehat{t_j},\ldots,t_n) \right.\right.$$

$$\left.\left. - \frac{(t_j^2-1)^3}{t_j^2} \frac{\partial}{\partial t_j} F_{g,n-1}^C(t_2,\ldots,t_n) \right) \right]$$

$$- \frac{1}{16}\sum_{j=2}^n \frac{(t_1^2-1)^2}{t_1^2} \frac{\partial}{\partial t_1} F_{g,n-1}^C(t_1,\ldots,\widehat{t_j},\ldots,t_n)$$

$$- \frac{1}{32} \frac{(t_1^2-1)^3}{t_1^2} \left[\frac{\partial^2}{\partial u_1 \partial u_2} F_{g-1,n+1}^C(u_1,u_2,t_2,t_3,\ldots,t_n) \right]\Bigg|_{u_1=u_2=t_1}$$

$$- \frac{1}{32} \frac{(t_1^2-1)^3}{t_1^2} \sum_{\substack{g_1+g_2=g \\ I\sqcup J=\{2,3,\ldots,n\}}}^{\text{stable}} \frac{\partial}{\partial t_1} F_{g_1,|I|+1}^C(t_1,t_I) \frac{\partial}{\partial t_1} F_{g_2,|J|+1}^C(t_1,t_J).$$

For a subset $I \subset \{1,2,\ldots,n\}$, we denote $t_I = (t_i)_{i\in I}$. The "stable" summation means $2g_1+|I|-1 > 0$ and $2g_2+|J|-1 > 0$.

The differential recursion uniquely determines all $F_{g,n}^C(t_1,\ldots,t_n)$ by integrating the right-hand side of (2.25) from -1 to t_1 with respect to the variable t_1. The initial conditions are

$$(2.26) \qquad F_{1,1}^C(t) = -\frac{1}{384}\frac{(t+1)^4}{t^2}\left(t-4+\frac{1}{t}\right)$$

and

$$F_{0,3}^C(t_1, t_2, t_3) = -\frac{1}{16}(t_1 + 1)(t_2 + 1)(t_3 + 1)\left(1 + \frac{1}{t_1 \, t_2 \, t_3}\right). \tag{2.27}$$

Remark 2.9. Theorem 2.8 is proved by Mincheng Zhou in his senior thesis [93]. It is the result of a rather difficult calculation of the Laplace transform of the Catalan recursion (2.8).

Remark 2.10. Theorem 2.7 has never been stated in this format before. Its proof follows from the results of [25, 27, 65, 68], based on induction using (2.25). The essential point of the discovery of Theorem 2.7 is the use of the normalization coordinate t of (2.19). The authors almost accidentally found the coordinate transformation

$$z(x) = \frac{t+1}{t-1}$$

in [27]. Then in [25], we noticed that this coordinate was exactly the normalization coordinate that was naturally obtained in the blow-up process (2.16).

The uniqueness of the solution of (2.25) allows us to identify the solution $F_{g,n}^C$ with the Laplace transform of the number of lattice points in $\mathcal{M}_{g,n}$, as we see later in this section. Through this identification, (2.22) and (2.23) become obvious. The asymptotic behavior (2.24) follows from the lattice point approximation of the Euclidean volume of polytopes, and Kontsevich's theorem that identifies the Euclidean volume of $\mathcal{M}_{g,n}$ and the intersection numbers (1.11) on $\overline{\mathcal{M}}_{g,n}$.

2.4. **The unstable geometries and the initial value of the topological recursion.**
The actual computation of the Laplace transform equation (2.25) from (2.8) requires the evaluation of the Laplace transforms of $C_{0,1}(\mu_1)$ and $C_{0,2}(\mu_1, \mu_2)$. It is done as follows.

Since a degree 0 vertex is allowed for the $(g, n) = (0, 1)$ unstable geometry, we do not have the corresponding $F_{0,1}^C(x)$ in (2.20). Since

$$d_{x_1} \cdots d_{x_n} F_{g,n}^C = \sum_{\mu_1 \geq 1, \ldots, \mu_n \geq 1} (-1)^n C_{g,n}(\mu_1, \ldots, \mu_n) x_1^{-\mu_1 - 1} \cdots x_n^{-\mu_n - 1} dx_1 \cdots dx_n,$$

we could choose

$$d_x F_{0,1}^C(x) = -\big(z(x) - x^{-1}\big) dx$$

as a defining equation for $F_{0,1}^C$.

In the light of (2.14), $ydx = -zdx$ is a natural global holomorphic 1-form on $T^*\mathbb{P}^1$, called the *tautological* 1-form. Its exterior differentiation $d(ydx) = dy \wedge dx$ defines the canonical holomorphic symplectic structure on $T^*\mathbb{P}^1$. Since we are interested in the *quantization* of the Hitchin spectral curve, we need a symplectic structure here, which is readily available for our use from ydx.

Therefore, it is reasonable for us to define the 'function' $F_{0,1}$ by

$$dF_{0,1} = ydx, \qquad \text{or} \qquad F_{0,1} = \int ydx. \tag{2.28}$$

Although this equation does not determine the constant term, it does not play any role for our purposes. Here, we choose the constant of integration to be 0. Since the symplectic structure on $T^*\mathbb{P}^1$ is non-trivial, we need to interpret the equation (2.28) being defined on the spectral curve, and be prepared that we may not obtain any meromorphic function on

the spectral curve. For the Catalan case, we have to use the spectral curve coordinate z or t as a variable of $F_{g,n}^C$, instead of x. Since

$$-zdx = -zdz + \frac{dz}{z},$$

we conclude

(2.29)
$$F_{0,1}^C(z) := -\frac{1}{2}z^2 + \log z$$
$$= -\frac{1}{2}\left(\frac{t+1}{t-1}\right)^2 + \log\left(\frac{t+1}{t-1}\right).$$

We encounter a non-algebraic function here, indeed.

For the computation of the Laplace transform $F_{0,2}^C$, we need an explicit formula for $C_{0,2}(\mu_1, \mu_2)$. Luckily, such computation has been carried out in [54], fully utilizing the technique of the *dispersionless Toda lattice hierarchy*. It is surprising to see how much integrable system consideration is involved in computing such a simple quantity as $C_{0,2}(\mu_1, \mu_2)$! The result is the following.

Theorem 2.11 (Calculation of the 2-point Catalan numbers, [54]). *For every $\mu_1, \mu_2 > 0$, the genus 0, 2-point Catalan numbers are given by*

(2.30)
$$C_{0,2}(\mu_1, \mu_2) = 2\frac{\lfloor\frac{\mu_1+1}{2}\rfloor\lfloor\frac{\mu_2+1}{2}\rfloor}{\mu_1 + \mu_2}\binom{\mu_1}{\lfloor\frac{\mu_1}{2}\rfloor}\binom{\mu_2}{\lfloor\frac{\mu_2}{2}\rfloor}.$$

We refer to [27] for the derivation of $F_{0,2}^C$. The result is

(2.31)
$$F_{0,2}^C(z_1, z_2) = -\log(1 - z_1 z_2)$$
$$= -\log\left(-2(t_1 + t_2)\right) + \log(t_1 - 1) + \log(t_2 - 1).$$

For the purpose of later use, we note that

(2.32)
$$d_{t_1} d_{t_2} F_{0,2}^C(t_1, t_2) = \frac{dt_1 \cdot dt_2}{(t_1 + t_2)^2}$$
$$= \frac{dt_1 \cdot dt_2}{(t_1 - t_2)^2} - (\tilde{\pi} \times \tilde{\pi})^* \frac{dx_1 \cdot dx_2}{(x_1 - x_2)^2},$$

where $\tilde{\pi} : \widetilde{\Sigma} \longrightarrow \mathbb{P}^1$ is the projection of (2.16), i.e., the variable transformation (2.19).

2.5. **Geometry of the topological recursion.** Computation of a multi-resolvent (2.5) is one thing. What we have in front of us is a different story. We wish to compute an asymptotic expansion of a solution to the differential equation that is defined on the base curve C and gives the quantization of the Hitchin spectral curve of a meromorphic Higgs bundle (E, ϕ). The expansion has to be done at the essential singularity of the solution.

Question 2.12. *Is there a mathematical framework suitable for such problems?*

The discovery of [24, 25] gives an answer: **Generalize the formalism of Eynard and Orantin of [34] to the geometric situation of meromorphic Higgs bundles. Then this generalized topological recursion computes the asymptotic expansion in question.**

We are now ready to present the topological recursion, continuing our investigation of the particular example of Catalan numbers. The point here is that **the topological recursion is a universal formula depending only on geometry**. Therefore, we can arrive at the general formula from any example. *One example rules them all!*

Theorem 2.13 (The topological recursion for the generalized Catalan numbers, [27]).
Define symmetric n-linear differential forms on $(\widetilde{\Sigma})^n$ for $2g - 2 + n > 0$ by

$$(2.33) \qquad W_{g,n}^C(t_1, \ldots, t_n) := d_{t_1} \cdots d_{t_n} F_{g,n}^C(t_1, \ldots, t_n),$$

and for $(g, n) = (0, 2)$ by

$$(2.34) \qquad W_{0,2}^C(t_1, t_2) := \frac{dt_1 \cdot dt_2}{(t_1 - t_2)^2}.$$

Then these differential forms satisfy the following integral recursion equation, called the
topological recursion.

$$(2.35) \quad W_{g,n}^C(t_1, \ldots, t_n) = -\frac{1}{64} \frac{1}{2\pi i} \int_\gamma \left(\frac{1}{t + t_1} + \frac{1}{t - t_1} \right) \frac{(t^2 - 1)^3}{t^2} \cdot \frac{1}{dt} \cdot dt_1$$

$$\times \left[\sum_{j=2}^n \left(W_{0,2}^C(t, t_j) W_{g,n-1}^C(-t, t_2, \ldots, \widehat{t_j}, \ldots, t_n) + W_{0,2}^C(-t, t_j) W_{g,n-1}^C(t, t_2, \ldots, \widehat{t_j}, \ldots, t_n) \right) \right.$$

$$\left. + W_{g-1,n+1}^C(t, -t, t_2, \ldots, t_n) + \sum_{\substack{g_1+g_2=g \\ I \sqcup J = \{2,3,\ldots,n\}}}^{stable} W_{g_1,|I|+1}^C(t, t_I) W_{g_2,|J|+1}^C(-t, t_J) \right].$$

The last sum is restricted to the stable geometries, i.e., the partitions should satisfies $2g_1 - 1 + |I| > 0$ and $2g_2 - 1 + |J|$, as in (2.25). The contour integral is taken with respect to t on the exactly the same cycle defined by Figure 1.4 as before, where t is the normalization coordinate of (2.19). Note that the second and the third lines of (2.35) is a quadratic differential in the variable t.

Remark 2.14. The notation $\frac{1}{dt}$ requires a justification. We note that the ratio of two global meromorphic sections of the same line bundle is a global meromorphic function. Here we are taking the ratio of two meromorphic 1-forms on the factor $\widetilde{\Sigma}$ corresponding to the t-variable. Thus after taking this ratio, the integrand becomes a meromorphic 1-form in $(-t)$-variable, which is integrated along the cycle γ.

Remark 2.15. The recursion (2.35) is a genuine induction formula with respect to $2g-2+n$. Thus from $W_{0,2}^C$, we can calculate all $W_{g,n}^C$s one by one. This is a big difference between (2.35) and (2.8). The latter relation contains terms with $C_{g,n}$ in the right-hand side as well, therefore, $C_{g,n}$ is not determined as a function by an induction procedure.

Remark 2.16. If we apply (2.33) to $F_{0,2}^C$ of (2.31), then we obtain (2.32), not (2.34). The difference is the pull-back of the 2-form $\frac{dx_1 \cdot dx_2}{(x_1 - x_2)^2}$. This difference does not affect the recursion formula (2.35) for $2g - 2 + n > 1$. The only case affected is $(g, n) = (1, 1)$. The above recursion allows us to calculate $W_{1,1}^C$ from $W_{0,2}^C$ as we see below, but we cannot use (2.32) in place of $W_{0,2}^C$ for this case because the specialization $t_1 = t, t_2 = -t$ does not make sense in $d_{t_1} d_{t_2} F_{0,2}(t_1, t_2)$.

Remark 2.17. In Subsection 3.2, we will formulate the topological recursion as a universal formalism for the context of Hitchin spectral curves in a coordinate-free manner depending only on the geometric setting. There we will explain the meaning of $W_{0,2}$, and the formula for the topological recursion. At this moment, we note that the origin of the topological recursion is the edge-contraction mechanism of the Catalan recursion (2.8). There is a surprising relation between the edge-contraction operations and two dimensional topological quantum field theories. We refer to [26] for more detail.

Remark 2.18. The integral recursion (2.35) and the PDE recursion (2.25) are equivalent in the range of $2g - 2 + n \geq 2$, since if we know $W_{g,n}^C$ from the integral recursion, then we can calculate $F_{g,n}^C$ by the integration

$$F_{g,n}^C(t_1, \ldots, t_n) = \int_{-1}^{t_1} \cdots \int_{-1}^{t_n} W_{g,n}^C.$$

But the differential recursion does not provide any mechanism to calculate $F_{1,1}^C$ and $F_{0,3}^C$.

To see how the topological recursion works, let us compute $W_{1,1}^C(t_1)$ from (2.35).

$$
\begin{aligned}
W_{1,1}^C(t_1) &= -\frac{1}{64}\frac{1}{2\pi i}\left[\int_\gamma \left(\frac{1}{t+t_1} + \frac{1}{t-t_1} \right) \frac{(t^2-1)^3}{t^2} \cdot \frac{1}{dt} \cdot W_{0,2}^C(t,-t) \right] dt_1 \\
&= -\frac{1}{64}\frac{1}{2\pi i}\left[\int_\gamma \left(\frac{1}{t+t_1} + \frac{1}{t-t_1} \right) \frac{(t^2-1)^3}{t^2} \cdot \frac{1}{dt} \cdot \left(\frac{-(dt)^2}{(2t)^2} \right) \right] dt_1 \\
&= \frac{1}{256}\frac{1}{2\pi i}\left[\int_\gamma \left(\frac{1}{t+t_1} + \frac{1}{t-t_1} \right) \frac{(t^2-1)^3}{t^2} \cdot \frac{1}{t^2} \cdot dt \right] dt_1 \\
&= -\frac{1}{128}\frac{(t_1^2-1)^3}{t_1^4}.
\end{aligned}
$$

Here, we changed the contour integral to the negative of the residue calculations at $t = \pm t_1$, as indicated in Figure 1.4. From (2.26), we find that indeed

$$F_{1,1}^C(t_1) = \int_{-1}^{t_1} W_{1,1}^C.$$

As explained in [27], we can calculate $W_{0,3}^C$ from $W_{0,2}^C$ as well, which recovers the initial condition (3.39).

To understand the geometry behind the topological recursion, we need to identify each term of the formula. First, recall the normalization morphism

$$\tilde{\pi} : \widetilde{\Sigma} \longrightarrow \mathbb{P}^1$$

of (2.16). We note that the transformation $t \longmapsto -t$ appearing in the recursion formula is the **Galois conjugation** of the global Galois covering $\tilde{\pi} : \widetilde{\Sigma} \longrightarrow \mathbb{P}^1$. From (2.19), we see that this transformation is induced by the the involution $y \longmapsto -y$ of $T^*\mathbb{P}^1$. The fixed point set of the Galois conjugation is the set of ramification points of the covering $\tilde{\pi}$, and the residue integration of (2.35) is taken around the two ramification points.

We claim that $W_{0,2}^C(t_1, t_2)$ is the *Cauchy differentiation kernel* on $\widetilde{\Sigma}$. This comes from the intrinsic geometry of the curve $\widetilde{\Sigma}$. The Cauchy differentiation kernel on $\mathbb{P}^1$ is the unique meromorphic symmetric bilinear differential form

$$(2.36) \qquad\qquad\qquad c(t_1, t_2) := \frac{dt_1 \cdot dt_2}{(t_1 - t_2)^2}$$

on $\mathbb{P}^1 \times \mathbb{P}^1$ such that

$$(2.37) \qquad\qquad\qquad df(t_2) = q_*\big(c(t_1, t_2) p^* f(t_1) \big)$$

for every rational function f on $\mathbb{P}^1$. Here, p and q are the projection maps

(2.38)

$$
\begin{array}{ccc}
 & \mathbb{P}^1 \times \mathbb{P}^1 & \\
\swarrow^{p} & & \searrow^{q} \\
\mathbb{P}^1 & & \mathbb{P}^1
\end{array}
$$

to the first and second factors. The push-forward q_* is defined by the integral

$$
q_*\big(c(t_1, t_2) p^* f(t_1)\big) = \frac{1}{2\pi i} \oint c(t_1, t_2) f(t_1)
$$

along a small loop in the fiber $q^*(t_2)$ that is centered at its intersection with the diagonal of $\Delta \subset \mathbb{P}^1 \times \mathbb{P}^1$. In terms of the other affine coordinate $u_i = 1/t_i$ of $\mathbb{P}^1$, we have

$$
\frac{dt_1 \cdot dt_2}{(t_1 - t_2)^2} = \frac{du_1 \cdot du_2}{(u_1 - u_2)^2} = -\frac{dt_1 \cdot du_2}{(t_1 u_2 - 1)^2} = -\frac{du_1 \cdot dt_2}{(u_1 t_2 - 1)^2}.
$$

Therefore, $c(t_1, t_2)$ is a globally defined bilinear meromorphic form on $\mathbb{P}^1 \times \mathbb{P}^1$ with poles only along the diagonal Δ.

Remark 2.19. The Cauchy integration formula

$$
f(w) = \frac{1}{2\pi i} \oint_{|z-w|<\epsilon} f(z) \frac{dz}{z-w}
$$

is the most useful formula in complex analysis. The Cauchy integration kernel $dz/(z-w)$ is a globally defined meromorphic 1-form on $\mathbb{C}$ that has only one pole, and its order is 1. It has to be noted that on a compact Riemann surface Σ, we do not have such a form. The best we can so is the meromorphic 1-form $\omega_\Sigma^{a-b}(t)$ with the following properties:

- It is a globally defined meromorphic 1-form with a pole of order 1 and residue 1 at a and a pole of order 1 and residue -1 at b for some pair (a, b) of distinct points of Σ.
- It is holomorphic in t except for $t = a$ and $t = b$.

Since we can add any holomorphic 1-form to $\omega_\Sigma^{a-b}(t)$ without violating the characteristic properties, the ambiguity of this form is exactly $H^0(\Sigma, K_\Sigma)$. Therefore, it is unique on $\Sigma = \mathbb{P}^1$, and is given by

(2.39)
$$
\omega_{\mathbb{P}^1}^{a-b}(t) = \left(\frac{1}{t-a} - \frac{1}{t-b} \right) dt.
$$

Note that

$$
d_{t_2} \omega_{\mathbb{P}^1}^{t_2-b}(t_1) = c(t_1, t_2).
$$

for any b.

Let us go back to the topological recursion for the Catalan numbers (2.35). Since all $W_{g,n}^C$ are determined by $W_{0,2}^C$ using the recursion, the remaining quantity we need to identify is the *integration kernel*. Recall the tautological 1-form

$$
\eta_{\mathbb{P}^1} := y\, dx
$$

on the tangent bundle $T^*\mathbb{P}^1$. We can see from (2.16) that its pull-back to the normalized spectral curve $\widetilde{\Sigma}$ is given by

$$
\begin{aligned}
\eta(t) &= -\left(\frac{t+1}{t-1}\right) d\left(\frac{4}{t^2-1}\right) \\
&= \frac{8t}{(t+1)(t-1)^3} \, dt.
\end{aligned}
$$

(2.40)

The factor

$$
-\frac{1}{64}\left(\frac{1}{t+t_1}+\frac{1}{t-t_1}\right)\frac{(t^2-1)^3}{t^2}\cdot\frac{1}{dt}\cdot dt_1
$$

in (2.35) is called the integration kernel, which is equal to

(2.41)
$$
\frac{1}{2}\frac{\omega_{\mathbb{P}^1}^{(-t)-t}(t_1)}{\eta(-t)-\eta(t)}.
$$

The integration kernel appears in this form in more general situations.

2.6. **The quantum curve for Catalan numbers.** There is yet another important role the differential recursion (2.25) plays. This is the derivation of the quantum curve equation for the Catalan case. In terms of the coordinates (x,y) of (2.13), the spectral curve is $y^2+xy+1=0$. The generating function $z(x)$ of (2.10), the normalization coordinate t of (2.19), and the base curve coordinate x on $\mathbb{P}^1$ are related by

(2.42)
$$
t=t(x):=\frac{z(x)+1}{z(x)-1},
$$

which we consider as a function of x that gives the branch of $\widetilde{\Sigma}$ determined by

$$
\lim_{x\to+\infty} t(x) = -1.
$$

Theorem 2.20 (Quantum curve for generalized Catalan numbers, [68]). *Define*

(2.43)
$$
\Psi(t,\hbar):=\exp\left(\sum_{g=0}^{\infty}\sum_{n=1}^{\infty}\frac{1}{n!}\hbar^{2g-2+n}F_{g,n}^C(t,t,\ldots,t)\right),
$$

incorporating (2.25), (2.29), and (2.31). Then as a function in x through (2.42), $\Psi\big(t(x),\hbar\big)$ satisfies the following differential equation

(2.44)
$$
\left[\left(\hbar\frac{d}{dx}\right)^2+x\left(\hbar\frac{d}{dx}\right)+1\right]\Psi\big(t(x),\hbar\big)=0.
$$

The semi-classical limit of (2.44) using $S_0(x)=F_{0,1}^C\big(t(x)\big)$ coincides with the spectral curve

$$
y^2+xy+1=0
$$

of (2.13).

Remark 2.21. Since $t=t(x)$ is a complicated function, it is surprising to see that $\Psi(t,\hbar)$ satisfies such a simple equation as (2.44).

Remark 2.22. The definition (2.43) and the meaning of (2.44) is the same as in the situation we have explained in the Introduction. First, we define

(2.45)
$$
S_m(x):=\sum_{2g-2+n=m-1}\frac{1}{n!}F_{g,n}^C\big(t(x),t(x),\ldots,t(x)\big)
$$

for every $m \geq 0$. We then interpret (2.44) as

$$(2.46) \qquad \left[e^{-\frac{1}{\hbar} S_0(x)} \left(\left(\hbar \frac{d}{dx} \right)^2 + x \left(\hbar \frac{d}{dx} \right) + 1 \right) e^{\frac{1}{\hbar} S_0(x)} \right] \exp \left(\sum_{m=1}^{\infty} \hbar^{m-1} S_m(x) \right) = 0.$$

To prove Theorem 2.20, let us recall a lemma from [68]:

Lemma 2.23. *Consider an open coordinate neighborhood $U \subset \Sigma$ of a projective algebraic curve Σ with a coordinate t. Let $f(t_1, \ldots, t_n)$ be a meromorphic symmetric function in n variables defined on Σ^n. Then on the coordinate neighborhood U^n, we have*

$$\frac{d}{dt} f(t, t, \ldots, t) = n \left[\frac{\partial}{\partial u} f(u, t, \ldots, t) \right] \bigg|_{u=t};$$

$$(2.47) \qquad \frac{d^2}{dt^2} f(t, t, \ldots, t) = n \left[\frac{\partial^2}{\partial u^2} f(u, t, \ldots, t) \right] \bigg|_{u=t}$$

$$+ n(n-1) \left[\frac{\partial^2}{\partial u_1 \partial u_2} f(u_1, u_2, t, \ldots, t) \right] \bigg|_{u_1 = u_2 = t}.$$

For a meromorphic function in one variable $f(t)$ on Σ, we have

$$(2.48) \qquad \lim_{t_2 \to t_1} \left[\omega_\Sigma^{t_2 - b}(t_1) \big(f(t_1) - f(t_2) \big) \right] = d_1 f(t_1),$$

where $\omega_\Sigma^{t_2 - b}(t_1)$ is the 1-form of Remark 2.19.

Proof of Theorem 2.20. Differential forms can be restricted to a subvariety, but partial differential equations cannot be restricted to a subvariety in general. Therefore, it is nontrivial that (2.25) has any meaningful restriction to the diagonal of $(\mathbb{P}^1)^n$. Our strategy is the following. First, we expand (2.46) with respect to powers of $\hbar$, and derive an infinite system of ordinary differential equations for a finite collection of $S_m(x)$'s. (This method is known as the WKB analysis.) We then prove that these ODEs are exactly the **principal specialization** of (2.25), using (2.47).

Let

$$(2.49) \qquad F(x, \hbar) := \sum_{m=0}^{\infty} \hbar^{m-1} S_m(x).$$

Unlike the ill-defined expression (2.43), $F(x, \hbar)$ is just a generating function of $S_m(x)$'s. Then (2.44), interpreted as (2.46), is equivalent to

$$(2.50) \qquad \hbar^2 \frac{d^2}{dx^2} F + \hbar^2 \frac{dF}{dx} \frac{dF}{dx} + x \hbar \frac{dF}{dx} + 1 = 0.$$

The $\hbar$-expansion of (2.50) gives

$$(2.51) \qquad \hbar^0\text{-terms}: \quad (S_0'(x))^2 + x S_0'(x) + 1 = 0,$$

$$(2.52) \qquad \hbar^1\text{-terms}: \quad 2 S_0'(x) S_1'(x) + S_0''(x) + x S_1'(x) = 0,$$

$$(2.53) \qquad \hbar^{m+1}\text{-terms}: \quad S_m''(x) + \sum_{a+b=m+1} S_a'(x) S_b'(x) + x S_{m+1}'(x) = 0, \quad m \geq 1,$$

where $'$ denotes the x-derivative. The WKB method is to solve these equations iteratively and find $S_m(x)$ for all $m \geq 0$. Here, (2.51) is the **semi-classical limit** of (2.44), and (2.52) is the *consistency condition* we need for solving the WKB expansion. Since $dS_0(x) = y\,dx$, we have $y = S_0'(x)$. Thus (2.51) follows from the spectral curve equation (2.13) and the definition of $F_{0,1}^C(t)$ of (2.29).

Recalling $x = z + 1/z$, we obtain

$$\frac{d}{dx} = \frac{z^2}{z^2 - 1}\frac{d}{dz}.$$

In z-coordinate, $F_{0,2}^C(z, z) = -\log(1 - z^2)$, which follows from (2.31). Therefore,

$$S_1'(x) = -\frac{1}{2}\frac{z^2}{z^2 - 1}\frac{d}{dz}\log(1 - z^2) = -\frac{z^3}{(z^2 - 1)^2}.$$

We can then calculate

$$2S_0'(x)S_1'(x) + S_0''(x) + xS_1'(x) = 2z\frac{z^3}{(z^2 - 1)^2} - \frac{z^2}{z^2 - 1} - \left(z + \frac{1}{z}\right)\frac{z^3}{(z^2 - 1)^2}$$

$$= \frac{1}{(z^2 - 1)^2}\left(2z^4 - z^2(z^2 - 1) - z^4 - z^2\right)$$

$$= 0.$$

Therefore, (2.52) holds. We refer to [25] for the proof of (2.53) from (2.25) and (2.23). $\square$

Remark 2.24. The solution $\Psi(t(x), \hbar)$ is a formal section of $K_{\mathbb{P}^1}^{-\frac{1}{2}}$, as before. Note that the quantum curve (2.44) has an irregular singularity at $x = \infty$, and hence its solution has an essential singularity at infinity. The expression (2.43) gives the asymptotic expansion of a solution around

$$t \to -1 \Longleftrightarrow x \to \infty.$$

The expansion in $1/x$ for $\hbar > 0$ is given in (3.54) below, using the Tricomi confluent hypergeometric function.

2.7. **Counting lattice points on the moduli space $\mathcal{M}_{g,n}$.** The topological recursion (2.35) is a consequence of (2.25), and the PDE recursion (2.25) is essentially the Laplace transform of the combinatorial formula (2.8). Then from where does the relation to the intersection numbers (2.24) arise?

To see this relation, we need to consider the dual of the cell graphs. They are commonly known as Grothendieck's dessins d'enfants (see for example, [79]), or ribbon graphs, as mentioned earlier. Recall that a ribbon graph has unlabeled vertices and edges, but faces are labeled. A *metric* ribbon graph is a ribbon graph with a positive real number (the length) assigned to each edge. For a given ribbon graph Γ with $e = e(\Gamma)$ edges, the space of metric ribbon graphs is $\mathbb{R}_+^{e(\Gamma)}/\mathrm{Aut}(\Gamma)$, where the automorphism group acts by permutations of edges (see [64, Section 1]). We restrict ourselves to the case that $\mathrm{Aut}(\Gamma)$ fixes each 2-cell of the cell-decomposition. We also require that every vertex of a ribbon graph has degree (i.e., valence) 3 or more. Using the canonical holomorphic coordinate systems on a topological surface of [64, Section 4] and the Strebel differentials [82], we have an isomorphism of topological orbifolds

$$(2.54) \qquad\qquad \mathcal{M}_{g,n} \times \mathbb{R}_+^n \cong RG_{g,n}.$$

Here

$$RG_{g,n} = \coprod_{\substack{\Gamma \text{ ribbon graph} \\ \text{of type } (g,n)}} \frac{\mathbb{R}_+^{e(\Gamma)}}{\mathrm{Aut}(\Gamma)}$$

is the orbifold consisting of metric ribbon graphs of a given topological type (g, n). The gluing of orbi-cells is done by making the length of a non-loop edge tend to 0. The space $RG_{g,n}$ is a smooth orbifold (see [64, Section 3] and [81]). We denote by $\pi : RG_{g,n} \longrightarrow \mathbb{R}_+^n$

the natural projection via (2.54), which is the assignment of the collection of perimeter lengths of each boundary to a given metric ribbon graph.

Consider a ribbon graph Γ whose faces are labeled by $[n] = \{1, 2 \ldots, n\}$. For the moment let us give a label to each edge of Γ by an index set $[e] = \{1, 2, \ldots, e\}$. The edge-face incidence matrix is then defined by

$$A_\Gamma = \left[a_{i\eta} \right]_{i \in [n], \, \eta \in [e]};$$

$$a_{i\eta} = \text{ the number of times edge } \eta \text{ appears in face } i.$$

Thus $a_{i\eta} = 0, 1$, or 2, and the sum of the entries in each column is always 2. The Γ contribution of the space $\pi^{-1}(p_1, \ldots, p_n) = RG_{g,n}(\mathbf{p})$ of metric ribbon graphs with a prescribed perimeter $\mathbf{p} = (p_1, \ldots, p_n)$ is the orbifold polytope

$$P_\Gamma(\mathbf{p})/\mathrm{Aut}(\Gamma), \qquad P_\Gamma(\mathbf{p}) = \{\mathbf{x} \in \mathbb{R}^e_+ \mid A_\Gamma \mathbf{x} = \mathbf{p}\},$$

where $\mathbf{x} = (\ell_1, \ldots, \ell_e)$ is the collection of edge lengths of the metric ribbon graph Γ. We have

$$\sum_{i \in [n]} p_i = \sum_{i \in [n]} \sum_{\eta \in [e]} a_{i\eta} \ell_\eta = 2 \sum_{\eta \in [e]} \ell_\eta.$$

We recall the topological recursion for the number of metric ribbon graphs $RG_{g,n}^{\mathbb{Z}_+}$ whose edges have integer lengths, following [15]. We call such a ribbon graph an *integral ribbon graph*. We can interpret an integral ribbon graph as Grothendieck's *dessin d'enfant* by considering an edge of integer length as a chain of edges of length one connected by bivalent vertices, and reinterpreting the notion of $\mathrm{Aut}(\Gamma)$ suitably. Since we do not go into the number theoretic aspects of dessins, we stick to the more geometric notion of integral ribbon graphs.

Definition 2.25. The weighted number $\left| RG_{g,n}^{\mathbb{Z}_+}(\mathbf{p}) \right|$ of integral ribbon graphs with prescribed perimeter lengths $\mathbf{p} \in \mathbb{Z}_+^n$ is defined by

$$(2.55) \qquad N_{g,n}(\mathbf{p}) = \left| RG_{g,n}^{\mathbb{Z}_+}(\mathbf{p}) \right| = \sum_{\substack{\Gamma \text{ ribbon graph} \\ \text{of type } (g,n)}} \frac{\left| \{\mathbf{x} \in \mathbb{Z}_+^{e(\Gamma)} \mid A_\Gamma \mathbf{x} = \mathbf{p}\} \right|}{|\mathrm{Aut}(\Gamma)|}.$$

Since the finite set $\{\mathbf{x} \in \mathbb{Z}_+^{e(\Gamma)} \mid A_\Gamma \mathbf{x} = \mathbf{p}\}$ is a collection of lattice points in the polytope $P_\Gamma(\mathbf{p})$ with respect to the canonical integral structure $\mathbb{Z} \subset \mathbb{R}$ of the real numbers, $N_{g,n}(\mathbf{p})$ can be thought of counting the number of *lattice points* in $RG_{g,n}(\mathbf{p})$ with a weight factor $1/|\mathrm{Aut}(\Gamma)|$ for each ribbon graph. The function $N_{g,n}(\mathbf{p})$ is a symmetric function in $\mathbf{p} = (p_1, \ldots, p_n)$ because the summation runs over all ribbon graphs of topological type (g, n).

Remark 2.26. Since the integral vector $\mathbf{x}$ is restricted to take strictly positive values, we would have $N_{g,n}(\mathbf{p}) = 0$ if we were to substitute $\mathbf{p} = 0$. This normalization is natural from the point of view of lattice point counting and Grothendieck's *dessins d'enphants*. However, we do not make such a substitution in these lectures because we consider $\mathbf{p}$ as a strictly positive integer vector. This situation is similar to Hurwitz theory [33, 69], where a partition μ is a strictly positive integer vector that plays the role of our $\mathbf{p}$. We note that a different assignment of values was suggested in [72, 73].

For brevity of notation, we denote by $p_I = (p_i)_{i \in I}$ for a subset $I \in [n] = \{1, 2 \ldots, n\}$. The cardinality of I is denoted by $|I|$. The following topological recursion formula was proved in [15] using the idea of ciliation of a ribbon graph.

Theorem 2.27 ([15]). *The number of integral ribbon graphs with prescribed boundary lengthes satisfies the topological recursion formula*

$$(2.56) \quad p_1 N_{g,n}(p_{[n]}) = \frac{1}{2} \sum_{j=2}^{n} \left[\sum_{q=0}^{p_1+p_j} q(p_1 + p_j - q) N_{g,n-1}(q, p_{[n]\setminus\{1,j\}}) \right.$$

$$+ H(p_1 - p_j) \sum_{q=0}^{p_1-p_j} q(p_1 - p_j - q) N_{g,n-1}(q, p_{[n]\setminus\{1,j\}})$$

$$\left. - H(p_j - p_1) \sum_{q=0}^{p_j-p_1} q(p_j - p_1 - q) N_{g,n-1}(q, p_{[n]\setminus\{1,j\}}) \right]$$

$$+ \frac{1}{2} \sum_{0 \le q_1 + q_2 \le p_1} q_1 q_2 (p_1 - q_1 - q_2) \left[N_{g-1,n+1}(q_1, q_2, p_{[n]\setminus\{1\}}) \right.$$

$$\left. + \sum_{\substack{g_1+g_2=g \\ I \sqcup J=[n]\setminus\{1\}}}^{\text{stable}} N_{g_1,|I|+1}(q_1, p_I) N_{g_2,|J|+1}(q_2, p_J) \right].$$

Here

$$H(x) = \begin{cases} 1 & x > 0 \\ 0 & x \le 0 \end{cases}$$

is the Heaviside function, and the last sum is taken for all partitions $g = g_1 + g_2$ and $I \sqcup J = \{2, 3, \ldots, n\}$ subject to the stability conditions $2g_1 - 1 + I > 0$ and $2g_2 - 1 + |J| > 0$.

For a fixed (g, n) in the stable range, *i.e.*, $2g - 2 + n > 0$, we choose n variables $t_1, t_2, \ldots, t_n$, and define the function

$$(2.57) \qquad z(t_i, t_j) = \frac{(t_i + 1)(t_j + 1)}{2(t_i + t_j)}.$$

An edge η of a ribbon graph Γ bounds two faces, say i_η and j_η. These two faces may be actually the same. Now we define the *Poincaré polynomial* [65] of $RG_{g,n}$ in the z-variables by

$$(2.58) \qquad F_{g,n}^P(t_1, \ldots, t_n) = \sum_{\substack{\Gamma \text{ ribbon graph} \\ \text{of type } (g,n)}} \frac{(-1)^{e(\Gamma)}}{|\text{Aut}(\Gamma)|} \prod_{\substack{\eta \text{ edge} \\ \text{of } \Gamma}} z(t_{i_\eta}, t_{j_\eta}),$$

which is a polynomial in $z(t_i, t_j)$ but actually a symmetric rational function in $t_1, \ldots, t_n$.

Let us consider the *Laplace transform*

$$(2.59) \qquad L_{g,n}(w_1, \ldots, w_n) \overset{\text{def}}{=} \sum_{\mathbf{p} \in \mathbb{Z}_+^n} N_{g,n}(\mathbf{p}) e^{-\langle \mathbf{p}, w \rangle}$$

of the number of integral ribbon graphs $N_{g,n}(\mathbf{p})$, where $\langle \mathbf{p}, w \rangle = p_1 w_1 + \cdots + p_n w_n$, and the summation is taken over all integer vectors $\mathbf{p} \in \mathbb{Z}_+^n$ of strictly positive entries. We shall prove that after the coordinate change of [15] from the w-coordinates to the t-coordinates defined by

$$(2.60) \qquad e^{-w_j} = \frac{t_j + 1}{t_j - 1}, \qquad j = 1, 2, \ldots, n,$$

the Laplace transform $L_{g,n}(w_{[n]})$ becomes the Poincaré polynomial

$$F_{g,n}^P(t_1, \ldots, t_n) = L_{g,n}\big(w_1(t), \ldots, w_n(t)\big).$$

The Laplace transform $L_{g,n}(w_N)$ can be evaluated using the definition of the number of integral ribbon graphs (2.55). Let a_η be the η-th column of the incidence matrix A_Γ so that

$$A_\Gamma = \big[a_1 \big| a_2 \big| \cdots \big| a_{e(\Gamma)}\big].$$

Then

$$(2.61) \quad L_{g,n}(w_{[n]}) = \sum_{\mathbf{p} \in \mathbb{Z}_+^n} N_{g,n}(\mathbf{p}) e^{-\langle \mathbf{p}, w \rangle}$$

$$= \sum_{\substack{\Gamma \text{ ribbon graph} \\ \text{of type } (g,n)}} \sum_{\mathbf{p} \in \mathbb{Z}_+^n} \frac{1}{|\mathrm{Aut}(\Gamma)|} \big|\{\mathbf{x} \in \mathbb{Z}_+^{e(\Gamma)} \mid A_\Gamma \mathbf{x} = \mathbf{p}\}\big| e^{-\langle \mathbf{p}, w \rangle}$$

$$= \sum_{\substack{\Gamma \text{ ribbon graph} \\ \text{of type } (g,n)}} \frac{1}{|\mathrm{Aut}(\Gamma)|} \sum_{\mathbf{x} \in \mathbb{Z}_+^{e(\Gamma)}} e^{-\langle A_\Gamma \mathbf{x}, w \rangle}$$

$$= \sum_{\substack{\Gamma \text{ ribbon graph} \\ \text{of type } (g,n)}} \frac{1}{|\mathrm{Aut}(\Gamma)|} \prod_{\substack{\eta \text{ edge} \\ \text{of } \Gamma}} \sum_{\ell_\eta=1}^{\infty} e^{-\langle a_\eta, w \rangle \ell_\eta}$$

$$= \sum_{\substack{\Gamma \text{ ribbon graph} \\ \text{of type } (g,n)}} \frac{1}{|\mathrm{Aut}(\Gamma)|} \prod_{\substack{\eta \text{ edge} \\ \text{of } \Gamma}} \frac{e^{-\langle a_\eta, w \rangle}}{1 - e^{-\langle a_\eta, w \rangle}}.$$

Every edge η bounds two faces, which we call face i_η^+ and face i_η^-. When $a_{i\eta} = 2$, these faces are the same. We then calculate

$$(2.62) \qquad \frac{e^{-\langle a_\eta, w \rangle}}{1 - e^{-\langle a_\eta, w \rangle}} = -z\Big(t_{i_\eta^+}, t_{i_\eta^-}\Big).$$

This follows from (2.60) and

$$\frac{e^{-(w_i + w_j)}}{1 - e^{-(w_i + w_j)}} = \frac{\frac{(t_i+1)(t_j+1)}{(t_i-1)(t_j-1)}}{1 - \frac{(t_i+1)(t_j+1)}{(t_i-1)(t_j-1)}} = -\frac{(t_i+1)(t_j+1)}{2(t_i + t_j)} = -z(t_i, t_j),$$

$$\frac{e^{-2w_i}}{1 - e^{-2w_i}} = -\frac{(t_i+1)^2}{4t_i} = -z(t_i, t_i).$$

Note that since $z(t_i, t_j)$ is a symmetric function, which face is named i_η^+ or i_η^- does not matter. From (2.61) and (2.62), we have established

Theorem 2.28 (The Poincaré polynomials and the Laplace transform, [65]). *The Laplace transform $L_{g,n}(w_{[n]})$ in terms of the t-coordinates (2.60) is the Poincaré polynomial*

$$(2.63) \qquad F_{g,n}^P(t_{[n]}) = \sum_{\substack{\Gamma \text{ ribbon graph} \\ \text{of type } (g,n)}} \frac{(-1)^{e(\Gamma)}}{|\mathrm{Aut}(\Gamma)|} \prod_{\substack{\eta \text{ edge} \\ \text{of } \Gamma}} z\Big(t_{i_\eta^+}, t_{i_\eta^-}\Big).$$

Corollary 2.29. *The evaluation of $F_{g,n}^P(t_{[n]})$ at $t_1 = \cdots = t_n = 1$ gives the Euler characteristic of $RG_{g,n}$*

$$(2.64) \qquad F_{g,n}^P(1, 1 \ldots, 1) = \chi(RG_{g,n}) = (-1)^n \chi(\mathcal{M}_{g,n}).$$

Furthermore, if we evaluate at $t_j = -1$ for any j, then we have

(2.65)
$$F_{g,n}^P(t_1, t_2 \ldots, t_n)\big|_{t_j=-1} = 0$$

as a function in the rest of the variables $t_{[n]\setminus\{j\}}$.

Proof. The Euler characteristic calculation immediately follows from $z(1,1) = 1$.

Consider a ribbon graph Γ of type (g, n). Its j-th face has at least one edge on its boundary. Therefore,

$$\prod_{\eta \text{ edge of } \Gamma} z\left(t_{i_\eta^+}, t_{i_\eta^-}\right)$$

has a factor $(t_j + 1)$ by (2.57). It holds for every ribbon graph Γ in the summation of (2.63). Therefore, (2.65) follows. $\square$

The following theorem is established in [65]. Its proof is the computation of the Laplace transform of the lattice point recursion (2.56).

Theorem 2.30 (Differential recursion for the Poincaré polynomials, [65])**.** *The Poincaré polynomials $F_{g,n}^P(t_1, \ldots, t_n)$ satisfy exactly the same differential recursion (2.25) with the same initial values of $F_{1,1}^P$ and $F_{0,3}^P$.*

Since the recursion uniquely determines all $F_{g,n}$'s for $2g - 2 + n > 0$, we have the following:

Corollary 2.31. *For every (g, n) in the stable range $2g - 2 + n > 0$, the two functions are identical:*

(2.66)
$$F_{g,n}^C(t_1, \ldots, t_n) = F_{g,n}^P(t_1, \ldots, t_n).$$

Because of this identification, we see that (2.64) implies (2.23), and (2.22) follows from (2.65). We can also see how (2.24) holds.

The limit $t_i \to \infty$ corresponds to $x_i \to 2$ through the normalization coordinate of (2.19), and $x_i = 2$ corresponds to a branch point of $\tilde{\pi} : \widetilde{\Sigma} \longrightarrow \mathbb{P}^1$ of (2.16). The defining equation (2.20) of $F_{g,n}^C$ does not tell us what we obtain by taking the limit $x_i \to 2$. The geometric meaning of the $t_i \to \infty$ limit becomes crystal clear by the equality (2.66). Let us take a look at the definition of the Poincaré polynomial (2.58). The fact that it is a Laurent polynomial follows from the recursion (2.25) by induction. If $|t_i| > |t_j| \gg 1$, then

$$z(t_i, t_j) = \frac{(t_i + 1)(t_j + 1)}{2(t_i + t_j)} \sim \frac{1}{2} t_j.$$

Therefore, the highest degree part of $F_{g,n}^P$ comes from the graphs of type (g, n) with the largest number of edges. Denoting the number of vertices of a ribbon graph Γ by $v(\Gamma)$, we have

$$2 - 2g - n = v(\Gamma) - e(\Gamma).$$

To maximize $e(\Gamma)$, we need to maximize $v(\Gamma)$, which is achieved by taking a trivalent graph (since we do not allow degree 1 and 2 vertices in our ribbon graph). By counting the number of half-edges of a trivalent graph, we obtain $2e(\Gamma) = 3v(\Gamma)$. Hence we have

(2.67)
$$e(\Gamma) = 6g - 6 + 3n.$$

This is the degree of $F_{g,n}^P$, which agrees with the degree of (2.24), and also consistent with the dimension of (2.54).

Fix a point $\mathbf{p} \in \mathbb{Z}_+^n$, and scale it by a large integer $\lambda \gg 1$. Then from (2.55) we see that the number of lattice points in the polytope $P_\Gamma(\lambda\mathbf{p})/\mathrm{Aut}(\Gamma)$ that is counted as a part of

$N_{g,n}(\lambda\mathbf{p})$ is the same as the number of scaled lattice points $\mathbf{x} \in \frac{1}{\lambda}\mathbb{Z}_+^n$ in $P_\Gamma(\mathbf{p})/\operatorname{Aut}(\Gamma)$. As $\lambda \to \infty$, the number of lattice points can be approximated by the Euclidean volume of the polytope (cf. theory of Ehrhart polynomials).

For a fixed $(w_1, \ldots, w_n)$ with $Re(w_j) > 0$, the contribution from large $\mathbf{p}$'s in the Laplace transform $L_{g,n}(w_1, \ldots, w_n)$ of (2.59) is small. The asymptotic behavior of $L_{g,n}$ as $w \to 0$ picks up the large perimeter contribution of $N_{g,n}(\mathbf{p})$, or the counting of the lattice points of smaller and smaller mesh size. Since

$$t_j \to \infty \Longleftrightarrow w_j \to 0,$$

the large t_j behavior of $F_{g,n}^P$, which is a homogeneous polynomial of degree $6g - 6 + 3n$, reflects the information of the volume of $\mathcal{M}_{g,n}$ in its coefficients. From Kontsevich [55], we learn that the volume is exactly the intersection number appearing in (2.24).

The topological recursion for the Airy case (1.42) is the $t \to \infty$ limit of the Catalan topological recursion (2.35), as we see from the limit

$$\frac{(t^2 - 1)^3}{t^2} \longrightarrow t^4.$$

Since the integrand of (1.42) has no poles at $t = 0$, the small circle of the contour γ does not contribute any residue. Thus we have derived the Airy topological recursion from the Catalan topological recursion.

3. Quantization of spectral curves

Quantum curves assemble information of quantum invariants in a compact manner. The global nature of quantum curves is not well understood at this moment of writing. In this section, we focus on explaining the relation between the **PDE version** of topological recursion discovered in [24, 25], and the *local* expression of quantum curves, suggested for example, in [44]. We give the precise definition of the PDE topological recursion in a geometric and coordinate-free manner. The discovery of [24, 25] is to connect the ideas from topological recursion with the Higgs bundle theory for the first time. Global definition of the quantum curves is being established by the authors, based on a recent work [23], and will be reported elsewhere.

As we have seen in the previous sections, for the examples of topological recursion such as the Catalan numbers and the Airy function case, the integral topological recursion is always a consequence of a corresponding recursion of free energies $F_{g,n}$ in the form of partial differential equations. Although we do not discuss them in these lectures, the situation is also true for the case of various Hurwitz numbers [10, 33, 69]. Since quantum curve is a differential equation, it is more natural to expect that the PDE recursion is directly related to quantum curves than the integral topological recursion.

This consideration motivates the authors' discovery of PDE topological recursion [24, 25]. We find that the most straightforward quantization of Hitchin spectral curves is obtained from the PDE recursion. Here, it has to be remarked that if one uses the integral topological recursion for Hitchin spectral curves, that is also introduced in [24, 25], then the quantization process produces a differential equation whose coefficients depend on all powers of $\hbar$, and thus the result is totally different from what we achieve. This shows that the integral topological recursion, which is closer to the original idea of [34], and the PDE topological recursion of [24, 25] are inequivalent for the case of Hitchin spectral curves of genus greater than 0.

From a geometric point of view, our quantization is a natural notion. Therefore, we believe the introduction of PDE topological recursion is crucial for building a theory of

quantum curves. It is also consistent from a physics point of view. Teschner [83] relates quantization of Hitchin moduli spaces with the quantization of Hitchin spectral curves in the way we do here.

3.1. **Geometry of non-singular Hitchin spectral curves of rank** 2. We wish to transplant the idea of [34] to Hitchin spectral curves. Our first task is to determine the differential form $W_{0,2}$ that gives the initial value of the topological recursion. It can reflect many aspects of the Hitchin spectral curve. In these lectures, we choose the Cauchy differentiation kernel as $W_{0,2}$, which depends only on the intrinsic geometry of the Hitchin spectral curve, for our purpose. Of course one can make other choices for different purposes, such as the kernel associated with a connection of a spectral line bundle on the Hitchin spectral curve.

Even for the Cauchy differentiation kernel, there is no canonical choice. It depends on a symplectic basis for the homology of the spectral curve. We give one particular choice in this subsection.

In this subsection, we consider a smooth projective curve C of genus $g(C) \geq 2$ defined over $\mathbb{C}$. As before, K_C is the canonical bundle on C. The cotangent bundle $\pi : T^*C \longrightarrow C$ is the total space of K_C, and there is the tautological section

$$(3.1) \qquad\qquad \eta \in H^0(T^*C, \pi^*K_C),$$

which is a globally defined holomorphic 1-form on T^*C.

$$(3.2) \qquad\qquad\qquad\qquad\qquad \pi^*(K_C)$$

Choose a *generic* quadratic differential $s \in H^0\big(C, K_C^{\otimes 2}\big)$, so that the spectral curve

$$(3.3) \qquad\qquad\qquad\qquad \Sigma_s \subset T^*C$$

that is defined by

$$(3.4) \qquad\qquad\qquad\qquad \eta^{\otimes 2} + \pi^* s = 0$$

is non-singular. Our spectral curve $\Sigma = \Sigma_s$ is a double sheeted ramified covering of C defined by (3.4). The genus of the spectral curve is $g(\Sigma) = 4g(C) - 3$. This is because a generic s has $\deg(K_C^{\otimes 2}) = 4g(C) - 4$ simple zeros, which correspond to branch points of the covering $\pi : \Sigma \longrightarrow C$. Thus the genus is calculated by the Riemann Hurwitz formula

$$2\big(2 - 2g(C) - \big(4g(C) - 4\big)\big) = 2 - 2g(\Sigma) - \big(4g(C) - 4\big).$$

The cotangent bundle T^*C has a natural involution

$$(3.5) \qquad\qquad \sigma : T^*C \supset T_x^*C \ni (x, y) \longmapsto (x, -y) \in T_x^*C \subset T^*C,$$

which preserves the spectral curve Σ. The action of this involution is the deck-transformation. Indeed, the covering $\pi : \Sigma \longrightarrow C$ is a **Galois covering** with the Galois group $\mathbb{Z}/2\mathbb{Z} = \langle \sigma \rangle$.

Let $R \subset \Sigma$ denote the ramification divisor of this covering. Because Σ is non-singular, R is supported at $4g(C) - 4$ distinct points that are determined by $s = 0$ on C. We consider C as the 0-section of T^*C. Thus both C and Σ are divisors of T^*C. Hence R is also defined as $C \cdot \Sigma$ in T^*C supported on $C \cap \Sigma$. Note that η vanishes only along $C \subset T^*C$. As a

holomorphic 1-form on Σ, $\iota^*\eta$ has $2g(\Sigma) - 2 = 8g(C) - 8$ zeros on Σ. Thus it has a degree 2 zero at each point of $\mathrm{supp}(R)$.

As explained in the earlier sections using examples, we wish to choose a differential form $W_{0,2}$ for our geometric situation. For this purpose, let us recall the differential form ω_Σ^{a-b} of Remark 2.19. Since we can add any holomorphic 1-form to ω_Σ^{a-b}, we need to impose $g(\Sigma)$ independent conditions to make it unique. If we have a principal polarization of the period matrix for Σ, then one obvious choice would be to impose

$$(3.6) \qquad \oint_{a_j} \omega_\Sigma^{a-b} = 0$$

for all "A-cycles" a_j, $j = 1, 2, \ldots, g(\Sigma)$, following Riemann himself. The reason for this canonical choice is that we can make it for a family of spectral curves

$$\{\Sigma_s\}_{s \in U},$$

where $U \subset H^0\left(C, K_C^{\otimes 2}\right)$ is a contractible open neighborhood of s. Since the base curve C is fixed, we can choose and fix a symplectic basis for $H_1(C, \mathbb{Z})$:

$$(3.7) \qquad \langle A_1, \ldots, A_g; B_1, \ldots, B_g \rangle = H_1(C, \mathbb{Z}).$$

From this choice, we construct a canonical symplectic basis for $H^1(\Sigma, \mathbb{Z})$ as follows. Let us label points of $R = \{p_1, p_2, \ldots, p_{4g-4}\}$, where $g = g(C)$. We can connect p_{2i} and p_{2i+1}, $i = 1, \ldots, 2g - 3$, with a simple path on Σ that is mutually non-intersecting so that $\pi^*(\overline{p_{2i} p_{2i+1}})$, $i = 1, \ldots, 2g-3$, form a part of the basis for $H_1(\Sigma, \mathbb{Z})$. We denote these cycles by $\alpha_1, \ldots, \alpha_{2g-3}$. Since π is locally homeomorphic away from R, we have g cycles $a_1, \ldots, a_g$ on Σ_s so that $\pi_*(a_j) = A_j$ for $j = 1, \ldots, g$, where A_j's are the A-cycles of C chosen as (3.7). We define the A-cycles of Σ to be the set

$$(3.8) \qquad \{a_1, \ldots, a_g, \sigma_*(a_1), \ldots, \sigma_*(a_g), \alpha_1, \ldots, \alpha_{2g-3}\} \subset H_1(\Sigma, \mathbb{Z}),$$

where σ is the Galois conjugation (see Figure 3.1). Clearly, this set can be extended into a symplectic basis for $H_1(\Sigma, \mathbb{Z})$. This choice of the symplectic basis trivializes the homology bundle

$$\left\{H_1(\Sigma_s, \mathbb{Z})\right\}_{s \in U} \longrightarrow U \subset H^0(C, K_C^{\otimes 2})$$

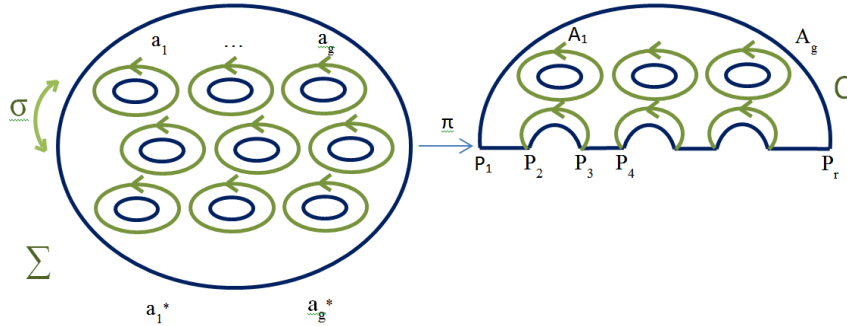locally on the contractible neighborhood U.

FIGURE 3.1. The choice of a symplectic basis for $H_1(\Sigma, \mathbb{Z})$ from that of $H_1(C, \mathbb{Z})$.

3.2. **The generalized topological recursion for the Hitchin spectral curves.** The topological recursion of [11, 34], that has initiated the explosive developments on this subject in recent years, is restricted to non-singular spectral curves in $\mathbb{C}^2$ or $(\mathbb{C}^*)^2$. A systematic generalization of the formalism to the case of Hitchin spectral curves is given for the first time in [24, 25]. First, we gave the definition for non-singular Hitchin spectral curves associated with holomorphic Higgs bundles in [24]. We then extended the consideration to meromorphic Higgs bundles and singular spectral curves in [25]. In each case, however, the actual evaluation of the generalized topological recursion for the purpose of quantization of the Hitchin spectral curves is limited to Higgs bundles of rank 2. This is due to many technical difficulties, and at this point we still do not have a better understanding of the theory in its full generality.

The purpose of this subsection is thus to present the theory in the way we know as of now, with the scope limited to what seems to work. Many aspects of the story can be immediately generalized. Mainly for the sake of simplicity of presentation, we concentrate on the case of rank 2 Higgs bundles.

The geometric setup is the following. We have a smooth projective curve C defined over $\mathbb{C}$ of an arbitrary genus, and a meromorphic Higgs bundle (E, ϕ). Here, the vector bundle E of rank 2 is a special one of the form

$$(3.9) \qquad\qquad E = K_C^{\frac{1}{2}} \oplus K_C^{-\frac{1}{2}}.$$

As a meromorphic Higgs field, we use

$$(3.10) \qquad\qquad \phi = \begin{bmatrix} -s_1 & s_2 \\ 1 & \end{bmatrix} : E \longrightarrow K_C(D) \otimes E$$

with poles at an effective divisor D of C, where

$$s_1 \in H^0\big(C, K_C(D)\big),$$
$$s_2 \in H^0\big(C, K_C^{\otimes 2}(D)\big).$$

Although ϕ involves a quadratic differential in its coefficient, since

$$\begin{bmatrix} -s_1 & s_2 \\ 1 & \end{bmatrix} : K_C^{\frac{1}{2}} \oplus K_C^{-\frac{1}{2}} \longrightarrow \left(K_C^{\frac{3}{2}} \oplus K_C^{\frac{1}{2}} \right) \otimes \mathcal{O}_C(D) \longrightarrow K_C(D) \otimes \left(K_C^{\frac{1}{2}} \oplus K_C^{-\frac{1}{2}} \right),$$

we have $\phi \in H^0\big(C, K_C(D) \otimes \mathrm{End}(E)\big)$.

Remark 3.1. We use a particular section (E, ϕ) of the *Hitchin fibration* given by the form of (3.10). This section is often called a *Hitchin section*. This choice is suitable for the WKB analysis explained below. The result of our quantization through WKB constructs a **formal** section of $K_C^{-\frac{1}{2}}$, and the relation to an $\hbar$-connection on C makes our choice necessary. The theory being developed as of now (relation to opers [23]) also requires the choice of a Hitchin section.

Denote by

$$(3.11) \qquad\qquad \overline{T^*C} := \mathbb{P}(K_C \oplus \mathcal{O}_C) \xrightarrow{\ \pi\ } C$$

the compactified cotangent bundle of C (see [5, 57]), which is a ruled surface on the base C. The Hitchin spectral curve

$$(3.12) \qquad \Sigma \xrightarrow{\;i\;} \overline{T^*C}$$

(commutative diagram with π from Σ to C and π from $\overline{T^*C}$ to C)

for a meromorphic Higgs bundle is defined as the divisor of zeros on $\overline{T^*C}$ of the characteristic polynomial of ϕ:

$$(3.13) \qquad \Sigma = (\det(\eta - \pi^*\phi))_0\,,$$

where $\eta \in H^0(T^*C, \pi^*K_C)$ is the tautological 1-form on T^*C extended as a meromorphic 1-form on the compactification $\overline{T^*C}$.

Definition 3.2 (Integral topological recursion for a degree 2 covering)**.** Let $\tilde{\pi} : \widetilde{\Sigma} \longrightarrow C$ be a degree 2 covering of C by a non-singular curve $\widetilde{\Sigma}$. We denote by R the ramification divisor of $\tilde{\pi}$. In this case the covering $\tilde{\pi}$ is a Galois covering with the Galois group $\mathbb{Z}/2\mathbb{Z} = \langle \tilde{\sigma} \rangle$, and R is the fixed-point divisor of the involution $\tilde{\sigma}$. The **integral topological recursion** is an inductive mechanism of constructing meromorphic differential forms $W_{g,n}$ on the Hilbert scheme $\widetilde{\Sigma}^{[n]}$ of n-points on $\widetilde{\Sigma}$ for all $g \geq 0$ and $n \geq 1$ in the *stable range* $2g - 2 + n > 0$, from given initial data $W_{0,1}$ and $W_{0,2}$.

- $W_{0,1}$ is a meromorphic 1-form on $\widetilde{\Sigma}$.
- $W_{0,2}$ is defined to be

$$(3.14) \qquad W_{0,2}(z_1, z_2) = d_1 d_2 \log E_{\widetilde{\Sigma}}(z_1, z_2),$$

where $E_{\widetilde{\Sigma}}(z_1, z_2)$ is the normalized Riemann prime form on $\widetilde{\Sigma} \times \widetilde{\Sigma}$ (see [24, Section 2]).

Let $\omega^{a-b}(z)$ be a normalized Cauchy kernel on $\widetilde{\Sigma}$ of Remark 2.19, which has simple poles at $z = a$ of residue 1 and at $z = b$ of residue -1. Then

$$d_1 \omega^{z_1-b}(z_2) = W_{0,2}(z_1, z_2).$$

Define

$$(3.15) \qquad \Omega := \tilde{\sigma}^* W_{0,1} - W_{0,1}.$$

Then $\tilde{\sigma}^* \Omega = -\Omega$, hence $\operatorname{supp}(R) \subset \operatorname{supp}(\Omega)$, where $\operatorname{supp}(\Omega)$ denotes the support of both zero and pole divisors of Ω. The inductive formula of the topological recursion is then given by the following:

$$(3.16) \quad W_{g,n}(z_1, \ldots, z_n) = \frac{1}{2} \frac{1}{2\pi\sqrt{-1}} \sum_{p \in \operatorname{supp}(\Omega)} \oint_{\gamma_p} \omega^{\tilde{z}-z}(z_1)$$

$$\times \frac{1}{\Omega(z)} \left[W_{g-1,n+1}(z, \tilde{z}, z_2, \ldots, z_n) + \sum_{\substack{g_1+g_2=g \\ I \sqcup J = \{2,\ldots,n\}}}^{\text{No } (0,1)} W_{g_1,|I|+1}(z, z_I) W_{g_2,|J|+1}(\tilde{z}, z_J) \right].$$

Here,

- γ_p is a positively oriented small loop around a point $p \in \operatorname{supp}(\Omega)$;
- the integration is taken with respect to $z \in \gamma_p$ for each $p \in \operatorname{supp}(\Omega)$;
- $\tilde{z} = \tilde{\sigma}(z)$ is the Galois conjugate of $z \in \widetilde{\Sigma}$;

- the ratio of two global meromorphic 1-forms on the same curve makes sense as a global meromorphic function. The operation $1/\Omega$ applied to a meromorphic 1-form produces this ratio;
- "No $(0,1)$" means that $g_1 = 0$ and $I = \emptyset$, or $g_2 = 0$ and $J = \emptyset$, are excluded in the summation;
- the sum runs over all partitions of g and set partitions of $\{2, \ldots, n\}$, other than those containing the $(0,1)$ geometry;
- $|I|$ is the cardinality of the subset $I \subset \{2, \ldots, n\}$; and
- $z_I = (z_i)_{i \in I}$.

The main idea of [25] for dealing with singular spectral curve is the following.

- The integral topological recursion of [24, 34] is extended to the curve Σ of (3.13), as (3.16). For this purpose, we blow up $\overline{T^*C}$ several times as in (3.19) below to construct the normalization $\widetilde{\Sigma}$. The blown-up $Bl(\overline{T^*C})$ is the minimal resolution of the support $\Sigma \cup C_\infty$ of the *total* divisor

$$(3.17) \qquad \Sigma - 2C_\infty = (\det(\eta - \pi^*\phi))_0 - (\det(\eta - \pi^*\phi))_\infty$$

of the characteristic polynomial, where

$$(3.18) \qquad C_\infty := \mathbb{P}(K_C \oplus \{0\}) = \overline{T^*C} \setminus T^*C$$

is the divisor at infinity. Therefore, in $Bl(\overline{T^*C})$, the proper transform $\widetilde{\Sigma}$ of Σ is smooth and does not intersect with the proper transform of C_∞.

$$(3.19)$$

- The genus of the normalization $\widetilde{\Sigma}$ is given by

$$g(\widetilde{\Sigma}) = 2g(C) - 1 + \frac{1}{2}\delta,$$

where δ is the sum of the number of cusp singularities of Σ and the ramification points of $\pi : \Sigma \longrightarrow C$.

- The generalized topological recursion Definition 3.2 requires a globally defined meromorphic 1-form $W_{0,1}$ on $\widetilde{\Sigma}$ and a symmetric meromorphic 2-form $W_{0,2}$ on the product $\widetilde{\Sigma} \times \widetilde{\Sigma}$ as the initial data. We choose

$$(3.20) \qquad \begin{cases} W_{0,1} = \tilde{i}^*\nu^*\eta \\ W_{0,2} = d_1 d_2 \log E_{\widetilde{\Sigma}}, \end{cases}$$

where $E_{\widetilde{\Sigma}}$ is a normalized Riemann prime form on $\widetilde{\Sigma}$. The form $W_{0,2}$ depends only on the intrinsic geometry of the smooth curve $\widetilde{\Sigma}$. The geometry of (3.19) is encoded in $W_{0,1}$. The integral topological recursion produces a symmetric meromorphic n-linear differential form $W_{g,n}(z_1, \ldots, z_n)$ on $\widetilde{\Sigma}$ for every (g, n) subject to $2g-2+n > 0$ from the initial data (3.20).

The key discovery of [24, (4.7)] is that we should use a partial differential equation version of the topological recursion, instead of (3.16), to construct a quantization of Σ.

3.3. **Quantization of Hitchin spectral curves.** The passage from the geometry of Hitchin spectral curve Σ of (3.13) to the quantum curve

$$(3.21) \qquad \left(\left(\hbar \frac{d}{dx} \right)^2 - \operatorname{tr} \phi(x) \left(\hbar \frac{d}{dx} \right) + \det \phi(x) \right) \Psi(x, \hbar) = 0$$

is a system of PDE recursion replacing the integration formula (3.16).

Definition 3.3 (Free energies). The **free energy** of type (g, n) is a function $F_{g,n}(z_1, \ldots, z_n)$ defined on the universal covering $\mathcal{U}^n$ of $\widetilde{\Sigma}^n$ such that

$$d_1 \cdots d_n F_{g,n} = W_{g,n}.$$

Remark 3.4. The free energies may contain logarithmic singularities, since it is an integral of a meromorphic function. For example, $F_{0,2}$ is the Riemann prime form itself considered as a function on $\mathcal{U}^2$, which has logarithmic singularities along the diagonal [24, Section 2].

Definition 3.5 (Differential recursion for a degree 2 covering). The **differential recursion** is the following partial differential equation for all (g, n) subject to $2g - 2 + n \geq 2$:

$$(3.22) \quad d_1 F_{g,n}(z_1, \ldots, z_n)$$

$$= \sum_{j=2}^{n} \left[\frac{\omega^{z_j - \sigma(z_j)}(z_1)}{\Omega(z_1)} \cdot d_1 F_{g,n-1}\big(z_{[\hat{j}]}\big) - \frac{\omega^{z_j - \sigma(z_j)}(z_1)}{\Omega(z_j)} \cdot d_j F_{g,n-1}\big(z_{[\hat{1}]}\big) \right]$$

$$+ \frac{1}{\Omega(z_1)} d_{u_1} d_{u_2} \left[F_{g-1,n+1}\big(u_1, u_2, z_{[\hat{1}]}\big) + \sum_{\substack{g_1+g_2=g \\ I \sqcup J = [\hat{1}]}}^{\text{stable}} F_{g_1, |I|+1}(u_1, z_I) F_{g_2, |J|+1}(u_2, z_J) \right] \Bigg|_{\substack{u_1 = z_1 \\ u_2 = z_1}}.$$

Here, $1/\Omega$ is again the ratio operation, and the index subset $[\hat{j}]$ denotes the exclusion of $j \in \{1, 2, \ldots, n\}$. The Cauchy integration kernel $\omega^{a-b}(z)$ on the spectral curve $\widetilde{\Sigma}$ is normalized differently than the A-cycle normalization we did earlier. This time we impose that

$$(3.23) \qquad \lim_{a \to b} \omega^{a-b}(z) = 0.$$

Remark 3.6. As pointed out in [24, Remark 4.8], (3.22) is a globally defined coordinate-free equation, written in terms of exterior differentiations and the ratio operation, on $\widetilde{\Sigma}$.

Theorem 3.7 (The relation between the differential recursion and the integral recursion, [25]). *Suppose that $F_{g,n}$ for $2g - 2 + n > 0$ are globally meromorphic on $\widetilde{\Sigma}^{[n]}$ with poles located only along the divisor of $\widetilde{\Sigma}^{[n]}$ when one of the factors lies in the zeros of Ω. Define $W_{g,n} := d_1 \cdots d_n F_{g,n}$ for $2g - 2 + n > 0$, and use (3.14) and (3.15) for (g, n) in the unstable range. If $F_{g,n}s$ are symmetric and satisfy the differential recursion (3.22), and if $W_{1,1}$ and $W_{0,3}$ satisfy the initial equations of the integral topological recursion (3.16), then $W_{g,n}s$ for all valued of (g, n) satisfy the integral topological recursion.*

Remark 3.8. The assumption of the theorem holds for $g(C) = 0$, and therefore, for all the examples we discuss in these lectures. But we are not establishing the general **equivalence** of the integral topological recursion (3.16) and the PDE recursion (3.22). They are never

equivalent when $g(\widetilde{\Sigma}) > 0$. Actually, what is assumed in the above theorem is that $W_{g,n}$ is exact, i.e., integrable by definition. In particular, this implies that $W_{g,n}$ has 0-period for every topological cycle. This does not happen if we start with (3.16) in general. Therefore, the above theorem serves only as a heuristic motivation for our discovery of (3.22) in [24, 25].

Proof. Although the context of the statement is slightly different, the proof is essentially the same as that of [24, Theorem 4.7]. The crucial assumption we have made is that $\tilde{\pi} : \widetilde{\Sigma} \longrightarrow C$ is a Galois covering. Therefore, the Galois conjugation $\tilde{\sigma} : \widetilde{\Sigma} \longrightarrow \widetilde{\Sigma}$ is a globally defined holomorphic mapping. To calculate the residues in the integral recursion (3.16), we need the global analysis of

$$\omega^{\tilde{z}-z}(z_1) \in H^0\big(\widetilde{\Sigma} \times \widetilde{\Sigma}, q_2^* K_{\widetilde{\Sigma}} \boxtimes q_2^* \mathcal{O}_{\widetilde{\Sigma}}(\tilde{z} + z)\big),$$

where q_1 and q_2 are projections

(3.24)
$$
\begin{array}{ccc}
 & \widetilde{\Sigma} \times \widetilde{\Sigma} & \\
 {}^{q_1}\swarrow & & \searrow{}^{q_2} \\
z \in \widetilde{\Sigma} & & \widetilde{\Sigma} \ni z_1.
\end{array}
$$

The residue integration is done at each point $p \in \mathrm{supp}(\Omega)$. The poles of the integrand of (3.16) that are enclosed in the union

$$\gamma = \bigcup_{p \in \mathrm{supp}(\Omega)} \gamma_p$$

of the contours on the complement of $\mathrm{supp}(\Omega)$ are located at

  (1) $z = z_1, z = \tilde{z}_1$ from $\omega^{\tilde{z}-z}(z_1)$; and
  (2) $z = z_j, z = \tilde{z}_j$, $j = 2, \dots, n$, from $W_{0,2}(z, z_j)$ and $W_{0,2}(\tilde{z}, z_j)$ that appear in the second line of (3.16).

The integrand has other poles at $\mathrm{supp}(\Omega)$ that includes the ramification divisor R, but they are not enclosed in γ. The local behavior of $\omega^{\tilde{z}-z}(z_1)$ at $z = z_1, z = \tilde{z}_1$ is well understood, and residues of the integrand of (3.16) are simply the evaluation of the differential form at $z = z_1, z = \tilde{z}_1$. The double poles coming from $W_{0,2}(z, z_j)$ and $W_{0,2}(\tilde{z}, z_j)$ contribute as differentiation of the factor it is multiplied to. Adding all contributions from the poles, we obtain (3.22). $\qquad\square$

Now let us consider a spectral curve $\Sigma \subset \overline{T^*C}$ of (3.12) defined by a pair of meromorphic sections $s_1 = -\mathrm{tr}\,\phi$ of K_C and $s_2 = \det\phi$ of $K_C^{\otimes 2}$. Let $\widetilde{\Sigma}$ be the desingularization of Σ in (2.16). We apply the topological recursion (3.16) to the covering $\tilde{\pi} : \widetilde{\Sigma} \longrightarrow C$. The geometry of the spectral curve Σ provides us with a canonical choice of the initial differential forms (3.20). At this point we pay special attention to the fact that the topological recursions (3.16) and (3.22) are both defined on the spectral curve $\widetilde{\Sigma}$, while we wish to construct a differential equation on C. Since the free energies are defined on the universal covering of $\widetilde{\Sigma}$, we need to have a mechanism to relate a coordinate on the desingularized spectral curve and that of the base curve C.

To analyze the singularity structure of Σ, let us consider the discriminant of the defining equation (3.4) of the spectral curve.

Definition 3.9 (Discriminant divisor)**.** The **discriminant divisor** of the spectral curve

(3.25)
$$\eta^{\otimes 2} + \pi^* s_1 \eta + \pi^* s_2 = 0$$

is a divisor on C defined by

(3.26)
$$\Delta := \left(\frac{1}{4}s_1^2 - s_2\right) = \Delta_0 - \Delta_\infty.$$

Here,

(3.27)
$$\Delta_0 = \sum_{i=1}^{m} m_i q_i, \quad m_i > 0, \quad q_i \in C,$$

is the divisor of zeros, and

(3.28)
$$\Delta_\infty = \sum_{j=1}^{n} n_j p_j, \quad n_j > 0, \quad p_j \in C,$$

is the divisor of ∞.

Since $\frac{1}{4}s_1^2 - s_2$ is a meromorphic section of $K_C^{\otimes 2}$, we have

(3.29)
$$\deg \Delta = \sum_{i=1}^{m} m_i - \sum_{j=1}^{n} n_j = 4g - 4.$$

Theorem 3.10 (Geometric genus formula, [25]). *Let us define an invariant of the discriminant divisor by*

(3.30)
$$\delta = |\{i \mid m_i \equiv 1 \mod 2\}| + |\{j \mid n_j \equiv 1 \mod 2\}|.$$

Then the geometric genus of the spectral curve Σ of (3.25) is given by

(3.31)
$$g(\widetilde{\Sigma}) := p_g(\Sigma) = 2g - 1 + \frac{1}{2}\delta.$$

We note that (3.29) *implies* $\delta \equiv 0 \mod 2$.

Take an arbitrary point $p \in C \setminus \mathrm{supp}(\Delta)$, and a local coordinate x around p. By choosing a small disc V around p, we can make the inverse image of $\tilde{\pi} : \widetilde{\Sigma} \longrightarrow C$ consist of two isomorphic discs. Since V is away from the critical values of $\tilde{\pi}$, the inverse image consists of two discs in the original spectral curve Σ. Note that we choose an eigenvalue α of ϕ on V in our main construction. We are thus specifying one of the inverse image discs here. Let us name the disc V_α that corresponds to α.

At this point apply the WKB analysis to the differential equation (3.21) with the WKB expansion of the solution

(3.32)
$$\Psi^\alpha(x, \hbar) = \exp\left(\sum_{m=0}^{\infty} \hbar^{m-1} S_m\big(x(z)\big)\right) = \exp F^\alpha(x, \hbar),$$

where we choose a coordinate z of V_α so that the function $x = x(z)$ represents the projection $\pi : V_\alpha \longrightarrow V$. The equation $P\Psi^\alpha = Pe^{F^\alpha} = 0$ reads

(3.33)
$$\hbar^2 \frac{d^2}{dx^2} F^\alpha + \hbar^2 \frac{dF^\alpha}{dx} \frac{dF^\alpha}{dx} + s_1 \hbar \frac{dF^\alpha}{dx} + s_2 = 0.$$

The $\hbar$-expansion of (3.33) gives

(3.34) $\hbar^0$-terms : $\quad (S_0'(x))^2 + s_1 S_0'(x) + s_2 = 0,$

(3.35) $\hbar^1$-terms : $\quad 2S_0'(x)S_1'(x) + S_0''(x) + s_1 S_1'(x) = 0,$

(3.36) $\hbar^{m+1}$-terms : $\quad S_m''(x) + \sum_{a+b=m+1} S_a'(x)S_b'(x) + s_1 S_{m+1}'(x) = 0, \quad m \geq 1,$

where $'$ denotes the x-derivative. The WKB method is to solve these equations iteratively and find $S_m(x)$ for all $m \geq 0$. Here, (3.34) is the **semi-classical limit** of (3.21), and (3.35) is the *consistency condition* we need to solve the WKB expansion, the same as before. Since the 1-form $dS_0(x)$ is a local section of T^*C, we identify $y = S_0'(x)$. Then (3.34) is the local expression of the spectral curve equation (3.4). This expression is the same everywhere for $p \in C \setminus \operatorname{supp}(\Delta)$. We note s_1 and s_2 are globally defined. Therefore, we recover the spectral curve Σ from the differential operator of (3.21).

Theorem 3.11 (Main theorem)**.** *The differential topological recursion provides a formula for each $S_m(x)$, $m \geq 2$, and constructs a formal solution to the quantum curve (3.21).*

- *The quantum curve associated with the Hitchin spectral curve Σ is defined as a differential equation on C. On each coordinate neighborhood $U \subset C$ with coordinate x, a generator of the quantum curve is given by*

$$P(x, \hbar) = \left(\hbar \frac{d}{dx} \right)^2 - \operatorname{tr} \phi(x) \left(\hbar \frac{d}{dx} \right) + \det \phi(x).$$

  *In particular, the **semi-classical limit** of the quantum curve recovers the singular spectral curve Σ, not its normalization $\widetilde{\Sigma}$.*

- *The **all-order WKB expansion***

$$(3.37) \qquad \Psi(x, \hbar) = \exp \left(\sum_{m=0}^{\infty} \hbar^{m-1} S_m(x) \right)$$

  of a solution to the Schrödinger equation

$$\left(\left(\hbar \frac{d}{dx} \right)^2 - \operatorname{tr} \phi(x) \left(\hbar \frac{d}{dx} \right) + \det \phi(x)) \right) \Psi(x, \hbar) = 0,$$

  *near each critical value of $\pi : \Sigma \longrightarrow C$, can be obtained by the **principal specialization** of the differential recursion (3.22), after determining the first three terms. The procedure is the following. We determine $S_0, S_1,$ and S_2 by solving*

$$(3.38) \qquad \begin{aligned} \left(S_0'(x) \right)^2 - \operatorname{tr}\phi(x) S_0'(x) + \det \phi(x) &= 0, \\ 2S_0'(x) S_1'(x) + S_0''(x) - \operatorname{tr}\phi(x) S_1'(x) &= 0, \\ S_1''(x) + \sum_{a+b=2} S_a'(x) S_b'(x) - \operatorname{tr}\phi(x) S_2'(x) &= 0. \end{aligned}$$

  Then find $F_{1,1}(z)$ and $F_{0,3}(z_1, z_2, z_3)$ so that

$$S_2(x) = F_{1,1}\big(z(x) \big) + \frac{1}{6} F_{0,3}\big(z(x), z(x), z(x) \big).$$

  This can be achieved as follows. First integrate $W_{1,1}(z)$ of (3.16) to construct $F_{1,1}(z)$. We do the same for the solution $W_{0,3}(z_1, z_2, z_3)$ of (3.16). We now define

$$(3.39) \qquad F_{0,3}(z_1, z_2, z_3) = \iiint W_{0,3}(z_1, z_2, z_3) + 2 \left(f(z_1) + f(z_2) + f(z_3) \right),$$

  where

$$(3.40) \qquad f(z) := \widetilde{S}_2(z) - \left(F_{1,1}(z) + \frac{1}{6} \int^z \int^z \int^z W_{0,3}(z_1, z_2, z_3) \right),$$

  and $\widetilde{S}_2(z)$ is the lift of $S_2(x)$ to $\widetilde{\Sigma}$.

- *Suppose we have symmetric meromorphic functions $F_{g,n}(z_1, \ldots, z_n)$ that solve the differential recursion (3.22) on the universal covering $\varpi : \mathcal{U} \longrightarrow \widetilde{\Sigma}$ with these $F_{1,1}$ and $F_{0,3}$ as initial values.*
- *Let*

$$(3.41) \qquad S_m(x) = \sum_{2g-2+n=m-1} \frac{1}{n!} F_{g,n}\big(z(x)\big), \qquad m \geq 3,$$

  *where $F_{g,n}\big(z(x)\big)$ is the principal specialization of $F_{g,n}(z_1, \ldots, z_n)$ evaluated at a local section $z = z(x)$ of $\tilde{\pi} : \widetilde{\Sigma} \longrightarrow C$. Then the **wave function** $\Psi(x, \hbar)$, a formal section of the line bundle $K_C^{-\frac{1}{2}}$ on C, solves (3.21).*
- *The canonical ordering of the quantization of the local functions on T^*C is automatically chosen in (3.22) and the principal specialization (3.41). This selects the canonical ordering in (3.21).*

Remark 3.12. We do not have a closed formula for $F_{1,1}$ and $F_{0,3}$ from the given geometric data. Except for the case of $g(C) = 0$, they are **not** given by integrating $W_{1,1}$ and $W_{0,3}$ of the integral topological recursion.

Remark 3.13. The differential recursion (3.22) assumes $F_{1,1}$ and $F_{0,3}$ as the initial values. The equation itself does not give any condition for them. The discovery of [24, 25] is that the WKB equations for $S_m(x)$ are consequences of (3.22) for all $m \geq 2$. We note that there is an alternative way of constructing a quantization of the spectral curve. From the geometric data, first choose $W_{0,1}$ and $W_{0,2}$ as in (3.20), and solve the integral topological recursion (3.16). Then define a set of alternative free energies by

$$(3.42) \qquad F_{g,n}^{\mathrm{alt}}(z_1, \ldots, z_n) = \int \cdots \int W_{g,n}(z_1, \ldots, z_n)$$

for all values of (g, n). Then use the same (3.41) and (3.37) to define a wave function $\Psi^{\mathrm{alt}}(x, \hbar)$. The differential equation that annihilates this alternative wave function is another quantum curve. We emphasize that $\Psi^{\mathrm{alt}}(x, \hbar)$ **does not satisfy** our quantum curve equation (3.21). It is obvious because our definition (3.40) of S_2 is different. The alternative quantum curve is a second order differential equation, but it cannot be given by a closed formula, unlike (3.21). It is also noted that every coefficient of this alternative differential operator contains terms depending on all orders of $\hbar$. Therefore, the mechanism described in these lecture notes provides a totally different notion of quantum curves. We have shown that the differential recursion (3.22) is the passage from the starting spectral curve to the quantum curve (3.21). This picture is consistent with the construction of opers in [23] and a physics point of view [83].

3.4. **Classical differential equations.** If quantum curves are natural objects, then where do we see them in classical mathematics? Indeed, they appear as classical differential equations. Riemann and Poincaré found the interplay between algebraic geometry of curves in a ruled surface and the asymptotic expansion of an analytic solution to a differential equation defined on the base curve of the ruled surface. We look at these classical subjects from a new point of view. Let us now recall the definition of regular and irregular singular points of a second order differential equation.

Definition 3.14. Let

$$(3.43) \qquad \left(\frac{d^2}{dx^2} + s_1(x) \frac{d}{dx} + s_2(x) \right) \Psi(x) = 0$$

be a second order differential equation defined around a neighborhood of $x = 0$ on a small disc $|x| < \epsilon$ with meromorphic coefficients $s_1(x)$ and $s_2(x)$ with poles at $x = 0$. Denote by k (reps. ℓ) the order of the pole of $s_1(x)$ (resp. $s_2(x)$) at $x = 0$. If $k \leq 1$ and $\ell \leq 2$, then (3.43) has a **regular singular point** at $x = 0$. Otherwise, consider the *Newton polygon* of the order of poles of the coefficients of (3.43). It is the upper part of the convex hull of three points $(0,0), (1,k), (2,\ell)$. As a convention, if $s_j(x)$ is identically 0, then we assign $-\infty$ as its pole order. Let $(1, r)$ be the intersection point of the Newton polygon and the line $x = 1$. Thus

$$(3.44) \qquad r = \begin{cases} k & 2k \geq \ell, \\ \frac{\ell}{2} & 2k \leq \ell. \end{cases}$$

The differential equation (3.43) has an **irregular singular point of class $r - 1$** at $x = 0$ if $r > 1$.

To illustrate the scope of interrelations among the geometry of meromorphic Higgs bundles, their spectral curves, the singularities of quantum curves, $\hbar$-connections, and the quantum invariants, let us tabulate five examples here (see Table 3.4). The differential operators of these equations are listed in the third column. In the first three rows, the quantum curves are examples of classical differential equations known as Airy, Hermite, the Gauß hypergeometric equations. The fourth and the fifth rows are added to show that it is *not* the singularity of the spectral curve that determines the singularity of the quantum curve. In each example, the Higgs bundle (E, ϕ) we are considering consists of the base curve $C = \mathbb{P}^1$ and the rank 2 vector bundle E on $\mathbb{P}^1$ of (1.21). For this situation, the two topological recursions (3.16) and (3.22) are equivalent.

The first column of the table shows the Higgs field $\phi : E \longrightarrow K_{\mathbb{P}^1}(5) \otimes E$. Here, x is the affine coordinate of $\mathbb{P}^1 \setminus \{\infty\}$. Since our vector bundle is a specific direct sum of line bundles, the quantization is simple in each case, due to the fact that the $\hbar$-deformation $E_\hbar$ of E satisfies the condition as described in (1.46). Thus our quantum curves are equivalent to $\hbar$-connections in the trivial bundle. Except for the Gauß hypergeometric case, the connections are given by

$$(3.45) \qquad \nabla^\hbar = \hbar d - \phi,$$

where d is the exterior differentiation operator acting on the trivial bundle $E_\hbar$, $\hbar \neq 0$.

For the third example of a Gauß hypergeometric equation, we use a particular choice of parameters so that the $\hbar$-connection becomes an $\hbar$-deformed Gauß-Manin connection of (3.47). More precisely, for every $x \in \mathcal{M}_{0,4}$, we consider the elliptic curve $E(x)$ ramified over $\mathbb{P}^1$ at four points $\{0, 1, x, \infty\}$, and its two periods given by the elliptic integrals [58]

$$(3.46) \qquad \omega_1(x) = \int_1^\infty \frac{ds}{\sqrt{s(s-1)(s-x)}}, \qquad \omega_2(x) = \int_x^1 \frac{ds}{\sqrt{s(s-1)(s-x)}}.$$

The quantum curve in this case is an **$\hbar$-deformed meromorphic Gauß-Manin connection**

$$(3.47) \qquad \nabla_{GM}^\hbar = \hbar d - \begin{bmatrix} \left(-\frac{2x-1}{x(x-1)} + \frac{\hbar}{x} \right) dx & -\frac{(dx)^2}{4(x-1)} \\ & \\ \frac{1}{x} & \end{bmatrix}$$

in the $\hbar$-deformed vector bundle $K_{\overline{\mathcal{M}}_{0,4}}^{\frac{1}{2}} \oplus K_{\overline{\mathcal{M}}_{0,4}}^{-\frac{1}{2}}$ of rank 2 over $\overline{\mathcal{M}}_{0,4}$. Here, d again denotes the exterior differentiation acting on this trivial vector bundle. The restriction ∇_{GM}^1 of the

Higgs Field	Spectral Curve	Quantum Curve
$\begin{bmatrix} & x(dx)^2 \\ 1 & \end{bmatrix}$	$y^2 - x = 0$ $w^2 - u^5 = 0$ $\Sigma = 2C_0 + 5F$ $p_a = 2, p_g = 0$	Airy $\left(\hbar \frac{d}{dx}\right)^2 - x$ Class $\frac{3}{2}$ irregular singularity at ∞
$\begin{bmatrix} -xdx & -(dx)^2 \\ 1 & \end{bmatrix}$	$y^2 + xy + 1 = 0$ $w^2 - uw + u^4 = 0$ $\Sigma = 2C_0 + 4F$ $p_a = 1, p_g = 0$	Hermite $\left(\hbar \frac{d}{dx}\right)^2 + x\hbar \frac{d}{dx} + 1$ Class 2 irregular singularity at ∞
$\begin{bmatrix} \frac{2x-1}{x(1-x)}dx & \frac{(dx)^2}{4(1-x)} \\ \frac{1}{x} & \end{bmatrix}$	$y^2 + \frac{2x-1}{x(x-1)}y + \frac{1}{4x(x-1)} = 0$ $w^2 + 4(u-2)uw$ $-4u^2(u-1) = 0$ $\Sigma = 2C_0 + 4F$ $p_a = 1, p_g = 0$	Gauß Hypergeometric $\left(\hbar \frac{d}{dx}\right)^2 + \frac{2x-1}{x(x-1)}\hbar \frac{d}{dx} + \frac{1}{4x(x-1)}$ Regular singular points at $x = 0, 1, \infty$
$\begin{bmatrix} -dx & -\frac{(dx)^2}{x+1} \\ 1 & \end{bmatrix}$	$y^2 + y + \frac{1}{x+1} = 0$ $w^2 - u(u+1)w$ $+u^3(u+1) = 0$ $\Sigma = 2C_0 + 4F$ $p_a = 1, p_g = 0$	$\left(\hbar \frac{d}{dx}\right)^2 + \hbar \frac{d}{dx} + \frac{1}{x+1}$ Regular singular point at $x = -1$ and a class 1 irregular singularity at $x = \infty$
$\begin{bmatrix} -\frac{2x^2}{x^2-1}dx & \frac{(dx)^2}{x^2-1} \\ 1 & \end{bmatrix}$	$(x^2-1)y^2 + 2x^2y - 1 = 0$ non-singular $\Sigma = 2C_0 + 4F$ $p_a = p_g = 1$	$\left(\hbar \frac{d}{dx}\right)^2 + 2\frac{x^2}{x^2-1}\hbar \frac{d}{dx} - \frac{1}{x^2-1}$ Regular singular points at $x = \pm 1$ and a class 1 irregular singularity at $x = \infty$

TABLE 1. Examples of quantum curves.

connection at $\hbar = 1$ is equivalent to the Gauß-Manin connection that characterizes the two periods of (3.46), and the Higgs field is the classical limit of the connection matrix at $\hbar \to 0$:

$$(3.48) \qquad \phi = \begin{bmatrix} -\frac{2x-1}{x(x-1)}dx & -\frac{(dx)^2}{4(x-1)} \\ \frac{1}{x} & \end{bmatrix}.$$

The spectral curve $\Sigma \subset \overline{T^*\mathcal{M}_{0,4}}$ as a moduli space consists of the data $\left(E(x), \alpha_1(x), \alpha_2(x)\right)$, where $\alpha_1(x)$ and $\alpha_2(x)$ are the two eigenvalues of the Higgs field ϕ. The spectral curve $\Sigma \subset \overline{T^*\mathcal{M}_{0,4}} = \mathbb{F}^2$ as a divisor in the Hirzebruch surface is determined by the characteristic equation

$$(3.49) \qquad y^2 + \frac{2x-1}{x(x-1)}y + \frac{1}{4x(x-1)} = 0$$

of the Higgs field. Geometrically, Σ is a singular rational curve with one ordinary double point at $x = \infty$. The quantum curve is a **quantization** of the characteristic equation (3.49) for the eigenvalues $\alpha_1(x)$ and $\alpha_2(x)$ of $\phi(x)$. It is an $\hbar$-deformed **Picard-Fuchs equation**

$$\left(\left(\hbar \frac{d}{dx}\right)^2 + \frac{2x-1}{x(x-1)}\left(\hbar \frac{d}{dx}\right) + \frac{1}{4x(x-1)}\right)\omega_i(x, \hbar) = 0,$$

and its semi-classical limit agrees with the singular spectral curve Σ. As a second order differential equation, the quantum curve has two independent solutions corresponding to the two eigenvalues. At $\hbar = 1$, these solutions are exactly the two periods $\omega_1(x)$ and $\omega_2(x)$ of the Legendre family of elliptic curves $E(x)$. The topological recursion produces asymptotic expansions of these periods as functions in $x \in \overline{\mathcal{M}}_{0,4}$, at which the elliptic curve $E(x)$ degenerates to a nodal rational curve.

This is a singular connection with simple poles at $0, 1, \infty$, and has an explicit $\hbar$-dependence in the connection matrix. The Gauß-Manin connection ∇^1_{GM} at $\hbar = 1$ is equivalent to the Picard-Fuchs equation that characterizes the periods (3.46) of the Legendre family of elliptic curves $E(x)$ defined by the cubic equation

$$(3.50) \qquad t^2 = s(s-1)(s-x), \qquad x \in \mathcal{M}_{0,4} = \mathbb{P}^1 \setminus \{0, 1, \infty\}.$$

The second column gives the spectral curve of the Higgs bundle (E, ϕ). Since the Higgs fields have poles, the spectral curves are no longer contained in the cotangent bundle $T^*\mathbb{P}^1$. We need the compactified cotangent bundle

$$\overline{T^*\mathbb{P}^1} = \mathbb{P}(K_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}) = \mathbb{F}_2,$$

which is a Hirzebruch surface. The parameter y is the fiber coordinate of the cotangent line $T^*_x\mathbb{P}^1$. The first line of the second column is the equation of the spectral curve in the (x, y) affine coordinate of $\mathbb{F}_2$. All but the last example produce a singular spectral curve. Let (u, w) be a coordinate system on another affine chart of $\mathbb{F}_2$ defined by

$$\begin{cases} x = 1/u \\ y\,dx = v\,du, \qquad w = 1/v. \end{cases}$$

The singularity of Σ in the (u, w)-plane is given by the second line of the second column. The third line of the second column gives $\Sigma \in \mathrm{NS}(\mathbb{F}_2)$ as an element of the Néron-Severy group of $\mathbb{F}_2$. Here, C_0 is the class of the zero-section of $T^*\mathbb{P}^1$, and F represents the fiber class of $\pi : \mathbb{F}_2 \longrightarrow \mathbb{P}^1$. We also give the arithmetic and geometric genera of the spectral curve.

A solution $\Psi(x, \hbar)$ to the first example is given by the **Airy function**

$$(3.51) \qquad Ai(x, \hbar) = \frac{1}{2\pi} \hbar^{-\frac{1}{6}} \int_{-\infty}^{\infty} \exp\left(\frac{ipx}{\hbar^{2/3}} + i\frac{p^3}{3} \right) dp,$$

which is an entire function in x for $\hbar \neq 0$, as discussed earlier in these lectures. The expansion coordinate $x^{\frac{3}{2}}$ of (1.38) indicates the class of the irregular singularity of the Airy differential equation.

The solutions to the second example are given by confluent hypergeometric functions, such as $_1F_1\left(\frac{1}{2\hbar}; \frac{1}{2}; -\frac{x^2}{2\hbar} \right)$, where

$$(3.52) \qquad _1F_1(a; c; z) := \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} \frac{z^n}{n!}$$

is the **Kummer confluent hypergeomtric function**, and the **Pochhammer symbol** $(a)_n$ is defined by

$$(3.53) \qquad (a)_n := a(a+1)(a+2)\cdots(a+n-1).$$

For $\hbar > 0$, the topological recursion determines the asymptotic expansion of a particular entire solution known as a **Tricomi confluent hypergeomtric function**

$$\Psi^{\text{Catalan}}(x, \hbar)$$

$$= \left(-\frac{1}{2\hbar}\right)^{\frac{1}{2\hbar}} \left(\frac{\Gamma[\frac{1}{2}]}{\Gamma[\frac{1}{2\hbar} + \frac{1}{2}]} {}_1F_1\left(\frac{1}{2\hbar}; \frac{1}{2}; -\frac{x^2}{2\hbar}\right) + \frac{\Gamma[-\frac{1}{2}]}{\Gamma[\frac{1}{2\hbar}]} \sqrt{-\frac{x^2}{2\hbar}} {}_1F_1\left(\frac{1}{2\hbar} + \frac{1}{2}; \frac{3}{2}; -\frac{x^2}{2\hbar}\right)\right).$$

The expansion is given in the form

$$\Psi^{\text{Catalan}}(x, \hbar) = \left(\frac{1}{x}\right)^{\frac{1}{\hbar}} \sum_{n=0}^{\infty} \frac{\hbar^n \left(\frac{1}{\hbar}\right)_{2n}}{(2n)!!} \cdot \frac{1}{x^{2n}}$$

(3.54)

$$= \exp\left(\sum_{g=0}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n!} \hbar^{2g-2+n} F_{g,n}^C(x, \ldots, x)\right),$$

where $F_{g,n}^C$ is defined by (2.20), in terms of generalized Catalan numbers. The expansion variable x^2 in (3.54) indicates the class of irregularity of the Hermite differential equation at $x = \infty$. The cases for $(g, n) = (0, 1)$ and $(0, 2)$ require again a special treatment, as we discussed earlier.

The Hermite differential equation becomes simple for $\hbar = 1$, and we have the asymptotic expansion

(3.55) $\quad i\sqrt{\frac{\pi}{2}} e^{-\frac{1}{2}x^2} \left[1 - \text{erf}\left(\frac{ix}{\sqrt{2}}\right)\right] = \sum_{n=0}^{\infty} \frac{(2n-1)!!}{x^{2n+1}}$

$$= \exp\left(\sum_{2g-2+n \geq -1} \frac{1}{n!} \sum_{\mu_1, \ldots, \mu_n > 0} \frac{C_{g,n}(\mu_1, \ldots, \mu_n)}{\mu_1 \cdots \mu_n} \prod_{i=1}^{n} x^{-(\mu_1 + \cdots + \mu_n)}\right).$$

Here, $\text{erf}(x) := \frac{2}{\sqrt{\pi}} \int_0^x e^{-z^2} dz$ is the Gauß error function.
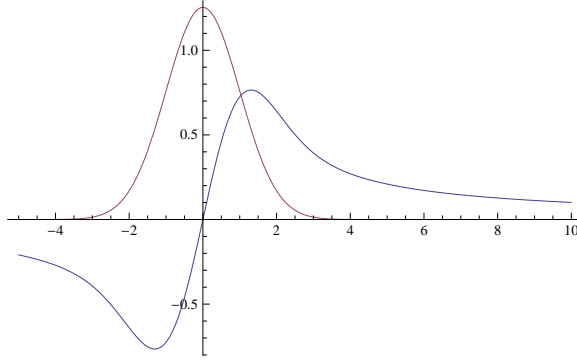
FIGURE 3.2. The imaginary part and the real part of $\Psi^{\text{Catalan}}(x, 1)$. For $x >> 0$, the imaginary part dies down, and only the real part has a non-trivial asymptotic expansion. Thus (3.55) is a series with real coefficients.

One of the two independent solutions to the third example, the Gauß hypergeometric equation, that is holomorphic around $x = 0$, is given by

(3.56)

$$\Psi^{\text{Gauß}}(x, \hbar) = {}_2F_1\left(-\frac{\sqrt{(h-1)(h-3)}}{2h} + \frac{1}{h} - \frac{1}{2}, \frac{\sqrt{(h-1)(h-3)}}{2h} + \frac{1}{h} - \frac{1}{2}; \frac{1}{h}; x\right),$$

where

$$(3.57) \qquad {}_2F_1(a,b;c;x) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!}$$

is the **Gauß hypergeometric function**. The topological recursion calculates the B-model genus expansion of the periods of the Legendre family of elliptic curves (3.50) at the point where the elliptic curve degenerates to a nodal rational curve. For example, the procedure applied to the spectral curve

$$y^2 + \frac{2x-1}{x(x-1)} y + \frac{1}{4x(x-1)} = 0$$

with a choice of

$$\eta = \frac{-(2x-1) - \sqrt{3x^2 - 3x + 1}}{2x(x-1)} dx,$$

which is an eigenvalue $\alpha_1(x)$ of the Higgs field ϕ, gives a genus expansion at $x = 0$:

$$(3.58) \qquad \Psi^{\text{Gauß}}(x, \hbar) = \exp\left(\sum_{g=0}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n!} \hbar^{2g-2+n} F_{g,n}^{\text{Gauß}}(x) \right).$$

At $\hbar = 1$, we have a topological recursion expansion of the period $\omega_1(x)$ defined in (3.46):

$$(3.59) \qquad \frac{\omega_1(x)}{\pi} = \Psi^{\text{Gauß}}(x, 1) = \exp\left(\sum_{g=0}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n!} F_{g,n}^{\text{Gauß}}(x) \right).$$

A subtle point we notice here is that while the Gauß hypergeometric equation has regular singular points at $x = 0, 1, \infty$, the Hermite equation has an irregular singular point of class 2 at ∞. The spectral curve of each case has an ordinary double point at $x = \infty$. But the crucial difference lies in the intersection of the spectral curve Σ with the divisor C_∞. For the Hermite case we have $\Sigma \cdot C_\infty = 4$ and the intersection occurs all at once at $x = \infty$. For the Gauß hypergeometric case, the intersection $\Sigma \cdot C_\infty = 4$ occurs once each at $x = 0, 1$, and twice at $x = \infty$. This *confluence* of regular singular points is the source of the irregular singularity in the Hermite differential equation.

The fourth row indicates an example of a quantum curve that has one regular singular point at $x = -1$ and one irregular singular point of class 1 at $x = \infty$. The spectral curve has an ordinary double point at $x = \infty$, the same as the Hermite case. As Figure 3.3 shows, the class of the irregular singularity at $x = \infty$ is determined by how the spectral curve intersects with C_∞.
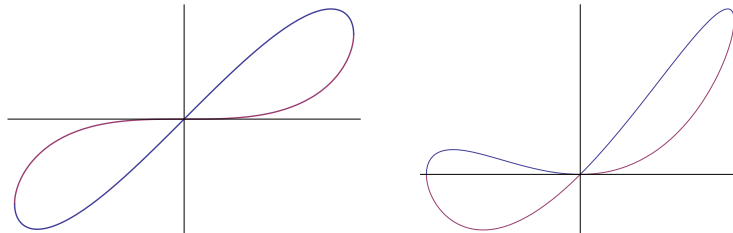

FIGURE 3.3. The spectral curves of the second and the fourth examples. The horizontal line is the divisor C_∞, and the vertical line is the fiber class F at $x = \infty$. The spectral curve intersects with C_∞ a total of four times. The curve on the right has a triple intersection at $x = \infty$, while the one on the left intersects all at once.

The existence of the irregular singularity in the quantum curve associated with a spectral curve has nothing to do with the singularity of the spectral curve. The fifth example shows a non-singular spectral curve of genus 1 (Figure 3.4), for which the quantum curve has a class 1 irregular singularity at $x = \infty$.
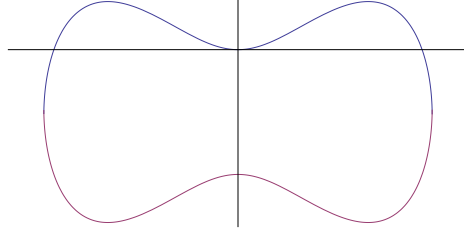

FIGURE 3.4. The spectral curve of the fifth example, which is non-singular. The corresponding quantum curve has two regular singular points at $x = \pm 1$, and a class 1 irregular singular point at $x = \infty$.

4. DIFFERENCE OPERATORS AS QUANTUM CURVES

Quantum curves often appear as infinite-order differential operators, or *difference* operators. In this section we present three typical examples: simple Hurwitz numbers, special double Hurwitz numbers, and the Gromov-Witten invariants of $\mathbb{P}^1$. These examples do not come from the usual Higgs bundle framework, because the rank of the Higgs bundle corresponds to the order of the quantum curves as a differential operator. Therefore, we ask:

Question 4.1. *What is the geometric structure generalizing the Hitchin spectral curves that correspond to difference operators as their quantization?*

In these lectures, we do not address this question, leaving it for a future investigation. We are content with giving examples here. We refer to [53] for a new and different perspective for the notion of quantum curves for difference operators.

4.1. **Simple and orbifold Hurwitz numbers.** The *simple Hurwitz number* $H_{g,n}(\vec{\mu})$ counts the automorphism weighted number of the topological types of simple Hurwitz covers of $\mathbb{P}^1$ of type $(g, \vec{\mu})$. A holomorphic map $\varphi : C \longrightarrow \mathbb{P}^1$ is a *simple Hurwitz cover* of type $(g, \vec{\mu})$ if C is a complete nonsingular algebraic curve defined over $\mathbb{C}$ of genus g, φ has n labeled poles of orders $\vec{\mu} = (\mu_1, \ldots, \mu_n)$, and all other critical points of φ are unlabeled simple ramification points.

In a similar way, we consider the *orbifold Hurwitz number* $H_{g,n}^{(r)}(\vec{\mu})$ for every positive integer $r > 0$ to be the automorphism weighted count of the topological types of smooth orbifold morphisms $\varphi : C \longrightarrow \mathbb{P}^1[r]$ with the same pole structure as the simple Hurwitz number case. Here, C is a connected 1-dimensional orbifold (a *twisted* curve) modeled on a nonsingular curve of genus g with $(\mu_1 + \cdots + \mu_n)/r$ stacky points of the type $[p/(\mathbb{Z}/r\mathbb{Z})]$. We impose that the inverse image of the morphism φ of the unique stacky point $[0/(\mathbb{Z}/r\mathbb{Z})] \in \mathbb{P}^1[r]$ coincides with the set of stacky points of C. When $r = 1$ we recover the simple Hurwitz number: $H_{g,n}^{(1)}(\vec{\mu}) = H_{g,n}(\vec{\mu})$.

Theorem 4.2 (Cut-and-join equation, [10]). *The orbifold Hurwitz numbers $H_{g,n}^{(r)}(\mu_1, \ldots, \mu_n)$ satisfy the following equation.*

$$(4.1) \quad sH_{g,n}^{(r)}(\mu_1,\ldots,\mu_n) = \frac{1}{2}\sum_{i\neq j}(\mu_i+\mu_j)H_{g,n-1}^{(r)}\left(\mu_i+\mu_j,\mu_{[\hat{i},\hat{j}]}\right)$$

$$+ \frac{1}{2}\sum_{i=1}^{n}\sum_{\alpha+\beta=\mu_i}\alpha\beta\left[H_{g-1,n+1}^{(r)}\left(\alpha,\beta,\mu_{[\hat{i}]}\right) + \sum_{\substack{g_1+g_2=g \\ I\sqcup J=[\hat{i}]}}H_{g_1,|I|+1}^{(r)}(\alpha,\mu_I)H_{g_2,|J|+1}^{(r)}(\beta,\mu_J)\right].$$

Here

$$(4.2) \qquad\qquad s = s(g,\vec{\mu}) = 2g - 2 + n + \frac{\mu_1+\cdots+\mu_n}{r}$$

is the number of simple ramification point given by the Riemann-Hurwitz formula. As before, we use the convention that for any subset $I \subset [n] = \{1, 2, \ldots, n\}$, $\mu_I = (\mu_i)_{i\in I}$. The hat notation $\hat{i}$ indicates that the index i is removed. The last summation is over all partitions of g and set partitions of $[\hat{i}] = \{1,\ldots, i-1, i+1, \ldots, n\}$.

Remark 4.3. There is a combinatorial description, in the same manner we have done for the Catalan numbers in Section 2, for simple and orbifold Hurwitz numbers. The cut-and-join equation is derived as the edge-contraction formula, exactly in the same way for the Catalan recursion (2.8). See [26] for more detail.

We regard $H_{g,n}^{(r)}(\vec{\mu})$ as a function in n integer variables $\vec{\mu} \in \mathbb{Z}_+^n$. Following the recipe of [27, 33, 68] that is explained in the earlier sections, we define the *free energies* as the Laplace transform

$$(4.3) \qquad\qquad F_{g,n}^{(r)}(z_1,\ldots,z_n) = \sum_{\vec{\mu}\in\mathbb{Z}_+^n}H_{g,n}^{(r)}(\vec{\mu})\,e^{-\langle\vec{w},\vec{\mu}\rangle}.$$

Here, $\vec{w} = (w_1,\ldots,w_n)$ is the vector of the Laplace dual coordinates of $\vec{\mu}$, $\langle\vec{w},\vec{\mu}\rangle = w_1\mu_1 + \cdots + w_n\mu_n$, and variables z_i and w_i for each i are related by the *r-Lambert* function

$$(4.4) \qquad\qquad e^{-w} = ze^{-z^r}.$$

It is often convenient to use a different variable $x = ze^{-z^r}$, with which the plane analytic curve called the *r-**Lambert curve*** is given by

$$(4.5) \qquad\qquad \begin{cases} x = ze^{-z^r} \\ y = z^r. \end{cases}$$

Then the free energies $F_{g,n}^{(r)}$ of (4.3) are generating functions of the orbifold Hurwitz numbers. By abuse of notation, we also write

$$(4.6) \qquad F_{g,n}^{(r)}(x_1,\ldots,x_n) = \sum_{\vec{\mu}\in\mathbb{Z}_+^n}H_{g,n}^{(r)}(\vec{\mu})\prod_{i=1}^{n}x_i^{\mu_i} = \sum_{\vec{\mu}\in\mathbb{Z}_+^n}H_{g,n}^{(r)}(\vec{\mu})\prod_{i=1}^{n}\left(z_ie^{-z_i^r}\right)^{\mu_i}.$$

For every (g,n), the power series (4.6) in $(x_1,\ldots,x_n)$ is convergent and defines an analytic function.

Theorem 4.4 (Differential recursion for Hurwitz numbers, [10]). *In terms of the z-variables, the free energies are calculated as follows.*

$$(4.7) \qquad\qquad F_{0,1}^{(r)}(z) = \frac{1}{r}z^r - \frac{1}{2}z^{2r},$$

$$(4.8) \qquad\qquad F_{0,2}^{(r)}(z_1, z_2) = \log\frac{z_1 - z_2}{x_1 - x_2} - (z_1^r + z_2^r),$$

where $x_i = z_i e^{-z_i^r}$. For (g,n) in the stable range, i.e., when $2g - 2 + n > 0$, the free energies satisfy the differential recursion equation

$$(4.9) \quad \left(2g - 2 + n + \frac{1}{r} \sum_{i=1}^{n} z_i \frac{\partial}{\partial z_i} \right) F_{g,n}^{(r)}(z_1, \ldots, z_n)$$

$$= \frac{1}{2} \sum_{i \neq j} \frac{z_i z_j}{z_i - z_j} \left[\frac{1}{(1 - r z_i^r)^2} \frac{\partial}{\partial z_i} F_{g,n-1}^{(r)}\left(z_{[\hat{j}]}\right) - \frac{1}{(1 - r z_j^r)^2} \frac{\partial}{\partial z_j} F_{g,n-1}^{(r)}\left(z_{[\hat{i}]}\right) \right]$$

$$+ \frac{1}{2} \sum_{i=1}^{n} \frac{z_i^2}{(1 - r z_i^r)^2} \frac{\partial^2}{\partial u_1 \partial u_2} F_{g-1,n+1}^{(r)}\left(u_1, u_2, z_{[\hat{i}]}\right) \Big|_{u_1 = u_2 = z_i}$$

$$+ \frac{1}{2} \sum_{i=1}^{n} \frac{z_i^2}{(1 - r z_i^r)^2} \sum_{\substack{g_1+g_2=g \\ I \sqcup J = [\hat{i}]}}^{\text{stable}} \left(\frac{\partial}{\partial z_i} F_{g_1,|I|+1}^{(r)}(z_i, z_I) \right) \left(\frac{\partial}{\partial z_i} F_{g_2,|J|+1}^{(r)}(z_i, z_J) \right).$$

Remark 4.5. Since $F_{g,n}^{(r)}(z_1, \ldots, z_n)\big|_{z_i=0} = 0$ for every i, the differential recursion (4.9), which is a linear first order partial differential equation, uniquely determines $F_{g,n}^{(r)}$ inductively for all (g,n) subject to $2g - 2 + n > 0$. This is a generalization of the result of [69] to the orbifold case.

Remark 4.6. The differential recursion of Theorem 4.4 is obtained by taking the Laplace transform of the cut-and-join equation for $H_{g,n}^{(r)}(\vec{\mu})$. The r-Lambert curve itself, (4.5), is obtained by computing the Laplace transform of $H_{0,1}^{(r)}(\mu)$, and solving the differential equation that arises from the cut-and-join equation. See also [26] for a different formulation of the r-Hurwitz numbers using the graph enumeration formulation and a universal mechanism to obtain the spectral curve.

The differential recursion produces two results, as we have seen for the case of the Catalan numbers. One is the quantum curve by taking the principal specialization, and the other the topological recursion of [34].

Theorem 4.7 (Quantum curves for r-Hurwitz numbers, [10]). *We introduce the partition function, or the wave function, of the orbifold Hurwitz numbers as*

$$(4.10) \qquad \Psi^{(r)}(z, \hbar) = \exp\left(\sum_{g=0}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n!} \hbar^{2g-2+n} F_{g,n}^{(r)}(z, z, \ldots, z) \right).$$

It satisfies the following system of (an infinite-order) linear differential equations.

$$(4.11) \qquad \left(\hbar D - e^{r\left(-w + \frac{r-1}{2}\hbar\right)} e^{r\hbar D} \right) \Psi^{(r)}(z, \hbar) = 0,$$

$$(4.12) \qquad \left(\frac{\hbar}{2} D^2 - \left(\frac{1}{r} + \frac{\hbar}{2} \right) D - \hbar \frac{\partial}{\partial \hbar} \right) \Psi^{(r)}(z, \hbar) = 0,$$

where

$$D = \frac{z}{1 - r z^r} \frac{\partial}{\partial z} = x \frac{\partial}{\partial x} = -\frac{\partial}{\partial w}.$$

Let the differential operator of (4.11) *(resp.* (4.12)*) be denoted by* P *(resp.* Q*). Then we have the commutator relation*

$$(4.13) \qquad [P, Q] = P.$$

The semi-classical limit of each of the equations (4.11) *or* (4.12) *recovers the r-Lambert curve* (4.5).

Remark 4.8. The Schrödinger equation (4.11) is first established in [67].

Remark 4.9. The above theorem is a generalization of [68, Theorem 1.3] for an arbitrary $r > 0$. The restriction $r = 1$ reduces to the simple Hurwitz case.

Remark 4.10. Unlike the situation of Hitchin spectral curves, the results of the quantization of the analytic spectral curves are a **difference-differential equation**, and a PDE containing the differentiation with respect to the deformation parameter $\hbar$.

Now let us define

$$(4.14) \qquad W_{g,n}^{(r)}(z_1,\ldots,z_n) := d_1 d_2 \cdots d_n F_{g,n}^{(r)}(z_1,\ldots,z_n).$$

Then we have

Theorem 4.11 (Topological recursion for orbifold Hurwitz numbers, [10])**.** *For the stable range* $2g - 2 + n > 0$, *the symmetric differential forms* (4.14) *satisfy the following integral recursion formula.*

$$(4.15) \quad W_{g,n}^{(r)}(z_1,\ldots,z_n) = \frac{1}{2\pi i} \sum_{j=1}^{r} \oint_{\gamma_j} K_j(z,z_1) \Bigg[W_{g-1,n+1}^{(r)}(z, s_j(z), z_2, \ldots, z_n)$$

$$+ \sum_{i=2}^{n} \left(W_{0,2}^{(r)}(z, z_i) \otimes W_{g,n-1}^{(r)}(s_j(z), z_{[\hat{1},\hat{i}]}) + W_{0,2}^{(r)}(s_j(z), z_i) \otimes W_{g,n-1}^{(r)}(z, z_{[\hat{1},\hat{i}]}) \right)$$

$$+ \sum_{\substack{g_1+g_2=g \\ I \sqcup J = \{2,\ldots,n\}}}^{\text{stable}} W_{g_1,|I|+1}^{(r)}(z, z_I) \otimes W_{g_2,|J|+1}^{(r)}(s_j(z), z_J) \Bigg].$$

Here, the integration is taken with respect to z *along a small simple closed loop* γ_j *around* p_j, *and* $\{p_1,\ldots,p_r\}$ *are the critical points of the r-Lambert function* $x(z) = ze^{-z^r}$ *at* $1 - rz^r = 0$. *Since* $dx = 0$ *has a simple zero at each* p_j, *the map* $x(z)$ *is locally a double-sheeted covering around* $z = p_j$. *We denote by* s_j *the deck transformation on a small neighborhood of* p_j. *Finally, the integration kernel is defined by*

$$(4.16) \qquad K_j(z,z_1) = \frac{1}{2} \frac{1}{W_{0,1}^{(r)}(s_j(z_1)) - W_{0,1}^{(r)}(z_1)} \otimes \int_z^{s_j(z)} W_{0,2}^{(r)}(\,\cdot\,, z_1).$$

Remark 4.12. As mentioned earlier, the significance of the integral formalism is its universality. The differential equation (4.9) takes a different form depending on the counting problem, whereas the integral formula (4.15) depends only on the choice of the spectral curve.

Remark 4.13. The proof is based on the idea of [33]. The notion of the *principal part* of meromorphic differentials plays a key role in converting the Laplace transform of the cut-and-join equation into a residue formula.

4.2. **Gromov-Witten invariants of the projective line.** Hurwitz numbers and Gromov-Witten invariants of $\mathbb{P}^1$ are closely related [75]. However, their relations to the topological recursion is rather different. For example, the topological recursion for stationary Gromov-Witten invariants of $\mathbb{P}^1$ was conjectured by Norbury and Scott [74] as a concrete formula, but its proof [30] is done in a very different way than that of [33, 69]. This is based on

the fact that we do not have a counterpart of the cut-and-join equation for the case of the Gromov-Witten invariants of $\mathbb{P}^1$. Nonetheless, the quantum curve exists.

Let $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1, d)$ denote the moduli space of stable maps of degree d from an n-pointed genus g curve to $\mathbb{P}^1$. This is an algebraic stack of dimension $2g - 2 + n + 2d$. The dimension reflects the fact that a generic map from an algebraic curve to $\mathbb{P}^1$ has only simple ramifications, and the number of such ramification points, which we can derive from the Riemann-Hurwitz formula, gives the dimension of this stack. The descendant Gromov-Witten invariants of $\mathbb{P}^1$ are defined by

$$(4.17) \qquad \left\langle \prod_{i=1}^{n} \tau_{b_i}(\alpha_i) \right\rangle_{g,n}^{d} := \int_{[\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1,d)]^{vir}} \prod_{i=1}^{n} \psi_i^{b_i} ev_i^*(\alpha_i),$$

where $[\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1, d)]^{vir}$ is the virtual fundamental class of the moduli space,

$$ev_i : \overline{\mathcal{M}}_{g,n}(\mathbb{P}^1, d) \longrightarrow \mathbb{P}^1$$

is a natural morphism defined by evaluating a stable map at the i-th marked point of the source curve, $\alpha_i \in H^*(\mathbb{P}^1, \mathbb{Q})$ is a cohomology class of the target $\mathbb{P}^1$, and ψ_i is the tautological cotangent class in $H^2(\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1, d), \mathbb{Q})$. We denote by 1 the generator of $H^0(\mathbb{P}^1, \mathbb{Q})$, and by $\omega \in H^2(\mathbb{P}^1, \mathbb{Q})$ the Poincaré dual to the point class. We assemble the Gromov-Witten invariants into particular generating functions as follows. For every (g, n) in the stable sector $2g - 2 + n > 0$, we define the *free energy* of type (g, n) by

$$(4.18) \qquad F_{g,n}(x_1, \ldots, x_n) := \left\langle \prod_{i=1}^{n} \left(-\frac{\tau_0(1)}{2} - \sum_{b=0}^{\infty} \frac{b! \tau_b(\omega)}{x_i^{b+1}} \right) \right\rangle_{g,n}.$$

Here the degree d is determined by the dimension condition of the cohomology classes to be integrated over the virtual fundamental class. We note that (4.18) contains the class $\tau_0(1)$. For unstable geometries, we introduce two functions

$$(4.19) \qquad S_0(x) := x - x \log x + \sum_{d=1}^{\infty} \left\langle -\frac{(2d-2)! \tau_{2d-2}(\omega)}{x^{2d-1}} \right\rangle_{0,1}^{d},$$

$$(4.20) \qquad S_1(x) := -\frac{1}{2} \log x + \frac{1}{2} \sum_{d=0}^{\infty} \left\langle \left(-\frac{\tau_0(1)}{2} - \sum_{b=0}^{\infty} \frac{b! \tau_b(\omega)}{x^{b+1}} \right)^2 \right\rangle_{0,2}^{d},$$

utilizing an earlier work of [27]. Then we have

Theorem 4.14 (The quantum curve for the Gromov-Witten invariants of $\mathbb{P}^1$, [29]). *The wave function*

$$(4.21) \qquad \Psi(x, \hbar) := \exp\left(\frac{1}{\hbar} S_0(x) + S_1(x) + \sum_{2g-2+n>0} \frac{\hbar^{2g-2+n}}{n!} F_{g,n}(x, \ldots, x) \right)$$

satisfies the quantum curve equation of an infinite order

$$(4.22) \qquad \left[\exp\left(\hbar \frac{d}{dx} \right) + \exp\left(-\hbar \frac{d}{dx} \right) - x \right] \Psi(x, \hbar) = 0.$$

Moreover, the free energies $F_{g,n}(x_1, \ldots, x_n)$ as functions in n-variables, and hence all the Gromov-Witten invariants (4.17), can be recovered from the equation (4.22) alone, using the mechanism of the **topological recursion** *of [34].*

Remark 4.15. The appearance of the extra terms in S_0 and S_1, in particular, the $\log x$ terms, is trickier than the cases studied in these lectures. We refer to [29, Section 3].

Remark 4.16. Put

$$(4.23) \qquad S_m(x) := \sum_{2g-2+n=m-1} \frac{1}{n!} F_{g,n}(x, \dots, x).$$

Then our wave function is of the form

$$(4.24) \qquad \Psi(x, \hbar) = \exp\left(\sum_{m=0}^{\infty} \hbar^{m-1} S_m(x)\right),$$

which provides the WKB approximation of the quantum curve equation (4.22). Thus the significance of (4.18) is that the exponential generating function (4.21) of the descendant Gromov-Witten invariants of $\mathbb{P}^1$ gives the solution to the WKB analysis in a closed formula for the difference equation (4.22).

Remark 4.17. For the case of Hitchin spectral curves [24, 25], the Schrödinger-like equation (4.22) is a direct consequence of the generalized topological recursion. In the $GW(\mathbb{P}^1)$ context, the topological recursion does not play any role in establishing (4.22).

We can recover the classical mechanics corresponding to (4.22) by taking its semi-classical limit, which is the singular perturbation limit

$$(4.25) \quad \lim_{\hbar \to 0} \left(e^{-\frac{1}{\hbar}S_0(x)} \left[\exp\left(\hbar \frac{d}{dx}\right) + \exp\left(-\hbar \frac{d}{dx}\right) - x\right] e^{\frac{1}{\hbar}S_0(x)} e^{\sum_{m=1}^{\infty} \hbar^{m-1} S_m(x)}\right)$$

$$= \left(e^{S_0'(x)} + e^{-S_0'(x)} - x\right) e^{S_1(x)} = 0.$$

In terms of new variables $y(x) = S_0'(x)$ and $z(x) = e^{y(x)}$, the semi-classical limit gives us an equation for the spectral curve

$$z \in \Sigma = \mathbb{C}^* \subset \mathbb{C} \times \mathbb{C}^* \xleftarrow{\exp} T^*\mathbb{C} = \mathbb{C}^2 \ni (x, y)$$

by

$$(4.26) \qquad \begin{cases} x = z + \frac{1}{z} \\ y = \log z \end{cases}.$$

This is the reason we consider (4.22) as the quantization of the Laudau-Ginzburg model

$$x = z + \frac{1}{z}.$$

It was conjectured in [74] that the stationary Gromov-Witten theory of $\mathbb{P}^1$ should satisfy the topological recursion with respect to the spectral curve (4.26). The conjecture is solved in [30] as a corollary to its main theorem, which establishes the correspondence between the topological recursion and the Givental formalism.

The key discovery of [29] is that the quantum curve equation (4.22) is equivalent to a recursion equation

$$(4.27) \qquad \frac{x}{\hbar}\left(e^{-\hbar \frac{d}{dx}} - 1\right) X_d(x, \hbar) + \frac{1}{1 + \frac{x}{\hbar}} e^{\hbar \frac{d}{dx}} X_{d-1}(x, \hbar) = 0$$

for a rational function

$$(4.28) \qquad X_d(x, \hbar) = \sum_{\lambda \vdash d} \left(\frac{\dim \lambda}{d!} \right)^2 \prod_{i=1}^{\ell(\lambda)} \frac{x + (i - \lambda_i)\hbar}{x + i\hbar}.$$

Here λ is a partition of $d \geq 0$ with parts λ_i and $\dim \lambda$ denotes the dimension of the irreducible representation of the symmetric group S_d characterized by λ.

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