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The spectral curve of the Eynard-Orantin recursion via the Laplace transform

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Abstract. The Eynard-Orantin recursion formula provides an effective tool for certain enumeration problems in geometry. The formula requires a *spectral curve* and the recursion kernel. We present a uniform construction of the spectral curve and the recursion kernel from the unstable geometries of the original counting problem. We examine this construction using four concrete examples: Grothendieck's dessins d'enfants (or higher-genus analogue of the Catalan numbers), the intersection numbers of tautological cotangent classes on the moduli stack of stable pointed curves, single Hurwitz numbers, and the stationary Gromov-Witten invariants of the complex projective line.

1. Introduction

What is the mirror dual object of the Catalan numbers? We wish to make sense of this question in the present paper. Since homological mirror symmetry is a categorial equivalence, it does not require the existence of underlying spaces to which the categories are associated. By identifying the Catalan numbers with a counting problem similar to Gromov-Witten theory, we come up with an *equation*

$$x = z + \frac{1}{z} \tag{1.1}$$

as their mirror dual. It is not a coincidence that (1.1) is the Landau-Ginzburg model in one variable [**2**, **41**]. Once the mirror dual object is identified, we can calculate the *higher-genus analogue* of the Catalan numbers using the *Eynard-Orantin topological recursion formula*. This recursion therefore provides a mechanism for calculating the higher-order quantum corrections term by term.

The purpose of this paper is to present a systematic construction of genus 0 spectral curves of the Eynard-Orantin recursion formula [**26**, **28**]. Suppose we have a symplectic space X on the A-model side. If the Gromov-Witten theory of X is controlled by an integrable system, then the homological mirror dual of X is expected to be a family of *spectral curves* Σ. Let us consider the descendant Gromov-Witten invariants of X as a function in integer variables. The *Laplace transform* of these functions can be viewed as symmetric meromorphic functions defined on the products of Σ. We expect that they satisfy the Eynard-Orantin topological recursion on the B-model side defined on the curve Σ.

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More specifically, we construct the spectral curve using the Laplace transform of the descendant Gromov-Witten type invariants for the *unstable* geometries $(g, n) = (0, 1)$ and $(0, 2)$. We give four concrete examples in this paper:

- The number of dessins d'enfants of Grothendieck, which can be thought of as higher-genus analogue of the Catalan numbers.
- The ψ-class intersection numbers $\langle \tau_{d_1} \cdots \tau_{d_n} \rangle_{g,n}$ on the moduli space $\overline{\mathcal{M}}_{g,n}$ of pointed stable curves [**4, 11, 16, 26, 47, 79**].
- Single Hurwitz numbers [**8, 25, 57**].
- The stationary Gromov-Witten invariants of $\mathbb{P}^1$ [**63, 67**].

The spectral curves we construct are listed in Table 1. The Eynard-Orantin recursion formula for the single Hurwitz numbers [**5, 8, 25, 58**] and the ψ-class intersection numbers [**26**] are known. Norbury and Scott conjecture that the stationary Gromov-Witten invariants of $\mathbb{P}^1$ also satisfy the Eynard-Orantin recursion [**63**]. A similar statement for the number of dessins d'enfants does not seem to be known. We give a full proof of this fact in the paper.

Grothendieck's Dessins	$\begin{cases} x = z + \frac{1}{z} \\ y = -z \end{cases}$
$\langle \tau_{d_1} \cdots \tau_{d_n} \rangle_{g,n}$	$\begin{cases} x = z^2 \\ y = -z \end{cases}$
Single Hurwitz Numbers	$\begin{cases} x = ze^{1-z} \\ y = e^{z-1} \end{cases}$
Stationary GW Invariants of $\mathbb{P}^1$	$\begin{cases} x = z + \frac{1}{z} \\ y = -\log(1 + z^2) \end{cases}$

TABLE 1. Examples of spectral curves.

Let $D_{g,n}(\mu_1, \ldots, \mu_n)$ denote the weighted count of clean Belyi morphisms of smooth connected algebraic curves of genus g with n poles of order $(\mu_1, \ldots, \mu_n)$. We first prove

THEOREM 1.1. *For $2g - 2 + n \geq 0$ and $n \geq 1$, the number of clean Belyi morphisms satisfies the following equation:*

$$\mu_1 D_{g,n}(\mu_1, \ldots, \mu_n) = \sum_{j=2}^{n} (\mu_1 + \mu_j - 2) D_{g,n-1}(\mu_1 + \mu_j - 2, \mu_{[n]\setminus\{1,j\}}) \tag{1.2}$$

$$+ \sum_{\alpha+\beta=\mu_1-2} \alpha\beta \left[D_{g-1,n+1}(\alpha, \beta, \mu_{[n]\setminus\{1\}}) + \sum_{\substack{g_1+g_2=g \\ I \sqcup J = \{2,\ldots,n\}}} D_{g_1,|I|+1}(\alpha, \mu_I) D_{g_2,|J|+1}(\beta, \mu_J) \right],$$

where $\mu_I = (\mu_i)_{i \in I}$ for a subset $I \subset [n] = \{1, 2, \ldots, n\}$.

The simplest case

$$D_{0,1}(2m) = \frac{1}{2m} C_m$$

is given by the Catalan number $C_m = \frac{1}{m+1}\binom{2m}{m}$. The next case $D_{0,2}(\mu_1, \mu_2)$ is calculated in [**45, 46**]. Note that the (g, n)-term appears also on the right-hand side of (1.2). Therefore, this is merely an equation, not an effective recursion formula.

Define the Eynard-Orantin differential form by

$$W^D_{g,n}(t_1,\dots,t_n) = d_1\cdots d_n \sum_{\mu_1,\dots,\mu_n>0} D_{g,n}(\mu_1,\dots,\mu_n)e^{-(\mu_1 w_1+\cdots+\mu_n w_n)},$$

where the w_j-coordinates and t_j-coordinates are related by

$$e^{w_j} = \frac{t_j+1}{t_j-1}+\frac{t_j-1}{t_j+1}.$$

Then

THEOREM 1.2. *The Eynard-Orantin differential forms for $2g-2+n>0$ satisfy the following topological recursion formula*

$$\begin{aligned}(1.3)\quad & W^D_{g,n}(t_1,\dots,t_n)\\ &= -\frac{1}{64}\frac{1}{2\pi i}\int_\gamma \left(\frac{1}{t+t_1}+\frac{1}{t-t_1}\right)\frac{(t^2-1)^3}{t^2}\cdot\frac{1}{dt}\cdot dt_1\Bigg[W^D_{g-1,n+1}(t,-t,t_2,\dots,t_n)\\ &\qquad + \sum_{j=2}^n \Big(W^D_{0,2}(t,t_j)W_{g,n-1}(-t,t_2,\dots,\widehat{t_j},\dots,t_n)\\ &\qquad + W^D_{0,2}(-t,t_j)W_{g,n-1}(t,t_2,\dots,\widehat{t_j},\dots,t_n)\Big)\\ &\qquad + \sum_{\substack{g_1+g_2=g\\ I\sqcup J=\{2,3,\dots,n\}}}^{stable} W^D_{g_1,|I|+1}(t,t_I)W^D_{g_2,|J|+1}(-t,t_J)\Bigg].\end{aligned}$$

This is now a recursion formula, since the topological type (g',n') of the Belyi morphisms appearing on the right-hand side satisfy

$$2g'-2+n' = (2g-2+n)-1,$$

counting the contributions from the disjoint union of the domain curves additively. A corollary to the recursion formula is a combinatorial identity between the number of clean Belyi morphisms and the number of lattice points on the moduli space $\mathcal{M}_{g,n}$ that has been studied in [**11**, **56**, **60**–**62**].

COROLLARY 1.3.
(1.4)

$$D_{g,n}(\mu_1,\dots,\mu_n) = \sum_{\ell_1>\frac{\mu_1}{2}}\cdots\sum_{\ell_n>\frac{\mu_n}{2}}\prod_{i=1}^n \frac{2\ell_i-\mu_i}{\mu_i}\binom{\mu_i}{\ell_i}N_{g,n}(2\ell_1-\mu_i,\cdots,2\ell_n-\mu_n),$$

where $N_{g,n}(\mu_1,\dots,\mu_n)$ is defined by (5.4)*.*

The recursion formula (1.3) is a typical example of the *Eynard-Orantin recursion* we discuss in this paper. We establish this theorem by taking the Laplace transform of (1.2). This is indeed a general theme. For every known case of the Eynard-Orantin recursion appearing in an enumerative or geometric problem, the proof has been established by taking the Laplace transform of a counting formula like (1.2). For example, for the cases of single Hurwitz numbers [**25**, **58**] and open Gromov-Witten invariants of $\mathbb{C}^3$ [**82**, **83**], the counting formulas similar to (1.2) are called the *cut-and-join* equations [**30**, **50**, **78**, **80**, **81**].

The Laplace transform plays a mysterious role in Gromov-Witten theory. We notice its appearance in Kontsevich's work [**47**] that relates the Euclidean volume of $\mathcal{M}_{g,n}$ and the intersection numbers on $\overline{\mathcal{M}}_{g,n}$, and also in the work of Okounkov-Pandharipande [**66**] that relates the single Hurwitz numbers and the enumeration of topological graphs. It has been proved that in these two cases the Laplace transform of the quantities in question satisfies the Eynard-Orantin recursion [**11**, **25**, **27**, **56**, **58**] for a particular choice of the spectral curve.

Then what is the role of the Laplace transform here? The answer we propose in this paper is that *the Laplace transform defines the spectral curve*. Since the spectral curve is a B-model object, **the Laplace transform plays the role of mirror symmetry**.

The Eynard-Orantin recursion formula is an effective tool in certain geometric enumeration. The formula originated in random matrix theory as a mechanism to compute the expectation value of a product of the resolvent of random matrices ([**1**], [**22**]). In [**26**, **28**] Eynard and Orantin propose a novel point of view, considering the recursion as a means of *defining* meromorphic symmetric differential forms $W_{g,n}$ on the product Σ^n of a Riemann surface Σ for every $g \geq 0$ and $n > 0$. They derive in [**26**, **28**] many beautiful properties that these quantities satisfy, including modularity and relations to integrable systems.

The effectiveness of the topological recursion in string theory is immediately noticed [**15**, **24**, **52**, **71**]. A remarkable discovery, connecting the recursion formula and geometry, is made by Mariño [**52**] and Bouchard, Klemm, Mariño and Pasquetti [**7**]. It is formulated as the *Remodeling Conjecture*. This conjecture covers many aspects of both closed and open Gromov-Witten invariants of arbitrary toric Calabi-Yau threefolds. One of their statements says the following. Let X be an arbitrary toric Calabi-Yau threefold, and Σ its mirror curve. Apply the Eynard-Orantin recursion formula to Σ. Then $W_{g,n}$ calculates the open Gromov-Witten invariants of X. The validity of the topological recursion of [**26**, **28**] is not limited to Gromov-Witten invariants. It has been applied to the HOMFLY polynomials of torus knots [**10**], and understanding the role of quantum Riemann surfaces and certain Seiberg-Witten invariants [**36**]. A speculation also suggests its relation to colored Jones polynomials and the hyperbolic volume conjecture of knot complements [**14**].

From the very beginning, the effectiveness of the Eynard-Orantin recursion in enumerative geometry was suggested by physicists. Bouchard and Mariño conjecture in [**8**] that particular generating functions of single Hurwitz numbers satisfy the Eynard-Orantin topological recursion. They have come up to this conjecture as the limiting case of the remodeling conjecture for $\mathbb{C}^3$ when the framing parameter tends to ∞. The spectral curve for this scenario is the *Lambert curve* $x = ye^{-y}$. The Bouchard-Mariño conjecture is solved in [**5**, **25**, **58**]. The work [**25**] also influenced the solutions to the remodeling conjecture for $\mathbb{C}^3$ itself. The statement on the open Gromov-Witten invariants was proved in [**12**, **82**, **83**], and the closed case was proved in [**6**, **84**].

The Eynard-Orantin topological recursion *starts* with a spectral curve Σ. Thus it is reasonable to propose the recursion formalism whenever there is a natural curve in the problem we study. Such curves may include the mirror curve of a toric Calabi-Yau threefold [**7**, **52**], the zero locus of an A-polynomial [**14**, **36**], the Seiberg-Witten curves [**36**], the torus on which a knot is drawn [**10**], and the character variety of the fundamental group of a knot complement relative to $SL(2, \mathbb{C})$ [**14**]. Now we ask the opposite question.

QUESTION 1.4. *If an enumerative geometry problem is given, then how do we find the spectral curve, with which the Eynard-Orantin formalism may provide a solution?*

In every work of [**6, 11, 12, 25–28, 56, 58, 61, 63, 82, 83**], the spectral curve is considered to be given. *How do we know that the particular choice of the spectral curve is correct?* Our proposal provides an answer to this question: *the Laplace transform of the unstable geometries* $(g, n) = (0, 1)$ *and* $(0, 2)$ *determines the spectral curve, and the topological recursion formula itself.* The key ingredients of the topological recursion are the spectral curve and the recursion kernel that is determined by the differential forms $W_{0,1}$ and $W_{0,2}$. In the literature starting from [**26**], the word "Bergman kernel" is used for the differential form $W_{0,2}$. But $W_{0,2}$ has indeed nothing to do with the classical *Bergman kernel* in complex analysis. It is a universally given 2-form depending only on the geometry of the spectral curve. We would rather emphasize in this paper that this "kernel" is the Laplace transform of the *annulus* amplitude, which should be determined by the counting problem we start with.

Although it is still vague, our proposal is the following

CONJECTURE 1.5 (The Laplace transform conjecture). *If the unstable geometries* $(g, n) = (0, 1)$ *and* $(0, 2)$ *make sense in a counting problem on the A-model side, then the Laplace transform of the solution to these cases determines the spectral curve and the recursion kernel of the Eynard-Orantin formalism, which is a B-model theory. Thus the Laplace transform plays a role of mirror symmetry. The recursion then determines the solution to the original counting problem for all* (g, n).

The Eynard-Orantin recursion is a process of *quantization* [**10, 36**]. Thus the implication of the conjecture is that *quantum invariants are uniquely determined by the disk and annulus amplitudes.* For example, single Hurwitz numbers $h_{g,\mu}$ are all determined by the first two cases $h_{0,(\mu_1)}$ and $h_{0,(\mu_1,\mu_2)}$. The present paper and our previous work [**25, 58**] establish this fact. The Lambert curve is the mirror dual of the number of trees.

The organization of this paper is the following. In Section 2 we present the Eynard-Orantin recursion formalism for the case of a genus 0 spectral curve. Higher genus situations will be discussed elsewhere. Sections 3 and 4 deal with the counting problem of Grothendieck's dessins d'enfants. We present our new results on this problem, which are Theorem 1.1 and Theorem 1.2. We are inspired by Kodama's beautiful talk [**45**] (that is based on [**46**]) to come up with the generating function of the Catalan numbers as the spectral curve for this problem. We are grateful to G. Gliner for drawing our attention to [**45**]. The counting problem of the lattice points on $\mathcal{M}_{g,n}$ of [**11, 56, 60, 61**] is closely related to the counting of dessins, which is also treated in Section 4. The Eynard-Orantin recursion becomes identical to the Virasoro constraint condition for the ψ-class intersection numbers on $\overline{\mathcal{M}}_{g,n}$. We discuss this relation in Section 6, using Kontsevich's idea that the intersection numbers on $\overline{\mathcal{M}}_{g,n}$ are essentially the same as the Euclidean volume of $\mathcal{M}_{g,n}$. Section 7 is devoted to single Hurwitz numbers. In our earlier work [**25, 58**] we used the Lambert curve as given. Here we reexamine the Hurwitz counting problem and derive the Lambert curve from the unstable geometries. We then consider the Norbury-Scott conjecture [**63**] in Section 8, which states that the generating functions of stationary Gromov-Witten invariants of $\mathbb{P}^1$ satisfy the Eynard-Orantin recursion. We are unable to prove this conjecture. What we establish in this section is *why* the spectral curve of [**63**] is the right choice for this problem.

The subject of this paper is closely related to random matrix theory. Since the matrix model side of the story has been extensively discussed by the original authors [**28**], we do not deal with that aspect in the current paper.

2. The Eynard-Orantin differential forms and the topological recursion

We use the following mathematical definition for the topological recursion of Eynard-Orantin for a genus 0 spectral curve. The differences between our definition and the original formulation found in [**26, 28**] are philosophical in nature. Indeed, the original formula and ours produce the exact same answer in all examples we examine in this paper.

DEFINITION 2.1. We start with $\mathbb{P}^1$ with a choice of coordinate t. Let $S \subset \mathbb{P}^1$ be a finite collection of points and compact real curves such that the complement $\Sigma = \mathbb{P}^1 \setminus S$ is connected. The *spectral curve* of genus 0 is the data (Σ, π) consisting of a Riemann surface Σ and a simply ramified holomorphic map

$$\pi : \Sigma \ni t \longmapsto \pi(t) = x \in \mathbb{P}^1 \tag{2.1}$$

so that its differential dx has only simple zeros. Let us denote by $R = \{p_1, \dots, p_r\} \subset \Sigma$ the ramification points, and by

$$U = \sqcup_{j=1}^r U_j$$

the disjoint union of small neighborhood U_j around each p_j such that $\pi : U_j \to \pi(U_j) \subset \mathbb{P}^1$ is a double-sheeted covering ramified only at p_j. We denote by $\bar{t} = s(t)$ the local Galois conjugate of $t \in U_j$. The canonical sheaf of Σ is denoted by $\mathcal{K}$. Because of our choice of the coordinate t, we have a preferred basis dt for $\mathcal{K}$ and $\partial/\partial t$ for $\mathcal{K}^{-1}$. The meromorphic differential forms $W_{g,n}(t_1, \dots, t_n)$, $g = 0, 1, 2, \dots, n = 1, 2, 3, \dots$, are said to satisfy the **Eynard-Orantin topological recursion** if the following conditions are satisfied:

(1) $W_{0,1}(t) \in H^0(\Sigma, \mathcal{K})$.
(2) $W_{0,2}(t_1, t_2) = \frac{dt_1 \cdot dt_2}{(t_1 - t_2)^2} - \pi^* \frac{dx_1 \cdot dx_2}{(x_1 - x_2)^2} \in H^0(\Sigma \times \Sigma, \mathcal{K}^{\otimes 2}(2\Delta))$, where Δ is the diagonal of $\Sigma \times \Sigma$.
(3) The recursion kernel $K_j(t, t_1) \in H^0(U_j \times C, (\mathcal{K}_{U_j}^{-1} \otimes \mathcal{K})(\Delta))$ for $t \in U_j$ and $t_1 \in C$ is defined by

$$K_j(t, t_1) = \frac{1}{2} \frac{\int_t^{\bar{t}} W_{0,2}(\cdot, t_1)}{W_{0,1}(\bar{t}) - W_{0,1}(t)}. \tag{2.2}$$

The kernel is an algebraic operator that multiplies dt_1 while contracts dt.
(4) The general differential forms $W_{g,n}(t_1, \dots, t_n) \in H^0(\Sigma^n, \mathcal{K}(*R)^{\otimes n})$ are meromorphic symmetric differential forms with poles only at the ramification points R for $2g - 2 + n > 0$, and are given by the recursion formula

$$W_{g,n}(t_1, t_2, \dots, t_n) = \frac{1}{2\pi i} \sum_{j=1}^r \oint_{U_j} K_j(t, t_1) \Big[W_{g-1,n+1}(t, \bar{t}, t_2, \dots, t_n) \tag{2.3}$$

$$+ \sum_{\substack{g_1+g_2=g \\ I \sqcup J=\{2,3,\ldots,n\}}}^{\text{No } (0,1) \text{ terms}} W_{g_1,|I|+1}(t, t_I) W_{g_2,|J|+1}(\bar{t}, t_J) \Bigg].$$

Here the integration is taken with respect to $t \in U_j$ along a positively oriented simple closed loop around p_j, and $t_I = (t_i)_{i \in I}$ for a subset $I \subset \{1, 2, \ldots, n\}$.

(5) The differential form $W_{1,1}(t_1)$ requires a separate treatment because $W_{0,2}(t_1, t_2)$ is regular at the ramification points but has poles elsewhere.

$$\begin{aligned} W_{1,1}(t_1) &= \frac{1}{2\pi i} \sum_{j=1}^{r} \oint_{U_j} K_j(t, t_1) \left[W_{0,2}(u, v) + \pi^* \frac{dx(u) \cdot dx(v)}{(x(u) - x(v))^2} \right] \Bigg|_{\substack{u=t \\ v=\bar{t}}} \\ &= \frac{1}{2\pi i} \sum_{j=1}^{r} \oint_{U_j} K_j(t, t_1) \left[\frac{dt \cdot d\bar{t}}{(t - \bar{t})^2} \right]. \end{aligned} \tag{2.4}$$

Let $y : \Sigma \longrightarrow \mathbb{C}$ be a holomorphic function defined by the equation

$$W_{0,1}(t) = y(t)dx(t). \tag{2.5}$$

Equivalently, we can define the function by contraction $y = i_{\mathcal{X}} W_{0,1}$, where $\mathcal{X}$ is the vector field on Σ dual to $dx(t)$ with respect to the coordinate t. Then we have an embedding

$$\Sigma \ni t \longmapsto (x(t), y(t)) \in \mathbb{C}^2.$$

(6) If the spectral curve has at most two branch points then we choose a *preferred coordinate* t with the branch points located at $t = \infty$ and $t = 0$. This results in differentials $W_{g,n}$ that are Laurent polynomials in t and serves to simplify many of the residue calculations.

REMARK 2.2. The recursion (2.3) also applies to $(g, n) = (0, 3)$, which gives $W_{0,3}$ in terms of $W_{0,2}$. In [**26**, Theorem 4.1] an equivalent but often more useful formula for $W_{0,3}$ is given:

$$W_{0,3}(t_1, t_2, t_3) = \frac{1}{2\pi i} \sum_{j=1}^{r} \oint_{U_j} \frac{W_{0,2}(t, t_1) W_{0,2}(t, t_2) W_{0,2}(t, t_3)}{dx(t) \cdot dy(t)}. \tag{2.6}$$

The philosophy being presented is that given an A-model type counting problem, the spectral curve describing the mirror B-model invariants is obtained by taking the Laplace transform of the unstable geometries – the so called disk and annulus amplitudes of the A-model. The mechanism by which this occurs is illustrated by several examples in the subsequent sections.

3. Counting Grothendieck's dessins d'enfants

The A-model side of the problem we consider in this section is the counting problem of Grothendieck's *dessins d'enfants* (see for example, [**72**, **73**]) for a fixed topological type of Belyi morphisms [**3**]. We define functions $D_{g,n}(\mu_1, \ldots, \mu_n)$ which, in brief, are weighted counts of dessins d'enfants having n vertices of valence $\mu_1, \ldots, \mu_n$. We find the spectral curve (3.13) by taking the Laplace transform of the unstable functions $D_{0,1}$ and $D_{0,2}$. The section closes with the derivation of recursion equation (3.16) satisfied by $D_{g,n}$ which comes from studying the edge contraction operation on graphs.

One motivation for studying dessins d'enfant using the framework of Eynard-Orantin recursion is the following. Gromov-Witten theory of an algebraic variety X is an intersection theory of naturally defined cycles on the moduli stack $\overline{\mathcal{M}}_{g,n}(X)$ of stable morphisms from n-pointed algebraic curves of genus g to the target variety X. Since we are considering tautological cycles, their 0-dimensional intersection points are also *natural*. These points determine a finite set on $\overline{\mathcal{M}}_{g,n}$ via the stabilization morphism. If we expect that the Gromov-Witten theory of X satisfies the Eynard-Orantin recursion, then we should also expect that the counting problem of naturally defined finite sets of points on $\overline{\mathcal{M}}_{g,n}$ may satisfy the Eynard-Orantin recursion.

Pointed curves defined over $\overline{\mathbb{Q}}$ form a dense subset of $\overline{\mathcal{M}}_g$. Using the natural correspondence between curves defined over $\overline{\mathbb{Q}}$ and Belyi morphisms, we have marked points on each such curve coming from the branch points above ∞ of the morphism. By fixing the profiles over the branch points we arrive at a canonically defined finite set of points on $\overline{\mathcal{M}}_{g,n}$.

More specifically, consider a Belyi morphism

$$b : C \longrightarrow \mathbb{P}^1 \tag{3.1}$$

of a smooth algebraic curve C of genus g. This means b is branched only over $0, 1, \infty \in \mathbb{P}^1$. By Belyi's Theorem [**3**], C is defined over $\overline{\mathbb{Q}}$. Let $q_1, \dots, q_n$ be the poles of b with orders $(\mu_1, \dots, \mu_n) \in \mathbb{Z}_+^n$ respectively. This vector of positive integers is the *profile* of b at ∞. In our enumeration we *label* all poles of b. Therefore, an automorphism of a Belyi morphism preserves the set of poles point-wise.

A *clean* Belyi morphism is a special class of Belyi morphism of even degree that has profile $(2, 2, \dots, 2)$ over the branch point $1 \in \mathbb{P}^1$. We note that a complex algebraic curve is defined over $\overline{\mathbb{Q}}$ if and only if it admits a clean Belyi morphism. Let us denote by $D_{g,n}(\mu_1, \dots, \mu_n)$ the weighted count of the number of genus g clean Belyi morphisms of profile $(\mu_1, \dots, \mu_n)$ at $\infty \in \mathbb{P}^1$. This is the number we study in this section.

Grothendieck visualized the clean Belyi morphism by considering the inverse image

$$\Gamma = b^{-1}([0, 1]) \tag{3.2}$$

of the closed interval $[0, 1] \subset \mathbb{P}^1$ by b (see his "Esquisse d'un programme" reprinted in [**73**]). It is a topological graph drawn on the algebraic curve C being considered as a Riemann surface. We call each pre-image of $0 \in \mathbb{P}^1$ by b a *vertex* of Γ. Since b has profile $(2, \dots, 2)$ over $1 \in \mathbb{P}^1$, a pre-image of 1 is the midpoint of an *edge* of Γ. The complement $C \setminus \Gamma$ of Γ in C is the disjoint union of n disks centered at each q_i. By abuse of terminology we call each disk a *face* of Γ. Then by Euler's formula we have

$$2 - 2g = |b^{-1}(0)| - |b^{-1}(1)| + n.$$

The added structure obtained by its inclusion in an oriented surface make the graph into a ribbon graph. A *ribbon graph* of topological type (g, n) is the 1-skeleton of a cell-decomposition of a closed oriented topological surface C of genus g that decomposes the surface into a disjoint union of 0-cells, 1-cells, and 2-cells. The number of 2-cells is n. Alternatively, a ribbon graph can be defined as a graph with a cyclic order assigned to the incident half-edges at each vertex.

The concrete construction of [**55**] gives a Belyi morphism to any given ribbon graph. Thus the enumeration of clean Belyi morphism is equivalent to the enumeration of ribbon graphs. Grothendieck's original motivation for studying ribbon graphs lies in the fact that the absolute Galois group $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ acts faithfully on the set of ribbon graphs.

An alternative description of a Belyi morphism is to use the dual graph

$$\check{\Gamma} = b^{-1}([1, i\infty]), \tag{3.3}$$

where

$$[1, i\infty] = \{1 + iy \mid 0 \le y \le \infty\} \subset \mathbb{P}^1$$

is the vertical half-line on $\mathbb{P}^1$ with real part 1. This time the graph $\check{\Gamma}$ has n labeled vertices of degrees $(\mu_1, \ldots, \mu_n)$. Since we consider ribbon graphs in the context of canonical cell-decomposition of the moduli space $\mathcal{M}_{g,n}$, we use the terminology *dessin d'enfant* (or just dessin) for a graph $\check{\Gamma}$ dual to a ribbon graph Γ. This distinction is important, because when we count the number of ribbon graphs, we consider the automorphism of a graph that preserves each face, while the automorphism group of the dual graph, i.e., a dessin, preserves each vertex point-wise, but can permute faces. We note that this terminology is different from that presented in "Esquisse d'un programme," where ribbon graphs were referred to as dessins d'enfant (and dual graphs were not considered). In this dual picture, we define the number of dessins with the automorphism factor by

$$D_{g,n}(\mu_1, \ldots, \mu_n) = \sum_{\substack{\check{\Gamma} \text{ dessin of} \\ \text{type } (g,n)}} \frac{1}{|\operatorname{Aut}_D(\check{\Gamma})|}, \tag{3.4}$$

where $\check{\Gamma}$ is a dessin of genus g with n labeled vertices with prescribed degrees $(\mu_1, \ldots, \mu_n)$, and $\operatorname{Aut}_D(\check{\Gamma})$ is the automorphism of $\check{\Gamma}$ preserving each vertex point-wise.

Our theme is to find the spectral curve of the theory by looking at the problem for unstable curves $(g, n) = (0, 1)$ and $(0, 2)$. The dessins counted in $D_{0,1}(\mu)$ for an integer $\mu \in \mathbb{Z}_+$ are spherical graphs that contain only one vertex of degree μ. Since any edge of this graph has to start and end with the same vertex, it is a loop, and thus μ is even. So let us put $\mu = 2m$. Each graph contributes with the weight $1/|\operatorname{Aut}_D(\check{\Gamma})|$ in the enumeration of the number $D_{0,1}(\mu)$. This automorphism factor makes counting more difficult. Note that the automorphism group of a spherical dessin with a single vertex is a subgroup of $\mathbb{Z}/(2m)\mathbb{Z}$ that preserves the graph. If we place an outgoing arrow to one of the $2m$ half-edges incident to the unique vertex (see Figure 3.1), then we can kill the automorphism altogether. Since there are $2m$ choices of placing such an arrow, the number of arrowed graphs is $2mD_{0,1}(2m)$. This is now an integer. With one cyclically ordered vertex, placing an arrow on a half-edge is equivalent to choosing a total ordering for the half-edges which is consistent with the cyclic ordering. Pairing the half-edges to form a graph with non-intersecting edges is then equivalent to an arrangement of m pairs of parentheses. The number of such arrangements is given by

$$2mD_{0,1}(2m) = C_m = \frac{1}{m+1}\binom{2m}{m}, \tag{3.5}$$

where C_m is the m-th *Catalan number*. We note that the Catalan numbers appear in the same context of counting graphs in [**38**].

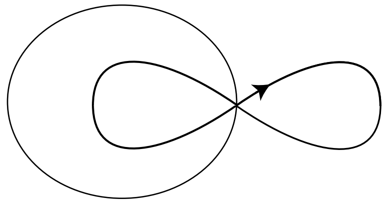

FIGURE 3.1. An arrowed dessin d'enfant of genus 0 with one vertex.

Define the Laplace transform of $D_{0,1}(\mu)$ by

$$\widetilde{F}^D_{0,1} = \sum_{m=1}^{\infty} D_{0,1}(2m)e^{-2mw}. \tag{3.6}$$

Then the Eynard-Orantin differential

$$\widetilde{W}^D_{0,1} = d\widetilde{F}^D_{0,1} = -\sum_{m=1}^{\infty} 2mD_{0,1}(2m)e^{-2mw}dw = -\sum_{m=1}^{\infty} C_m e^{-2mw}dw$$

is a generating function of the Catalan numbers. Actually a better choice is (see [**45**, **46**])

$$z(x) = \sum_{m=0}^{\infty} C_m \frac{1}{x^{2m+1}} = \frac{1}{x} + \frac{1}{x^3} + \frac{2}{x^5} + \frac{5}{x^7} + \frac{14}{x^9} + \frac{42}{x^{11}} + \cdots. \tag{3.7}$$

The radius of convergence of this infinite Laurent series is 2, hence the series converges absolutely for $|x| > 2$. The inverse function of $z = z(x)$ near $(x, z) = (\infty, 0)$ is given by

$$x = z + \frac{1}{z}. \tag{3.8}$$

This can be easily seen by solving the quadratic equation $z^2 - xz + 1 = 0$ with respect to z, which is equivalent to the quadratic recursion

$$C_{m+1} = \sum_{i+j=m} C_i \cdot C_j$$

of Catalan numbers. To take advantage of these simple formulas, let us *define*

$$x = e^w \tag{3.9}$$

and allow the $m = 0$ term in the Eynard-Orantin differential:

$$W^D_{0,1} = -\sum_{m=0}^{\infty} C_m \frac{dx}{x^{2m+1}}. \tag{3.10}$$

Accordingly the Laplace transform of $D_{0,1}(2m)$ needs to be modified:

$$F^D_{0,1} = \sum_{m=1}^{\infty} D_{0,1}(2m)\, e^{-2mw} - w = \sum_{m=1}^{\infty} D_{0,1}(2m)\, \frac{1}{x^{2m}} - \log x. \tag{3.11}$$

From (3.7) and (3.10), we obtain

$$W^D_{0,1} = -z(x)\, dx. \tag{3.12}$$

In light of (2.5), we have identified the spectral curve for the counting problem of dessins $D_{g,n}(\mu)$. It is given by

$$\begin{cases} x = z + \frac{1}{z} \\ y = -z \end{cases} . \tag{3.13}$$

We note that the spectral curve has branch points at $z = \pm 1$, hence we introduce our preferred coordinate t through the equation

$$z = \frac{t+1}{t-1}, \tag{3.14}$$

which will simplify the residue calculations in Section 4 and Appendix A.

To compute the recursion kernel of (2.2), we need to identify $D_{0,2}(\mu_1, \mu_2)$ for the other unstable geometry $(g, n) = (0, 2)$. In the dual graph picture, $D_{0,2}(\mu_1, \mu_2)$ counts the number of spherical dessins $\check{\Gamma}$ with two vertices of degree μ_1 and μ_2, counted with the weight of $1/|\operatorname{Aut}_D(\check{\Gamma})|$. The computation was done by Kodama and Pierce in [**46**, Theorem 3.1]. We also refer to a beautiful lecture by Kodama [**45**].

PROPOSITION 3.1 ([**46**]). *The number of connected spherical dessins $\check{\Gamma}$ with two vertices of degrees j and k, counted with the weight of $1/|\operatorname{Aut}_D(\check{\Gamma})|$, is given by the following formula.*

$$D_{0,2}(\mu_1, \mu_2) = \begin{cases} \frac{1}{4}\, \frac{1}{j+k} \binom{2j}{j}\binom{2k}{k} & \mu_1 = 2j \neq 0, \mu_2 = 2k \neq 0 \\ \\ \frac{1}{j+k+1} \binom{2j}{j}\binom{2k}{k} & \mu_1 = 2j+1, \mu_2 = 2k+1 \end{cases} . \tag{3.15}$$

In all other cases with $\mu_i > 0$, $D_{0,2}(\mu_1, \mu_2) = 0$. Here the automorphism group $\operatorname{Aut}_D(\check{\Gamma})$ is the topological graph automorphisms that fix each vertex, but may permute faces.

The number of dessins satisfies the following:

THEOREM 3.2. *For $g \geq 0$ and $n \geq 1$ subject to $2g - 2 + n \geq 0$, the number of dessins (3.4) satisfies the equation*

$$\mu_1 D_{g,n}(\mu_1, \dots, \mu_n) = \sum_{j=2}^{n} (\mu_1 + \mu_j - 2) D_{g,n-1}\big(\mu_1 + \mu_j - 2, \mu_{[n]\setminus\{1,j\}}\big) \tag{3.16}$$

$$+ \sum_{\alpha+\beta=\mu_1-2} \alpha\beta \left[D_{g-1,n+1}(\alpha, \beta, \mu_{[n]\setminus\{1\}}) + \sum_{\substack{g_1+g_2=g \\ I \sqcup J = \{2,\dots,n\}}} D_{g_1,|I|+1}(\alpha, \mu_I) D_{g_2,|J|+1}(\beta, \mu_J) \right],$$

where $\mu_I = (\mu_i)_{i\in I}$ for a subset $I \subset [n] = \{1, 2, \dots, n\}$. The last sum is over all partitions of the genus g and the index set $\{2, 3, \dots, n\}$ into two pieces.

REMARK 3.3. Note that when $g_1 = 0$ and $I = \emptyset$, $D_{g,n}$ appears in the right-hand side of (3.16). Therefore, this is an equation of the number of dessins, not a recursion formula.

PROOF. Consider the collection of genus g dessins with n vertices labeled by the index set $[n] = \{1, 2, \dots, n\}$ and of degrees $(\mu_1, \dots, \mu_n)$. The left-hand side of (3.16) is the number of dessins with an outward arrow placed on one of the incident edges at the vertex 1. The equation is based on contracting this edge to a point. There are two cases.

CASE 1. The arrowed edge connects the vertex 1 and vertex $j > 1$. We then contract the edge and put the two vertices 1 and j together as shown in Figure 3.2. The resulting dessin has one less vertex, but the genus is the same as before. The degree of the newly created vertex is $\mu_1 + \mu_j - 2$, while the degrees of all other vertices are unaffected. It is natural to mark the edge that was immediately counterclockwise of the contracted edge, as indicated in Figure 3.2.

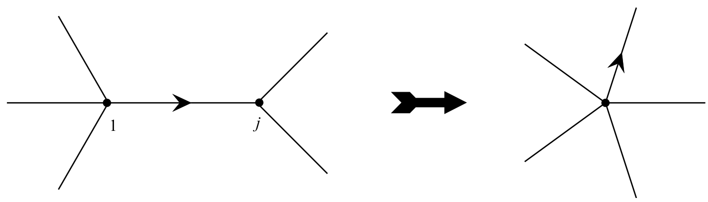

FIGURE 3.2. The operation that shrinks the arrowed edge to a point and joins two vertices labeled by 1 and j together.

To make the bijection argument, we need to be able to reconstruct the original dessin from the new one. Since both μ_1 and μ_j are given as the input value, we have to specify which edges go to vertex 1 and which go to j when we separate the vertex of degree $\mu_1 + \mu_j - 2$. For this purpose, what we need is a marker on one of the incident edges. We group the marked edge and $\mu_i - 2$ edges following it according to the cyclic order. The rest of the $\mu_j - 1$ incident edges are also grouped. Then we insert an edge and separate the vertex into two vertices, 1 and j, so that the first group of edges are incident to vertex 1 and the second group is incident to j, honoring their cyclic orders. In other words, the arrow in Figure 3.2 can be reversed. The contribution from this case is therefore

$$\sum_{j=2}^{n}(\mu_1 + \mu_j - 2)D_{g,n-1}\big(\mu_1 + \mu_j - 2, \mu_{[n]\setminus\{1,j\}}\big).$$

CASE 2. The arrowed edge forms a loop that is attached to vertex 1. We remove this loop from the dessin, and separate the vertex into two vertices. The loop classifies all incident half-edges, except for the loop itself, into two groups: the ones that follow the arrowed half-edge in the cyclic order but before the incoming end of the loop, and all others (see Figure 3.3). Let α be the number of half-edges in the first group, and β the rest. Then $\alpha + \beta = \mu_1 - 2$, and we have created two vertices of degrees α and β.

To recover the original dessin from the new one, we need to mark a half-edge from each vertex so that we can put the loop back to the original place. The number of choices of these markings is $\alpha\beta$.

The operation of the removal of the loop and the separation of the vertex into two vertices certainly increases the number of vertices from n to $n + 1$. This operation also affects the genus of the dessin. If the resulting dessin is connected, then g goes down to $g - 1$. If the result is the disjoint union of two dessins of genera

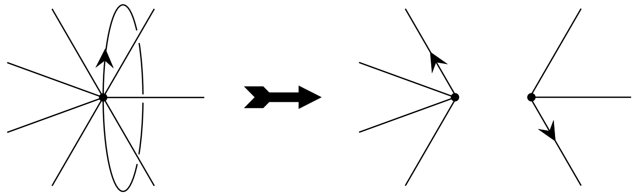

FIGURE 3.3. The operation that removes a loop, and separates the incident vertex into two vertices.

g_1 and g_2, then we have $g = g_1 + g_2$. Altogether the contribution from this case is

$$\sum_{\alpha+\beta=\mu_1-2} \alpha\beta \left[D_{g-1,n+1}(\alpha, \beta, \mu_{[n]\setminus\{1\}}) + \sum_{\substack{g_1+g_2=g \\ I \sqcup J=\{2,\dots,n\}}} D_{g_1,|I|+1}(\alpha, \mu_I) D_{g_2,|J|+1}(\beta, \mu_J) \right].$$

Note that the outward arrow we place defines the two groups of incident half-edges uniquely, since one is after and the other before the arrowed half-edge according to the cyclic order. Thus we do not need to symmetrize α and β. Indeed, if the arrow is placed in the other end of the loop, then α and β are interchanged.

The right-hand side of the equation (3.16) is the sum of the above two contributions. □

REMARK 3.4. The equation (3.16) is considerably simpler, compared to the recursion formula for the number of ribbon graphs with integral edge lengths that is proved in [**11**, Theorem 3.3]. The edge removal operation of [**11**] is the dual operation of the edge shrinking operations of Case 1 and Case 2 above, and the placement of an arrow corresponds to the *ciliation* of [**11**]. In the dual picture, the graphs enumerated in [**11**] are more restrictive than arbitrary clean dessins, which makes the equation more complicated. We also note that [**11**, Theorem 3.3] is a recursion formula, not just a mere relation like what we have in (3.16). In this regard, (3.16) is indeed similar to the cut-and-join equation (7.28) of [**30**, **78**]. We will come back to this point in Section 7.

The relation (3.16) becomes an effective recursion formula after taking the Laplace transform.

4. The Laplace transform of the number of dessins

In this section we derive the Eynard-Orantin recursion formula for the generating functions of the number of dessins. The key technique is the Laplace transform.

Note that recursion equation (3.16) does not provide an effective recursion formula, because $D_{g,n}(\mu_1, \dots, \mu_n)$ appears in the equation in a complicated manner. Our strategy is to compute the Laplace transform

$$F^D_{g,n}(w_1, \dots, w_n) = \sum_{\mu_1,\dots,\mu_n>0} D_{g,n}(\mu_1, \dots, \mu_n)\, e^{-(\mu_1 w_1 + \cdots + \mu_n w_n)},$$

and rewrite the recursion equation in terms of the Laplace transformed functions. We then show that the symmetric differential forms

$$W^D_{g,n} = d_1 \cdots d_n F^D_{g,n}$$

satisfy the Eynard-Orantin recursion formula. This time it is an effective recursion formula for the generating functions $W^D_{g,n}$ of the number $D_{g,n}(\mu_1,\dots,\mu_n)$ of clean Belyi morphisms.

Since the projection $x = z+1/z$ of the spectral curve to the x-coordinate plane has two ramification points $z = \pm 1$, it is natural to introduce a coordinate that has these ramification points at 0 and ∞. So we define

$$z = \frac{t+1}{t-1}. \tag{4.1}$$

PROPOSITION 4.1. *The Laplace transform of $D_{0,2}(\mu_1,\mu_2)$ is given by*

$$\begin{aligned} F^D_{0,2}(t_1,t_2) &\overset{\text{def}}{=} \sum_{\mu_1,\mu_2>0} D_{0,2}(\mu_1,\mu_2)\, e^{-(\mu_1 w_1+\mu_2 w_2)} = -\log\big(1 - z(x_1)z(x_2)\big) \\ &= \log(t_1-1)+\log(t_2-1)-\log(-2(t_1+t_2)), \end{aligned} \tag{4.2}$$

where $z(x)$ is the generating function of the Catalan numbers (3.7), and the variables t, w, x, z are related by (3.9), (3.13), and (4.1). We then have

$$W^D_{0,2}(t_1,t_2) = d_1 d_2 F^D_{0,2}(t_1,t_2) = \frac{dt_1\cdot dt_2}{(t_1-t_2)^2} - \frac{dx_1\cdot dx_2}{(x_1-x_2)^2} = \frac{dt_1\cdot dt_2}{(t_1+t_2)^2}. \tag{4.3}$$

PROOF. In terms of $x = e^w$, the Laplace transform (4.2) is given by

$$\begin{aligned} &\sum_{\mu_1,\mu_2>0} D_{0,2}(\mu_1,\mu_2)\, e^{-(\mu_1 w_1+\mu_2 w_2)} \\ &= \frac{1}{4}\sum_{j,k=1}^{\infty} \frac{1}{j+k}\binom{2j}{j}\binom{2k}{k}\frac{1}{x_1^{2j}}\,\frac{1}{x_2^{2k}} + \sum_{j,k=0}^{\infty}\frac{1}{j+k+1}\binom{2j}{j}\binom{2k}{k}\frac{1}{x_1^{2j+1}}\,\frac{1}{x_2^{2k+1}}. \end{aligned} \tag{4.4}$$

Since

$$dx = \left(1-\frac{1}{z^2}\right) dz, \tag{4.5}$$

we have

$$x\frac{d}{dx} = \frac{z+\frac{1}{z}}{1-\frac{1}{z^2}}\,\frac{d}{dz} = \frac{z(z^2+1)}{z^2-1}\,\frac{d}{dz}. \tag{4.6}$$

To make the computation simpler, let us introduce

$$\xi_0(x) = \sum_{m=0}^{\infty}\binom{2m}{m}\frac{1}{x^{2m+1}}. \tag{4.7}$$

This will also be used in Section 8. In terms of z and t we have

$$\begin{aligned} \xi_0(x) &= \frac{1}{2}\left(1 - x\frac{d}{dx}\right)\sum_{m=0}^{\infty}\frac{1}{m+1}\binom{2m}{m}\frac{1}{x^{2m+1}} \\ &= \frac{1}{2}\left(1-\frac{z(z^2+1)}{z^2-1}\,\frac{d}{dz}\right) z = -\frac{z}{z^2-1} = -\frac{t^2-1}{4t}. \end{aligned} \tag{4.8}$$

Note that

$$-\left(x_1\frac{d}{dx_1}+x_2\frac{d}{dx_2}\right)\left(\frac{1}{4}\sum_{j,k=1}^{\infty}\frac{1}{j+k}\binom{2j}{j}\binom{2k}{k}\frac{1}{x_1^{2j}}\,\frac{1}{x_2^{2k}}\right.$$

$$+\sum_{j,k=0}^{\infty}\frac{1}{j+k+1}\binom{2j}{j}\binom{2k}{k}\frac{1}{x_1^{2j+1}}\frac{1}{x_2^{2k+1}}\Bigg)$$

$$=\frac{1}{2}\,(x_1\xi_0(x_1)-1)(x_2\xi_0(x_2)-1)+2\xi_0(x_1)\xi_0(x_2)$$

$$=2z_1z_2\frac{1+z_1z_2}{(z_1^1-1)(z_2^2-1)}$$

$$=-\left(\frac{z_1(z_1^2+1)}{z_1^2-1}\frac{d}{dz_1}+\frac{z_2(z_2^2+1)}{z_2^2-1}\frac{d}{dz_2}\right)(-\log(1-z_1z_2))\,.$$

In other words, we have a partial differential equation

$$\left(x_1\frac{d}{dx_1}+x_2\frac{d}{dx_2}\right)\big(F_{0,2}^D(t_1,t_2)+\log(1-z_1z_2)\big)=0$$

for a holomorphic function in x_1 and x_2 defined for $|x_1|\gg 2$ and $|x_2|\gg 2$. Since the first few terms of the Laurent expansions of $-\log\big(1-z(x_1)z(x_2)\big)$ using (3.7) agree with the first few terms of the sums of (4.4), we have the initial condition for the above differential equation. By the uniqueness of the solution to the Euler differential equation with the initial condition, we obtain (4.2). Equation (4.3) follows from differentiation of (4.2). □

In terms of the t-coordinate of (4.1), the Galois conjugate of $t\in\Sigma$ under the projection $x:\Sigma\longrightarrow\mathbb{C}$ is $-t$. Therefore, the recursion kernel for counting of dessins is given by

(4.9)

$$K^D(t,t_1)=\frac{1}{2}\frac{\int_t^{-t}W_{0,2}^D(\cdot,t_1)}{W_{0,1}^D(-t)-W_{0,1}^D(t)}=\frac{1}{2}\left(\frac{1}{t+t_1}+\frac{1}{t-t_1}\right)\frac{1}{\frac{t+1}{t-1}-\frac{t-1}{t+1}}\cdot\frac{1}{dx}\cdot dt_1$$

$$=-\frac{1}{64}\left(\frac{1}{t+t_1}+\frac{1}{t-t_1}\right)\frac{(t^2-1)^3}{t^2}\cdot\frac{1}{dt}\cdot dt_1.$$

One of the first two stable cases (2.4) gives us

$$(4.10)\quad W_{1,1}^D(t_1)=\frac{1}{2\pi i}\int_\gamma K^D(t,t_1)\left[W_{0,2}^D(t,-t)+\frac{dx\cdot dx_1}{(x-x_1)^2}\right]$$

$$=-\frac{1}{2\pi i}\int_\gamma K^D(t,t_1)\frac{dt\cdot dt}{4t^2}=-\frac{1}{128}\,\frac{(t_1^2-1)^3}{t_1^4}dt_1,$$

where the integration contour γ consists of two concentric circles of a small radius and a large radius centered around $t=0$, with the inner circle positively and the outer circle negatively oriented (Figure 4.1). The $(g,n)=(0,3)$ case is given by

$$(4.11)\quad W_{0,3}^D(t_1,t_2,t_3)=\frac{1}{2\pi i}\int_\gamma\frac{W_{0,2}^D(t,t_1)W_{0,2}^D(t,t_2)W_{0,2}^D(t,t_3)}{dx(t)\cdot dy(t)}$$

$$=-\frac{1}{16}\left[\frac{1}{2\pi i}\int_\gamma\frac{(t^2-1)^2(t-1)^2}{(t+t_1)^2(t+t_2)^2(t+t_3)^2}\cdot\frac{dt}{t}\right]dt_1dt_2dt_3$$

$$=-\frac{1}{16}\left(1-\frac{1}{t_1^2\,t_2^2\,t_3^2}\right)dt_1dt_2dt_3.$$

REMARK 4.2. The general formula (2.3) for $(g,n)=(0,3)$ also gives the same answer. This is because $W^D_{0,2}$ acts as the Cauchy differentiation kernel.

$$W^D_{0,3}(t_1,t_2,t_3) = \frac{1}{2\pi i}\int_\gamma K^D(t,t_1)\Big[W^D_{0,2}(t,t_2)W^D_{0,2}(-t,t_3)+W^D_{0,2}(t,t_3)W^D_{0,2}(-t,t_2)\Big]$$
$$= \frac{1}{64}\left[\frac{1}{2\pi i}\int_\gamma\left(\frac{1}{t+t_1}+\frac{1}{t-t_1}\right)\frac{(t^2-1)^3}{t^2}\left(\frac{1}{(t+t_2)^2(t-t_3)^2}+\frac{1}{(t+t_3)^2(t-t_2)^2}\right)dt\right] \cdot dt_1dt_2dt_3$$
$$= \Bigg[-\frac{1}{32}\frac{(t_1^2-1)^3}{t_1^2}\left(\frac{1}{(t_1+t_2)^2(t_1-t_3)^2}+\frac{1}{(t_1+t_3)^2(t_1-t_2)^2}\right) - \frac{1}{16}\frac{\partial}{\partial t_2}\left(\frac{t_2}{t_2^2-t_1^2}\frac{(t_2^2-1)^3}{t_2^2}\frac{1}{(t_2+t_3)^2}\right)$$
$$-\frac{1}{16}\frac{\partial}{\partial t_3}\left(\frac{t_3}{t_3^2-t_1^2}\frac{(t_3^2-1)^3}{t_3^2}\frac{1}{(t_2+t_3)^2}\right)\Bigg]dt_1dt_2dt_3 = -\frac{1}{16}\left(1-\frac{1}{t_1^2\,t_2^2\,t_3^2}\right)dt_1dt_2dt_3.$$

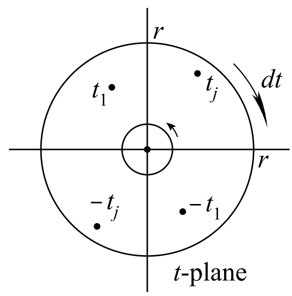

FIGURE 4.1. The integration contour γ. This contour encloses an annulus bounded by two concentric circles centered at the origin. The outer one has a large radius $r > \max_{j\in N}|t_j|$ and the negative orientation, and the inner one has an infinitesimally small radius with the positive orientation.

THEOREM 4.3. *Let us define the Laplace transform of the number of Grothendieck's dessins by*

$$F^D_{g,n}(t_1,\dots,t_n) = \sum_{\mu\in\mathbb{Z}^n_+} D_{g,n}(\mu)e^{-(\mu_1w_1+\cdots+\mu_nw_n)}, \tag{4.12}$$

where the coordinate t_i is related to the Laplace conjugate coordinate w_j by

$$e^{w_j} = \frac{t_j+1}{t_j-1}+\frac{t_j-1}{t_j+1}.$$

Then the differential forms

$$W^D_{g,n}(t_1,\dots,t_n) = d_1\cdots d_nF^D_{g,n}(t_1,\dots,t_n) \tag{4.13}$$

satisfy the Eynard-Orantin topological recursion

$$W^D_{g,n}(t_1,\dots,t_n) = -\frac{1}{64}\frac{1}{2\pi i}\int_\gamma\left(\frac{1}{t+t_1}+\frac{1}{t-t_1}\right)\frac{(t^2-1)^3}{t^2}\cdot\frac{1}{dt}\cdot dt_1 \tag{4.14}$$

$$\times\Bigg[\sum_{j=2}^{n}\Big(W^D_{0,2}(t,t_j)W_{g,n-1}(-t,t_2,\dots,\widehat{t_j},\dots,t_n)+W^D_{0,2}(-t,t_j)W_{g,n-1}(t,t_2,\dots,\widehat{t_j},\dots,t_n)\Big)$$
$$+W^D_{g-1,n+1}(t,-t,t_2,\dots,t_n)+\sum_{\substack{g_1+g_2=g\\ I\sqcup J=\{2,3,\dots,n\}}}^{stable} W^D_{g_1,|I|+1}(t,t_I)W^D_{g_2,|J|+1}(-t,t_J)\Bigg].$$

The last sum is restricted to the stable geometries. In other words, the partition should satisfy $2g_1-1+|I|>0$ and $2g_2-1+|J|>0$. The spectral curve Σ of the Eynard-Orantin recursion is given by

$$\begin{cases} x = z+\frac{1}{z}\\ y=-z\end{cases}$$

with the preferred coordinate t given by

$$t=\frac{z+1}{z-1}.$$

We give the proof of this theorem in the appendix.

5. Counting lattice points in moduli spaces of curves

The problem of counting dessins is closely related to the counting problem of the lattice points of the moduli space $\mathcal{M}_{g,n}$ of smooth n-pointed algebraic curves of genus g studied in [**60**, **61**]. Let us briefly recall the combinatorial model for the moduli space $\mathcal{M}_{g,n}$ due to Thurston (see for example, [**74**]), Harer [**37**], Mumford [**59**], and Strebel [**76**], following [**55**, **56**]. For a given ribbon graph Γ with $e=e(\Gamma)$ edges, the space of metric ribbon graphs is $\mathbb{R}_+^{e(\Gamma)}/\operatorname{Aut}(\Gamma)$, where the automorphism group acts by permutations of edges (see [**55**, Section 1]). When we consider *ribbon graph automorphisms*, we restrict ourselves to automorphisms that fix each 2-cell of the cell-decomposition. We also require that every vertex of a ribbon graph has degree 3 or more. Using the canonical holomorphic coordinate system on a topological surface of [**55**, Section 4] corresponding to a metric ribbon graph, realized through Strebel differentials [**76**], we have an isomorphism of topological orbifolds [**37**, **59**]

$$\mathcal{M}_{g,n}\times\mathbb{R}_+^n\cong R_{g,n} \tag{5.1}$$

for (g,n) in the stable range. Here

$$R_{g,n}=\coprod_{\substack{\Gamma \text{ boundary labeled}\\ \text{ribbon graph}\\ \text{of type } (g,n)}}\frac{\mathbb{R}_+^{e(\Gamma)}}{\operatorname{Aut}(\Gamma)}$$

is an orbifold parametrizing metric ribbon graphs of a given topological type (g,n). The gluing of orbi-cells is done by making the length of a non-loop edge tend to 0. The space $R_{g,n}$ is a smooth orbifold (see [**55**, Section 3] and [**74**]). We denote by $\pi: R_{g,n}\longrightarrow\mathbb{R}_+^n$ the natural projection via (5.1), which is the assignment of the perimeter length of each boundary to a given metric ribbon graph.

Take a boundary labeled ribbon graph Γ, with labels chosen from $[n]=\{1,2\dots,n\}$. For the moment let us give a label to each edge of Γ by an index set $[e]=\{1,2,\dots,e\}$. The edge-face incidence matrix is defined by

$$
\begin{aligned}
A_\Gamma &= \big[a_{i\eta}\big]_{i\in[n],\ \eta\in[e]};\\
a_{i\eta} &= \text{ the number of times edge } \eta \text{ appears in face } i.
\end{aligned}
\tag{5.2}
$$

Thus $a_{i\eta}=0,1$, or 2, and the sum of the entries in each column is always 2. The Γ contribution of the space $\pi^{-1}(\mu_1,\dots,\mu_n)=R_{g,n}(\mu)$ of metric ribbon graphs with a prescribed perimeter $\mu=(\mu_1,\dots,\mu_n)\in\mathbb{R}_+^n$ is the orbifold polytope

$$\frac{\{\mathbf{x}\in\mathbb{R}_+^e \mid A_\Gamma\mathbf{x}=\mu\}}{\mathrm{Aut}(\Gamma)},$$

where $\mathbf{x}=(\ell_1,\dots,\ell_e)$ is the collection of edge lengths of the metric ribbon graph Γ. We have

$$\sum_{i\in[n]}\mu_i=\sum_{i\in[n]}\sum_{\eta\in[e]}a_{i\eta}\ell_\eta=2\sum_{\eta\in[e]}\ell_\eta. \tag{5.3}$$

Now let $\mu\in\mathbb{Z}_+^n$ be a vector consisting of positive integers. The lattice point counting function we consider is defined by

$$N_{g,n}(\mu)=\sum_{\substack{\Gamma \text{ ribbon graph}\\ \text{of type } (g,n)}}\frac{\big|\{\mathbf{x}\in\mathbb{Z}_+^n \mid A_\Gamma\mathbf{x}=\mu\}\big|}{|\mathrm{Aut}(\Gamma)|} \tag{5.4}$$

for (g,n) in the stable range (**[11, 56, 60, 61]**).

To find the spectral curve for lattice point counting, we need to identify the unstable moduli $\mathcal{M}_{0,1}$ and the ribbon graph space $R_{0,1}$. We recall that the orbifold isomorphism (5.1) holds for (g,n) in the stable range by *defining* $R_{g,n}$ as the space of metric ribbon graphs of type (g,n) *without* vertices of degrees 1 and 2. For $(g,n)=(0,1)$, there are no ribbon graphs satisfying these conditions. Let v_j denote the number of degree j vertices in a ribbon graph Γ of type (g,n). Then we have

$$\sum_{j\ge1}jv_j=2e,\qquad \sum_{j\ge1}v_j=v,$$

where v is the total number of vertices of Γ. Hence

$$2(2g-2+n)=2e-2v=\sum_{j\ge1}(j-2)v_j=-v_1+\sum_{j\ge3}(j-2)v_j. \tag{5.5}$$

It follows that the number of degree 1 vertices v_1 is positive when $(g,n)=(0,1)$. In other words, $N_{0,1}(\mu)=0$. Thus we conclude that *there is no spectral curve for this counting problem.*

Still we can consider the Laplace transform of the number (5.4) of lattice points of the moduli space $\mathcal{M}_{g,n}$ with a prescribed perimeter length. We define for every stable (g,n)

$$F^L_{g,n}(t_1,\dots,t_n)=\sum_{\mu\in\mathbb{Z}_+^n}N_{g,n}(\mu)\prod_{i=1}^n\frac{1}{z_i^{\mu_i}}, \tag{5.6}$$

where

$$z=\frac{t+1}{t-1},$$

and the Eynard-Orantin differential forms by

$$W^L_{g,n}(t_1,\dots,t_n)=d_1\cdots d_nF^L_{g,n}(t_1,\dots,t_n). \tag{5.7}$$

The following result is proven in [**11**], with inspiration from [**61**].

THEOREM 5.1 ([**11**]). *The differential forms $W^L_{g,n}(t_1,\dots,t_n)$ satisfy the Eynard-Orantin topological recursion with respect to the same spectral curve (3.13) and the recursion kernel (4.9) as the dessins counting problem, starting with exactly the same first two stable cases*

$$W^L_{1,1}(t_1) = -\frac{1}{128}\frac{(t_1^2-1)^3}{t_1^4}dt_1, \tag{5.8}$$

and

$$W^L_{0,3}(t_1,t_2,t_3) = -\frac{1}{16}\left(1-\frac{1}{t_1^2\,t_2^2\,t_3^2}\right)dt_1dt_2dt_3. \tag{5.9}$$

REMARK 5.2. It is somewhat surprising, because the spectral curve (3.13) has nothing to do with the lattice point counting problem. As we have mentioned, the $(g,n)=(0,1)$ and $(0,2)$ considerations for this problem do not produce the spectral curve. This example illustrates that our philosophy is only a partial understanding of Eynard-Orantin recursion, and a different approach is needed for A-model invariants which do not have unstable geometric information.

In the next section, we will be studying Eynard-Orantin recursion for ψ-class intersections on moduli spaces of stable curves. The spectral curve can be obtained by a scaling limit from the lattice-point counting curve, with the link being provided by the following theorem, which was established in [**56**].

THEOREM 5.3 ([**56**]). *The functions $F^L_{g,n}(t_1,\dots,t_n)$ of (5.6) for the stable range $2g-2+n>0$ are uniquely determined by the following differential recursion formula from the initial values $F^L_{0,3}(t_1,t_2,t_3)$ and $F^L_{1,1}(t_1)$.*

$$\begin{aligned}
&F^L_{g,n}(t_1,\dots,t_n) \\
&= -\frac{1}{16}\int_{-1}^{t_1}\Bigg[\sum_{j=2}^n \frac{t_j}{t^2-t_j^2}\left(\frac{(t^2-1)^3}{t^2}\frac{\partial}{\partial t}F^L_{g,n-1}(t,t_{[n]\setminus\{1,j\}}) - \frac{(t_j^2-1)^3}{t_j^2}\frac{\partial}{\partial t_j}F^L_{g,n-1}(t_{[n]\setminus\{1\}})\right) \\
&\qquad + \sum_{j=2}^n \frac{(t^2-1)^2}{t^2}\frac{\partial}{\partial t}F^L_{g,n-1}(t,t_{[n]\setminus\{1,j\}}) \\
&\qquad + \frac{1}{2}\frac{(t^2-1)^3}{t^2}\frac{\partial^2}{\partial u_1\partial u_2}\Bigg(F^L_{g-1,n+1}(u_1,u_2,t_{[n]\setminus\{1\}}) \\
&\qquad + \sum_{\substack{g_1+g_2=g\\ I\sqcup J=[n]\setminus\{1\}}}^{\text{stable}} F^L_{g_1,|I|+1}(u_1,t_I)F_{g_2,|J|+1}(u_2,t_J)\Bigg)\Bigg|_{u_1=u_2=t}\Bigg]dt.
\end{aligned} \tag{5.10}$$

Here $[n]=\{1,2,\dots,n\}$ is an index set, and the last sum is taken over all partitions $g_1+g_2=g$ and set partitions $I\sqcup J=[n]\setminus\{1\}$ subject to the stability conditions $2g_1-1+|I|>0$ and $2g_2-1+|J|>0$. The initial values are given by

$$F^L_{1,1}(t_1) = -\frac{1}{384}\frac{(t+1)^4}{t^2}\left(t-4+\frac{1}{t}\right) \tag{5.11}$$

and

$$F^L_{0,3}(t_1,t_2,t_3) = -\frac{1}{16}(t_1+1)(t_2+1)(t_3+1)\left(1+\frac{1}{t_1\,t_2\,t_3}\right). \tag{5.12}$$

In the stable range $F^L_{g,n}(t_1,\dots,t_n)$ *is a Laurent polynomial of degree* $3(2n-2+n)$ *and satisfies the reciprocity relation*

$$F^L_{g,n}(1/t_1,\dots,1/t_n) = F^L_{g,n}(t_1,\dots,t_n). \tag{5.13}$$

The leading terms of $F^L_{g,n}(t_1,\dots,t_n)$ *form a homogeneous polynomial of degree* $3(2g-2+n)$, *and is given by*

(5.14)

$$F^K_{g,n}(t_1,\dots,t_n) \overset{def}{=} \frac{(-1)^n}{2^{2g-2+n}} \sum_{\substack{d_1+\cdots+d_n\\=3g-3+n}} \langle\tau_{d_1}\cdots\tau_{d_n}\rangle_{g,n} \prod_{j=1}^n (2d_j-1)!! \left(\frac{t_j}{2}\right)^{2d_j+1},$$

where

$$\langle\tau_{d_1}\cdots\tau_{d_n}\rangle_{g,n} = \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1}\cdots\psi_n^{d_n}$$

is the ψ*-class intersection number (see Section 7 for more detail about intersection numbers). The special value at* $t_i=1$ *gives*

$$F^L_{g,n}(1,1,\dots,1) = (-1)^n\chi(\mathcal{M}_{g,n}). \tag{5.15}$$

Corollary 5.4. *For every* (g,n) *with* $2g-2+n>0$, *we have the identity*

$$W^D_{g,n}(t_1,\dots,t_n) = W^L_{g,n}(t_1,\dots,t_n). \tag{5.16}$$

The differential form $W^D_{g,n}(t_1,\dots,t_n)$ *is a Laurent polynomial in* $t_1^2,\dots,t_n^2$ *of degree* $2(3g-3+n)$, *with a reciprocity property*

$$W^D_{g,n}(1/t_1,\dots,1/t_n) = (-1)^n t_1^2\cdots t_n^2\, W^D_{g,n}(t_1,\dots,t_n). \tag{5.17}$$

The numbers of dessins can be expressed in terms of the number of lattice points:

(5.18)

$$D_{g,n}(\mu_1,\dots,\mu_n) = \sum_{\ell_1>\frac{\mu_1}{2}}\cdots\sum_{\ell_n>\frac{\mu_n}{2}} \prod_{i=1}^n \frac{2\ell_i-\mu_i}{\mu_i}\binom{\mu_i}{\ell_i} N_{g,n}(2\ell_1-\mu_1,\cdots,2\ell_n-\mu_n).$$

Remark 5.5. The relation (5.18) appears in [**63**, Section 2.1] in an abstract setting.

Proof. The Eynard-Orantin topological recursion uniquely determines the differential forms for all (g,n). Since $W^D_{1,1}(t) = W^L_{1,1}(t)$ and $W^D_{0,3}(t_1,t_2,t_3) = W^L_{0,3}(t_1,t_2,t_3)$, we conclude that $W^D_{g,n}(t_1,\dots,t_n) = W^L_{g,n}(t_1,\dots,t_n)$ for $2g-2+n > 0$.

By induction on $2g-2+n$ we can show that $W^D_{g,n}(t_1,\dots,t_n)$ is a Laurent polynomial in $t_1^2,\dots,t_n^2$. The statement is true for the initial cases (4.10) and (4.11). The integral transformation formula (4.14) is a residue calculation at $t=0$ and $t=\infty$. By the induction hypothesis, the right-hand side of (4.14) becomes

$$\begin{aligned}
&-\frac{1}{64}\,\frac{1}{2\pi i}\int_\gamma \left(\frac{1}{t+t_1}+\frac{1}{t-t_1}\right)\frac{(t^2-1)^3}{t^2}\cdot\frac{1}{dt}\cdot dt_1\\
&\times\Bigg[\sum_{j=2}^n\Big(W^D_{0,2}(t,t_j)W_{g,n-1}(-t,t_2,\dots,\widehat{t_j},\dots,t_n)+W^D_{0,2}(-t,t_j)W_{g,n-1}(t,t_2,\dots,\widehat{t_j},\dots,t_n)\Big)\\
&\qquad + W^D_{g-1,n+1}(t,-t,t_2,\dots,t_n) + \sum_{\substack{g_1+g_2=g\\ I\sqcup J=\{2,3,\dots,n\}}}^{\text{stable}} W^D_{g_1,|I|+1}(t,t_I)W^D_{g_2,|J|+1}(-t,t_J)\Bigg]
\end{aligned}$$

$$= \frac{1}{32}\frac{1}{2\pi i}\int_\gamma \frac{(t^2-1)^3}{t^2-t_1^2}\frac{1}{t}\cdot\frac{1}{dt}\cdot dt_1 \left[\sum_{j=2}^{n}\frac{2(t^2+t_j^2)}{(t^2-t_j^2)^2}\,W_{g,n-1}(t,t_2,\dots,\widehat{t_j},\dots,t_n)\,dt\cdot dt_j \right.$$
$$\left. + W^D_{g-1,n+1}(t,t,t_2,\dots,t_n) + \sum_{\substack{g_1+g_2=g\\ I\sqcup J=\{2,3,\dots,n\}}}^{\text{stable}} W^D_{g_1,|I|+1}(t,t_I)W^D_{g_2,|J|+1}(t,t_J)\right].$$

Clearly the residues at $t=0$ and $t=\infty$ are Laurent polynomials in $t_1^2,\dots,t_n^2$.

Because of (5.16), we have

(5.19)

$$\sum_{\mu\in\mathbb{Z}_+^n} D_{g,n}(\mu)\prod_{i=1}^{n} d\left(\frac{1}{x_i^{\mu_i}}\right) = \sum_{\nu\in\mathbb{Z}_+^n} N_{g,n}(\nu)\prod_{i=1}^{n} d\left(\frac{1}{z_i^{\nu_i}}\right) = (-1)^n\sum_{\nu\in\mathbb{Z}_+^n} N_{g,n}(\nu)\prod_{i=1}^{n} dz_i^{\nu_i},$$

where $x_i = z_i + 1/z_i$. The Galois conjugation $t\to -t$ corresponds to $z\to 1/z$. Since

$$W^N_{g,n}(t_1,\dots,t_n) = (-1)^n W^N_{g,n}(-t_1,\dots,-t_n),$$

the second equality of (5.19) follows. Multiply (5.19) by $x_1^{\mu_1}\cdots x_n^{\mu_n}$ and take residues at $x_i=\infty$ for $i=1,\dots,n$ (which corresponds with residues at $z_i=0$ on the right-hand side). Then for every $(\mu_1,\dots,\mu_n)\in\mathbb{Z}_+^n$ we have

$$(5.20)\quad D_{g,n}(\mu_1,\dots,\mu_n)\mu_1\cdots\mu_n$$
$$= \left(\frac{1}{2\pi i}\right)^n \int_{|z_1|=\epsilon}\cdots\int_{|z_n|=\epsilon} x_1^{\mu_1}\cdots x_n^{\mu_n}\sum_{\nu\in\mathbb{Z}_+^n} N_{g,n}(\nu)\prod_{i=1}^{n} dz_i^{\nu_i}.$$

Since

$$\left(z_i+\frac{1}{z_i}\right)^{\mu_i} = \sum_{\ell_i=0}^{\mu_i}\binom{\mu_i}{\ell_i} z_i^{\mu_i-2\ell_i},$$

the residue of (5.20) comes from the term $\mu_i - 2\ell_i + \nu_i = 0$, and we have

$$D_{g,n}(\mu_1,\dots,\mu_n)\mu_1\cdots\mu_n$$
$$= \sum_{\ell_1>\mu_1/2}\cdots\sum_{\ell_n>\mu_n/2}\prod_{i=1}^{n}(2\ell_i-\mu_i)\binom{\mu_i}{\ell_i} N_{g,n}(2\ell_1-\mu_1,\dots,2\ell_n-\mu_n).$$

The reciprocity relation, and the degree of the Laurent polynomial, is a consequence of Theorem 5.3. This completes the proof of Corollary 5.4. □

6. The ψ-class intersection numbers on $\overline{\mathcal{M}}_{g,n}$

The crucial discovery of Konstevich [**47**] is the equality between the intersection numbers on the compact moduli space $\overline{\mathcal{M}}_{g,n}$ and the Euclidean volume of the moduli space $\mathcal{M}_{g,n}$ of smooth curves using isomorphism (5.1). The Feynman diagram expansion of the Kontsevich matrix integral relates the Euclidean volume with a τ-function of the KdV equations. The Eynard-Orantin recursion for the ψ-class intersection numbers is precisely the Dijkgraaf-Verlinde-Verlinde formula [**16**] of the intersection numbers.

In this section we take a scaling limit of the spectral curve obtained in Section 5 for the lattice point count and argue that it is the spectral curve which determines intersection numbers on $\overline{\mathcal{M}}_{g,n}$. We then show that this same spectral curve can be obtained by taking the Laplace transform of the unstable intersection numbers. Due to its simple form, we are able to explicitly evaluate the residues involved in the Eynard-Orantin recursion formula. We prove that it is equivalent to the DVV formula [**16**] for the intersection numbers of ψ-classes on $\overline{\mathcal{M}}_{g,n}$.

As we have noted, the derivative of the recursion formula (5.10) is *not* the Eynard-Orantin recursion because the spectral curve is not defined by the unstable geometries. Indeed, we have $dF^L_{0,1} \equiv 0$. However, when we associate the number of lattice points with the ψ-class intersection numbers on $\overline{\mathcal{M}}_{g,n}$ through a scaling limit, we arrive in a setting where the unstable geometries do make sense. In particular, there are coherent definitions for $\int_{\overline{\mathcal{M}}_{0,1}} \psi^d$ and $\int_{\overline{\mathcal{M}}_{0,2}} \psi_1^{d_1}\psi_2^{d_2}$ which, using the Laplace transform philosophy of the present work, generate the spectral curve independent of the lattice point count argument.

Let us recall a computation in [**56**, Section 4].

(6.1)

$$\begin{aligned}
\sum_{\mu\in\mathbb{Z}_+^n} N_{g,n}(\mu)e^{-\langle\mu,w\rangle} &= \sum_{\substack{\Gamma \text{ ribbon graph}\\ \text{of type } (g,n)}} \sum_{\mu\in\mathbb{Z}_+^n} \frac{1}{|\operatorname{Aut}(\Gamma)|}\left|\{\mathbf{x}\in\mathbb{Z}_+^{e(\Gamma)} \mid A_\Gamma\mathbf{x}=\mu\}\right| e^{-\langle\mu,w\rangle}\\
&= \sum_{\substack{\Gamma \text{ ribbon graph}\\ \text{of type } (g,n)}} \frac{1}{|\operatorname{Aut}(\Gamma)|} \sum_{\mathbf{x}\in\mathbb{Z}_+^{e(\Gamma)}} e^{-\langle A_\Gamma\mathbf{x},w\rangle}\\
&= \sum_{\substack{\Gamma \text{ ribbon graph}\\ \text{of type } (g,n)}} \frac{1}{|\operatorname{Aut}(\Gamma)|} \prod_{\substack{\eta \text{ edge}\\ \text{of } \Gamma}} \sum_{\ell_\eta=1}^{\infty} e^{-\langle a_\eta,w\rangle \ell_\eta}\\
&= \sum_{\substack{\Gamma \text{ ribbon graph}\\ \text{of type } (g,n)}} \frac{1}{|\operatorname{Aut}(\Gamma)|} \prod_{\substack{\eta \text{ edge}\\ \text{of } \Gamma}} \frac{e^{-\langle a_\eta,w\rangle}}{1-e^{-\langle a_\eta,w\rangle}},
\end{aligned}$$

where A_Γ is the incidence matrix of (5.2), a_η is the η-th column of A_Γ, and $\langle\mu,w\rangle = \mu_1w_1+\cdots+\mu_nw_n$. By comparing (5.6) and (6.1), we see that we are substituting $e^{w_i}=z_i$ in this computation. Therefore, we obtain

$$F^L_{g,n}(t_1,\dots,t_n) = \sum_{\substack{\Gamma \text{ ribbon graph}\\ \text{of type } (g,n)}} \frac{1}{|\operatorname{Aut}(\Gamma)|} \prod_{\substack{\eta \text{ edge}\\ \text{of } \Gamma}} \frac{1}{\prod_{i=1}^n z_i^{a_{i\eta}}-1}. \tag{6.2}$$

Thus the series (5.6) in z_i converges for $|z_i|>1$. Since $z_i = \frac{t_i+1}{t_1-1}$, the $t_i\to\infty$ limit picks up the limit of (5.6) as $z_i\to 1$, and hence the information of $N_{g,n}(\mu)$ as $\mu_i\to\infty$. Since the orbifold isomorphism (5.1) is scale invariant under the action of $\mathbb{R}_+$, making the perimeter length μ large is the same as making the mesh small in the lattice point counting. Hence at the limit we obtain the Euclidean volume of $\mathcal{M}_{g,n}$ considered by Kontsevich in [**47**]. This is why we expect that (5.14) holds. Let us now consider the limit of the spectral curve (3.13) as $t\to\infty$. First we have

$$\begin{aligned}
x &= z+\frac{1}{z} = 2+\frac{4}{t^2-1}\\
y &= -z = -1-\frac{2}{t-1}.
\end{aligned}$$

Ignoring the constant shifts of x and y, we obtain for a large t

$$\begin{cases} x = \frac{4}{t^2}\\ y = -\frac{2}{t}. \end{cases} \tag{6.3}$$

Hence the spectral curve is given by the equation $x = y^2$. We use t as the preferred coordinate.

We now compare the Eynard-Orantin recursion with respect to this spectral curve and the Witten-Kontsevich theory. We use (5.14) and define

(6.4)
$$\begin{aligned} W^K_{g,n}(t_1,\dots,t_n) &= d_1\cdots d_n F^K_{g,n}(t_1,\dots,t_n) \\ &= \frac{(-1)^n}{2^{2g-2+n}} \sum_{\substack{d_1+\cdots+d_n\\ =3g-3+n}} \langle \tau_{d_1}\cdots\tau_{d_n}\rangle_{g,n} \prod_{j=1}^n (2d_j+1)!! \left(\frac{t_j}{2}\right)^{2d_j} d\left(\frac{t_j}{2}\right) \\ &= \frac{(-1)^n}{16^{2g-2+n}}\, w^K_{g,n}(t_1,\dots,t_n)\, dt_1\cdots dt_n, \end{aligned}$$

where $w^K_{g,n}(t_1,\dots,t_n)$ is the coefficient of the Eynard-Orantin differential form normalized by the constant factor $\frac{(-1)^n}{16^{2g-2+n}}$. Note that $w^K_{g,n}(t_1,\dots,t_n)$ is a polynomial in t_i^2's with positive rational coefficients for (g,n) in the stable range.

Recall that in genus 0, the intersection numbers are determined by the formula

$$\langle \tau_{d_1}\cdots\tau_{d_n}\rangle_{0,n} = \binom{n-3}{d_1,\dots,d_n}, \tag{6.5}$$

provided $\sum d_i = n-3$. For $(g,n)=(0,1)$ and $(0,2)$, we have

$$\langle \tau_k\rangle_{0,1} = \delta_{k+2,0} \tag{6.6}$$

$$\langle \tau_{k_1}\tau_{k_2}\rangle_{0,2} = (-1)^{k_1}, \qquad k_1+k_2=-1. \tag{6.7}$$

Therefore,

$$W^K_{0,1}(t) = \frac{-1}{16^{-1}}\langle\tau_{-2}\rangle(-3)!!\, t^{-4}dt = \frac{16}{t^4}\,dt = y\,dx, \tag{6.8}$$

in agreement with the spectral curve $x=y^2$ (6.3). Similarly, we have

$$\begin{aligned} F^K_{0,2}(t_1,t_2) &= \sum_{d=0}^\infty (-1)^d(2d-1)!!(-2d-3)!! \left(\frac{t_1}{2}\right)^{2d+1}\left(\frac{t_2}{2}\right)^{-2d-1} \\ &= -\sum_{d=0}^\infty \frac{1}{2d+1}\left(\frac{t_1}{t_2}\right)^{2d+1} = \log\left(1-\frac{t_1}{t_2}\right) - \frac{1}{2}\log\left(1-\frac{t_1^2}{t_2^2}\right), \end{aligned} \tag{6.9}$$

and hence

$$W^K_{0,2}(t_1,t_2) = \frac{dt_1\cdot dt_2}{(t_1-t_2)^2} - \frac{1}{2}\,\frac{dx_1\cdot dx_2}{(x_1-x_2)^2}. \tag{6.10}$$

As a consequence, the recursion kernel is given by

$$K^K(t,t_1) = -\frac{1}{2}\left(\frac{1}{t+t_1}+\frac{1}{t-t_1}\right)\frac{t^4}{32}\,\frac{1}{dt}\,dt_1, \tag{6.11}$$

since $\frac{dx_1\cdot dx_2}{(x_1-x_2)^2}$ does not contribute to the kernel (being even in t and the kernel involves an integral from t to $-t$). The Eynard-Orantin recursion for the Euclidean volume then becomes

(6.12) $W^K_{g,n}(t_1,\dots,t_n)$
$$= -\frac{1}{2\pi i}\int_{\gamma_\infty}\left(\frac{1}{t+t_1}+\frac{1}{t-t_1}\right)\frac{t^4}{64}\,\frac{1}{dt}\,dt_1\Bigg[W^K_{g-1,n+1}(t,-t,t_2,\dots,t_n)$$

$$+\sum_{j=2}^{n}\left(\frac{dt\cdot dt_j}{(t-t_j)^2}\,W^K_{g,n-1}(-t,t_2,\dots,\widehat{t_j},\dots,t_n)\right.$$
$$\left.-\frac{dt\cdot dt_j}{(t+t_j)^2}\,W^K_{g,n-1}(t,t_2,\dots,\widehat{t_j},\dots,t_n)\right)$$
$$\left.+\sum_{\substack{g_1+g_2=g\\ I\sqcup J=\{2,\dots,n\}}}^{\text{stable}} W^K_{g_1,|I|+1}(t,t_I)W^K_{g_2,|J|+1}(-t,t_J)\right],$$

where the integral is taken with respect to a large negatively oriented circle γ_∞ that encloses any of $\pm t_1,\dots,\pm t_n$. This is the larger circle of Figure 4.1. Here again $\frac{dx_1\cdot dx_2}{(x_1-x_2)^2}$ does not contribute in the formula. Since the coefficients $w^K_{g,n}(t_1,\dots,t_n)$ in the stable range are polynomials, the poles of the integrand of (6.12) in the integration coutour are at $t=\pm t_i$'s. Therefore, we can perform the integral in terms of the residue calculus at poles $t=\pm t_i$. First let us get rid of the factor $1/16^{2g-2+n}$ from (6.12). Since the recursion is an induction on $2g-2+n$, we have an overall factor 16 adjustment on the right-hand side. The integration contour is negatively oriented, so the residue calculation at $t=\pm t_i$ receives universally the negative sign. This sign is exactly cancelled by the choice of the sign of $w^K_{g,n}$ in (6.4). Thus the result of residue evaluation of (6.12) is

$$\text{(6.13)}\quad w^K_{g,n}(t_1,\dots,t_n)=\frac{1}{2}\,t_1^4 w^K_{g-1,n+1}(t_1,t_1,t_2,\dots,t_n)$$
$$+\frac{1}{2}\,t_1^4\sum_{\substack{g_1+g_2=g\\ I\sqcup J=\{2,\dots,n\}}}^{\text{stable}} w^K_{g_1,|I|+1}(t_1,t_I)w^K_{g_2,|J|+1}(t_1,t_J)$$
$$+t_1^4\sum_{j=2}^{n}\frac{t_1^2+t_j^2}{(t_1^2-t_j^2)^2}\,w^K_{g,n-1}(t_1,\dots,\widehat{t_j},\dots,t_n)$$
$$+\frac{1}{2}\sum_{j=2}^{n}\left(\frac{\partial}{\partial t}\bigg|_{t=t_j}+\frac{\partial}{\partial t}\bigg|_{t=-t_j}\right)\left(\frac{1}{t^2-t_1^2}\,t^5 w^K_{g,n-1}(t,t_2,\dots,\widehat{t_j},\dots,t_n)\right)$$
$$=\frac{1}{2}\,t_1^4\left[w^K_{g-1,n+1}(t_1,t_1,t_2,\dots,t_n)+\sum_{\substack{g_1+g_2=g\\ I\sqcup J=\{2,\dots,n\}}}^{\text{stable}} w^K_{g_1,|I|+1}(t_1,t_I)w^K_{g_2,|J|+1}(t_1,t_J)\right]$$
$$+\sum_{j=2}^{n}\frac{\partial}{\partial t_j}\left[\frac{t_j}{t_1^2-t_j^2}\left(t_1^4 w^K_{g,n-1}(t_{[n]\setminus\{j\}})-t_j^4 w^K_{g,n-1}(t_{[n]\setminus\{1\}})\right)\right].$$

This is the same as [**11**, Theorem 5.2], and with a different choice of preferred coordinate, [**6**, Lemma 6.1].

Let us adopt the normalized notation

$$\text{(6.14)}\qquad \langle\sigma_{d_1}\cdots\sigma_{d_n}\rangle_{g,n}=\langle\tau_{d_1}\cdots\tau_{d_n}\rangle_{g,n}\prod_{i=1}^{n}(2d_i+1)!!$$

to make the formula shorter. Then

$$w^K_{g,n}(t_1,\dots,t_n) = \sum_{d_1,\dots,d_n} \langle \sigma_{d_1}\cdots\sigma_{d_n}\rangle_{g,n} \prod_{j=1}^n t_j^{2d_j}. \tag{6.15}$$

The DVV formula [**16**] for the Virasoro constraint condition on the ψ-class intersection numbers on $\overline{\mathcal{M}}_{g,n}$ reads

$$\begin{aligned}\langle \sigma_k \prod_{i=2}^n \sigma_{d_i}\rangle_{g,n} &= \frac{1}{2}\sum_{a+b=k-2} \langle \sigma_a\sigma_b \prod_{i=2}^n \sigma_{d_i}\rangle_{g-1,n+1} \\ &+ \frac{1}{2}\sum_{a+b=k-2} \sum^{\text{stable}}_{\substack{g_1+g_2=g\\ I\sqcup J=\{2,\dots,n\}}} \langle \sigma_a \prod_{i\in I}\sigma_{d_i}\rangle_{g_1,|I|+1}\cdot\langle \sigma_b \prod_{j\in J}\sigma_{d_j}\rangle_{g_2,|J|+1} \\ &\qquad + \sum_{j=2}^n (2d_j+1)\langle \sigma_{k+d_j-1}\prod_{i\neq 1,j}\sigma_{d_i}\rangle_{g,n-1}.\end{aligned} \tag{6.16}$$

We thus recover the discovery of [**26**]:

THEOREM 6.1. *The Eynard-Orantin recursion formula for the spectral curve $x=y^2$ is the Dijkgraaf-Verlinde-Verlinde formula* [**16**] *for the intersection numbers $\langle \tau_{d_1}\cdots\tau_{d_n}\rangle_{g,n}$ on the moduli space $\overline{\mathcal{M}}_{g,n}$ of pointed stable curves.*

PROOF. We extract the coefficient of

$$t_1^{2k}\prod_{j=2}^n t_j^{2d_j} \tag{6.17}$$

in (6.13) and compare the result with (6.16). It is obvious that the fifth line of (6.13) produces the first and second lines of (6.16).

To compare the last lines of (6.13) and (6.16), we consider the case $|t_j|<|t_1|$ for all $j\geq 2$ in (6.13). We then have the expansion

$$\frac{1}{t_1^2-t_j^2} = \frac{1}{t_1^2}\,\frac{1}{1-\frac{t_j^2}{t_1^2}} = \frac{1}{t_1^2}\sum_{m=0}^\infty \left(\frac{t_j^2}{t_1^2}\right)^m .$$

The (6.17)-term of the last line of (6.13) has two contributions. The first one comes from

$$\frac{\partial}{\partial t_j}\left(t_1^2 t_j \sum_{m=0}^\infty \left(\frac{t_j^2}{t_1^2}\right)^m w^K_{g,n-1}(t_1,t_2,\dots,\widehat{t_j},\dots,t_n)\right).$$

Since $w^K_{g,n-1}(t_1,t_2,\dots,\widehat{t_j},\dots,t_n)$ does not contain t_j, we set $m=d_j$ to produce the right power $2d_j$ of t_j. The power of t_1 has to be $2k$. Thus from $w^K_{g,n-1}$ we take the term of $t_1^{2k+2d_j-2}$, whose coefficient is $\langle \sigma_{k+d_j-1}\prod_{i\neq 1,j}\sigma_{d_i}\rangle$. The total contribution from the first kind comes from the differentiation, which gives $2m+1=2d_j+1$.

The second possible contribution for the (6.17)-term may come from

$$-\frac{\partial}{\partial t_j}\left(\frac{t_j^5}{t_1^2} \sum_{m=0}^\infty \left(\frac{t_j^2}{t_1^2}\right)^m w^K_{g,n-1}(t_2,\dots,t_n)\right).$$

However, this term does not produce t_1^{2k}, and hence does not contribute to the (6.17)-term. This completes the proof of Theorem 6.1. $\square$

7. Single Hurwitz numbers

What is the mirror dual of the number of trees? The answer we wish to present in this section is that it is the *Lambert curve*. This analytic curve serves as the spectral curve for the Hurwitz counting problem, and comes from the the unstable geometries $(g, n) = (0, 1)$ and $(0, 2)$ via the Laplace transform.

A *Hurwitz* cover is a holomorphic mapping $f : C \to \mathbb{P}^1$ from a connected nonsingular projective algebraic curve C of genus g to the projective line $\mathbb{P}^1$ with only simple ramifications except for $\infty \in \mathbb{P}^1$. Such a cover is further refined by specifying its *profile*, which is a partition $\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n > 0)$ of the degree of the covering $d = |\mu| = \mu_1 + \cdots + \mu_n$. The length $\ell(\mu) = n$ of this partition is the number of points in the inverse image $f^{-1}(\infty) = \{p_1, \ldots, p_n\}$ of ∞. Each part μ_i gives a local description of the map f, which is given by $u \longmapsto u^{-\mu_i}$ in terms of a local coordinate u of C around p_i. The number $h_{g,\mu}$ of the topological types of Hurwitz covers of a given genus g and a profile μ, counted with the weight factor $1/|\operatorname{Aut} f|$, is the *single Hurwitz number* we shall deal with in this section.

Another natural way of encoding single Hurwitz numbers is through the functions

$$H_g(\mu) = \frac{|\operatorname{Aut}(\mu)|}{(2g - 2 + n + |\mu|)!} \cdot h_{g,\mu}. \tag{7.1}$$

Here,

$$r = r(g, \mu) \overset{\text{def}}{=} 2g - 2 + n + |\mu| \tag{7.2}$$

is the number of simple ramification points of f by the Riemann-Hurwitz formula, and $\operatorname{Aut}(\mu)$ is the group of permutations of equal parts of the partition μ. Note that multiplication by $\operatorname{Aut}(\mu)$ is equivalent to counting Hurwitz covers where the preimages of ∞ on C are marked.

One reason that explains why single Hurwitz numbers are interesting is a remarkable formula due to Ekedahl, Lando, Shapiro and Vainshtein [**21**, **34**, **49**, **66**] that relates Hurwitz numbers and Gromov-Witten invariants. For genus $g \geq 0$ and a partition μ of length $\ell(\mu) = n$ subject to the stability condition $2g - 2 + n > 0$, the ELSV formula states that

$$\begin{aligned} H_g(\mu) &= \prod_{i=1}^{n} \frac{\mu_i^{\mu_i}}{\mu_i!} \int_{\overline{\mathcal{M}}_{g,n}} \frac{\Lambda_g^\vee(1)}{\prod_{i=1}^{n}(1 - \mu_i \psi_i)} \\ &= \sum_{j=0}^{g} (-1)^j \sum_{k_1, \ldots, k_n \geq 0} \langle \tau_{k_1} \cdots \tau_{k_n} c_j(\mathbb{E}) \rangle \prod_{i=1}^{n} \frac{\mu_i^{\mu_i + k_i}}{\mu_i!}, \end{aligned} \tag{7.3}$$

where $\overline{\mathcal{M}}_{g,n}$ is the Deligne-Mumford moduli stack of stable algebraic curves of genus g with n distinct smooth marked points, $\Lambda_g^\vee(1) = 1 - c_1(\mathbb{E}) + \cdots + (-1)^g c_g(\mathbb{E})$ is the alternating sum of the Chern classes of the Hodge bundle $\mathbb{E}$ on $\overline{\mathcal{M}}_{g,n}$, ψ_i is the i-th tautological cotangent class, and

$$\langle \tau_{k_1} \cdots \tau_{k_n} c_j(\mathbb{E}) \rangle = \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{k_1} \cdots \psi_\ell^{k_n} c_j(\mathbb{E}) \tag{7.4}$$

is the linear Hodge integral, which is 0 unless $k_1 + \cdots + k_n + j = 3g - 3 + n$.

The Deligne-Mumford stack $\overline{\mathcal{M}}_{g,n}$ is defined as the moduli space of *stable* curves satisfying the stability condition $2 - 2g - n < 0$. However, single Hurwitz numbers are well defined for *unstable* geometries $(g,n) = (0,1)$ and $(0,2)$, and their values are

$$H_0((d)) = \frac{d^{d-3}}{(d-1)!} = \frac{d^{d-2}}{d!} \qquad \text{and} \qquad H_0((\mu_1,\mu_2)) = \frac{1}{\mu_1+\mu_2}\cdot\frac{\mu_1^{\mu_1}}{\mu_1!}\cdot\frac{\mu_2^{\mu_2}}{\mu_2!}. \tag{7.5}$$

The ELSV formula remains valid for unstable cases by *defining*

$$\int_{\overline{\mathcal{M}}_{0,1}} \frac{\Lambda_0^\vee(1)}{1-d\psi} = \frac{1}{d^2}, \tag{7.6}$$

$$\int_{\overline{\mathcal{M}}_{0,2}} \frac{\Lambda_0^\vee(1)}{(1-\mu_1\psi_1)(1-\mu_2\psi_2)} = \frac{1}{\mu_1+\mu_2}. \tag{7.7}$$

Let us examine the $(g,n) = (0,1)$ case. We wish to count the number of Hurwitz covers $f : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ of degree d with profile $\mu = (d)$. If $d = 2$, then $f(u) = u^2$ is the only map, since $r = 1$ and the two ramification points can be placed at $u = 0$ and $u = \infty$. The automorphism of this map is $\mathbb{Z}/2\mathbb{Z}$. We now consider the case when $d \geq 3$. First we label all branch points. One is ∞, so let us place all others, the images of simple ramification points, at the r-th roots of unity. Here $r = d-1$. We label these points with indices $[r] = \{1, 2, \ldots, r\}$. Connect each r-th root of unity with the origin by a straight line (see Figure 7.1). Let $*$ denote this star-like shape, which has one vertex at the center and r half-edges. Then the inverse image $f^{-1}(*)$ is a tree-like shape with d vertices and rd half-edges. Here we call each inverse image of 0 a vertex of $f^{-1}(*)$. If f is simply ramified at p, then two half-edges are connected at p and form a real edge that is incident to two vertices. Since $f(p)$ is one of the r-th root of unity, we give the same label to p. Thus all simple ramification points are labeled with the index set $[r]$. Now we remove all half-edges from $f^{-1}(*)$ that are not made into an edge, and denote it by T. It is a tree on $\mathbb{P}^1$ that has d vertices and $r = d-1$ edges. Note that except for the case $d = 2$, the edge labeling gives a labeling of vertices. For example, if a vertex x is incident to edges $i_1 < i_2 < \cdots < i_k$, then x is labeled by $i_1 i_2 \cdots i_k$.

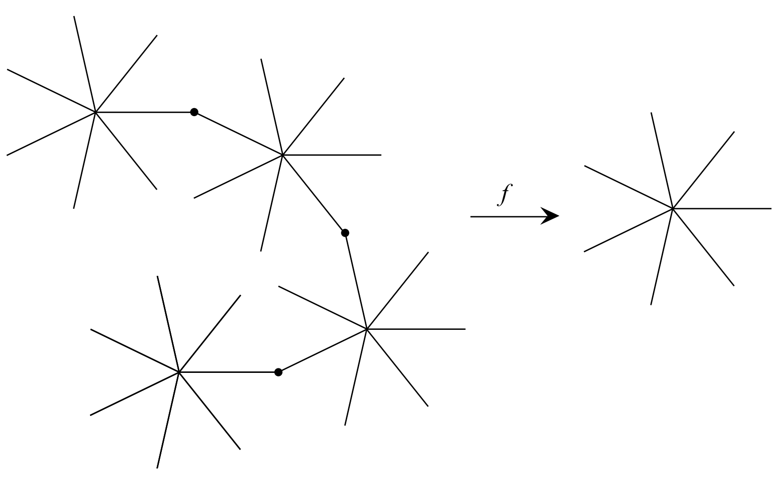

FIGURE 7.1. Counting the genus 0 single Hurwitz numbers with the total ramification at ∞.

Conversely, suppose we are given a tree with d labeled vertices by the index set $[d] = \{1, 2, \dots, d\}$ and $r = d - 1$ edges. At each vertex we can give a cyclic order to incident edges by aligning them in the increasing order of the labels of the other ends of the edges. Thus the tree becomes a *ribbon graph* (see Section 3), and hence it can be placed on $\mathbb{P}^1$. Then by choosing the midpoint of each edge as a simple ramification point and each vertex as a zero of f, we can construct a Hurwitz cover. Recall that the number of trees with d labeled vertices is d^{d-2}. Therefore,

$$H_0((d)) = \frac{d^{d-2}}{d!}$$

is the number of trees with d *unlabeled* vertices.

Fix an $n \geq 1$, and consider a partition μ of length n as an n-dimensional vector

$$\mu = (\mu_1, \dots, \mu_n) \in \mathbb{Z}_+^n$$

consisting of positive integers. The Laplace transform of $H_g(\mu)$ as a function in μ,

$$H_{g,n}(w_1, \dots, w_n) = \sum_{\mu \in \mathbb{Z}_+^n} H_g(\mu) e^{-(\mu_1(w_1+1)+\cdots+\mu_n(w_n+1))}, \tag{7.8}$$

is the function we wish to compute. Note that the automorphism group $\mathrm{Aut}(\mu)$ acts trivially on the function $e^{-(\mu_1(w_1+1)+\cdots+\mu_n(w_n+1))}$, which explains its appearance in (7.1). The reason for shifting the variables $w_i \longmapsto w_i + 1$ is due to the asymptotic behavior

$$\frac{\mu^{\mu+k}}{\mu!} e^{-\mu} \sim \frac{1}{\sqrt{2\pi}}\, \mu^{k-\frac{1}{2}}$$

as μ approaches to ∞. These asymptotics also suggests that the holomorphic function $H_{g,n}(w_1, \dots, w_n)$ is actually defined on a double-sheeted overing on the w_i-plane, since $\sqrt{w_i}$ behaves better as a holomorphic coordinate.

In order to recover the spectral curve for single Hurwitz numbers, we must take the Laplace transform of the unstable geometries. As was done with the count of dessins, we introduce new parameters z and x, related through the $(0,1)$ geometry:

$$z = \sum_{\mu=1}^{\infty} \mu H_0(\mu) e^{-\mu} x^{\mu}.$$

We note that there are other, equally valid choices for the expansion of z in terms of x,but the one presented here results in a function whose inverse has a closed form expression. Following [**25, 58**], the Laplace transform calculations are simplified by introducing a series of polynomials $\hat{\xi}_n(t)$ of degree $2n + 1$ in t for $n \geq 0$ by the recursion formula

$$\hat{\xi}_n(t) = t^2(t-1)\frac{d}{dt}\hat{\xi}_{n-1}(t) \tag{7.9}$$

with the initial condition $\hat{\xi}_0(t) = t - 1$. This differential operator appears in [**32**]. The functions $\hat{\xi}_{-1}(t)$ and $\hat{\xi}_0(t)$ were also used by Zvonkine [**85**] as the two fundamental functions that generate his algebra $\mathcal{A}$.

Proposition 7.1 ([**13, 25**]). *Let*

$$x = e^{-w}, \qquad z = \sum_{\mu=1}^{\infty} \frac{\mu^{\mu-1}}{\mu!}\, e^{-\mu}\, x^{\mu}, \qquad t - 1 = \sum_{\mu=1}^{\infty} \frac{\mu^{\mu}}{\mu!}\, e^{-\mu}\, x^{\mu}. \tag{7.10}$$

Then the inverse function of $z = z(x)$ is given by

$$x = ze^{1-z}, \tag{7.11}$$

and the variables z and t are related by

$$z = \frac{t-1}{t}. \tag{7.12}$$

Moreover, we have

$$\hat{\xi}_n(t) = \sum_{\mu=1}^{\infty} \frac{\mu^{\mu+n}}{\mu!}\, e^{-\mu(w+1)} = \sum_{\mu=1}^{\infty} \frac{\mu^{\mu+n}}{\mu!}\, e^{-\mu}\, x^{\mu} \tag{7.13}$$

for $n \geq 0$.

Proof. The infinite series (7.13) has the radius of convergence 1, and for $|x| < 1$, we can apply the Lagrange inversion formula to obtain (7.11). Since the application of

$$-\frac{d}{dw} = x\,\frac{d}{dx} = t^2(t-1)\frac{d}{dt}$$

n-times to $\sum_{\mu=1}^{\infty} \frac{\mu^{\mu}}{\mu!}\, e^{-\mu(w+1)}$ produces $\sum_{\mu=1}^{\infty} \frac{\mu^{\mu+n}}{\mu!}\, e^{-\mu(w+1)}$, we obtain (7.9). If we extend (7.13) formally to $n = -1$, then we have $z = \hat{\xi}_{-1}(t)$. To obtain the expression of z as a function of t, we need to solve the differential equation

$$t^2(t-1)\frac{d}{dt} \cdot z = t - 1.$$

Its solution is $z = c - \frac{1}{t}$. Since $x = 0 \Longleftrightarrow z = 0$ and $x = 0 \Longrightarrow t = 1$, we conclude that the constant of integration is $c = 1$. Thus $z = 1 - 1/t$. □

Remark 7.2. The relation between our z as a function in x and the classical Lambert W-function (see for example, [**13**]) is

$$z(x) = -W(-x/e).$$

Because of the ELSV formula (7.1), the Laplace transform of $H_g(\mu)$ becomes a polynomial in $t_i, \dots, t_n$ for (g, n) in the stable range. The result is

$$\begin{aligned} F^H_{g,n}(t_1, \dots, t_n) &= H_{g,n}(w(t_1), \dots, w(t_n)) \\ &= \sum_{\mu \in \mathbb{Z}_+^n} H_g(\mu) e^{-(\mu_1(w_1+1)+\cdots+\mu_n(w_n+1))} \\ &= \sum_{\mu \in \mathbb{Z}_+^n} \sum_{k_1+\cdots+k_n \leq 3g-3+n} \langle \tau_{k_1} \cdots \tau_{k_n} \Lambda_g^{\vee}(1) \rangle \prod_{i=1}^{n} \frac{\mu_i^{\mu_i+k_i}}{\mu_i!} e^{-(\mu_1(w_1+1)+\cdots+\mu_n(w_n+1))} \\ &= \sum_{k_1+\cdots+k_n \leq 3g-3+n} \langle \tau_{k_1} \cdots \tau_{k_n} \Lambda_g^{\vee}(1) \rangle \prod_{i=1}^{n} \hat{\xi}_{k_i}(t_i). \end{aligned} \tag{7.14}$$

The Laplace transform (7.14) is no longer a polynomial for the unstable geometries $(g, n) = (0, 1)$ and $(0, 2)$. We use (7.5) to calculate $F^H_{0,1}$ and $F^H_{0,2}$.

Theorem 7.3. *The Laplace transform of the unstable cases $(g, n) = (0, 1)$ and $(0, 2)$ are given by*

$$F^H_{0,1}(t) = \frac{1}{2}\left(1 - \frac{1}{t^2}\right) \tag{7.15}$$

and

$$F^H_{0,2}(t_1,t_2) = \log\left(\frac{z_1-z_2}{x_1-x_2}\right) - (z_1+z_2)+1, \tag{7.16}$$

where t_i, x_i, z_i are related by (7.11) and (7.12).

PROOF. The $(0,1)$ case is a straightforward computation.

$$F^H_{0,1}(t) = \sum_{k=d}^{\infty} H_0((d))\, e^{-d}x^d = \sum_{d=1}^{\infty}\frac{d^{d-2}}{d!}\, e^{-d}x^d = \hat{\xi}_{-2}(t).$$

This is a solution to the differential equation

$$t^2(t-1)\frac{d}{dt}\,\hat{\xi}_{-2}(t) = \hat{\xi}_{-1}(t) = z = \frac{t-1}{t}.$$

Therefore, $\hat{\xi}_{-2}(t) = c - \frac{1}{2}\,\frac{1}{t^2}$ for a constant of integration c. Here again we note

$$t=1 \Longrightarrow z=0 \Longrightarrow x=0 \Longrightarrow \hat{\xi}_{-2}(t)=0.$$

This determines that $c=\frac{1}{2}$. Thus we have established (7.15).

Since

$$F^H_{0,2}(t_1,t_2) = \sum_{\mu_1,\mu_2\geq 1}\frac{1}{\mu_1+\mu_2}\cdot\frac{\mu_1^{\mu_1}}{\mu_1!}\,e^{-\mu_1}\cdot\frac{\mu_2^{\mu_2}}{\mu_2!}\,e^{-\mu_2}\cdot x_1^{\mu_1}\,x_2^{\mu_2}$$

and since $z=\hat{\xi}_{-1}(t)$, (7.16) is equivalent to

$$\sum_{\substack{\mu_1,\mu_2\geq 0\\(\mu_1,\mu_2)\neq(0,0)}}\frac{1}{\mu_1+\mu_2}\cdot\frac{\mu_1^{\mu_1}}{\mu_1!}\,e^{-\mu_1}\cdot\frac{\mu_2^{\mu_2}}{\mu_2!}\,e^{-\mu_2}\cdot x_1^{\mu_1}x_2^{\mu_2} = \log\left(e\sum_{k=1}^{\infty}\frac{k^{k-1}}{k!}\,e^{-k}\cdot\frac{x_1^k-x_2^k}{x_1-x_2}\right), \tag{7.17}$$

where $|x_1|<1, |x_2|<1$, and $0<|x_1-x_2|<1$ so that the formula is an equation of holomorphic functions in x_1 and x_2. Define

$$\phi(x_1,x_2) \stackrel{\text{def}}{=} \sum_{\substack{\mu_1,\mu_2\geq 0\\(\mu_1,\mu_2)\neq(0,0)}}\frac{1}{\mu_1+\mu_2}\cdot\frac{\mu_1^{\mu_1}}{\mu_1!}\,e^{-\mu_1}\cdot\frac{\mu_2^{\mu_2}}{\mu_2!}\,e^{-\mu_2}\cdot x_1^{\mu_1}x_2^{\mu_2} - \log\left(\sum_{k=1}^{\infty}\frac{k^{k-1}}{k!}\,e^{1-k}\cdot\frac{x_1^k-x_2^k}{x_1-x_2}\right).$$

Then

$$\begin{aligned}\phi(x,0) &= \sum_{\mu_1\geq 1}\frac{\mu_1^{\mu_1-1}}{\mu_1!}\,e^{-\mu_1}x^{\mu_1} - \log\left(\sum_{k=1}^{\infty}\frac{k^{k-1}}{k!}\,e^{-k}\cdot x^{k-1}\right) - 1\\ &= \hat{\xi}_{-1}(t) - \log\left(\frac{\hat{\xi}_{-1}(t)}{x}\right) - 1 = 1-\frac{1}{t} - \log\left(1-\frac{1}{t}\right) + \log x - 1\\ &= -\frac{1}{t} - \log\left(1-\frac{1}{t}\right) - w = 0\end{aligned}$$

because

$$x = e^{-w} = ze^{1-z} = \left(1-\frac{1}{t}\right)e^{\frac{1}{t}}.$$

Here t is restricted on the domain $Re(t)>1$. Since

$$
\begin{aligned}
x_1\frac{\partial}{\partial x_1}\log\left(e\sum_{k=1}^{\infty}\frac{k^{k-1}}{k!}\,e^{-k}\cdot\frac{x_1^k-x_2^k}{x_1-x_2}\right)
&= t_1^2(t_1-1)\frac{\partial}{\partial t_1}\log\left(\hat{\xi}_{-1}(t_1)-\hat{\xi}_{-1}(t_2)\right)-x_1\frac{\partial}{\partial x_1}\log(x_1-x_2)\\
&= t_1^2(t_1-1)\frac{\partial}{\partial t_1}\log\left(-\frac{1}{t_1}+\frac{1}{t_2}\right)-\frac{x_1}{x_1-x_2}\\
&= \frac{t_1t_2(t_1-1)}{t_1-t_2}-\frac{x_1}{x_1-x_2},
\end{aligned}
$$

we have

$$
\begin{aligned}
\left(x_1\frac{\partial}{\partial x_1}+x_2\frac{\partial}{\partial x_2}\right)\log\left(e\sum_{k=1}^{\infty}\frac{k^{k-1}}{k!}\,e^{-k}\cdot\frac{x_1^k-x_2^k}{x_1-x_2}\right)
&= \frac{t_1t_2(t_1-1)-t_1t_2(t_2-1)}{t_1-t_2}-\frac{x_1-x_2}{x_1-x_2}\\
&= t_1t_2-1=\hat{\xi}_0(t_1)\hat{\xi}_0(t_2)+\hat{\xi}_0(t_1)+\hat{\xi}_0(t_2).
\end{aligned}
$$

On the other hand, we also have

$$
\begin{aligned}
\left(x_1\frac{\partial}{\partial x_1}+x_2\frac{\partial}{\partial x_2}\right)&\sum_{\substack{\mu_1,\mu_2\geq 0\\(\mu_1,\mu_2)\neq(0,0)}}\frac{1}{\mu_1+\mu_2}\cdot\frac{\mu_1^{\mu_1}}{\mu_1!}\,e^{-\mu_1}\cdot\frac{\mu_2^{\mu_2}}{\mu_2!}\,e^{-\mu_2}\cdot x_1^{\mu_1}x_2^{\mu_2}\\
&= \sum_{\substack{\mu_1,\mu_2\geq 0\\(\mu_1,\mu_2)\neq(0,0)}}\frac{\mu_1+\mu_2}{\mu_1+\mu_2}\cdot\frac{\mu_1^{\mu_1}}{\mu_1!}\,e^{-\mu_1}\cdot\frac{\mu_2^{\mu_2}}{\mu_2!}\,e^{-\mu_2}\cdot x_1^{\mu_1}x_2^{\mu_2}\\
&= \hat{\xi}_0(t_1)\hat{\xi}_0(t_2)+\hat{\xi}_0(t_1)+\hat{\xi}_0(t_2).
\end{aligned}
$$

Therefore,

$$
\left(x_1\frac{\partial}{\partial x_1}+x_2\frac{\partial}{\partial x_2}\right)\phi(x_1,x_2)=0. \tag{7.18}
$$

Note that $\phi(x_1,x_2)$ is a holomorphic function in x_1 and x_2. Therefore, it has a series expansion in homogeneous polynomials around $(0,0)$. Since a homogeneous polynomial in x_1 and x_2 of degree n is an eigenvector of the differential operator $x_1\frac{\partial}{\partial x_1}+x_2\frac{\partial}{\partial x_2}$ belonging to the eigenvalue n, the only holomorphic solution to the Euler differential equation (7.18) is a constant. But since $\phi(x_1,0)=0$, we conclude that $\phi(x_1,x_2)=0$. This completes the proof of (7.17), and hence Theorem 7.3. $\square$

Definition 7.4. We define the symmetric differential forms for all $g\geq 0$ and $n>0$ by

$$
W_{g,n}^H(t_1,\dots,t_n)=d_1\cdots d_nF_{g,n}^H(t_1,\dots,t_n), \tag{7.19}
$$

and call them the Hurwitz differential forms.

The unstable cases are given by

$$
W_{0,1}^H(t_1)=d_1F_{0,1}^H(t_1)=\frac{1}{t_1^3}\,dt_1=\frac{z}{x}\,dx, \tag{7.20}
$$

and

$$
W_{0,2}^H(t_1,t_2)=d_1d_2F_{0,2}^H(t_1,t_2)=d_1d_2\left[\log(z_1-z_2)-\log(x_1-x_2)\right] \tag{7.21}
$$

$$= d_1 d_2 \left[\log\left(\frac{1}{t_2} - \frac{1}{t_1}\right) - \log(x_1 - x_2)\right] = d_1 d_2 \left[\log(t_1 - t_2) - \log(x_1 - x_2)\right]$$
$$= \frac{dt_1 \cdot dt_2}{(t_1 - t_2)^2} - \frac{dx_1 \cdot dx_2}{(x_1 - x_2)^2}.$$

We note that all quantities are expressible in terms of z, or equivalently, in t. Now Definition 2.1 tells us that the spectral curve Σ of the single Hurwitz number is

$$\begin{cases} x = ze^{1-z} \\ y = \frac{z}{x} = e^{z-1}. \end{cases} \tag{7.22}$$

The *Lambert curve* Σ defined by $x = ze^{1-z}$, which is obtained by the Laplace transform of the number of trees, is an analytic curve and its x-projection has a simple ramification point at $z = 1$, since

$$dx = (1 - z)e^{1-z}\, dz.$$

The t-coordinate brings this ramification point to $t = \infty$. Let $\bar{z}$ (resp. $\bar{t}$) denote the unique local Galois conjugate of z (reps. t). We also use

$$\bar{t} = s(t), \tag{7.23}$$

which is defined by the functional equation

$$\left(1 - \frac{1}{t}\right) e^{\frac{1}{t}} = \left(1 - \frac{1}{s(t)}\right) e^{\frac{1}{s(t)}}. \tag{7.24}$$

Although the Galois conjugate is only locally defined near the branched point $t = \infty$, we consider $s(t)$ as a global holomorphic function via analytic continuation. For $Re(t) > 1$, (7.24) implies

$$w(t) = -\log x = -\left(\frac{1}{t} - \sum_{n=1}^{\infty} \frac{1}{n} \frac{1}{t^n}\right) = \sum_{n=2}^{\infty} \frac{1}{t^n}.$$

When considered as a functional equation, (7.24) has exactly two solutions: t and

$$s(t) = -t + \frac{2}{3} + \frac{4}{135} t^{-2} + \frac{8}{405} t^{-3} + \frac{8}{567} t^{-4} + \cdots. \tag{7.25}$$

This is the deck-transformation of the projection $\pi : \Sigma \to \mathbb{C}$ near $t = \infty$ and satisfies the involution equation $s\big(s(t)\big) = t$. It is analytic on $\mathbb{C} \setminus [0, 1]$ and has logarithmic singularities at 0 and 1.

Let us calculate the recursion kernel. Since

$$\frac{dx}{x} = \frac{1-z}{z}\, dz = \frac{dt}{t^2(t-1)} = \frac{s'(t)dt}{s(t)^2(s(t)-1)},$$

we have

(7.26)

$$K^H(t, t_1) = \frac{1}{2} \frac{\int_t^{s(t)} W_{0,2}^H(\cdot, t_1)}{W_{0,1}(s(t)) - W_{0,1}(t)} = \frac{1}{2}\left(\frac{1}{t - t_1} - \frac{1}{s(t) - t_1}\right) \frac{t^2(t-1)}{\frac{1}{t} - \frac{1}{s(t)}} \cdot \frac{1}{dt} \cdot dt_1$$
$$= \frac{1}{2}\left(\frac{1}{t - t_1} - \frac{1}{s(t) - t_1}\right) \frac{ts(t)}{s(t) - t} \cdot \frac{t^2(t-1)}{dt} \cdot dt_1.$$

THEOREM 7.5 ([**25**,**58**]). *The Hurwitz differential forms (7.19) for $2g-2+n > 0$ satisfy the Eynard-Orantin recursion:*

$$W^H_{g,n}(t_1,\dots,t_n) = \frac{1}{2\pi i}\oint_{\gamma_\infty} K^H(t,t_1)\Bigg[W^H_{g-1,n+1}(t,s(t),t_2,\dots,t_n) + \sum_{\substack{g_1+g_2=g\\ I\sqcup J=\{2,\dots,n\}}}^{No\ (0,1)\text{-}terms} W^H_{g_1,|I|+1}(t,t_I)W^H_{g_2,|J|+1}(s(t),t_J)\Bigg], \tag{7.27}$$

where γ_∞ is a negatively oriented circle around ∞ whose radius is larger than any of $|t_j|$'s and $|s(t_j)|$'s.

REMARK 7.6. The recursion formula (7.27) was first conjectured by Bouchard and Mariño in [**8**]. Its proofs appear in [**5**, **25**, **58**]. The method of [**5**] is to use a matrix integral expression of the single Hurwitz numbers. The idea of [**25**, **58**] is that the Laplace transform of the *cut-and-join equation* of [**30**, **78**] is the Eynard-Orantin recursion. The cut-and-join equation takes the following form:

$$r(g,\mu)H_g(\mu) = \sum_{i<j}(\mu_i+\mu_j)H_g\big(\mu(\hat{i},\hat{j}),\mu_i+\mu_j\big) + \frac{1}{2}\sum_{i=1}^{n}\sum_{\alpha+\beta=\mu_i}\alpha\beta\left(H_{g-1}\big(\mu(\hat{i}),\alpha,\beta\big) + \sum_{\substack{g_1+g_2=g\\ \nu_1\sqcup\nu_2=\mu(\hat{i})}} H_{g_1}(\nu_1,\alpha)H_{g_2}(\nu_2,\beta)\right). \tag{7.28}$$

Here μ is a partition of length n, and $\mu(\hat{i})$ and $\mu(\hat{i},\hat{j})$ indicate the partition obtained by deleting parts of μ.

REMARK 7.7. As we have seen above, Hurwitz numbers for the unstable geometries determine the spectral curve and hence the shape of the recursion formula (7.27). Since the recursion gives the Hurwitz numbers for all (g,n), we have thus established that unstable $(g,n) = (0,1)$ and $(0,2)$ Hurwitz numbers determine all other single Hurwitz numbers.

It is important to check if the formulas (2.4) and (2.6) agree with the geometry. From definition (7.14) we calculate

$$F^H_{0,3}(t_1,t_2,t_3) = \langle\tau_0\tau_0\tau_0\rangle_{0,3}\hat{\xi}_0(t_1)\hat{\xi}_0(t_2)\hat{\xi}_0(t_3) = (t_1-1)(t_2-1)(t_3-1),$$

which yields

$$W^H_{0,3}(t_1,t_2,t_3) = dt_1dt_2dt_3. \tag{7.29}$$

Since

$$dx(z)\cdot dy(z) = (1-z)dz\cdot dz = \frac{dt\cdot dt}{t^5}$$

from (7.22) and (7.11), the general formula (2.6) yields

$$\begin{aligned} W^H_{0,3}(t_1,t_2,t_3) &= -\frac{1}{2\pi i}\oint_{\gamma_\infty}\frac{W^H_{0,2}(t,t_1)W^H_{0,2}(t,t_2)W^H_{0,2}(t,t_3)}{dx(t)\cdot dy(t)} \\ &= -\left[\frac{1}{2\pi i}\oint_{\gamma_\infty}\frac{t^5}{(t-t_1)^2(t-t_2)^2(t-t_3)^2}dt\right]dt_1dt_2dt_3 = dt_1dt_2dt_3, \end{aligned}$$

in agreement with geometry. Here we calculate the residue at $t = \infty$. Although

$$W^H_{0,2}(t, t_i) = \frac{dt \cdot dt_i}{(t - t_i)^2} - \frac{dx \cdot dx_i}{(x - x_i)^2},$$

the second term does not contribute to the integral. This is because as $t \to \infty$, we have $x \to 1$, and $dx \cdot dx_i/(x - x_i)^2$ has no pole at $x = 1$.

Similarly,

$$F^H_{1,1}(t_1) = \langle \tau_1 \rangle_{1,1} \hat{\xi}_1(t_1) - \langle \tau_0 \lambda_1 \rangle_{1,1} \hat{\xi}_0(t_1) = \frac{1}{24}(t_1^2 - 1)(t_1 - 1),$$

and thus we have

$$W^H_{1,1}(t_1) = \frac{1}{24}(t_1 - 1)(3t_1 + 1)dt_1. \tag{7.30}$$

On the other hand, the general formula (2.4) gives

$$\begin{aligned} W^H_{1,1}(t_1) &= \frac{1}{2\pi i} \oint_{\gamma_\infty} K^H(t, t_1) \left[W^H_{0,2}(u, v) + \frac{dx(u) \cdot dx(v)}{(x(u) - x(v))^2} \right] \Bigg|_{\substack{u=t \\ v=s(t)}} \\ &= \frac{1}{2\pi i} \oint_{\gamma_\infty} K^H(t, t_1) \frac{dt \cdot s'(t)dt}{(t - s(t))^2} \\ &= \left[\frac{1}{2\pi i} \oint_{\gamma_\infty} \frac{1}{2} \left(\frac{1}{t - t_1} - \frac{1}{s(t) - t_1} \right) \frac{ts(t)}{s(t) - t} t^2(t-1) \frac{s'(t)dt}{(t - s(t))^2} \right] dt_1 \\ &= \frac{t_1 s(t_1)}{(t_1 - s(t_1))^3} s(t_1)^2 (s(t_1) - 1)\, dt_1 \\ &\quad - \left[\frac{1}{2\pi i} \oint_{\gamma_{[0,1]}} \frac{1}{2} \left(\frac{1}{t - t_1} - \frac{1}{s(t) - t_1} \right) \frac{ts(t)}{s(t) - t} t^2(t-1) \frac{s'(t)dt}{(t - s(t))^2} \right] dt_1, \end{aligned}$$

where $\gamma_{[0,1]}$ is a contour circling around the slit $[0, 1]$ in the t-plane in the positive direction.

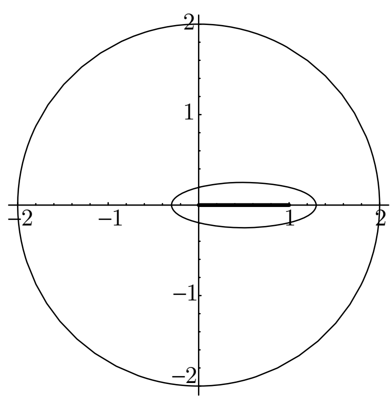

FIGURE 7.2. The contours of integration. The outer loop γ_∞ is the circle of a large radius oriented clock wise, and $\gamma_{[0,1]}$ is the thin loop surrounding the closed interval $[0, 1]$ in the positive direction.

Note that the integrand of the last integral is a holomorphic function in t on $\gamma_{[0,1]}$, hence it has a finite value. It is also clear that as $t_1 \to \infty$, this integral tends to 0, because $\gamma_{[0,1]}$ is a compact space. Therefore, we conclude that

$$W^H_{1,1}(t_1) = \frac{t_1 s(t_1)}{(t_1 - s(t_1))^3} s(t_1)^2 (s(t_1) - 1)\, dt_1 + O(1/t_1)$$
$$= \left(\frac{1}{8}\, t_1^2 - \frac{1}{12}\, t_1 - \frac{1}{24}\right) dt_1 + O(1/t_1),$$

since $s(t) = -t + 2/3 + O(1/t^2)$. It agrees with (7.30) because of the following

LEMMA 7.8. *A solution to the topological recursion (7.27) is a polynomial in t_1.*

PROOF. The t_1-dependence of $W^H_{g,n}(t_1, \dots, t_n)$ only comes from the factor

$$\left(\frac{1}{t - t_1} - \frac{1}{s(t) - t_1}\right)$$
$$= \frac{1}{t} + + \frac{1}{3}\, \frac{1}{t^2} + \left(t_1^2 - \frac{2}{3} t_1 + \frac{2}{9}\right) \frac{1}{t^3} + \left(t_1^2 - \frac{2}{3} t_1 + \frac{22}{135}\right) \frac{1}{t^4} + \cdots$$

in the recursion kernel (7.26). Since each coefficient of the t-expansion of $K^H(t, t_1)$ is a polynomial in t_1, the lemma follows. □

8. The stationary Gromov-Witten invariants of $\mathbb{P}^1$

In this section we study the generating functions of stationary Gromov-Witten invariants of $\mathbb{P}^1$. The conjectural relation between these invariants and the Eynard-Orantin topological recursion was first formulated in [**63**]. We identify the spectral curve and the recursion kernel using the unstable geometries.

So our main object of this section is the Laplace transform of the stationary Gromov-Witten invariants

$$F^{\mathbb{P}^1}_{g,n}(x_1, \dots, x_n) = \sum_{\mu_1, \dots, \mu_n = 0}^{\infty} \langle \tau_{\mu_1}(\omega) \cdots \tau_{\mu_n}(\omega) \rangle_{g,n} \prod_{i=1}^{n} \mu_i! \prod_{i=1}^{n} \frac{1}{x^{\mu_i + 1}}, \tag{8.1}$$

where $\omega \in A_0(\mathbb{P}^1)$ is the point class generator, and

$$\langle \tau_{\mu_1}(\omega) \cdots \tau_{\mu_n}(\omega) \rangle_{g,n} = \int_{[\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1, d)]^{\mathrm{virt}}} \psi_1^{\mu_1} ev_1^*(\omega) \cdots \psi_1^{\mu_n} ev_n^*(\omega) \tag{8.2}$$

is a stationary Gromov-Witten invariant of $\mathbb{P}^1$. More precisely, $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1, d)$ is the moduli stack of stable morphisms from a connected n-pointed curve $(C, p_1, \dots, p_n)$ into $\mathbb{P}^1$ of degree d such that $f(p_i)$, $i = 1, \dots, n$, are distinct, and ev_i is the natural evaluation morphism

$$ev_i : \overline{\mathcal{M}}_{g,n}(\mathbb{P}^1, d) \ni [f, (C, p_1, \dots, p_n)] \longmapsto f(p_i) \in \mathbb{P}^1.$$

The Gromov-Witten invariant (8.2) vanishes unless

$$2g - 2 + 2d = \mu_1 + \cdots + \mu_n. \tag{8.3}$$

The sum in (8.1) is the Laplace transform if we identity

$$x = e^w. \tag{8.4}$$

The extra numerical factor $\prod_{i=1}^n \mu_i!$ is included in (8.1) because of the polynomial growth order of

$$\langle \tau_{\mu_1}(\omega)\cdots\tau_{\mu_n}(\omega)\rangle_{g,n}\prod_{i=1}^n \mu_i! \tag{8.5}$$

for large μ that is established in [**67**]. Indeed (8.5) is essentially a special type of *Hurwitz number* that counts the number of certain coverings of $\mathbb{P}^1$.

To determine the spectral curve and the annulus amplitude, we need to consider unstable geometries $(g,n)=(0,1)$ and $(0,2)$. From [**67**] we learn

$$\langle \tau_{\mu_1}(\omega)\rangle_{0,1}=\langle \tau_{2d-2}(\omega)\rangle_{0,1}=\left(\frac{1}{d!}\right)^2. \tag{8.6}$$

To compute a closed formula for

$$F_{0,1}^{\mathbb{P}^1}(x)=\sum_{\mu_1=0}^\infty \langle \tau_{\mu_1}(\omega)\rangle_{0,1}\ \mu_1!\frac{1}{x^{\mu_1+1}}=\sum_{d=1}^\infty \frac{(2d-2)!}{d!d!}\ \frac{1}{x^{2d-1}},$$

we notice that the generating function of Catalan numbers (3.5)

$$z(x)=\sum_{m=0}^\infty C_m\frac{1}{x^{2m+1}}$$

provides again an effective tool. Thus we have

$$\begin{aligned}\left(x\frac{d}{dx}-1\right)F_{0,1}^{\mathbb{P}^1}(x)&=-2\sum_{d=1}^\infty \frac{(2d-2)!}{(d-1)!d!}\ \frac{1}{x^{2d-1}}\\ &=-2\sum_{m=0}^\infty\frac{(2m)!}{(m+1)!m!}\ \frac{1}{x^{2m+1}}=-2z(x).\end{aligned} \tag{8.7}$$

The advantage of using the Catalan series $z(x)$ is that we know its inverse function (3.8). Using (4.6), we see that (8.7) is equivalent to

$$\left(\frac{z^3+z}{z^2-1}\ \frac{d}{dz}-1\right)F_{0,1}^{\mathbb{P}^1}(z)=-2z. \tag{8.8}$$

The solution of (8.8) is given by

$$F_{0,1}^{\mathbb{P}^1}(z)=-\frac{2}{z}-\left(z+\frac{1}{z}\right)\log(1+z^2)+c\left(z+\frac{1}{z}\right),$$

with a constant of integration c. Since

$$z\to 0\Longrightarrow x\to\infty\Longrightarrow F_{0,1}^{\mathbb{P}^1}\to 0,$$

we conclude that $c=2$. We thus obtain

$$F_{0,1}^{\mathbb{P}^1}(z)=2z-\left(z+\frac{1}{z}\right)\log(1+z^2), \tag{8.9}$$

and therefore,

$$W_{0,1}^{\mathbb{P}^1}(z)=dF_{0,1}^{\mathbb{P}^1}(z)=-\log(1+z^2)\ d\left(z+\frac{1}{z}\right). \tag{8.10}$$

THEOREM 8.1. *The spectral curve for the stationary Gromov-Witten invariants of* $\mathbb{P}^1$ *is given by*

$$\begin{cases} x = z + \frac{1}{z} \\ y = -\log(1+z^2). \end{cases} \tag{8.11}$$

REMARK 8.2. Since $dx = 0$ has two zeros at $z = \pm 1$, we also use as our preferred coordinate

$$t = \frac{z+1}{z-1} \Longleftrightarrow z = \frac{t+1}{t-1}. \tag{8.12}$$

We obtain a well defined branch of the log function appearing in the spectral curve by removing $\{z = is \,|\, 1 \leq s < \infty\}$, which corresponds in the t-plane with the right semicircle of radius 1 connecting i to $-i$ (see Figure 8.1). The expression of $W_{0,1}^{\mathbb{P}^1}$ in terms of the preferred coordinate is

$$W_{0,1}^{\mathbb{P}^1}(t) = \frac{8t}{(t^2-1)^2} \log\left(\frac{2(t^2+1)}{(t-1)^2}\right) dt. \tag{8.13}$$

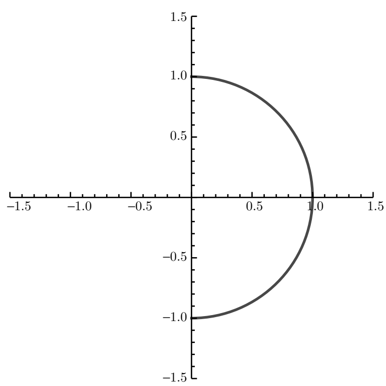

FIGURE 8.1. The spectral curve for the stationary Gromov-Witten invariants of $\mathbb{P}^1$ is the complex t-plane minus the semicircle.

REMARK 8.3. The function $x = z + \frac{1}{z}$ is expected here, since it is the Landau-Ginzburg model that is homologically mirror dual to $\mathbb{P}^1$ [**2**].

REMARK 8.4. The Galois conjugate of $x = z + \frac{1}{z}$ is globally defined, and is given by

$$t \longmapsto \bar{t} = -t. \tag{8.14}$$

REMARK 8.5. Since

$$\frac{1}{1-z(x)} = \sum_{k=0}^{\infty} z(x)^k = 1 + \sum_{n=0}^{\infty} \binom{n}{\lfloor \frac{n}{2} \rfloor} \frac{1}{x^{n+1}}, \tag{8.15}$$

we can express t in the branch near $t = -1$ as a function in x. The result is

$$t+1 = \frac{z(x)+1}{z(x)-1} + 1 = 2 - \frac{2}{1-z(x)} = -\sum_{n=0}^{\infty} 2\binom{n}{\lfloor \frac{n}{2} \rfloor} \frac{1}{x^{n+1}}, \tag{8.16}$$

which is also absolutely convergent for $|x| > 2$.

REMARK 8.6. We are using the normalized Gromov-Witten invariants (8.5) to compute the Laplace transform (8.1). If we did not include the $\mu!$ factor in our computation of the spectral curve, then we would have encountered the *modified Bessel function*

$$I_0(2x) = \sum_{m=1}^{\infty} \frac{1}{(m!)^2} x^{2m},$$

instead of $z(x)$, in computing (8.9). We note that $I_0(2x)$ appears in [**19**] in the exact same context of computing the Gromov-Witten invariants of $\mathbb{P}^1$. We prefer the Catalan number series $z(x)$ over the modified Bessel function mainly because the inverse function of $z(x)$ takes a simple form $x = z + \frac{1}{z}$.

Motivated by the technique developed in [**8**, **25**, **58**] for single Hurwitz numbers, let us define

$$\xi_n(t) = \sum_{k=0}^{\infty} \binom{2k}{k} k^n \frac{1}{x^{2k+1}}, \qquad n \geq 0, \tag{8.17}$$

and

$$\eta_n(t) = \sum_{k=0}^{\infty} \binom{2k+1}{k} k^n \frac{1}{x^{2k+2}}, \qquad n \geq 0. \tag{8.18}$$

We then have

$$\xi_{n+1}(t) = -\frac{1}{2}\left(x\frac{d}{dx} + 1\right)\xi_n(t) = \left(\frac{t^4-1}{8t}\frac{d}{dt} - \frac{1}{2}\right)\xi_n(t) \tag{8.19}$$

and

$$\eta_{n+1}(t) = -\frac{1}{2}\left(x\frac{d}{dx} + 2\right)\eta_n(t) = \left(\frac{t^4-1}{8t}\frac{d}{dt} - 1\right)\eta_n(t). \tag{8.20}$$

The initial values are computed as follows:

$$\begin{aligned} \xi_0(t) &= \frac{1}{2}\left(1 - x\frac{d}{dx}\right)\sum_{m=0}^{\infty} \frac{1}{m+1}\binom{2m}{m}\frac{1}{x^{2m+1}} \\ &= \frac{1}{2}\left(1 - \frac{z(z^2+1)}{z^2-1}\frac{d}{dz}\right) z = -\frac{z}{z^2-1} = -\frac{t^2-1}{4t}, \end{aligned} \tag{8.21}$$

and similarly

$$\eta_0(t) = -\frac{(t+1)^2}{4t}. \tag{8.22}$$

We note that $\xi_n(t)$ and $\eta_n(t)$ are Laurent polynomials of degree $2n+1$ for every $n \geq 0$. Since they are defined as functions in x, we have the reciprocity property

$$\begin{aligned} \xi_n(1/t) &= -\xi_n(t) \\ \eta_n(1/t) &= \eta_n(t). \end{aligned} \tag{8.23}$$

This follows from

$$t \longmapsto \frac{1}{t} \Longrightarrow x \longmapsto -x.$$

The annulus amplitude requires $(g,n) = (0,2)$ Gromov-Witten invariants. They can be calculated from the $(g,n) = (0,1)$ invariants using the Topological Recursion Relation [**29**]. The results are

(8.24)
$$\langle \tau_{\mu_1}(\omega)\tau_{\mu_2}(\omega)\rangle_{0,2} = \begin{cases} \frac{1}{(m_1!)^2(m_2!)^2}\ \frac{1}{(m_1+m_2+1)} & \mu_1 = 2m_1, \mu_2 = 2m_2 \\ \frac{1}{(m_1!)^2(m_2!)^2}\ \frac{1}{(m_1+m_2+2)} & \mu_1 = 2m_1+1, \mu_2 = 2m_2+1. \end{cases}$$

THEOREM 8.7. *The annulus amplitude is given by*

$$F_{0,2}^{\mathbb{P}^1}(z_1,z_2) = -\log(1-z_1z_2). \tag{8.25}$$

Hence we have

$$W_{0,2}^{\mathbb{P}^1}(t_1,t_2) = \frac{dt_1\cdot dt_2}{(t_1-t_2)^2} - \frac{dx_1\cdot dx_2}{(x_1-x_2)^2} = \frac{dt_1\cdot dt_2}{(t_1+t_2)^2}. \tag{8.26}$$

PROOF. From (8.24) we calculate

$$\begin{aligned} F_{0,2}^{\mathbb{P}^1}(z_1,z_2) &= \sum_{\mu_1,\mu_2=0}^{\infty} \langle \tau_{\mu_1}(\omega)\tau_{\mu_2}(\omega)\rangle_{0,2}\ \mu_1!\mu_2!\ \frac{1}{x_1^{\mu_1+1}}\ \frac{1}{x_2^{\mu_2+1}} \\ &= \sum_{m_1,m_2=0}^{\infty} \frac{1}{(m_1+m_2+1)}\binom{2m_1}{m_1}\binom{2m_2}{m_2}\frac{1}{x_1^{2m_1+1}}\ \frac{1}{x_2^{2m_2+1}} \\ &+ \sum_{m_1,m_2=0}^{\infty} \frac{1}{(m_1+m_2+2)}(2m_1+1)(2m_2+1)\binom{2m_1}{m_1}\binom{2m_2}{m_2}\frac{1}{x_1^{2m_1+2}}\ \frac{1}{x_2^{2m_2+2}}. \end{aligned}$$

Thus we have

$$\begin{aligned} &\left(x_1\frac{d}{dx_1} + x_2\frac{d}{dx_2}\right) F_{0,2}^{\mathbb{P}^1}(z_1,z_2) \\ &\quad = -2\sum_{m_1,m_2=0}^{\infty} \binom{2m_1}{m_1}\binom{2m_2}{m_2}\frac{1}{x_1^{2m_1+1}}\ \frac{1}{x_2^{2m_2+1}} \\ &\quad - 2\sum_{m_1,m_2=0}^{\infty} (2m_1+1)(2m_2+1)\binom{2m_1}{m_1}\binom{2m_2}{m_2}\frac{1}{x_1^{2m_1+2}}\ \frac{1}{x_2^{2m_2+2}} \\ &\quad = -2\xi_0(x_1)\xi_0(x_2) - 2z'(x_1)z'(x_2) \\ &\quad = -2\frac{z_1}{z_1^2-1}\ \frac{z_2}{z_2^2-1} - 2\frac{z_1^2}{z_1^2-1}\ \frac{z_2^2}{z_2^2-1} = -2\frac{z_1z_2(1+z_1z_2)}{(z_1^2-1)(z_2^2-1)}, \end{aligned}$$

where $\xi_0(x)$ is calculated in (8.21), and from (4.5) we know

$$z'(x) = \frac{dz}{dx} = \frac{z^2}{z^2-1}.$$

On the other hand,

$$\begin{aligned} &\left(x_1\frac{d}{dx_1} + x_2\frac{d}{dx_2}\right)(-\log(1-z_1z_2)) \\ &\quad = \left(\frac{z_1(z_1^2+1)}{z_1^2-1}\ \frac{d}{dz_1} + \frac{z_2(z_2^2+1)}{z_2^2-1}\ \frac{d}{dz_2}\right)(-\log(1-z_1z_2)) \\ &\quad = \left(\frac{(z_1^2+1)}{z_1^2-1} + \frac{(z_2^2+1)}{z_2^2-1}\right)\frac{z_1z_2}{1-z_1z_2} = -2\ \frac{z_1z_2(1+z_1z_2)}{(z_1^2-1)(z_2^2-1)}. \end{aligned}$$

Therefore,

$$\left(x_1\frac{d}{dx_1}+x_2\frac{d}{dx_2}\right)\left(F^{\mathbb{P}^1}_{0,2}(z_1,z_2)+\log(1-z_1z_2)\right)$$
$$=\left(x_1\frac{d}{dx_1}+x_2\frac{d}{dx_2}\right)\left(\sum_{\mu_1,\mu_2=0}^{\infty}\langle\tau_{\mu_1}(\omega)\tau_{\mu_2}(\omega)\rangle_{0,2}\ \mu_1!\mu_2!\ \frac{1}{x_1^{\mu_1+1}}\ \frac{1}{x_2^{\mu_2+1}}\right.$$
$$\left.-\sum_{n=1}^{\infty}\frac{1}{n}\left(\sum_{m=0}^{\infty}C_m\frac{1}{x_1^{2m+1}}\sum_{m=0}^{\infty}C_m\frac{1}{x_2^{2m+1}}\right)^n\right)=0.$$

Since the kernel of the Euler differential operator consists of constant functions, and since actual computation shows that the first few expansion terms of the Laurent series

$$\sum_{\mu_1,\mu_2=0}^{\infty}\langle\tau_{\mu_1}(\omega)\tau_{\mu_2}(\omega)\rangle_{0,2}\ \mu_1!\mu_2!\ \frac{1}{x_1^{\mu_1+1}}\ \frac{1}{x_2^{\mu_2+1}}$$
$$-\sum_{n=1}^{\infty}\frac{1}{n}\left(\sum_{m=0}^{\infty}C_m\frac{1}{x_1^{2m+1}}\sum_{m=0}^{\infty}C_m\frac{1}{x_2^{2m+1}}\right)^n$$

are 0, we complete the proof of (8.25). □

Using $\xi_n(t)$ and $\eta_n(t)$ of (8.17) and (8.18) and the classical topological recursion relation [**29**], we can systematically calculate the Laplace transform of stationary Gromov-Witten invariants. First let us consider $(g,n)=(0,3)$. Since the sum of the descendant indices of

$$\langle\tau_{\mu_1}(\omega)\tau_{\mu_2}(\omega)\tau_{\mu_3}(\omega)\rangle_{0,3}$$

is even, we have

$$\begin{aligned}\langle\tau_{2m_1}(\omega)\tau_{2m_2}(\omega)\tau_{2m_3}(\omega)\rangle_{0,3}&=\frac{1}{m_1^2m_2^2m_3^2},\\ \langle\tau_{2m_1}(\omega)\tau_{2m_2+1}(\omega)\tau_{2m_3+1}(\omega)\rangle_{0,3}&=\frac{(m_2+1)(m_3+1)}{m_1^2(m_2+1)^2(m_3+1)^2}.\end{aligned}\tag{8.27}$$

The Laplace transform is therefore

(8.28)
$$F^{\mathbb{P}^1}_{0,3}(t_1,t_2,t_3)=\sum_{\mu_1\mu_2\mu_3\ge 0}\langle\tau_{\mu_1}(\omega)\tau_{\mu_2}(\omega)\tau_{\mu_3}(\omega)\rangle_{0,3}\mu_1!\mu_2!\mu_3!\frac{1}{x_1^{\mu_1+1}}\cdot\frac{1}{x_2^{\mu_2+1}}\cdot\frac{1}{x_3^{\mu_3+1}}$$
$$=\sum_{m_1,m_2,m_3\ge 0}\binom{2m_1}{m_1}\binom{2m_2}{m_2}\binom{2m_3}{m_3}\frac{1}{x_1^{2m_1+1}}\cdot\frac{1}{x_2^{2m_2+1}}\cdot\frac{1}{x_3^{2m_3+1}}$$
$$+\sum_{m_1,m_2,m_3\ge 0}\binom{2m_1}{m_1}\binom{2m_2+1}{m_2}\binom{2m_3+1}{m_3}\frac{1}{x_1^{2m_1+1}}\cdot\frac{1}{x_2^{2m_2+2}}\cdot\frac{1}{x_3^{2m_3+2}}$$
$$+\sum_{m_1,m_2,m_3\ge 0}\binom{2m_1+1}{m_1}\binom{2m_2}{m_2}\binom{2m_3+1}{m_3}\frac{1}{x_1^{2m_1+2}}\cdot\frac{1}{x_2^{2m_2+1}}\cdot\frac{1}{x_3^{2m_3+2}}$$
$$+\sum_{m_1,m_2,m_3\ge 0}\binom{2m_1+1}{m_1}\binom{2m_2+1}{m_2}\binom{2m_3}{m_3}\frac{1}{x_1^{2m_1+2}}\cdot\frac{1}{x_2^{2m_2+2}}\cdot\frac{1}{x_3^{2m_3+1}}$$

$$= \xi_0(t_1)\xi_0(t_2)\xi_0(t_3) + \xi_0(t_1)\eta_0(t_2)\eta_0(t_3) + \eta_0(t_1)\xi_0(t_2)\eta_0(t_3) + \eta_0(t_1)\eta_0(t_2)\xi_0(t_3)$$
$$= -\frac{1}{16}(t_1+1)(t_2+1)(t_3+1)\left(1-\frac{1}{t_1t_2t_3}\right),$$

which is indeed a Laurent polynomial. Since it is an odd degree polynomial in $\xi_n(t)$'s, we have the reciprocity

$$F_{0,3}^{\mathbb{P}^1}(1/t_1, 1/t_2, 1/t_3) = -F_{0,3}^{\mathbb{P}^1}(t_1, t_2, t_3).$$

The $n = 1$ stationary invariants are concretely calculated in [**67**]. We have

$$\begin{aligned}
\langle \tau_{2d} \rangle_{1,1} &= \frac{1}{24}\left(\frac{1}{d!}\right)^2 (2d-1) \\
\langle \tau_{2d+2} \rangle_{2,1} &= \left(\frac{1}{d!}\right)^2 \left(\frac{1}{5!\,4^2}(2d-1) + \frac{1}{24^2}\binom{2d-1}{2}\right) \\
\langle \tau_{2g-2+2d} \rangle_{g,1} &= \left(\frac{1}{d!}\right)^2 \sum_{\ell=1}^{g} \binom{2d-1}{\ell} \sum_{\substack{k_i>0 \\ k_1+\cdots+k_\ell=g}} \prod_{i=1}^{\ell} \frac{1}{(2k_i+1)!\,4^{k_i}}.
\end{aligned} \tag{8.29}$$

We thus obtain

$$\begin{aligned}
F_{1,1}^{\mathbb{P}^1}(t_1) &= \frac{1}{24}\sum_{d=0}^{\infty}\binom{2d}{d}(2d-1)\frac{1}{x_1^{2d+1}} = \frac{1}{24}\left(2\xi_1(t_1) - \xi_0(t_1)\right) \\
&= -\frac{1}{384}\left(t_1^3 - 7t_1 + \frac{7}{t_1} - \frac{1}{t_1^3}\right).
\end{aligned} \tag{8.30}$$

To calculate the $g = 2$ case we need to do the following.

$$\begin{aligned}
F_{2,1}^{\mathbb{P}^1}(t_1) &= \sum_{d=0}^{\infty} \frac{(2d+2)!}{d!\,d!}\left(\frac{1}{5!\,4^2}(2d-1) + \frac{1}{24^2}\binom{2d-1}{2}\right)\frac{1}{x_1^{2d+3}} \\
&= \left(\frac{d}{dx_1}\right)^2 \sum_{d=0}^{\infty}\binom{2d}{d}\left(\frac{1}{5!\,4^2}(2d-1) + \frac{1}{24^2}\left(2d^2-3d+1\right)\right)\frac{1}{x_1^{2d+1}} \\
&= \left(-\frac{(t^2-1)^2}{8t}\frac{d}{dt}\right)^2 \left[\frac{1}{5!\,4^2}(2\xi_1(t_1) - \xi_0(t_1)) + \frac{1}{24^2}\left(2\xi_2(t_1) - 3\xi_1(t_1) + \xi_0(t_1)\right)\right] \\
&= -\frac{1}{2^{19}\cdot 3^2 \cdot 5}\,\frac{(t^2-1)^3}{t^9}\big(525t_1^{12} - 1470t_1^{10} + 1107t_1^8 + 527t_1^6 + 1107t_1^4 - 1470t_1^2 + 525\big).
\end{aligned} \tag{8.31}$$

PROPOSITION 8.8. *$F_{g,1}^{\mathbb{P}^1}(t_1)$ is a Laurent polynomial of degree $6g-3$ with the reciprocity*

$$F_{g,1}^{\mathbb{P}^1}(1/t_1) = -F_{g,1}^{\mathbb{P}^1}(t_1).$$

PROOF. First we calculate the binomial coefficient

$$\binom{2d-1}{\ell} = \frac{1}{\ell!}(2d-1)(2d-2)\cdots(2d-\ell) = \frac{1}{\ell!}\left(2^\ell d^\ell - \frac{\ell(\ell+1)}{2}d^{\ell-1} + \cdots + (-1)^\ell \ell!\right)$$

as a polynomial in d, and then replace each d^i with $\xi_i(t_1)$. The result is a linear combination of $\xi_0(t_1), \ldots, \xi_\ell(t_1)$. Let $\Xi_\ell(t_1)$ denote the resulting Laurent polynomial of degree $2\ell + 1$. Then we have an expression

(8.32)

$$F^{\mathbb{P}^1}_{g,1}(t_1) = \left(\frac{d}{dx_1}\right)^{2g-2} \sum_{d=0}^{\infty} \binom{2d}{d} \sum_{\ell=1}^{g} \binom{2d-1}{\ell} \sum_{\substack{k_i>0 \\ k_1+\cdots+k_\ell=g}} \prod_{i=1}^{\ell} \frac{1}{(2k_i+1)!\, 4^{k_i}} \frac{1}{x_1^{2d+1}}$$

$$= \left(-\frac{(t^2-1)^2}{8t}\frac{d}{dt}\right)^{2g-2} \left[\sum_{\ell=1}^{g} \Xi_\ell(t_1) \sum_{\substack{k_i>0 \\ k_1+\cdots+k_\ell=g}} \prod_{i=1}^{\ell} \frac{1}{(2k_i+1)!\, 4^{k_i}}\right],$$

which is a Laurent polynomial of degree $2(2g-2)+2g+1 = 6g-3$. The reciprocity property follows from (8.23) and the x_1 expression of (8.32), where x_1 changes to $-x_1$. In particular, the even order differentiation in x_1 is not affected by this change. ☐

The Eynard-Orantin recursion is for the differential forms $W^{\mathbb{P}^1}_{g,n}(t_1,\dots,t_n)$. In order to find the recursion kernel we use (8.13) and (8.26) to compute

$$K^{\mathbb{P}^1}(t,t_1) = \frac{1}{2}\frac{\int_t^{-t} W^{\mathbb{P}^1}_{0,2}(\cdot,t_1)}{W^{\mathbb{P}^1}_{0,1}(-t) - W^{\mathbb{P}^1}_{0,1}(t)} \tag{8.33}$$

$$= \frac{1}{2}\left(\frac{1}{t+t_1}+\frac{1}{t-t_1}\right)\frac{1}{\log\left(\frac{2(t^2+1)}{(t+1)^2}\right) - \log\left(\frac{2(t+1)}{(t-1)^2}\right)}\frac{(t^2-1)^2}{-8t\,dt}\cdot dt_1$$

$$= \frac{1}{16}\left(\frac{1}{t+t_1}+\frac{1}{t-t_1}\right)\frac{1}{\log\left(\frac{(t-1)^2}{(t+1)^2}\right)}\frac{(t^2-1)^2}{t\,dt}\cdot dt_1.$$

We note the reciprocity property of the kernel

$$K^{\mathbb{P}^1}(1/t,1/t_1) = -K^{\mathbb{P}^1}(t,t_1). \tag{8.34}$$

The topological recursion (2.3) becomes

$$W^{\mathbb{P}^1}_{g,n}(t_1,t_2,\dots,t_n) = \frac{1}{2\pi i}\oint_\gamma K^{\mathbb{P}^1}(t,t_1)\Bigg[W^{\mathbb{P}^1}_{g-1,n+1}(t,-t,t_2,\dots,t_n) + \sum_{\substack{g_1+g_2=g \\ I\sqcup J=\{2,3,\dots,n\}}}^{\text{No }(0,1)\text{ terns}} W^{\mathbb{P}^1}_{g_1,|I|+1}(t,t_I)W^{\mathbb{P}^1}_{g_2,|J|+1}(-t,t_J)\Bigg], \tag{8.35}$$

where the residue calculation is taken along the integration contour γ (see Figure 4.1) consisting of two concentric circles of radius ϵ and $1/\epsilon$ for a small ϵ centered around $t=0$, with the inner circle positively oriented and the outer circle negatively oriented. Since there is a log singularity in the complex t-plane, we cannot use the residue calculus method to evaluate the integral at $t=t_1$ and $t=-t_1$. Thus the residue calculation of (8.35) is performed around the neighborhood of $t=0$ and $t=\infty$.

So let us provide two expansion formulas for the kernel $K^{\mathbb{P}^1}(t,t_1)$, assuming that $t_1\in\mathbb{C}^*$ is away from the log singularity of Figure 8.1. The transcendental factor of $K^{\mathbb{P}^1}(t,t_1)$ has an expansion

$$\frac{4}{t\log\left(\frac{(t-1)^2}{(t+1)^2}\right)} = -\frac{1}{t^2}+\frac{1}{3}+\frac{4}{45}t^2+\frac{44}{945}t^4+\frac{428}{14175}t^6+\frac{10196}{467775}t^8+\cdots \tag{8.36}$$

around $t=0$. The denominator of the coefficient of t^{2k-2} is given by

$$\prod_{q=3,\ \text{prime}}^{2k+1} q^{\lfloor \frac{2k}{q-1} \rfloor} = 3^{\lfloor k \rfloor}\cdot 5^{\lfloor \frac{k}{2} \rfloor}\cdot 7^{\lfloor \frac{k}{3} \rfloor}\cdots,$$

which is the same as $\mu(L_k)$ of [**39**, Lemma 1.5.2]. The expansion of $\frac{1}{t+t_1}+\frac{1}{t-t_1}$ at $t=0$ is given by

$$\frac{1}{t+t_1}+\frac{1}{t-t_1} = -2t\,\frac{1}{t_1^2}\frac{1}{1-\frac{t^2}{t_1^2}} = -2\sum_{n=0}^{\infty}\frac{t^{2n+1}}{t_1^{2n+2}}.$$

From the expression (8.33) and the above consideration, we know that around $t=0$, $K^{\mathbb{P}^1}(t,t_1)$ starts from t^{-1}, and that the coefficient of t^{2n-1} is a Laurent polynomial in t_1^2 starting from $\frac{1}{32}t_1^{-(2n+2)}$ up to t_1^{-2} with rational coefficients. More concretely, we have

(8.37)

$$\begin{aligned}K^{\mathbb{P}^1}(t,t_1) = \Bigg[&\frac{1}{t}\left(\frac{1}{32}\frac{1}{t_1^2}\right)+t\left(\frac{1}{32}\frac{1}{t_1^4}-\frac{7}{96}\frac{1}{t_1^2}\right)+t^3\left(\frac{1}{32}\frac{1}{t_1^6}-\frac{7}{96}\frac{1}{t_1^4}+\frac{71}{1440}\frac{1}{t_1^2}\right)\\ &+t^5\left(\frac{1}{32}\frac{1}{t_1^8}-\frac{7}{96}\frac{1}{t_1^6}+\frac{71}{1440}\frac{1}{t_1^4}-\frac{191}{30240}\frac{1}{t_1^2}\right)\\ &+t^7\left(\frac{1}{32}\frac{1}{t_1^{10}}-\frac{7}{96}\frac{1}{t_1^8}+\frac{71}{1440}\frac{1}{t_1^6}-\frac{191}{30240}\frac{1}{t_1^4}-\frac{23}{28350}\frac{1}{t_1^2}\right)\\ &+t^9\left(\frac{1}{32}\frac{1}{t_1^{12}}-\frac{7}{96}\frac{1}{t_1^{10}}+\frac{71}{1440}\frac{1}{t_1^8}-\frac{191}{30240}\frac{1}{t_1^6}-\frac{23}{28350}\frac{1}{t_1^4}-\frac{233}{935550}\frac{1}{t_1^2}\right)+\cdots\Bigg]\frac{1}{dt}\cdot dt_1.\end{aligned}$$

Similarly, around $t=\infty$ we have

$$\begin{aligned}(8.38)\quad K^{\mathbb{P}^1}(t,t_1) = \Bigg[&-t^3\frac{1}{32}+t\left(-\frac{1}{32}t_1^2+\frac{7}{96}\right)+\frac{1}{t}\left(-\frac{1}{32}t_1^4+\frac{7}{96}t_1^2-\frac{71}{1440}\right)\\ &+\frac{1}{t^3}\left(-\frac{1}{32}t_1^6+\frac{7}{96}t_1^4-\frac{71}{1440}t_1^2+\frac{191}{30240}\right)\\ &+\frac{1}{t^5}\left(-\frac{1}{32}t_1^8+\frac{7}{96}t_1^6-\frac{71}{1440}t_1^4+\frac{191}{30240}t_1^2+\frac{23}{28350}\right)\\ &+\frac{1}{t^7}\left(-\frac{1}{32}t_1^{10}+\frac{7}{96}t_1^8-\frac{71}{1440}t_1^6+\frac{191}{30240}t_1^4+\frac{23}{28350}t_1^2+\frac{233}{935550}\right)+\cdots\Bigg]\frac{1}{dt}\cdot dt_1.\end{aligned}$$

THEOREM 8.9. *The Eynard-Orantin differential form $W_{g,n}^{\mathbb{P}^1}(t_1,\dots,t_n)$ is a Laurent polynomial in $t_1^2,t_2^2,\dots,t_n^2$ of degree $2(3g-3+n)$ in the stable range $2g-2+n>0$. It satisfies the reciprocity property*

$$W_{g,n}^{\mathbb{P}^1}(1/t_1,\dots,1/t_n) = (-1)^n W_{g,n}^{\mathbb{P}^1}(t_1,\dots,t_n) \tag{8.39}$$

as a meromorphic symmetric n-form. The highest degree terms form a homogeneous polynomial of degree $2(3g-3+n)$, which is given by

(8.40)

$$\widehat{W}_{g,n}^{\mathbb{P}^1}(t_1,\dots,t_n) = \frac{(-1)^n}{2^{2g-2+2n}}\sum_{k_1,\dots,k_n\ge 0}\langle\tau_{k_1}\cdots\tau_{k_n}\rangle_{g,n}\prod_{i=1}^{n}\left[(2k_1+1)!!\left(\frac{t_i}{2}\right)^{2k_i}dt_i\right].$$

Indeed it is the same as the generating function of the ψ-class intersection numbers (6.4).

PROOF. The statement is proved by induction on $2g-2+n$ using the recursion (8.35). The initial cases $(g,n)=(1,1)$ and $(g,n)=(0,3)$ are easily verified from the concrete calculations below. Since we are expanding $\frac{1}{t+t_1}+\frac{1}{t-t_1}$ around $t=0$ and $t=\infty$, it is obvious that the recursion produces a Laurent polynomial in $t_1^2, t_2^2, \dots, t_n^2$ as the result.

Equation (8.38) tells us that the residue calculation at infinity increases the degree by 4. This is because the leading term of the coefficient of $t^{-(2n+1)}$ is t_1^{2n+4}, and the residue calculation picks up the term t^{2n}. By the induction hypothesis, the right-hand side of (8.35) without the kernel term has homogenous degree $2(3g-3+n)-4$. The reciprocity property also follows by induction using (8.34).

The leading terms of $W_{g,n}^{\mathbb{P}^1}(t_1,\dots,t_n)$ satisfy a topological recursion themselves. We can extract the terms in the kernel that produce the leading terms of the differential forms from (8.36) or (8.38). The result is

$$K^{\mathrm{WK}}(t,t_1) = -\frac{1}{32}\, t^3 \sum_{k=0}^{\infty} \frac{t_1^{2n}}{t^{2n}}\frac{1}{dt}\cdot dt_1 = -\frac{1}{2}\left(\frac{1}{t-t_1}+\frac{1}{t+t_1}\right)\frac{1}{32}\, t^4\cdot\frac{1}{dt}\cdot dt_1, \tag{8.41}$$

which is identical to [**11**, Theorem 7.4], and also to (6.11). Since the topological recursion uniquely determines all the differential forms from the initial condition, and again since the $(g,n)=(0,3)$ and $(1,1)$ cases satisfy (8.40), by induction we obtain (8.40) for all stable values of (g,n). □

The $(g,n)=(1,1)$ Eynard-Orantin differential form is computed using (2.4).

(8.42)
$$\begin{aligned}
W_{1,1}^{\mathbb{P}^1}(t_1) &= \frac{1}{2\pi i}\int_\gamma K^{\mathbb{P}^1}(t,t_1)\left[W_{0,2}^{\mathbb{P}^1}(t,-t)+\frac{dx\cdot dx_1}{(x-x_1)^2}\right] = -\frac{1}{2\pi i}\int_\gamma K^{\mathbb{P}^1}(t,t_1)\frac{dt\cdot dt}{4t^2}\\
&= -\frac{1}{64}\left(\frac{1}{2\pi i}\int_\gamma\left(\frac{1}{t+t_1}+\frac{1}{t-t_1}\right)\frac{1}{\log\left(\frac{(t-1)^2}{(t+1)^2}\right)}\frac{(t^2-1)^2}{t^3}dt\right)dt_1\\
&= \left(-\frac{1}{128}t_1^2+\frac{7}{384}+\frac{7}{384}\frac{1}{t_1^2}-\frac{1}{128}\frac{1}{t_1^4}\right)dt_1.
\end{aligned}$$

This is in agreement with $W_{1,1}^{\mathbb{P}^1}(t_1)=dF_{1,1}^{\mathbb{P}^1}(t_1)$ and (8.30). From (8.35) we have

$$\begin{aligned}
&W_{0,3}^{\mathbb{P}^1}(t_1,t_2,t_3)\\
&= \frac{1}{2\pi i}\int_\gamma K^{\mathbb{P}^1}(t,t_1)\left[W_{0,2}^{\mathbb{P}^1}(t,t_2)W_{0,2}^{\mathbb{P}^1}(-t,t_3)+W_{0,2}^{\mathbb{P}^1}(t,t_3)W_{0,2}^{\mathbb{P}^1}(-t,t_2)\right]\\
&= -\frac{1}{16}\left(1+\frac{1}{t_1^2\,t_2^2\,t_3^2}\right)dt_1dt_2dt_3,
\end{aligned} \tag{8.43}$$

which is also in agreement with (8.28).

Norbury and Scott conjecture the following

CONJECTURE 8.10 (Norbury-Scott Conjecture [**63**]). *For (g,n) in the stable range we have*

$$W_{g,n}^{\mathbb{P}^1}(t_1,\dots,t_n) = d_1\cdots d_n F_{g,n}^{\mathbb{P}^1}(t_1,\dots,t_n). \tag{8.44}$$

The conjecture is verified for $g = 0$ and $g = 1$ cases in [**63**], with numerical evidence provided for higher degree cases. We recall that the Eyanrd-Orantin recursion for simple Hurwitz numbers is essentially the Laplace transform of the cut-and-join equation [**25**]. For the case of the counting problem of clean Belyi morphisms the recursion is the Laplace transform of the edge-contraction operation of Theorem 3.2.

QUESTION 8.11. *What is the equation among the stationary Gromov-Witten invariants of $\mathbb{P}^1$ whose Laplace transform is the Eynard-Orantin recursion (8.35)?*

Appendix A. Calculation of the Laplace transform

In this appendix we give the proof of Theorem 4.3.

PROPOSITION A.1. *Let us use the x_j-variables defined by $x_j = e^{w_j}$, and write*

$$W^D_{g,n}(t_1, \dots, t_n) = w_{g,n}(x_1, \dots, x_n)\, dx_1 \cdots dx_n.$$

Then the Laplace transform of the recursion formula (3.16) is the following differential recursion:

$$\begin{aligned}
\text{(A.1)}\quad & -x_1\, w_{g,n}(x_1, \dots, x_n) \\
&= \sum_{j=2}^{n} \frac{\partial}{\partial x_j}\left(\frac{1}{x_j - x_1}\left(w_{g,n-1}(x_2, \dots, x_n) - w_{g,n-1}(x_1, x_2, \dots, \widehat{x_j}, \dots, x_n)\right)\right) \\
&\quad + w_{g-1,n+1}(x_1, x_1, x_2, \dots, x_n) + \sum_{\substack{g_1+g_2=g \\ I \sqcup J = \{2,\dots,n\}}} w_{g_1,|I|+1}(x_1, x_I) w_{g_2,|J|+1}(x_1, x_J).
\end{aligned}$$

PROOF. The operation we wish to do is to apply

$$(-1)^n \sum_{\mu_1,\dots,\mu_n > 0} \mu_2 \cdots \mu_n \prod_{i=1}^{n} \frac{1}{x_i^{\mu_i+1}}$$

to each side of (3.16). Then by (4.13), the left-hand side becomes $w_{g,n}(x_1, \dots, x_n)$.

The second line of (3.16) is straightforward. Let us just consider the first term, since the computation of the second term is the same.

$$\begin{aligned}
&(-1)^n \sum_{\mu_1,\dots,\mu_n>0} \mu_2 \cdots \mu_n \prod_{i=1}^{n} \frac{1}{x_i^{\mu_i+1}} \sum_{\alpha+\beta=\mu_1-2} \alpha\beta D_{g-1,n+1}(\alpha, \beta, \mu_2, \dots, \mu_n) \\
&= -\frac{1}{x_1}(-1)^{n+1} \sum_{\mu_2,\dots,\mu_n>0} \sum_{\alpha,\beta>0} \alpha\beta\mu_2 \cdots \mu_n D_{g-1,n+1}(\alpha, \beta, \mu_2 \dots, \mu_n) \frac{1}{x_1^{\alpha+1}} \cdot \frac{1}{x_1^{\beta+1}} \prod_{i=2}^{n} \frac{1}{x_i^{\mu_i+1}} \\
&= -\frac{1}{x_1} w_{g-1,n+1}(x_1, x_1, x_2, \dots, x_n).
\end{aligned}$$

Thus the second line of (3.16) produces

$$-\frac{1}{x_1}\left(w_{g-1,n+1}(x_1, x_1, x_2, \dots, x_n) + \sum_{\substack{g_1+g_2=g \\ I \sqcup J = \{2,\dots,n\}}} w_{g_1,|I|+1}(x_1, x_I) w_{g_2,|J|+1}(x_1, x_J)\right).$$

To calculate the operation on the first line of (3.16), let us fix $j > 1$ and set $\nu = \mu_1 + \mu_j - 2 \geq 0$. Then

$$(A.2)\quad (-1)^n \sum_{\mu_1,\dots,\mu_n>0} \mu_2\cdots\mu_n(\mu_1+\mu_j-2)$$
$$\times D_{g,n-1}(\mu_1+\mu_j-2,\mu_2,\dots,\widehat{\mu_j},\dots,\mu_n)\prod_{i=1}^n \frac{1}{x_i^{\mu_i+1}}$$
$$= -\sum_{\nu=0}^{\infty}\sum_{\mu_2,\dots,\widehat{\mu_j},\dots,\mu_n>0}(-1)^{n-1}\nu\mu_2\cdots\widehat{\mu_j}\cdots\mu_n$$
$$\times D_{g,n-1}(\nu,\mu_2,\dots,\widehat{\mu_j},\dots,\mu_n)\frac{1}{x_1^{\nu+1}}\prod_{i\neq 1,j}\frac{1}{x_i^{\mu_i+1}}\sum_{\mu_j=1}^{\nu+1}\mu_j x_1^{\mu_j-2}\frac{1}{x_j^{\mu_j+1}}.$$

Assuming $|x_1|<|x_j|$, we calculate

(A.3)
$$\sum_{\mu_j=1}^{\nu+1}\mu_j x_1^{\mu_j-2}\frac{1}{x_j^{\mu_j+1}} = -\frac{1}{x_1^2}\frac{\partial}{\partial x_j}\sum_{\mu_j=0}^{\nu+1}\left(\frac{x_1}{x_j}\right)^{\mu_j} = -\frac{1}{x_1^2}\frac{\partial}{\partial x_j}\left(\frac{1}{1-\frac{x_1}{x_j}} - \frac{\left(\frac{x_1}{x_j}\right)^{\nu+2}}{1-\frac{x_1}{x_j}}\right)$$
$$= -\frac{1}{x_1^2}\frac{\partial}{\partial x_j}\left(\frac{1}{1-\frac{x_1}{x_j}}\right) + x_1^{\nu}\frac{\partial}{\partial x_j}\left(\frac{1}{x_j-x_1}\,\frac{1}{x_j^{\nu+1}}\right).$$

We then substitute (A.3) in (A.2) and obtain

$$(A.4)\quad = w_{g,n-1}(x_1,x_2,\dots,\widehat{x_j},\dots,x_n)\frac{1}{x_1^2}\frac{\partial}{\partial x_j}\left(\frac{1}{1-\frac{x_1}{x_j}}\right)$$
$$-\frac{1}{x_1}\frac{\partial}{\partial x_j}\left(\frac{1}{x_j-x_1}\,w_{g,n-1}(x_2,\dots,x_j,\dots,x_n)\right)$$
$$= -\frac{1}{x_1}\frac{\partial}{\partial x_j}\left(\frac{1}{x_j-x_1}\left(w_{g,n-1}(x_2,\dots,x_j,\dots,x_n) - w_{g,n-1}(x_1,x_2,\dots,\widehat{x_j},\dots,x_n)\right)\right).$$

This completes the proof. □

Proof of Theorem 4.3. When the curve is split into two pieces, the second term of the third line of (A.1) contains contributions from unstable geometries $(g,n)=(0,1)$ and $(0,2)$. We first separate them out. For $g_1=0$ and $I=\emptyset$, or $g_2=0$ and $J=\emptyset$, we have a contribution of

$$2w_{0,1}(x_1)w_{g,n}(x_1,x_2,\dots,x_n).$$

Similarly, for $g_1=0$ and $I=\{j\}$, or $g_2=0$ and $J=\{j\}$, we have

$$2\sum_{j=2}^n w_{0,2}(x_1,x_j)w_{g,n-1}(x_1,\dots,\widehat{x_j},\dots,x_n).$$

Since $W^D_{0,1}$ and $W^D_{0,2}$ are defined on the spectral curve, it is time for us to switch to the preferred coordinate t of (4.1) now. We thus introduce

(A.5)
$$W^D_{g,n}(t_1,\dots,t_n) = w^D_{g,n}(t_1,\dots,t_n)\,dt_1\cdots dt_n = w_{g,n}(x_1,\dots,x_n)\,dx_1\cdots dx_n.$$

Since $w_{0,1}(x)=-z(x)$, we have

$$w_{0,1}(x) = -\frac{t+1}{t-1}$$

$$w_{0,2}(x_1,x_2) = \frac{1}{(t_1+t_2)^2}\,\frac{(t_1^2-1)^2}{8t_1}\,\frac{(t_2^2-1)^2}{8t_2}$$
$$w_{g,n}(x_1,\dots,x_n) = (-1)^n w^D_{g,n}(t_1,\dots,t_n)\prod_{i=1}^n \frac{(t_i^2-1)^2}{8t_i}.$$

Thus (A.1) is equivalent to

$$\begin{aligned}
&2\left(\frac{t_1^2+1}{t_1^2-1}-\frac{t_1+1}{t_1-1}\right) w^D_{g,n}(t_1,\dots,t_n)\\
&\quad=\sum_{j=2}^n\left(\frac{(t_1^2-1)^2(t_j^2-1)^2}{16(t_1^2-t_j^2)^2}\,\frac{8t_j}{(t_j^2-1)^2}\,w^D_{g,n-1}(t_1,\dots,\widehat{t_j},\dots,t_n)\right.\\
&\qquad\left.+\frac{\partial}{\partial t_j}\left(\frac{(t_1^2-1)(t_j^2-1)}{4(t_1^2-t_j^2)}\,\frac{8t_1}{(t_1^2-1)^2}\,\frac{(t_j^2-1)^2}{8t_j}w^D_{g,n-1}(t_2,\dots,t_n)\right)\right)\\
&\quad+\frac{(t_1^2-1)^2}{8t_1}\left(w^D_{g-1,n+1}(t_1,t_1,t_2,\dots,t_n)+\sum^{\text{stable}}_{\substack{g_1+g_2=g\\ I\sqcup J=\{2,\dots,n\}}} w^D_{g_1,|I|+1}(t_1,t_I)w^D_{g_2,|J|+1}(t_1,t_J)\right)\\
&\qquad+2\sum_{j=2}^n\frac{1}{(t_1+t_j)^2}\,\frac{(t_1^2-1)^2}{8t_1}w^D_{g,n-1}(t_1,\dots,\widehat{t_j},\dots,t_n)\\
&\quad=\sum_{j=2}^n\left(\left(\frac{t_j(t_1^2-1)^2}{2(t_1^2-t_j^2)^2}+\frac{1}{(t_1+t_j)^2}\,\frac{(t_1^2-1)^2}{4t_1}\right)w^D_{g,n-1}(t_1,\dots,\widehat{t_j},\dots,t_n)\right.\\
&\qquad\left.+\frac{t_1}{t_1^2-1}\,\frac{\partial}{\partial t_j}\left(\frac{(t_j^2-1)^3}{4t_j(t_1^2-t_j^2)}\,w^D_{g,n-1}(t_2,\dots,t_n)\right)\right)\\
&\quad+\frac{(t_1^2-1)^2}{8t_1}\left(w^D_{g-1,n+1}(t_1,t_1,t_2,\dots,t_n)+\sum^{\text{stable}}_{\substack{g_1+g_2=g\\ I\sqcup J=\{2,\dots,n\}}} w^D_{g_1,|I|+1}(t_1,t_I)w^D_{g_2,|J|+1}(t_1,t_J)\right).
\end{aligned}$$

Since

$$2\left(\frac{t_1^2+1}{t_1^2-1}-\frac{t_1+1}{t_1-1}\right) = -\frac{4t_1}{t_1^2-1},$$

we obtain

$$\begin{aligned}
\text{(A.6)}\quad w^D_{g,n}(t_1,\dots,t_n) &= -\sum_{j=2}^n\left(\frac{\partial}{\partial t_j}\left(\frac{(t_j^2-1)^3}{16t_j(t_1^2-t_j^2)}\,w^D_{g,n-1}(t_2,\dots,t_n)\right)\right.\\
&\qquad\left.+\frac{(t_1^2-1)^3}{16t_1^2}\,\frac{t_1^2+t_j^2}{(t_1^2-t_j^2)^2}w^D_{g,n-1}(t_1,\dots,\widehat{t_j},\dots,t_n)\right)\\
&-\frac{(t_1^2-1)^3}{32t_1^2}\left(w^D_{g-1,n+1}(t_1,t_1,t_2,\dots,t_n)+\sum^{\text{stable}}_{\substack{g_1+g_2=g\\ I\sqcup J=\{2,\dots,n\}}} w^D_{g_1,|I|+1}(t_1,t_I)w^D_{g_2,|J|+1}(t_1,t_J)\right).
\end{aligned}$$

Now let us compute the integral

$$\text{(A.7)}\quad W^D_{g,n}(t_1,\dots,t_n) = -\frac{1}{64}\,\frac{1}{2\pi i}\int_\gamma\left(\frac{1}{t+t_1}+\frac{1}{t-t_1}\right)\frac{(t^2-1)^3}{t^2}\cdot\frac{1}{dt}\cdot dt_1$$

$$\times \left[\sum_{j=2}^{n} \left(W^D_{0,2}(t,t_j) W_{g,n-1}(-t,t_2,\ldots,\widehat{t_j},\ldots,t_n) \right. \right.$$
$$\left. + W^D_{0,2}(-t,t_j) W_{g,n-1}(t,t_2,\ldots,\widehat{t_j},\ldots,t_n) \right)$$
$$\left. + W^D_{g-1,n+1}(t,-t,t_2,\ldots,t_n) + \sum^{\text{stable}}_{\substack{g_1+g_2=g \\ I\sqcup J=\{2,3,\ldots,n\}}} W^D_{g_1,|I|+1}(t,t_I) W^D_{g_2,|J|+1}(-t,t_J) \right].$$

Recall that for $2g-2+n>0$, $w^D_{g,n}(t_1,\ldots,t_n)$ is a Laurent polynomial in $t_1^2,\ldots,t_n^2$. Thus the third line of (A.7) is immediately calculated because the integration contour γ of Figure 4.1 encloses $\pm t_1$ and contributes residues with the negative sign. The result is exactly the last line of (A.6). Similarly, since

$$W^D_{0,2}(t,t_j) W_{g,n-1}(-t,t_2,\ldots,\widehat{t_j},\ldots,t_n) + W^D_{0,2}(-t,t_j) W_{g,n-1}(t,t_2,\ldots,\widehat{t_j},\ldots,t_n)$$
$$= -\left(\frac{1}{(t+t_j)^2} + \frac{1}{(t-t_j)^2} \right) w^D_{g,n-1}(t,t_2,\ldots,\widehat{t_j},\ldots,t_n)\, dt\, dt\, dt_2 \cdots \widehat{dt_j} \cdots dt_n,$$

the residues at $\pm t_1$ contribute

$$-\frac{(t_1^2-1)^3(t_1^2+t_j^2)}{16 t_1^2 (t_1^2-t_j^2)^2}\, w^D_{g,n-1}(t_1,\ldots,\widehat{t_j},\ldots,t_n).$$

This is the same as the second line of the right-hand side of (A.6).

Within the contour γ, there are second order poles at $\pm t_j$ for each $j \geq 2$ that come from $W^D_{0,2}(\pm t, t_j)$. Note that $W^D_{0,2}(t,t_j)$ acts as the Cauchy differentiation kernel. We calculate

$$\frac{1}{64}\frac{1}{2\pi i}\int_\gamma \left(\frac{1}{t+t_1} + \frac{1}{t-t_1} \right) \frac{(t^2-1)^3}{t^2} \sum_{j=2}^{n} \left(w^D_{0,2}(t,t_j) w^D_{g,n-1}(-t,t_2,\ldots,\widehat{t_j},\ldots,t_n) \right.$$
$$\left. + w^D_{0,2}(-t,t_j) w_{g,n-1}(t,t_2,\ldots,\widehat{t_j},\ldots,t_n) \right)$$
$$= -\frac{1}{32}\frac{\partial}{\partial t_j} \left(\left(\frac{1}{t_j+t_1} + \frac{1}{t_j-t_1} \right) \frac{(t_j^2-1)^3}{t_j^2} w^D_{g,n-1}(t_j,t_2,\ldots,\widehat{t_j},\ldots,t_n) \right)$$
$$= -\frac{1}{16}\frac{\partial}{\partial t_j} \left(\frac{1}{t_j^2-t_1^2} \frac{(t_j^2-1)^3}{t_j} w^D_{g,n-1}(t_j,t_2,\ldots,\widehat{t_j},\ldots,t_n) \right).$$

This gives the first line of the right-hand side of (A.6). We have thus completed the proof of Theorem 4.3. □

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References

[1] A. Alexandrov, A. Mironov, and A. Morozov, *Unified description of correlators in non-Gaussian phases of Hermitian matrix model*, Internat. J. Modern Phys. A **21** (2006), no. 12, 2481–2517, DOI 10.1142/S0217751X06029375. MR2234724 (2007c:81138)

[2] Matthew Robert Ballard, *Meet homological mirror symmetry*, Modular forms and string duality, Fields Inst. Commun., vol. 54, Amer. Math. Soc., Providence, RI, 2008, pp. 191–224. MR2454326 (2009m:53225)

[3] G. V. Belyĭ, *Galois extensions of a maximal cyclotomic field*, Izv. Akad. Nauk SSSR Ser. Mat. **43** (1979), no. 2, 267–276, 479 (Russian). MR534593 (80f:12008)

[4] Julia Bennett, David Cochran, Brad Safnuk, and Kaitlin Woskoff, *Topological recursion for symplectic volumes of moduli spaces of curves*, Michigan Math. J. **61** (2012), no. 2, 331–358, DOI 10.1307/mmj/1339011530. MR2944483

[5] G. Borot, B. Eynard, M. Mulase and B. Safnuk, *Hurwitz numbers, matrix models and topological recursion*, arXiv:0906.1206 [math.Ph] (2009).

[6] Vincent Bouchard, Andrei Catuneanu, Olivier Marchal, and Piotr Sułkowski, *The Remodeling Conjecture and the Faber–Pandharipande Formula*, Lett. Math. Phys. **103** (2013), no. 1, 59–77, DOI 10.1007/s11005-012-0588-z. MR3004817

[7] Vincent Bouchard, Albrecht Klemm, Marcos Mariño, and Sara Pasquetti, *Remodeling the B-model*, Comm. Math. Phys. **287** (2009), no. 1, 117–178, DOI 10.1007/s00220-008-0620-4. MR2480744 (2010i:81352)

[8] Vincent Bouchard and Marcos Mariño, *Hurwitz numbers, matrix models and enumerative geometry*, From Hodge theory to integrability and TQFT tt*-geometry, Proc. Sympos. Pure Math., vol. 78, Amer. Math. Soc., Providence, RI, 2008, pp. 263–283. MR2483754 (2009m:14080)

[9] Andrea Brini, *The local Gromov-Witten theory of $\mathbb{CP}^1$ and integrable hierarchies*, Comm. Math. Phys. **313** (2012), no. 3, 571–605, DOI 10.1007/s00220-012-1517-9. MR2945617

[10] A. Brini, B. Eynard, and M. Mariño, *Torus knots and mirror symmetry* arXiv:1105.2012.

[11] K. Chapman, M. Mulase, and B. Safnuk, *Topological recursion and the Kontsevich constants for the volume of the moduli of curves*, Preprint (2010).

[12] L. Chen, *Bouchard-Klemm-Marino-Pasquetti Conjecture for $\mathbb{C}^3$*, arXiv:0910.3739 (2009).

[13] R. M. Corless, G. H. Gonnet, D. E. G. Hare, D. J. Jeffrey, and D. E. Knuth, *On the Lambert W function*, Adv. Comput. Math. **5** (1996), no. 4, 329–359, DOI 10.1007/BF02124750. MR1414285 (98j:33015)

[14] Robbert Dijkgraaf, Hiroyuki Fuji, and Masahide Manabe, *The volume conjecture, perturbative knot invariants, and recursion relations for topological strings*, Nuclear Phys. B **849** (2011), no. 1, 166–211, DOI 10.1016/j.nuclphysb.2011.03.014. MR2795276 (2012g:81196)

[15] R. Dijkgraaf and C. Vafa, *Two Dimensional Kodaira-Spencer Theory and Three Dimensional Chern-Simons Gravity*, arXiv:0711.1932 [hep-th].

[16] R. Dijkgraaf, E. Verlinde, and H. Verlinde, *Loop equations and Virasoro constraints in non-perturbative two-dimensional quantum gravity*, Nucl. Phys. **B348**, 435–456 (1991).

[17] Boris Dubrovin, *Geometry of 2D topological field theories*, Integrable systems and quantum groups (Montecatini Terme, 1993), Lecture Notes in Math., vol. 1620, Springer, Berlin, 1996, pp. 120–348, DOI 10.1007/BFb0094793. MR1397274 (97d:58038)

[18] Boris Dubrovin and Youjin Zhang, *Frobenius manifolds and Virasoro constraints*, Selecta Math. (N.S.) **5** (1999), no. 4, 423–466, DOI 10.1007/s000290050053. MR1740678 (2001k:53171)

[19] Boris Dubrovin and Youjin Zhang, *Virasoro symmetries of the extended Toda hierarchy*, Comm. Math. Phys. **250** (2004), no. 1, 161–193, DOI 10.1007/s00220-004-1084-9. MR2092034 (2005k:37169)

[20] P. Dunin-Barkowski, A. Mironov, A. Morozov, A. Sleptsov, and A. Smirnov. *Superpolynomials for toric knots from evolution induced by cut-and-join operators* , arXiv:1106.4305 [hep-th].

[21] Torsten Ekedahl, Sergei Lando, Michael Shapiro, and Alek Vainshtein, *Hurwitz numbers and intersections on moduli spaces of curves*, Invent. Math. **146** (2001), no. 2, 297–327, DOI 10.1007/s002220100164. MR1864018 (2002j:14034)

[22] Bertrand Eynard, *Topological expansion for the 1-Hermitian matrix model correlation functions*, J. High Energy Phys. **11** (2004), 031, 35 pp. (electronic) (2005), DOI 10.1088/1126-6708/2004/11/031. MR2118807 (2006c:81157)

[23] B. Eynard, *Recursion between volumes of moduli spaces*, arXiv:0706.4403 [math-ph].

[24] Bertrand Eynard, Nicolas Orantin, and Marcos Mariño, *Holomorphic anomaly and matrix models*, J. High Energy Phys. **6** (2007), 058, 20 pp. (electronic), DOI 10.1088/1126-6708/2007/06/058. MR2326593 (2009e:81219)

[25] B. Eynard, M. Mulase and B. Safnuk, *The Laplace transform of the cut-and-join equation and the Bouchard-Mariño conjecture on Hurwitz numbers*, arXiv:0907.5224 math.AG (2009).

[26] B. Eynard and N. Orantin, *Invariants of algebraic curves and topological expansion*, Commun. Number Theory Phys. **1** (2007), no. 2, 347–452. MR2346575 (2008m:14049)

[27] B. Eynard and N. Orantin, *Weil-Petersson volume of moduli spaces, Mirzakhani's recursion and matrix models*, arXiv:0705.3600 [math-ph] (2007).

[28] B. Eynard and N. Orantin, *Topological expansion of mixed correlations in the Hermitian 2-matrix model and x-y symmetry of the F_g algebraic invariants*, J. Phys. A **41** (2008), no. 1, 015203, 28, DOI 10.1088/1751-8113/41/1/015203. MR2450700 (2011b:14063)

[29] E. Getzler, *Topological recursion relations in genus* 2, Integrable systems and algebraic geometry (Kobe/Kyoto, 1997), World Sci. Publ., River Edge, NJ, 1998, pp. 73–106. MR1672112 (2000b:14028)

[30] I. P. Goulden and D. M. Jackson, *Transitive factorisations into transpositions and holomorphic mappings on the sphere*, Proc. Amer. Math. Soc. **125** (1997), no. 1, 51–60, DOI 10.1090/S0002-9939-97-03880-X. MR1396978 (97j:05007)

[31] I. P. Goulden, D. M. Jackson, and A. Vainshtein, *The number of ramified coverings of the sphere by the torus and surfaces of higher genera*, Ann. Comb. **4** (2000), no. 1, 27–46, DOI 10.1007/PL00001274. MR1763948 (2002b:14067)

[32] I. P. Goulden, D. M. Jackson, and R. Vakil, *The Gromov-Witten potential of a point, Hurwitz numbers, and Hodge integrals*, Proc. London Math. Soc. (3) **83** (2001), no. 3, 563–581, DOI 10.1112/plms/83.3.563. MR1851082 (2002h:14043)

[33] I. P. Goulden, D. M. Jackson, and R. Vakil, *A short proof of the λ_g-conjecture without Gromov-Witten theory: Hurwitz theory and the moduli of curves*, J. Reine Angew. Math. **637** (2009), 175–191, DOI 10.1515/CRELLE.2009.094. MR2599085 (2011f:14087)

[34] Tom Graber and Ravi Vakil, *Hodge integrals and Hurwitz numbers via virtual localization*, Compositio Math. **135** (2003), no. 1, 25–36, DOI 10.1023/A:1021791611677. MR1955162 (2004c:14108)

[35] Tom Graber and Ravi Vakil, *Relative virtual localization and vanishing of tautological classes on moduli spaces of curves*, Duke Math. J. **130** (2005), no. 1, 1–37, DOI 10.1215/S0012-7094-05-13011-3. MR2176546 (2006j:14035)

[36] S. Gukov and P. Sułkowski, *A-polynomial, B-model, and quantization*, arXiv:1108.0002v1 [hep-th].

[37] John L. Harer, *The cohomology of the moduli space of curves*, Theory of moduli (Montecatini Terme, 1985), Lecture Notes in Math., vol. 1337, Springer, Berlin, 1988, pp. 138–221, DOI 10.1007/BFb0082808. MR963064 (90a:32026)

[38] J. Harer and D. Zagier, *The Euler characteristic of the moduli space of curves*, Invent. Math. **85** (1986), no. 3, 457–485, DOI 10.1007/BF01390325. MR848681 (87i:32031)

[39] F. Hirzebruch, *Topological methods in algebraic geometry*, Third enlarged edition. New appendix and translation from the second German edition by R. L. E. Schwarzenberger, with an additional section by A. Borel. Die Grundlehren der Mathematischen Wissenschaften, Band 131, Springer-Verlag New York, Inc., New York, 1966. MR0202713 (34 #2573)

[40] Nigel Hitchin, *Lectures on special Lagrangian submanifolds*, Submanifolds (Cambridge, MA, 1999), AMS/IP Stud. Adv. Math., vol. 23, Amer. Math. Soc., Providence, RI, 2001, pp. 151–182. MR1876068 (2003f:53086)

[41] Kentaro Hori, Sheldon Katz, Albrecht Klemm, Rahul Pandharipande, Richard Thomas, Cumrun Vafa, Ravi Vakil, and Eric Zaslow, *Mirror symmetry*, Clay Mathematics Monographs, vol. 1, American Mathematical Society, Providence, RI, 2003. With a preface by Vafa. MR2003030 (2004g:14042)

[42] A. Hurwitz, *Über Riemann'sche Flächen mit gegebene Verzweigungspunkten*, Mathematische Annalen **39**, 1–66 (1891).

[43] Dominic Joyce, *Lectures on special Lagrangian geometry*, Global theory of minimal surfaces, Clay Math. Proc., vol. 2, Amer. Math. Soc., Providence, RI, 2005, pp. 667–695. MR2167283 (2006j:53077)

[44] R. Kaufmann, Yu. Manin, and D. Zagier, *Higher Weil-Petersson volumes of moduli spaces of stable n-pointed curves*, Comm. Math. Phys. **181** (1996), no. 3, 763–787. MR1414310 (98i:14029)

[45] Y. Kodama, *Combinatorics of the dispersionless Toda hierarchy*, Lecture delivered at the International Workshop on Nonlinear and Modern Mathematical Physics, Beijing Xiedao Group, Beijing, China, July 15–21, 2009.

[46] Yuji Kodama and Virgil U. Pierce, *Combinatorics of dispersionless integrable systems and universality in random matrix theory*, Comm. Math. Phys. **292** (2009), no. 2, 529–568, DOI 10.1007/s00220-009-0894-1. MR2544742 (2011a:82043)

[47] Maxim Kontsevich, *Intersection theory on the moduli space of curves and the matrix Airy function*, Comm. Math. Phys. **147** (1992), no. 1, 1–23. MR1171758 (93e:32027)

[48] Maxim Kontsevich, *Homological algebra of mirror symmetry*, 2 (Zürich, 1994), Birkhäuser, Basel, 1995, pp. 120–139. MR1403918 (97f:32040)

[49] Chiu-Chu Melissa Liu, *Lectures on the ELSV formula*, Transformation groups and moduli spaces of curves, Adv. Lect. Math. (ALM), vol. 16, Int. Press, Somerville, MA, 2011, pp. 195–216. MR2883688

[50] Chiu-Chu Melissa Liu, Kefeng Liu, and Jian Zhou, *A proof of a conjecture of Mariño-Vafa on Hodge integrals*, J. Differential Geom. **65** (2003), no. 2, 289–340. MR2058264 (2005m:14104)

[51] Marcos Mariño, *Chern-Simons theory, matrix models and topological strings*, International Series of Monographs on Physics, vol. 131, The Clarendon Press Oxford University Press, Oxford, 2005. MR2177747 (2006g:81169)

[52] Marcos Mariño, *Open string amplitudes and large order behavior in topological string theory*, J. High Energy Phys. **3** (2008), 060, 34, DOI 10.1088/1126-6708/2008/03/060. MR2391060 (2009i:81149)

[53] A. Mironov and A. Morozov, *Virasoro constraints for Kontsevich-Hurwitz partition function*, J. High Energy Phys. **2** (2009), 024, 52, DOI 10.1088/1126-6708/2009/02/024. MR2486405 (2010c:14064)

[54] Motohico Mulase, *Asymptotic analysis of a Hermitian matrix integral*, Internat. J. Math. **6** (1995), no. 6, 881–892, DOI 10.1142/S0129167X95000389. MR1353999 (96m:14035)

[55] M. Mulase and M. Penkava, *Ribbon graphs, quadratic differentials on Riemann surfaces, and algebraic curves defined over $\overline{\mathbf{Q}}$*, Asian J. Math. **2** (1998), no. 4, 875–919. Mikio Sato: a great Japanese mathematician of the twentieth century. MR1734132 (2001g:30028)

[56] Motohico Mulase and Michael Penkava, *Topological recursion for the Poincaré polynomial of the combinatorial moduli space of curves*, Adv. Math. **230** (2012), no. 3, 1322–1339, DOI 10.1016/j.aim.2012.03.027. MR2921181

[57] Motohico Mulase and Brad Safnuk, *Mirzakhani's recursion relations, Virasoro constraints and the KdV hierarchy*, Indian J. Math. **50** (2008), no. 1, 189–218. MR2379144 (2009b:14053)

[58] Motohico Mulase and Naizhen Zhang, *Polynomial recursion formula for linear Hodge integrals*, Commun. Number Theory Phys. **4** (2010), no. 2, 267–293. MR2725053 (2012a:14069)

[59] David Mumford, *Towards an enumerative geometry of the moduli space of curves*, Arithmetic and geometry, Vol. II, Progr. Math., vol. 36, Birkhäuser Boston, Boston, MA, 1983, pp. 271–328. MR717614 (85j:14046)

[60] Paul Norbury, *Counting lattice points in the moduli space of curves*, Math. Res. Lett. **17** (2010), no. 3, 467–481. MR2653682 (2011g:14068)

[61] Paul Norbury, *String and dilaton equations for counting lattice points in the moduli space of curves*, Trans. Amer. Math. Soc. **365** (2013), no. 4, 1687–1709, DOI 10.1090/S0002-9947-2012-05559-0. MR3009643

[62] P. Norbury and N. Scott, *Polynomials representing Eynard-Orantin invariants*, arXiv:1001.0449 (2010).

[63] P. Norbury and N. Scott, *Gromov-Witten invariants of $\mathbb{P}^1$ and Eynard-Orantin invariants*, arXiv:1106.1337 (2011).

[64] Andrei Okounkov, *Random matrices and random permutations*, Internat. Math. Res. Notices **20** (2000), 1043–1095, DOI 10.1155/S1073792800000532. MR1802530 (2002c:15045)

[65] Andrei Okounkov, *Toda equations for Hurwitz numbers*, Math. Res. Lett. **7** (2000), no. 4, 447–453. MR1783622 (2001i:14047)

[66] A. Okounkov and R. Pandharipande, *Gromov-Witten theory, Hurwitz numbers, and matrix models*, Algebraic geometry—Seattle 2005. Part 1, Proc. Sympos. Pure Math., vol. 80, Amer. Math. Soc., Providence, RI, 2009, pp. 325–414. MR2483941 (2009k:14111)

[67] A. Okounkov and R. Pandharipande, *Gromov-Witten theory, Hurwitz theory, and completed cycles*, Ann. of Math. (2) **163** (2006), no. 2, 517–560, DOI 10.4007/annals.2006.163.517. MR2199225 (2007b:14123)

[68] A. Okounkov and R. Pandharipande, *The equivariant Gromov-Witten theory of* $\mathbf{P}^1$, Ann. of Math. (2) **163** (2006), no. 2, 561–605, DOI 10.4007/annals.2006.163.561. MR2199226 (2006j:14075)

[69] A. Okounkov and R. Pandharipande, *Virasoro constraints for target curves*, Invent. Math. **163** (2006), no. 1, 47–108, DOI 10.1007/s00222-005-0455-y. MR2208418 (2006m:14077)

[70] A. Okounkov and R. Pandharipande, *Hodge integrals and invariants of the unknot*, Geom. Topol. **8** (2004), 675–699, DOI 10.2140/gt.2004.8.675. MR2057777 (2006c:14041)

[71] Hirosi Ooguri, Piotr Sułkowski, and Masahito Yamazaki, *Wall crossing as seen by matrix models*, Comm. Math. Phys. **307** (2011), no. 2, 429–462, DOI 10.1007/s00220-011-1330-x. MR2837121

[72] *The Grothendieck theory of dessins d'enfants*, London Mathematical Society Lecture Note Series, vol. 200, Cambridge University Press, Cambridge, 1994. Papers from the Conference on Dessins d'Enfant held in Luminy, April 19–24, 1993; Edited by Leila Schneps. MR1305390 (95f:11001)

[73] *Geometric Galois actions. 1*, London Mathematical Society Lecture Note Series, vol. 242, Cambridge University Press, Cambridge, 1997. Around Grothendieck's "Esquisse d'un programme"; Edited by Leila Schneps and Pierre Lochak. MR1483106 (98e:14003)

[74] Daniel D. Sleator, Robert E. Tarjan, and William P. Thurston, *Rotation distance, triangulations, and hyperbolic geometry*, J. Amer. Math. Soc. **1** (1988), no. 3, 647–681, DOI 10.2307/1990951. MR928904 (90h:68026)

[75] Richard P. Stanley, *Enumerative combinatorics. Vol. 2*, Cambridge Studies in Advanced Mathematics, vol. 62, Cambridge University Press, Cambridge, 1999. With a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin. MR1676282 (2000k:05026)

[76] Kurt Strebel, *Quadratic differentials*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 5, Springer-Verlag, Berlin, 1984. MR743423 (86a:30072)

[77] G. 't Hooft, *A planer diagram theory for strong interactions*, Nuclear Physics **B 72**, 461–473 (1974).

[78] R. Vakil, Harvard Thesis 1997.

[79] Edward Witten, *Two-dimensional gravity and intersection theory on moduli space*, Surveys in differential geometry (Cambridge, MA, 1990), Lehigh Univ., Bethlehem, PA, 1991, pp. 243–310. MR1144529 (93e:32028)

[80] Jian Zhou, *Some closed formulas and conjectures for Hodge integrals*, Math. Res. Lett. **10** (2003), no. 2-3, 275–286. MR1981904 (2004e:14046)

[81] J. Zhou, *On computations of Hurwitz-Hodge integrals*, arXiv:0710.1679 (2007).

[82] J. Zhou, *Local Mirror Symmetry for One-Legged Topological Vertex*, arXiv:0910.4320 (2009).

[83] J. Zhou, *Local Mirror Symmetry for the Topological Vertex* arXiv:0911.2343 (2009).

[84] S. Zhu, *On a proof of the Bouchard-Sułkowski conjecture*, arXiv:1108.2831 (2011).

[85] D. Zvonkine, *An algebra of power series arising in the intersection theory of moduli spaces of curves and in the enumeration of ramified coverings of the sphere*, arXiv:math.AG/0403092 (2004).

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