NORMALIZATION OF THE KRICHEVER DATA

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1. Introduction.

The purpose of this note is to give a canonical normalization of the Krichever data consisting of algebraic curves and torsion free sheaves on them. Generalizing the original Krichever data of Segal-Wilson [SW] in order to deal with the higher rank cases, the notion of *quintets* was introduced in [M1]. A quintet $(C, p, \pi, \mathcal{F}, \phi)$ is a set of geometric data consisting of a curve C, a point $p \in C$, a locally defined r-sheeted covering π of C ramified at p, a torsion free rank r sheaf $\mathcal{F}$, and a local trivialization ϕ of $\mathcal{F}$ near p. One can define a category $\mathcal{Q}$ of these quintets. The cohomology functors give rise to a fully-faithful contravariant functor χ between $\mathcal{Q}$ and a category $\mathcal{S}$ of some algebraic data [M1]. An object of $\mathcal{S}$ is a pair (A, W) consisting of a subring A of the formal Laurent series ring $k((z))$ and a vector subspace $W \subset k((z))$, which is commensurable with $k[z^{-1}]$ in $k((z))$, and satisfying that

$$A \cdot W \subset W \ .$$

The case that the sheaf $\mathcal{F}$ has rank one was studied extensively in [SW]. The higher rank cases were studied in [M1] in which a complete geometric classification of all commutative rings of ordinary differential operators was established.

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Segal and Wilson [SW] showed that a quintet with a nonsingular curve C and a line bundle $\mathcal{F}$ is in one-to-one correspondence with W, which can be identified with a point of certain infinite dimensional Grassmannian. In this case, the algebra A can be recovered simply by

$$A = A_W = \{v \in k((z)) \mid vW \subset W\} ,$$

which we call the maximal stabilizer of W. The algebraic nature of the above statement will be clarified in Lemma 3.1. We will show that if the pair (A, W) satisfies that

(1) W is a rank-one A-module, and
(2) A is a normal ring,

then A is a maximal stabilizer of W. In general, however, maximality does not imply normality. Thus we are led to study normalization of the pair (A, W).

Our result of this paper is that for every quintet $(C, p, \pi, \mathcal{F}, \phi)$, there is a unique quintet $(C', p', \pi', \mathcal{F}', \phi')$ and a canonical morphism

$$(C', p', \pi', \mathcal{F}', \phi') \longrightarrow (C, p, \pi, \mathcal{F}, \phi)$$

such that C' is a normalization of C and $\mathcal{F}'$ is a locally free sheaf on C' having the same rank of $\mathcal{F}$. We construct this normalization by a simple algebraic procedure on the pair (A, W).

A natural supersymmetric generalization of the theory has been obtained in [M2] and [MR]. In particular, a characterization of the Jacobians of algebraic super curves of dimension $1|1$ has been established in [M2].

2. The Krichever functor.

Throughout this paper, we work with a field k of an arbitrary characteristic. Let $V = k((z))$ be the set of all formal Laurent series in one variable z. This is the field of fractions of the ring $k[[z]]$ of formal power series. We denote by

$$V^{(\nu)} = k[[z]] \cdot z^{-\nu} ,$$

which is the set of all formal Laurent series whose pole order at $z = 0$ is less than or equal to $\nu \in \mathbb{Z}$. We say $v \in V$ has *order* ν if $v \in V^{(\nu)} \setminus V^{(\nu-1)}$. For every vector subspace W in V, let γ_W denote the natural map of W into $V/V^{(-1)}$ defined by

$$\begin{array}{ccc} V & \xrightarrow{\text{identity}} & V \\ {\scriptstyle\text{inclusion}}\uparrow & & \downarrow{\scriptstyle\text{projection}} \\ W & \xrightarrow{\gamma_W} & V/V^{(-1)} \end{array} .$$

When γ_W is Fredholm, we define the Fredholm index by $\operatorname{Index} \gamma_W = \dim_k \operatorname{Ker} \gamma_W - \dim_k \operatorname{Coker} \gamma_W$.

Definition 2.1. *We call the following set the Grassmannian of index* μ*:*

$$G(\mu) = \Big\{ \textit{vector subspace } W \,\Big|\, \gamma_W \textit{ is Fredholm of index } \mu \Big\}.$$

Note that $G(\mu)$ has a structure of the pro-algebraic variety in the sense of Grothendieck.

Definition 2.2. *Let* r *be a positive integer and* μ *an arbitrary integer. A pair* (A, W) *is said to be a Schur pair of rank* r *and index* μ *if the following conditions are satisfied:*

1. W *is a point of the Grassmannian* $G(\mu)$ *of index* μ.
2. $A \subset V$ *is a* k*-subalgebra of* V *such that* $k \subset A$, $k \neq A$, $AW \subset W$ *and*

$$r = \textit{rank}\, A = \textit{G.C.D.}\ \{ \textit{ord}\, a \,|\, a \in A \}.$$

We denote by $\mathcal{S}_r(\mu)$ *the set of all Schur pairs of rank* r *and index* μ.

Schur [S] showed that every commutative ring of ordinary differential operators can be embedded in the ring of pseudo-differential operators with constant coefficients. Our Schur pair is nothing but an algebraic abstraction of his situation. See [M1] for detail.

Remark 2.3. Let

$$A_W = \{ v \in V \,|\, vW \subset W \}.$$

If $k \neq A_W$, then (A_W, W) gives a Schur pair, which we call a *maximal Schur pair.* However, we have always $A_W = k$ for a generic W. In this case, W does not have any interesting geometric information.

Definition 2.4. *We define the category of Schur pairs* $\mathcal{S}$ *as follows:*

1. *The set of objects is defined by*

$$Ob(\mathcal{S}) = \bigcup_{\mu \in \mathbb{Z}} \bigcup_{r \in \mathbb{N}} \mathcal{S}_r(\mu).$$

2. *The set of morphisms* $Mor\big((A_2, W_2),\ (A_1, W_1)\big)$ *consists of*

$$(\alpha, \iota) : (A_2, W_2) \longrightarrow (A_1, W_1)\ ,$$

where $\alpha : A_2 \hookrightarrow A_1$ *and* $\iota : W_2 \hookrightarrow W_1$ *are injective homomorphisms.*

Next, let us define a category of geometric data consisting of algebraic curves and torsion free sheaves on them, and construct a contravariant functor from this category to the category of Schur pairs.

Definition 2.5. *Let r be a positive integer and μ an arbitrary integer. We call $(C, p, \pi, \mathcal{F}, \phi)$ a quintet of rank r and index μ if it consists of the following geometric data:*

(1) *C is a reduced irreducible complete algebraic curve defined over k.*
(2) *$p \in C$ is a smooth k-rational point.*
(3) *$\pi : U_0 \to U_p$ is an r-sheeted covering of U_p ramified at p, where U_0 is the formal completion of the affine line $\mathbb{A}^1_k = \mathbb{A}^1$ at the origin $0 \in \mathbb{A}^1$ and U_p is the formal completion of the curve C at p. Once for all, we choose a coordinate z on $\mathbb{A}^1$ and fix it throughout this paper so that we have $U_0 = Spec\, k[[z]]$.*
(4) *$\mathcal{F}$ is a torsion free sheaf of $\mathcal{O}_C$-modules on C of rank r satisfying*

$$\dim_k H^0(C, \mathcal{F}) - \dim_k H^1(C, \mathcal{F}) = \mu \ .$$

(5) *$\phi : \mathcal{F}_{U_p} \xrightarrow{\sim} \pi_* \mathcal{O}_{U_0}(-1)$ is an $\mathcal{O}_{U_p}$-module isomorphism between the formal completion $\mathcal{F}_{U_p}$ of $\mathcal{F}$ at $p \in C$, which is a free $\mathcal{O}_{U_p}$-module of rank r, and the direct image sheaf $\pi_* \mathcal{O}_{U_0}(-1)$.*

Two quintets $(C, p, \pi, \mathcal{F}, \phi)$ and $(C, p, \pi, \mathcal{F}, c\phi)$ are identified if $c \in k^\times$. We also identify $(C, p, \pi_1, \mathcal{F}, \phi_1)$ with $(C, p, \pi_2, \mathcal{F}, \phi_2)$ when the following diagram commutes:

$$\begin{array}{ccc} H^0(U_p, \mathcal{F}_{U_p}) & \xrightarrow{\phi_1} & H^0(U_p, \pi_{1*}\mathcal{O}_{U_0}(-1)) \\ {\scriptstyle \phi_2}\downarrow & & \downarrow \wr \\ H^0(U_p, \pi_{2*}\mathcal{O}_{U_0}(-1)) & \xrightarrow{\sim} & H^0(U_0, \mathcal{O}_{U_0}(-1)). \end{array}$$

The set of all quintets of rank r and index μ is denoted by $\mathcal{Q}_r(\mu)$.

Remark 2.6. When $r = 1$, π is an isomorphism $\pi : U_0 \xrightarrow{\sim} U_p$. Since we have chosen a coordinate z on U_0, π gives a local coordinate $y = \pi(z)$ on U_p. Thus our quintet $(C, p, \pi, \mathcal{F}, \phi)$ becomes $(C, p, y, \mathcal{F}, \phi)$ with a local parameter y around p and a local trivialization ϕ of $\mathcal{F}$ near the point p. This is the original Krichever data introduced by Segal-Wilson [SW].

Definition 2.7. *We define a category $\mathcal{Q}$ of quintets as follows:*

(1) *The set of objects is defined by*

$$Ob(\mathcal{Q}) = \bigcup_{\mu \in \mathbb{Z}} \bigcup_{r \in \mathbb{N}} \mathcal{Q}_r(\mu).$$

(2) *A morphism*

$$(\beta, \psi) : (C_1, p_1, \pi_1, \mathcal{F}_1, \phi_1) \longrightarrow (C_2, p_2, \pi_2, \mathcal{F}_2, \phi_2)$$

consists of a morphism $\beta : C_1 \to C_2$ *of curves and a homomorphism* $\psi : \mathcal{F}_2 \to \beta_* \mathcal{F}_1$ *of sheaves on* C_2 *such that*

$$\beta(p_1) = p_2,$$

$$\begin{array}{ccc} U_0 & = & U_0 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ U_{p_1} & \xrightarrow{\widehat{\beta}} & U_{p_2}, \end{array}$$

i.e. $\pi_2 = \widehat{\beta} \circ \pi_1$, *where* $\widehat{\beta}$ *is the morphism of formal schemes determined by* β, *and*

$$\begin{array}{ccc} \mathcal{F}_{2U_{p_2}} & \xrightarrow{\widehat{\psi}} & \widehat{\beta}_* \mathcal{F}_{1U_{p_1}} \\ {\scriptstyle \phi_2}\downarrow & & \downarrow{\scriptstyle \widehat{\beta}_*(\phi_1)} \\ \pi_{2*}\mathcal{O}_{U_0}(-1) & = & \widehat{\beta}_* \pi_{1*} \mathcal{O}_{U_0}(-1) \end{array},$$

where $\widehat{\psi}$ *is the homomorphism of sheaves on* U_{p_2} *defined by* ψ.

For a quintet $(C, p, \pi, \mathcal{F}, \phi)$ of rank r and index μ, we define

$$\begin{cases} A = \pi^* \big(H^0(C \setminus \{p\}, \mathcal{O}_C) \big) \\ W = \phi \big(H^0(C \setminus \{p\}, \mathcal{F}) \big) . \end{cases}$$

The identification $U_0 = \text{Spec } k[[z]]$ makes both A and W subsets of $k((z))$. Moreover, we can show that (A, W) is a Scur pair of rank r and index μ. Furthermore, a morphism

$$(\beta, \psi) : (C_1, p_1, \pi_1, \mathcal{F}_1, \phi_1) \longrightarrow (C_2, p_2, \pi_2, \mathcal{F}_2, \phi_2)$$

gives rise to a morphism

$$(\alpha, \iota) : (A_2, W_2) \longrightarrow (A_1, W_1) ,$$

where

$$\begin{cases} \alpha : \pi_2^* \big(H^0(C_2 \setminus \{p_2\}, \mathcal{O}_{C_2}) \big) \longrightarrow \pi_1^* \big(H^0(C_1 \setminus \{p_1\}, \mathcal{O}_{C_1}) \big) \\ \iota : \phi_2 \big(H^0(C_2 \setminus \{p_2\}, \mathcal{F}_2) \big) \longrightarrow \phi_1 \big(H^0(C_1 \setminus \{p_1\}, \mathcal{F}_1) \big) \end{cases}$$

are defined by the natural cohomology homomorphisms, and (A_i, W_i) is the Schur pair corresponding to the quintet $(C_i, p_i, \pi_i, \mathcal{F}_i, \phi_i)$ for $i = 1$, 2. It was established in [M1] that the above correspondence gives a fully-faithful contravariant functor

$$\chi : \mathcal{Q} \longrightarrow \mathcal{S}.$$

We call this anti-equivalence functor the *Krichever functor*.

3. Normalization of the Schur pairs and the quintets.

In this section, we study the Schur pairs and the quintets of rank one, and define the Krichever map. The injectivity of this map is proved by using a property of normal rings. We show that every rank one nonsingular quintet corresponds to a maximal Schur pair. We then study the *normalization* of the Schur pairs, and show that it corresponds to the geometric normalization of the algebraic curves.

Let us start with

Lemma 3.1. *Let (A, W) be a Schur pair. If A is a normal ring and rank $A = 1$, then A is maximal:*

$$A = A_W = \{v \in V_0 \mid v \cdot W \subset W\} .$$

Proof. Since $A \subset A_W$, the rank of A_W is also one. In particular, we have

$$\dim_k A_W/A < +\infty . \tag{3.2}$$

Let $a \in A_W \setminus A$ and consider the set $A \cdot a \subset A_W$. If $A \cdot a \cap A = 0$, then $A \oplus A \cdot a \subset A_W$ and it contradicts (3.2). Therefore, there are elements a_0 and a_1 in A such that $a = \frac{a_0}{a_1}$. Hence A_W is contained in the field $K(A)$ of fractions of A.

The condition (3.2) also shows that A_W is integral over A. But since A is integrally closed in $K(A)$, we can conclude that $A = A_W$. This completes the proof.

Theorem 3.3. *Let $\mathcal{M}_1(\mu)_{ns}$ be the set of isomorphism classes of quintets of rank one such that the algebraic curve C in the quintet is nonsingular. Then the Krichever functor χ gives an injective map*

$$\chi_1 : \mathcal{M}_1(\mu)_{ns} \longrightarrow G(\mu) .$$

Proof. A quintet in $\mathcal{M}_1(\mu)_{ns}$ corresponds to a Schur pair (A, W) of rank one. Since the isomorphism relation among quintets gives the equality of the Schur pairs, the isomorphism class of the quintets determines a unique Schur pair. The smoothness assumption of C implies that the affine coordinate ring A is normal, hence by Lemma 3.1, maximal. But since the maximal Schur pair (A_W, W) is in one-to-one correspondence with the point $W \in G(\mu)$ canonically, the image point W of χ_1 determines the isomorphism class of the quintets. This completes the proof.

Lemma 3.1 tells us that every rank one normal ring A is maximal. It is natural to ask if its converse is true: is the maximal stabilizer A_W of a point W of the Grassmannian a normal ring?

Let us consider an example:

$$\begin{cases} W = k[z^{-2}, z^{-3}] \in G(-1) \\ A = A_W = k[z^{-2}, z^{-3}] . \end{cases}$$

Certainly A is maximal and of rank one, but it is not a normal ring. This example leads us to the following:

Theorem 3.4. *For an arbitrary Schur pair (A, W) of rank r, let us denote by A' the integral closure of the ring A in the field $K(A)$ of fractions of A, and $W' = A' \cdot W$. Then*

1. *(A', W') is also a Schur pair of rank r, which we call the normalization of (A, W).*
2. *Let $(C, p, \pi, \mathcal{F}, \phi)$ and $(C', p', \pi', \mathcal{F}', \phi')$ be the quintets corresponding to the Schur pairs (A, W) and (A', W'), respectively. Then the morphism*

$$(\beta, \psi) : (C', p', \pi', \mathcal{F}', \phi') \longrightarrow (C, p, \pi, \mathcal{F}, \phi)$$

corresponding to $A \hookrightarrow A'$ and $W \hookrightarrow W'$ consists of a normalization

$$\beta : C' \longrightarrow C$$

of the curve C such that $p' = \beta^{-1}(p)$ and a sheaf homomorphism

$$\psi : \mathcal{F} \longrightarrow \beta_* \mathcal{F}' \ .$$

Here the curve C' is nonsingular and $\mathcal{F}'$ is a locally free sheaf of $\mathcal{O}_{C'}$-modules of rank r.

Proof. It is obvious that $A \subset A'$, $W \subset W'$ and

$$A' \cdot W' = A' \cdot A' \cdot W = A' \cdot W = W' \ .$$

Since the order of every element of $K(A)$ is a multiple of r, the rank of A' is also r. In order to show the well-definedness of (A', W') as a Schur pair, it suffices to establish the following:

Lemma 3.5. *Let (A, W) be a Schur pair of rank r, and A' the integral closure of A in $K(A)$. Then we have*

$$\begin{cases} A' \cap k[[z]] = k \\ A' \cap k[[z]] \cdot z = 0 \ . \end{cases}$$

Proof. Suppose that A' has an element y of negative order, say $-\ell r$. Consider the polynomial ring $A[y] \subset A'$. Since y is integral over A, there are m elements $f_1, f_2, \cdots, f_m \in A[y]$ which generate $A[y]$ over A. We can express the element y as $y = \frac{b}{a}$ with $a, b \in A$ and $a \neq 0$ because $A' \subset K(A)$. Thus there is a large positive integer N such that

$$a^N \cdot f_j \in A, \qquad j = 1, 2, \cdots, m.$$

Hence we have $a^N \cdot A[y] \subset A$. It means that

$$a^N \cdot y^n \in A$$

for every $n \geq 0$. So by taking n sufficiently large, we obtain

$$a^N \cdot y^n \in A \cap k[[z]] \cdot z \ .$$

But since we know $A \cap k[[z]] \cdot z = 0$ from [M1, Section 3], we conclude that $y = 0$.

Therefore, every element of A' has a nonnegative order. Since $k \subset A'$, the only possible order zero elements are nonzero constants. This completes the proof of the lemma.

By this lemma, we see that $W' = A' \cdot W$ satisfies the Fredholm condition. Therefore, (A', W') is a Schur pair.

Of course the normalization of a Schur pair corresponds to the normalization of the algebraic curve C. Since $\mathcal{F}'$ is a torsion free sheaf on the normalization C', it is locally free. This completes the proof of the theorem.

The normalization (A', W') is a maximal Schur pair if the rank r is equal to one. In general, however, A_W is not necessarily included in the field $K(A)$. This gives the difference between the normal Schur pairs and the maximal ones. It should be an interesting project to study geometry of maximal Schur pairs of higher ranks from the point of view of two-dimensional quantum gravity [Sc].

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