A NEW SUPER KP SYSTEM AND A CHARACTERIZATION OF THE JACOBIANS OF ARBITRARY ALGEBRAIC SUPER CURVES†

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ABSTRACT. A set of super-commuting vector fields is defined on the super Grassmannians. A characterization of the Jacobian varieties of super curves (super Schottky problem) is established in the following manner: Every finite dimensional integral manifold of these vector fields has a canonical structure of the Jacobian variety of an algebraic super curve, and conversely, the Jacobian variety of an arbitrary algebraic super curve is obtained in this way. The vector fields restricted on the super Grassmannian of *index* $0|0$ give a completely integrable system of partial super differential equations which gives a new supersymmetric generalization of the KP system. Thus every finite-dimensional solution of this new system gives rise to a Jacobian variety of an algebraic super curve. The correspondence between this super Grassmannian and the group of monic super pseudo-differential operators of order zero (the super Sato correspondence) is also established.

Table of Contents

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0. Introduction.

The purpose of this paper is to establish a characterization of the Jacobian varieties of arbitrary algebraic super curves defined over a field k of characteristic zero by using certain super-commuting vector fields on the super Grassmannians.

Since our main theorem is a supersymmetric generalization of the characterization theorem of usual Jacobian varieties obtained in [M2], let us sketch the nonsupersymmetric situation first. Consider the ring $k\big[[x]\big]$ of formal power series in one variable x with coefficients in the field k, and a formal pseudo-differential operator

$$S = 1 + s_1(x)\big(\frac{\partial}{\partial x}\big)^{-1} + s_2(x)\big(\frac{\partial}{\partial x}\big)^{-2} + \cdots \tag{0.1}$$

of order zero with coefficients in $k\big[[x]\big]$. The Kadomtsev-Petviashvili (KP) system is the following completely integrable system

$$\frac{\partial S}{\partial t_n} = -\,\big(S \cdot (\frac{\partial}{\partial x})^n \cdot S^{-1}\big)_- \cdot S,\ n = 1,\ 2,\ 3,\ \cdots \tag{0.2}$$

of nonlinear partial differential equations of the coefficients of S which also depend on parameters $t = (t_1,\ t_2,\ t_3,\ \cdots)$, where $(\bullet)_-$ denotes the negative power terms of $\frac{\partial}{\partial x}$. What makes this nonlinear system so interesting in pure mathematics is the Sato correspondence which assigns a point of an infinite dimensional Grassmannian to every pseudo-differential operator S. The Grassmannian we need here is the set $G(0,-1)$ of all vector subspaces W of the field $k\big((z)\big)$ of formal Laurent series in another variable z such that the natural map

$$\gamma_W : W \longrightarrow k\big((z)\big)/k\big[[z]\big]z \tag{0.3}$$

is Fredholm of index zero. (The new variable z can be thought of as the Fourier transform of $(\frac{\partial}{\partial x})^{-1}$.) Sato [S] discovered that there is a natural bijection between the group Γ_0 of all pseudo-differential operators of the form (0.1) and the big cell $G^+(0,-1)$ of the Grassmannian consisting of the points $W \in G(0,-1)$ such that the γ_W of (0.3) is an isomorphism. Thus one can interpret the KP system as a dynamical system, or a system of vector fields, on the Grassmannian $G(0,-1)$.

Theorem 0.1 [M2]**.** *A finite-dimensional algebraic variety M is the Jacobian variety of an algebraic curve C if and only if M can be an orbit of the KP system defined on the Grassmannian $G(0,-1)$.*

Thus the KP system characterizes the Jacobian varieties among everything else. If one incorporates the theory of τ-functions of Hirota and Sato, then one obtains a characterization of the Jacobians among the Abelian varieties in terms of theta functions by using Theorem 0.1.

Because of the success of the KP theory, it is natural to try to generalize the entire theory to the supersymmetric cases. The program of supersymmetrization was initiated by Manin and Radul [ManR]. They introduced a supersymmetric generalization of the KP system in the Lax formalism. The unique solvability of the initial value problem and the complete integrability of the super KP system of Manin-Radul was then established in [M3] as a corollary of the super Birkhoff decomposition of infinite-dimensional groups of super pseudo-differential operators. I did not write explicitly in that paper, but the exact solution I obtained in [M3, Section 5] turned out to be a super elliptic function of Rabin and Freund [RF]. Unfortunately, the case of genus one is an exceptional case. More general solutions of the Manin-Radul system have nothing to do with the super conformal structures on the algebraic super curves. It has also become clear that this system does not have any simple relation with the Jacobian varieties of algebraic super curves.

We present in this paper a new supersymmetric generalization of the KP system which enjoys the following properties:

(1) It is a completely integrable system of nonlinear partial super differential equations;
(2) The initial value problem is uniquely solvable;
(3) The even part of the equation recovers the original KP system; and
(4) Every finite-dimensional solution of this system gives rise to the Jacobian variety of an algebraic super curve.

Our version of the super KP system is described as follows. We consider the ring $k\big[[x,\xi]\big]$ of formal power series in the even variable x and the nilpotent odd variable ξ. This ring has a super derivation operator

$$\delta = \frac{\partial}{\partial \xi} + \xi \frac{\partial}{\partial x}$$

satisfying $\delta^2 = \frac{\partial}{\partial x}$. As in (0.1) above, we use a super pseudo-differential operator

$$S = 1 + s_1(x,\xi)\delta^{-1} + s_2(x,\xi)\delta^{-2} + \cdots \tag{0.4}$$

of order zero such that every s_{2n} is an even quantity and s_{2n+1} is an odd quantity. Now our new super KP system is introduced as follows:

$$\begin{cases} \frac{\partial S}{\partial t_{2n}} = -\,(S\cdot\partial^n\cdot S^{-1})_-\cdot S \\ \frac{\partial S}{\partial t_{2n+1}} = -\,(S\cdot\partial^n\cdot\partial_\xi\cdot S^{-1})_-\cdot S \end{cases} \qquad n \geq 1\,, \tag{0.5}$$

where $\partial = \frac{\partial}{\partial x}$, $\partial_\xi = \frac{\partial}{\partial \xi}$, t_{2n}'s are the usual even parameters and t_{2n+1}'s are the anti-commuting odd parameters.

It is obvious from the definition that this system is completely integrable and recovers the original KP system. The unique solvability of the Cauchy problem can be shown by the super Birkhoff decomposition of [M3, Theorem 3.4].

If one compares our system (0.5) with the Manin-Radul system

$$\begin{cases} \frac{\partial S}{\partial t_{2n}} = -\,(S\cdot\delta^{2n}\cdot S^{-1})_-\cdot S \\ \frac{\partial S}{\partial t_{2n-1}} = -\,\big(S\cdot(\delta^{2n-1}+\sum_{k=1}^\infty t_{2k-1}\delta^{2n+2k-2})\cdot S^{-1}\big)_-\cdot S\,, \end{cases}$$

then one realizes that the even part of the systems coincides because it is essentially the original KP system, but the odd part is far from similar. The term involving the infinite sum was necessary to make the Manin-Radul system completely integrable, but the term also made it quite difficult to understand its geometric meaning.

In order to study the geometric meaning of the new system, let us introduce another odd variable θ, which is the Fourier transform of ∂_ξ, and consider $V = k\big((z)\big) \oplus k\big((z)\big)\theta$. The super Grassmannian $G(0|0,-1)$ is the set of all super vector subspaces $W \subset V$ such that the natural map

$$\gamma_W : W \longrightarrow V/k\big[[z,\theta]\big]z \tag{0.6}$$

is Fredholm of index $0|0$. We have

Theorem of super Sato correspondence. *There is a canonical bijection between the group of all super pseudo-differential operators of the form of (0.4) and the big cell* $G^+(0|0,-1)$ *consisting of* $W \in G(0|0,-1)$ *such that* γ_W *of (0.6) gives an isomorphism.*

Thus the system (0.5) defines a system of super commutative vector fields (flows) on the super Grassmannian $G(0|0,-1)$. Now we have

Theorem 0.2. *Every finite-dimensional integral manifold of the flows on* $G(0|0,-1)$ *defined by the system (0.5) gives rise to the Jacobian variety of a certain algebraic super curve.*

But why does an integral manifold determine an algebraic super curve? Of course, we can ask the same question for the Theorem 0.1.

In the nonsupersymmetric case, the reason why an algebraic curve appears is because of the Krichever map. Let $(C, p, z, \mathcal{L}, \phi)$ be a quintuple consisting of an algebraic curve C of an arbitrary genus g, a smooth point $p \in C$, a local coordinate z around p, a line bundle $\mathcal{L}$ of degree $g-1$, and a local trivialization ϕ of $\mathcal{L}$ near p. Then the quintuple corresponds to a unique point W of the Grassmannian $G(0,-1)$. This correspondence was discovered and formulated in the above form by Segal and Wilson [SW].

What we need here is a supersymmetric generalization of the Krichever map. In the joint work with Rabin [MR], we discovered the following:

Theorem 0.3. *Let* $\big(C, p, (z,\theta), \mathcal{L}, \phi\big)$ *be the geometric data consisting of an arbitrary algebraic super variety* C *of dimension* $1|1$ *defined over a field* k *of arbitrary characteristic, a* $0|1$ *divisor* $p \subset C$*, a local coordinate* (z,θ) *which defines the divisor by* $p = \{z = 0\}$*, a line bundle* $\mathcal{L}$ *of rank* $1|0$*, and a "local trivialization"* ϕ *of* $\mathcal{L}$ *near the reduced point* p_{red}*. Then this set of data corresponds uniquely and injectively to a point of the super Grassmannian* $G(0|n,-1)$ *consisting of the super subspaces of* V *such that the natural map of (0.6) has index* $0|n$*. The number* n *is the degree of the odd line bundle* $\mathcal{N}$ *on* C_{red} *which defines the structure sheaf*

$$\mathcal{O}_C = \wedge^\bullet(\mathcal{N}) = \mathcal{O}_{C_{\text{red}}} \oplus \mathcal{N} \, . \tag{0.7}$$

Actually, we established in [MR] a much stronger theorem which includes not only line bundles but also arbitrary vector bundles, based on the construction of the Krichever functor of [M4].

Let $M \subset G(0|0,-1)$ be a finite-dimensional orbit of the flows defined by the new super KP system of (0.5). It can be shown that every point $W \in M$ corresponds to the geometric data $\big(C, p, (z,\theta), \mathcal{L}, \phi\big)$ and that the first three data depend only on the orbit itself and are independent of the specific point W. The algebraic super curve appearing here has the structure sheaf (0.7) given by a line bundle $\mathcal{N}$ of degree 0. The statement of the Theorem 0.2 can be refined as follows: M is canonically isomorphic to the Jacobian variety $\mathrm{Jac}(C)$ of C, where we define

$$\mathrm{Jac}(C) = H^1(C, \mathcal{O}_C)/H^1(C, \mathbb{Z}) \ .$$

More general algebraic super curves appear in the super Grassmannian $G(\mu_0|\mu_1,-1)$ of an arbitrary index $\mu_0|\mu_1$. In order to obtain the Jacobian varieties of all the algebraic super curves, we have to extend our vector fields to all the super Grassmannians. But how?

In order to define more general vector fields on the Grassmannian of an arbitrary index, we introduce the ring E of all super pseudo-differential operators and re-define the super vector space V as the representation module of E. Through the action on V, every super pseudo-differential operator induces a vector field on the super Grassmannian of an arbitrary index. Consider now the set

$$\big(\bigoplus_{n\geq 1} k\cdot\partial^n\big)\oplus\big(\bigoplus_{n\geq 1} k\cdot\partial^{2n+1}\cdot\partial_\xi\big)$$

of infinitely many super-commuting super differential operators. We have a corresponding set of super-commuting vector fields on the super Grassmannian

$$G(\mu_0|\mu_1,-1),$$

which we call the *Jacobian flows*.

Main Theorem. *Every finite-dimensional orbit of the Jacobian flows is canonically isomorphic to the Jacobian variety of an algebraic super curve. Conversely, the Jacobian variety of an arbitrary algebraic super curve defined over k is obtained as a finite-dimensional orbit of the Jacobian flows.*

Therefore, *a super manifold is the Jacobian variety of an algebraic super curve if and only if it can be a finite-dimensional orbit of the Jacobian flows on the super Grassmannians.* This is the characterization of Jacobian varieties of arbitrary algebraic super curves we are establishing in this paper as a supersymmetric generalization of the theory of [M2].

Of course, the restriction of the Jacobian flows on $G(0|0,-1)$ coincides with the flows determined by the equations (0.5).

The importance of the algebraic super geometry lies in the theory of families. The peculiar properties of algebraic super varieties come in when we define these varieties over super schemes. In this paper, however, we have to assume that everything is defined over a field k. This is an unfortunate restriction for the super geometry, but we cannot do better at this moment because the theories and techniques we need in this paper, which have been developed in [M1–4], are all based on a field. Further developments should be left to the (hopefully, near) future.

Another aspect which is missing from our current theory is the super τ-function of A. S. Schwarz [Sch]. It will be very interesting to study the super τ-function from the point of view of the nonlinear partial super differential equations of (0.5), but this will also be left to the future.

I do not know of any relation between our current theory and the very interesting work by Kac and van de Leur [KvdL]. It would be nice to provide a geometric framework for their work, but it is beyond the scope of this paper. There is a large literature on the current topics in the physics context. Since it is impossible to list them all and since I am not familiar with the physics literature, I do not cite them here.

We do not study the Manin-Radul system in this paper. Therefore, whenever we say "the super KP system," we mean the system (0.5). The geometric meaning of the Manin-Radul system has been studied by Rabin [R]. He has discovered a remarkable fact that the system mixes the deformations of the $1|0$ line bundles on the super curve and the deformations of the base manifold itself. Rabin has also arrived the system (0.5) as a deformation equation of line bundles.

For a necessary background of the algebraic super geometry, we refer to the fundamental literature by Manin [Man].

This paper is organized as follows. In Section 1, we define the ring of super pseudo-differential operators and its representation module. The super Grassmannians are defined using this module. The super Sato correspondence is formulated and proved in section 2. As far as I know, no precise statements or proofs of this correspondence have been proposed before, except for some speculations. In Section 3, we introduce our new supersymmetric generalization of the KP system. The unique solvability of the initial value problem of this system is established by using the super Birkhoff decomposition of [M3]. We state the main theorem of [MR] in Section 4 in a slightly more general form. The proof of anti-equivalence of the super Krichever functor is based on the algebro-geometric technique of [M4]. In Section 5, we define the Jacobian variety of an algebraic super curve, the Jacobian flows on the super Grassmannians, and prove the main theorem.

1. Super pseudo-differential operators and the super Grassmannians.

Let k be an arbitrary field of characteristic zero. In this section, we define the algebra E of all formal super pseudo-differential operators and construct a representation module V of this algebra. The filtration of E defined by the order of operators induces a filtration in this module, and we define the super Grassmannians classifying certain super vector subspaces of V by using the filtration.

Let us start with the definition of the super pseudo-differential operators following Manin–Radul [ManR] and [M3]. The function space we need is the super-commutative algebra

$$R = k\big[[x, \xi]\big] = k\big[[x]\big] \oplus k\big[[x]\big]\xi = R_0 \oplus R_1$$

of formal power series in an even variable x and an odd variable ξ. These variables satisfy $x \cdot \xi = \xi \cdot x$ and $\xi^2 = 0$. An element of R_0 (resp. R_1) is called a homogeneous element of degree 0 (resp. degree 1). The ring R has a super derivation operator

$$\delta = \frac{\partial}{\partial \xi} + \xi \frac{\partial}{\partial x} \tag{1.1}$$

which satisfies the super Leibniz rule

$$\delta(ab) = \delta(a) \cdot b + (-1)^{\tilde{a}} a \delta(b) \ ,$$

where a is a homogeneous element of R of $\mathbb{Z}_2$-degree $\tilde{a}$, and b is an arbitrary element of R. Note that we have $\delta^2 = \frac{\partial}{\partial x}$. We call an expression

$$P = \sum_{m=0}^{\infty} a_m \delta^{n-m} \tag{1.2}$$

a *super pseudo-differential operator* with coefficients in R if $a_m \in R$. The *order* of P is defined to be n only when $0 \neq a_0 \in R_0$. In particular, we do not assign any order to a nilpotent operator. The set of all super pseudo-differential operators with coefficients in R is denoted by E.

For an arbitrary integer ν and a nonnegative integer i, we define the super binomial coefficients following [ManR] by

$$\begin{bmatrix} \nu \\ i \end{bmatrix} = \begin{cases} 0 \ , & \text{if } 0 \leq \nu < i \text{ or } (\nu, i) \equiv (0,1) \bmod 2 \\ \binom{[\frac{\nu}{2}]}{[\frac{i}{2}]} \ , & \text{otherwise,} \end{cases} \tag{1.3}$$

where $[\alpha]$ is the largest integer not greater than α. The super binomial coefficients satisfy

$$\begin{bmatrix} \nu \\ i \end{bmatrix} + (-1)^{i+1} \begin{bmatrix} \nu \\ i+1 \end{bmatrix} = \begin{bmatrix} \nu+1 \\ i+1 \end{bmatrix} \tag{1.4}$$

and

$$\sum_{i=0}^{n} (-1)^{\frac{i(i+1)}{2} + i(\nu - n)} \begin{bmatrix} \nu \\ i \end{bmatrix} \begin{bmatrix} \nu - i \\ n - i \end{bmatrix} = 0 \qquad \text{for } n > 0. \tag{1.5}$$

The set E of the super pseudo-differential operators has a super algebra structure introduced by the generalized super Leibniz rule

$$\delta^\nu \cdot f = \sum_{i=0}^{\infty} (-1)^{\tilde{f}\cdot(\nu-i)} \begin{bmatrix} \nu \\ i \end{bmatrix} f^{[i]} \delta^{\nu-i} , \tag{1.6}$$

where ν is an arbitrary integer, f is a homogeneous element of R of degree $\tilde{f}$, and $f^{[i]} = \delta^i(f)$. We say that the operator P of (1.2) is in the *right nomal form*. The *left normal form* of P is given by

$$P = \sum_{m=0}^{\infty} \delta^{N-m} \cdot b_m , \tag{1.7}$$

where the coefficients b_m of (1.7) can be computed by the adjoint super Leibniz rule

$$f\delta^\nu = \sum_{i=0}^{\infty} (-1)^{\frac{i(i+1)}{2} + \tilde{f}\nu} \begin{bmatrix} \nu \\ i \end{bmatrix} \delta^{\nu-i} f^{[i]} \tag{1.8}$$

which follows from (1.4), (1.5) and (1.6).

Let $E^{(n)}$ denote the set of all super pseudo-differential operators of the form of (1.2). It is important to notice that the definition of $E^{(n)}$ does not depend on the choice of the normal form of operators. We have a natural filtration

$$\cdots \supset E^{(n+1)} \supset E^{(n)} \supset E^{(n-1)} \supset \cdots \tag{1.9}$$

of E which satisfies

$$\bigcup_{n\in\mathbb{Z}} E^{(n)} = E \quad \text{and} \quad \bigcap_{n\in\mathbb{Z}} E^{(n)} = \{0\} .$$

Thus E has the structure of a complete topological space. The expressions in (1.2), (1.6) – (1.8) are convergent series with respect to this topology of E. Let us define

$$\begin{aligned} E_0 &= \{\sum_\nu f_\nu \delta^\nu \mid \tilde{f}_{2\nu} = 0 \text{ and } \tilde{f}_{2\nu+1} = 1\} , \\ E_1 &= \{\sum_\nu f_\nu \delta^\nu \mid \tilde{f}_{2\nu} = 1 \text{ and } \tilde{f}_{2\nu+1} = 0\} . \end{aligned} \tag{1.10}$$

Then $E = E_0 \oplus E_1$, and hence E has also the structure of a super algebra. An element of E_0 (resp. E_1) is called a *homogeneous even* (resp. *odd*) operator.

Symbolically, we can write $E = R\big((\delta^{-1})\big)$, where $k\big((x)\big)$ is the standard notation for the field of quotients of the power series ring $k\big[[x]\big]$. Let us consider the other set

$R\big((\partial^{-1})\big) \oplus R\big((\partial^{-1})\big)\frac{\partial}{\partial\xi}$ of operators, where $\partial = \frac{\partial}{\partial x}$. Since $\partial = \delta^2$ and $\frac{\partial}{\partial\xi} = \delta - \xi\delta^2$, the new set of operators is included in $R\big((\delta^{-1})\big)$. On the other hand, we have

$$\begin{aligned}
\sum_\nu a_\nu \delta^\nu &= \sum_\mu a_{2\mu}\delta^{2\mu} + \sum_\mu a_{2\mu+1}\delta^{2\mu+1} \\
&= (\sum_\mu a_{2\mu}\delta^{2\mu} + \sum_\mu a_{2\mu+1}\xi\delta^{2\mu+2}) + \sum_\mu a_{2\mu+1}\delta^{2\mu}(\delta - \xi\delta^2) \\
&= (\sum_\mu a_{2\mu}\partial^\mu + \sum_\mu a_{2\mu+1}\xi\partial^{\mu+1}) + \sum_\mu a_{2\mu+1}\partial^\mu \frac{\partial}{\partial\xi} \ .
\end{aligned}$$

Therefore, we can conclude that

$$R\big((\delta^{-1})\big) = R\big((\delta^{-2})\big) \oplus R\big((\delta^{-2})\big)(\delta - \xi\delta^2) = R\big((\partial^{-1})\big) \oplus R\big((\partial^{-1})\big)\frac{\partial}{\partial\xi} \ .$$

We call the third line of the above expansion the *standard form* of the super pseudo-differential operator $\sum_\nu a_\nu\delta^\nu$.

In order to define a left E-module, let us consider the left ideal $E(x,\xi)$ of E generated by x and ξ. Note that it is not a maximal ideal of E. Now we define

$$(1.11) \qquad \begin{cases} z = \delta^{-2} \bmod E(x,\xi) & = \partial^{-1} \bmod E(x,\xi) \\ \theta = \delta \bmod E(x,\xi) & = \frac{\partial}{\partial\xi} \bmod E(x,\xi) \ . \end{cases}$$

We regard z as an even variable of order -2, and θ as an odd variable of order 1. Let us define $V = E/E(x,\xi)$ and denote the canonical projection by

$$(1.12) \qquad \rho : E \longrightarrow E/E(x,\xi) = V \ .$$

If we write elements of E in the standard form, then it is easy to see that there is a canonical isomorphism

$$(1.13) \qquad V = E/E(x,\xi) \cong k\big((z)\big) \oplus k\big((z)\big)\theta \ ,$$

which is given by (1.11). The filtration (1.9) introduces a filtration

$$(1.14) \qquad \cdots \supset V^{(n+1)} \supset V^{(n)} \supset V^{(n-1)} \supset \cdots$$

of V, where we define $V^{(n)} = \rho(E^{(n)})$. The filtration (1.14) satisfies

$$\bigcup_{n\in\mathbb{Z}} V^{(n)} = V \quad \text{and} \quad \bigcap_{n\in\mathbb{Z}} V^{(n)} = \{0\} \ ,$$

and defines a topology in V. Under the identification of (1.13), each $V^{(n)}$ has the following expression:

$$\begin{cases} V^{(2n+1)} & = \{v \in V \mid \operatorname{ord} v \le 2n+1\} = k\big[[z,\theta]\big]z^{-n} \ , \\ V^{(2n)} & = \{v \in V \mid \operatorname{ord} v \le 2n\} = k\big[[z,\theta]\big]z^{-n+1} \oplus kz^{-n} \ . \end{cases}$$

We also have the super space structure in V defined by

$$V = V_0 \oplus V_1 \ ,$$

where $V_0 = \rho(E_0)$ and $V_1 = \rho(E_1)$. Obviously, the identification (1.13) gives $V_0 = k\big((z)\big)$ and $V_1 = k\big((z)\big)\theta$.

A *super subspace* W of the super vector space V is a direct sum $W = W_0 \oplus W_1$ which satisfies $W_0 = W \cap V_0$ and $W_1 = W \cap V_1$. We call W_0 (resp. W_1) the even (resp. odd) part of W. For every super subspace $W \subset V$, we define the canonical map $\gamma_W(\nu) : W \longrightarrow V/V^{(\nu)}$ by

$$\begin{array}{ccc} V & = & V \\ {\scriptstyle\text{inclusion}}\uparrow & & \downarrow{\scriptstyle\text{projection}} \\ W & \xrightarrow{\gamma_W(\nu)} & V/V^{(\nu)} \ . \end{array} \tag{1.15}$$

We call $\gamma_W(\nu)$ *Fredholm* if both its kernel and the cokernel are finite-dimensional over k. For a Fredholm map γ, we define the *Fredholm index* by

$$\text{Index}\ \gamma = \dim\ \text{Ker}\ \gamma - \dim\ \text{Coker}\ \gamma \ ,$$

which is a pair $\mu_0|\mu_1$ of integers indicating the indices of the even part and the odd part.

Definition 1.1. *Let μ_0, μ_1 and ν be arbitrary integers. The super Grassmannian $G(\mu_0|\mu_1,\ 2\nu+1)$ of index $\mu_0|\mu_1$ and level $2\nu+1$ is the set of all super subspaces $W = W_0 \oplus W_1 \subset V$ such that the canonical map $\gamma_W(2\nu+1)$ is Fredholm of index $\mu_0|\mu_1$.*

Remark. Note that W_0 (resp. W_1) is subspace of V_0 (resp. V_1). Thus for every pair (U, U') of points $U \in G(\mu_0, \nu)$ and $U' \in G(\mu_1, \nu)$, the map $(U, U') \longmapsto W = U \oplus \theta \cdot U'$ gives a bijection

$$G(\mu_0, \nu) \times G(\mu_1, \nu) \cong G(\mu_0|\mu_1, 2\nu+1),$$

where $G(\mu, \nu) = \{$ subspace $U \subset k\big((z)\big) \mid U \to k\big((z)\big)/(k\big[[z]\big]z^{-\nu})$ is Fredholm of index $\mu\}$ is the Grassmannian of index μ and level ν studied in [M4]. Using this bijection we introduce the structure of a pro-algebraic variety of Grothendieck in our $G(\mu_0|\mu_1, 2\nu+1)$. The super manifold structure of our super Grassmannian is defined by the projective limit of the finite-dimensional super Grassmannians defined by Manin [Man].

In the nonsupersymmetric case, the index 0 and the level -1 is the standard choice for the Grassmannian and every point W of $G(0, -1)$ gives rise to the geometric data consisting of an arbitrary algebraic curve if it has a nontrivial stabilizer A_W. In the supersymmetric case, however, no single super Grassmannian can produce all the algebraic super curves. Especially, the obvious choice $G(0|0,0)$ or $G(0|0,-1)$ does not correspond to algebraic super curves with super conformal structures except for the genus 1 case (see Section 4). This is the reason why we need to consider all the super Grassmannians of arbitrary indices. The level of the Grassmannians can be fixed, for example to -1, without loss of generality.

If we imagine E as a generalization of a field, then the subring of E which corresponds to the integer ring is the set D of super differential operators. We call an element $P = \sum_\nu a_\nu \delta^\nu \in E$ a super differential operator if $a_\nu = 0$ for all negative ν. There is a natural (left, right or both-sided) R-module direct sum decomposition

$$E = D \oplus E^{(-1)} \tag{1.16}$$

which does not depend on the choice of the form of operators. According to (1.16), we write $P = P_+ + P_-$, where $P \in E$, $P_+ \in D$ and $P_- \in E^{(-1)}$. Since $D = D_0 \oplus D_1$ for $D_0 = D \cap E_0$ and $D_1 = D \cap E_1$, D is a super subalgebra of E.

2. The super Sato correspondence.

The supersymmetric generalization of the theorem of Sato [S] is proved in this section. In order to investigate further the relation between the super pseudo-differential operators and the super Grassmannians, we need an adic topology in R and a super analogue of the Taylor's expansion formula, which is also proved in this section.

Let $\mathrm{val} : R \longrightarrow \mathbb{N} \cup \{\infty\}$ be the valuation defined by

$$\begin{cases} \mathrm{val}\, \xi = 1 \ , \\ \mathrm{val}\, x = 2 \ , \end{cases}$$

where $\mathbb{N}$ denotes the set of all nonnegative integers. The valuation of an element of R is defined to be the valuation of its leading term. Let us denote by R_m the subset of R consisting of elements of valuation greater than or equal to m. Then we have

$$\bigcup_{m=0}^{\infty} R_m = R \quad \text{and} \quad \bigcap_{m=0}^{\infty} R_m = \{0\} \ , \tag{2.1}$$

and hence R becomes a complete topological ring with respect to this valuation.

The super analogue of the Taylor expansion takes its simplest form in terms of the new variable λ which is defined as follows:

$$\begin{cases} \lambda^{2m} = \frac{1}{m!} x^m & \in R_{2m} \text{ for } m \geq 0 \ , \\ \lambda^{2m+1} = \frac{1}{m!} x^m \xi & \in R_{2m+1} \text{ for } m \geq 0 \ . \end{cases} \tag{2.2}$$

Every element $f \in R$ has a unique expansion

$$f = \sum_{n=0}^{\infty} c_n \lambda^n, \quad c_n \in k$$

which is a convergent series with respect to the topology of R. Let us define $f(0) = c_0 \in k$. It is easy to show that

$$\delta^m(\lambda^n) = \begin{cases} \lambda^{n-m}, & \text{if } n \geq m \\ 0\ , & \text{otherwise} \end{cases}$$

which implies that $f^{[n]}(0) = c_n$. Thus we can establish the super Taylor formula

$$f(x,\xi) = f(\lambda) = \sum_{n=0}^{\infty} f^{[n]}(0)\lambda^n\ . \tag{2.3}$$

An element $f \in R$ is contained in R_m if and only if $f^{[\ell]}(0) = 0$ for all $0 \leq \ell < m$. Note also that $R \cap E(x,\xi) = R_1$.

In order to give the explicit formula for the projection ρ of (1.12), we need a new symbol ζ of order -1 defined by $\zeta^\ell = \delta^{-\ell} \bmod E(x,\xi)$. In terms of the variables z and θ, we have

$$\begin{cases} \zeta^{2m} = z^m \\ \zeta^{2m-1} = z^m\theta \end{cases} \tag{2.4}$$

for every integer m. Note that the order of the both hand sides of the above equations is consistent. Let us take an operator $P \in E$. First, we write it in the right normal form

$$P = \sum_{n=0}^{\infty} \delta^{N-n} \cdot f_n(\lambda)$$

with coefficients in the λ-expansion. Then we have

$$\rho(P) = \sum_{n=0}^{\infty} f_n(0) \cdot \zeta^{-N+n} \in V\ .$$

The left $P \in E$ action on V is given by

$$P : V \ni v = \rho(Q) \longmapsto Pv = \rho(PQ) \in V\ .$$

The following lemma gives an interesting characterization of the super differential operators in E.

Lemma 2.1. *A super pseudo-differential operator $P \in E$ is a super differential operator if and only if it preserves $\rho(D)$ in V, i.e.*

$$P\rho(D) \subset \rho(D)\ .$$

Proof. Every super differential operator $P \in D$ preserves $\rho(D)$ because $P\rho(Q) = \rho(PQ) \in \rho(D)$ for every $Q \in D$. In order to prove the converse, let P be a super pseudo-differential operator and let

$$P_- = \sum_{n=1}^{\infty} \delta^{-n} \cdot f_n(\lambda)$$

be the $E^{(-1)}$-part of P according to the decomposition of (1.16). The condition $P\rho(D) \subset \rho(D)$ implies that $PD \subset D + E(x,\xi)$, and hence

$$(PQ)_- \in E(x,\xi)$$

for every $Q \in D$. In particular, we have $P_- \in E(x,\xi)$ by taking $Q = 1 \in D$. Thus $f_n \in R_1$ for all $n \geq 1$. So let $\ell \geq 1$ be the largest integer such that $f_n \in R_\ell$ for all $n \geq 1$. Then we have

$$\begin{aligned}
(P \cdot \delta^\ell)_- &= (P_- \cdot \delta^\ell)_- \\
&= \left(\sum_{n=1}^{\infty} \delta^{-n} \cdot f_n(\lambda) \cdot \delta^\ell \right)_- \\
&= \left(\sum_{n=1}^{\infty} \sum_{i=0}^{\ell} \delta^{-n+\ell-i} (-1)^{\frac{i(i+1)}{2} + \tilde{f}_n \cdot \ell} \begin{bmatrix} \ell \\ i \end{bmatrix} f_n^{[i]}(\lambda) \right)_- \\
&= \left(\sum_{j=1}^{\infty} \delta^{\ell-j} \sum_{i=0}^{\ell} (-1)^{\frac{i(i+1)}{2} + \tilde{f}_{j-i} \cdot \ell} \begin{bmatrix} \ell \\ i \end{bmatrix} f_{j-i}^{[i]}(\lambda) \right)_- \\
&= \sum_{j=\ell+1}^{\infty} \delta^{\ell-j} \sum_{i=0}^{\ell} (-1)^{\frac{i(i+1)}{2} + \tilde{f}_{j-i} \cdot \ell} \begin{bmatrix} \ell \\ i \end{bmatrix} f_{j-i}^{[i]}(\lambda) .
\end{aligned}$$

Since $f_n^{[i]}(0) = 0$ for $0 \leq i < \ell$, we have

$$\begin{aligned}
\rho\big((P \cdot \delta^\ell)_-\big) &= \sum_{j=\ell+1}^{\infty} (-1)^{\frac{\ell(\ell+1)}{2} + \tilde{f}_{j-\ell} \cdot \ell} f_{j-\ell}^{[\ell]}(0) \cdot \zeta^{-\ell+j} \\
&= \sum_{n=1}^{\infty} (-1)^{\frac{\ell(\ell+1)}{2} + \tilde{f}_n \cdot \ell} f_n^{[\ell]}(0) \cdot \zeta^n ,
\end{aligned}$$

where we have used the fact that $\left[\begin{smallmatrix} \ell \\ \ell \end{smallmatrix}\right] = 1$ for $\ell \geq 0$. Since $\rho\big((P \cdot \delta^\ell)_-\big) = 0$, we have $f_n^{[\ell]}(0) = 0$ for all $n \geq 1$, i.e. $f_n \in R_{\ell+1}$. But this contradicts to our assumption that ℓ is the largest integer satisfying this condition. Therefore, $f_n \in R_m$ for all $n \geq 1$ and $m \geq 1$. By (2.1), we can conclude that $f_n = 0$ for all $n \geq 1$, which means that P is a differential operator. This completes the proof.

Definition 2.2. *The super Sato Grassmannian, which is denoted by SSG^+, is the set of right super D-submodules $J = J_0 \oplus J_1 \subset E$ (i.e. $JD \subset J$) such that $E = J \oplus E^{(-1)}$.*

The geometric counter part of this set is the *big cell* $G^+(0|0,-1)$ of the super Grassmannian of index $0|0$ and level -1 consisting of the super subspaces $W \subset V$ such that $W \oplus V^{(-1)} = V$. Note that $\rho(D) = k[z^{-1},\theta] \in G^+(0|0,-1)$.

We call an operator in E *monic* if its leading coefficient is 1.

Theorem 2.3.

(1) *Let $\Gamma_0 \subset E_0$ denote the group of homogeneous even monic super pseudo-differential operators of order zero, and let SSG^+ be the super Sato Grassmannian. Then there is a natural bijection $\sigma : \Gamma_0 \xrightarrow{\sim} SSG^+$ obtained by*

$$\Gamma_0 \ni S \overset{\sigma}{\longmapsto} \sigma(S) = S^{-1}D = J \in SSG^+ \ .$$

(2) *Let $G^+(0|0,-1)$ be the big cell of the Grassmannian of index $0|0$ and level -1. Then the natural projection $\rho : E \to V$ induces a bijection*

$$\rho : SSG^+ \xrightarrow{\sim} G^+(0|0,-1) \ .$$

Proof. (1) Well-definedness of σ:

Take an element $S \in \Gamma_0$ and define $J = J_0 \oplus J_1 = S^{-1}D_0 \oplus S^{-1}D_1$. Then J is a right super D-module which satisfies $E = J \oplus E^{(-1)}$, because $S^{-1}E = E$ and $S^{-1}E^{(-1)} = E^{(-1)}$. Therefore, $J \in SSG^+$.

Injectivity of σ:

Suppose we have two operators S_1 and S_2 such that $\sigma(S_1) = \sigma(S_2) = J$. This means that $S_1^{-1}D = S_2^{-1}D$, hence $S_1S_2^{-1}\cdot 1 = S_1S_2^{-1} \in D$. Therefore, $S_1 \cdot S_2^{-1} \in \Gamma_0 \cap D = \{1\}$, i.e. $S_1 = S_2$.

Subjectivity of σ:

Let $J \in SSG^+$ be an arbitrary element. Since $E = J \oplus E^{(-1)}$, we can choose a monic zero-th order operator S such that $S^{-1} \in J \cap \Gamma_0$. Then J contains the right super D-module $S^{-1}D$ generated by S^{-1} in E. Define

$$J^{(N)} = J \cap E^{(N)}$$

and take an arbitrary element $P \in J^{(N)}$ for $N \geq 0$. Since $S^{-1} \in J$ is monic of order 0, we have

$$P - S^{-1}Q_N \in J^{(N-1)} \ ,$$

where $Q_N \in D$ is the leading term of P. Similarly, there is a $Q_{N-1} \in D \cap E^{(N-1)}$ such that

$$P - S^{-1}Q_N - S^{-1}Q_{N-1} \in J^{(N-2)} \ .$$

If we repeat this process N-times, then we end up with

$$P - S^{-1} \sum_{n=0}^{N} Q_{N-n} \in J^{(-1)} = J \cap E^{(-1)} = \{0\} \; .$$

Therefore, $P = S^{-1} \sum\limits_{n=0}^{N} Q_{N-n} \in S^{-1}D$, i.e. $J \subset S^{-1}D$. Thus $J = S^{-1}D = \sigma(S)$.

(2) Well-definedness of ρ:

For every $J \in SSG^{+}$, we have an $S \in \Gamma_0$ such that $J = S^{-1}D$ by the above (1). Since $\rho(J) = S^{-1}\rho(D)$ and $S^{-1}V^{(-1)} = V^{(-1)}$, we have

$$V = S^{-1}\rho(D) \oplus V^{(-1)} \; .$$

Thus $\rho(J)$ is an element of $G^{+}(0|0,-1)$.

Injectivity of ρ:

Suppose that $S_1^{-1}\rho(D) = S_2^{-1}\rho(D)$. Then $S_1 S_2^{-1}\rho(D) = \rho(D)$, which means that $S_1 S_2^{-1} \in D$ by Lemma 2.1. Therefore, $S_1 S_2^{-1} \in D \cap \Gamma_0 = \{1\}$, namely, $S_1 = S_2$.

Surjectivity of ρ:

Let W be an arbitrary point of the big cell $G^{+}(0|0,-1)$. Since $V = W \oplus V^{(-1)}$ and $W = W_0 \oplus W_1$, we can choose a basis $\{w_n\}_{n \geq 0}$ for W in the following form for every $n \geq 0$:

$$\begin{cases} w_{2n} & = \zeta^{-2n} + \sum_{\ell=1}^{\infty} a_{2\ell}^{2n} \zeta^{2\ell} \\ w_{2n+1} & = \zeta^{-2n-1} + \sum_{\ell=1}^{\infty} a_{2\ell+1}^{2n+1} \zeta^{2\ell+1} \; . \end{cases}$$

Of course $\{w_{2n}\}_{n \geq 0}$ forms a basis for W_0 and $\{w_{2n+1}\}_{n \geq 0}$ spans W_1. For the convenience, let us define

$$a_\ell^n = \begin{cases} a_\ell^n & \text{if both } n \text{ and } \ell \text{ are even or odd,} \\ 0 & \text{otherwise.} \end{cases} \tag{2.5}$$

In order to construct a homogeneous even operator $S \in \Gamma_0$ which satisfies that $S^{-1}\rho(D) = W$, let us give S^{-1} in the right normal form

$$S^{-1} = \sum_{\ell=0}^{\infty} \delta^{-\ell} \cdot s_\ell(\lambda) \; ,$$

where $s_0(\lambda) \equiv 1$ and $s_\ell(\lambda) \in R$. The coefficients satisfy $\tilde{s}_{2\ell} = 0$ and $\tilde{s}_{2\ell+1} = 1$. Then the equation

$$w_0 = S^{-1} \cdot 1 = \rho(S^{-1}) = 1 + \sum_{\ell=1}^{\infty} s_\ell(0) \zeta^\ell$$

determines the constant terms $s_\ell(0)$ of the coefficients of S^{-1} by $s_\ell(0) = a_\ell^0$ for all $\ell \geq 1$.

Now let us assume that we know $s_\ell^{[i]}(0)$ for all $\ell \geq 1$ and $0 \leq i < n$. Note that we have

$$\begin{aligned}
S^{-1} \cdot \zeta^{-n} &= \rho(S^{-1} \cdot \delta^n) \\
&= \rho\left(\sum_{m=0}^{\infty} \delta^{-m} \cdot s_m(\lambda) \cdot \delta^n\right) \\
&= \rho\left(\sum_{m=0}^{\infty} \sum_{i=0}^{n} \delta^{n-m-i} \cdot (-1)^{\frac{i(i+1)}{2} + \tilde{s}_m \cdot n} \begin{bmatrix} n \\ i \end{bmatrix} s_m^{[i]}(\lambda)\right) \\
&= \rho\left(\delta^n + \sum_{\ell=1}^{\infty} \delta^{n-\ell} \sum_{i=0}^{n} (-1)^{\frac{i(i+1)}{2} + \tilde{s}_{\ell-i} \cdot n} \begin{bmatrix} n \\ i \end{bmatrix} s_{\ell-i}^{[i]}(\lambda)\right) \\
&= \zeta^{-n} + \sum_{\ell=1}^{n-1} \sum_{i=0}^{\ell} (-1)^{\frac{i(i+1)}{2} + \tilde{s}_{\ell-i} \cdot n} \begin{bmatrix} n \\ i \end{bmatrix} s_{\ell-i}^{[i]}(0) \zeta^{-n+\ell} \\
&\quad + \sum_{i=0}^{n-1} (-1)^{\frac{i(i+1)}{2} + \tilde{s}_{n-i} \cdot n} \begin{bmatrix} n \\ i \end{bmatrix} s_{n-i}^{[i]}(0) \\
&\quad + \sum_{\ell=1}^{\infty} \sum_{i=0}^{n} (-1)^{\frac{i(i+1)}{2} + \tilde{s}_{n+\ell-i} \cdot n} \begin{bmatrix} n \\ i \end{bmatrix} s_{n+\ell-i}^{[i]}(0) \zeta^{\ell} \ .
\end{aligned}$$

The nonnegative order terms of the above expression exactly coincides with

$$\begin{aligned}
w_n &+ \sum_{\ell=1}^{n-1} \sum_{i=0}^{\ell} (-1)^{\frac{i(i+1)}{2} + \tilde{s}_{\ell-i} \cdot n} \begin{bmatrix} n \\ i \end{bmatrix} s_{\ell-i}^{[i]}(0) w_{n-\ell} \\
&+ \sum_{i=0}^{n-1} (-1)^{\frac{i(i+1)}{2} + \tilde{s}_{n-i} \cdot n} \begin{bmatrix} n \\ i \end{bmatrix} s_{n-i}^{[i]}(0) w_0 \ ,
\end{aligned}$$

which contains only known quantities. Therefore, the equation

$$\begin{aligned}
S^{-1} \cdot \zeta^{-n} = w_n &+ \sum_{\ell=1}^{n-1} \sum_{i=0}^{\ell} (-1)^{\frac{i(i+1)}{2} + \tilde{s}_{\ell-i} \cdot n} \begin{bmatrix} n \\ i \end{bmatrix} s_{\ell-i}^{[i]}(0) w_{n-\ell} \\
&+ \sum_{i=0}^{n-1} (-1)^{\frac{i(i+1)}{2} + \tilde{s}_{n-i} \cdot n} \begin{bmatrix} n \\ i \end{bmatrix} s_{n-i}^{[i]}(0) w_0
\end{aligned}$$

determines $s_j^{[n]}(0)$ for all $j \geq 1$ by

$$
\begin{aligned}
(-1)^{\frac{n(n+1)}{2}+\tilde{s}_j \cdot n} s_j^{[n]}(0) = a_j^n &+ \sum_{\ell=1}^{n-1} \sum_{i=0}^{\ell} (-1)^{\frac{i(i+1)}{2}+\tilde{s}_{\ell-i} \cdot n} \begin{bmatrix} n \\ i \end{bmatrix} s_{\ell-i}^{[i]}(0) a_j^{n-\ell} \\
&+ \sum_{i=0}^{n-1} (-1)^{\frac{i(i+1)}{2}+\tilde{s}_{n-i} \cdot n} \begin{bmatrix} n \\ i \end{bmatrix} s_{n-i}^{[i]}(0) a_j^0 \\
&+ \sum_{i=0}^{n-1} (-1)^{\frac{i(i+1)}{2}+\tilde{s}_{n+j-i} \cdot n} \begin{bmatrix} n \\ i \end{bmatrix} s_{n+j-i}^{[i]}(0) \ .
\end{aligned}
$$

Thus we can determine the coefficients $s_\ell(\lambda) = \sum_{n=0}^{\infty} s_\ell^{[n]}(0)\lambda^n$ because of the super Taylor formula, and hence the operator $S^{-1} = \sum_{\ell=0}^{\infty} \delta^{-\ell} \cdot s_\ell(\lambda)$. It is easy to see from this construction that S satisfies $S^{-1}\rho(D) = W$ as required. The only remaining thing we have to show is that S is a homogeneous even operator. Since the coefficients of our operator are defined on the field k, S is even if and only if $s_j^{[n]}(0) = 0$ for (1) j is odd and n is even, or (2) j is even and n is odd. Using the mathematical induction and the property (2.5) of a_j^n, we can show the vanishing of the coefficient $s_j^{[n]}(0)$ in the both cases. This completes the proof.

3. A new supersymmetric generalization of the KP system.

In this section, we introduce a system of completely integrable nonlinear partial super differential equations which gives a supersymmetric generalization of the KP system. We prove the unique solvability of this system following the technique of [M3].

Let us recall the variable ζ of (2.4). If we further identify

$$
\begin{cases} z = \partial^{-1} \\ \theta = \partial_\xi = \frac{\partial}{\partial \xi} \end{cases}
$$

following (1.11), then we can use ζ to indicate

$$
\begin{cases} \zeta^{-2m} = \partial^m \\ \zeta^{-2m-1} = \partial^m \cdot \partial_\xi \ . \end{cases} \tag{3.1}
$$

In order to introduce the time evolution of the operator $S \in \Gamma_0$ of Section 2, let us define the set $\{t_{2n}\}_{n\geq 1}$ of infinitely many even variables and the set $\{t_{2n+1}\}_{n\geq 1}$ of infinitely many odd variables. The even variables commute with everything and the odd ones anti-commute one another. For differential forms, we use the convention that dt_{2n}'s are odd quantities which anti-commute one another and dt_{2n+1}'s are even quantities which commute with everything. Since the coefficients of the time evolution of S is a function in $t = (t_2,\ t_3,\ t_4,\ \cdots)$, we have to extend our function ring R to

$$\mathfrak{R} = R\big[[t_2,\ t_3,\ t_4,\ \cdots]\big] = \varprojlim_n R\big[[t_2,\ t_3,\ t_4,\ \cdots,\ t_n]\big]\ .$$

We need a new valuation $\mathrm{val}_t : \mathfrak{R} \to \mathbb{N} \cup \{\infty\}$ defined by $\mathrm{val}_t\ (R \setminus \{0\}) = 0$ and $\mathrm{val}_t\ t_n = n$. The set of all elements of $\mathfrak{R}$ whose valuation is greater than or equal to m is denoted by $\mathfrak{R}_m$. Note that $\mathfrak{R}/\mathfrak{R}_1 = R$. An expression

$$P = \sum_{\nu=-\infty}^{\infty} a_\nu(t)\delta^\nu \tag{3.2}$$

is called an *infinite order* super pseudo-differential operator if $a_\nu(t) \in \mathfrak{R}$ and there exist positive real numbers c_1, c_2 and c_3 depending on P such that

$$\mathrm{val}_t\ a_\nu(t) > c_1\nu - c_2$$

for all $\nu > c_3$. The set of all infinite order super pseudo-differential operators is denoted by $\mathfrak{E}$. Even though our P has infinitely many terms in both the positive and the negative directions, it is not so hard to show that $\mathfrak{E}$ forms an associative algebra. Like E, the extended algebra has a super algebra structure $\mathfrak{E} = \mathfrak{E}_0 \oplus \mathfrak{E}_1$ in an obvious way. If P of (3.2) has no negative power terms of δ, then we call it an infinite order super differential operator. We denote by $\mathfrak{D}$ the ring of all infinite order super differential operators. Define

$$\begin{aligned}
&\mathfrak{E}_0^\times = \{P \in \mathfrak{E} \mid P \bmod \mathfrak{R}_1 \in \Gamma_0\} \\
&\mathfrak{D}_0^\times = \{P \in \mathfrak{D} \mid P \bmod \mathfrak{R}_1 = 1\} \\
&\mathfrak{G} = \{1 + \sum_{n>0} s_n(t)\delta^{-n} \mid s_n(t) \in \mathfrak{R}\} \\
&\mathfrak{G}_0 = \mathfrak{G} \cap \mathfrak{E}_0\ .
\end{aligned} \tag{3.3}$$

It is established in [M3, Theorem 3.4] that these are infinite-dimensional groups and satisfy the super Birkhoff decomposition

$$\mathfrak{E}_0^\times = \mathfrak{G}_0 \cdot \mathfrak{D}_0^\times\ . \tag{3.4}$$

This is the group version of the module decomposition

$$\mathfrak{E}_0 = \mathfrak{E}_0^{(-1)} \oplus \mathfrak{D}_0\ ,$$

where $\mathfrak{E}^{(n)} = E^{(n)}\widehat{\otimes}\mathfrak{R}$ and $\mathfrak{E}_0^{(n)} = \mathfrak{E}^{(n)} \cap \mathfrak{E}_0$.

With these preparations, let us now introduce the *time evolution operator* of our super KP system by

$$H = \exp(\sum_{m\geq 2} t_m \cdot \zeta^{-m}) = (1 + \sum_{n\geq 1} t_{2n+1}\partial^n \cdot \partial_\xi) \cdot \exp(\sum_{n\geq 1} t_{2n} \cdot \partial^n) \in \mathfrak{D}_0^\times \ .$$

This operator defines a connection form

$$\Omega = dH \cdot H^{-1} = \sum_{n\geq 1} dt_{2n} \cdot \partial^n + \sum_{n\geq 1} dt_{2n+1} \cdot \partial^n \cdot \partial_\xi \ , \tag{3.5}$$

which satisfies the zero-curvature condition trivially:

$$d\Omega = \Omega \wedge \Omega = 0 \ .$$

Definition 3.1. *The new super KP system is the total differential equation for an even homogeneous monic pseudo-differential operator $S \in \mathfrak{G}_0$ of order zero:*

$$dS = -(S \cdot \Omega \cdot S^{-1})_- \cdot S \ ,$$

where $(\bullet)_-$ denotes the $\mathfrak{E}^{(-1)}$-part of the super pseudo-differential operator appearing in the coefficients of the differential form.

In the coordinate system t_n, the above system is given by

$$\begin{cases} \frac{\partial S}{\partial t_{2n}} = -(S \cdot \partial^n \cdot S^{-1})_- \cdot S \\ \frac{\partial S}{\partial t_{2n+1}} = -(S \cdot \partial^n \cdot \partial_\xi \cdot S^{-1})_- \cdot S \ . \end{cases} \tag{3.6}$$

Remark. If we apply the reduction modulo ξ to the super KP system, then the even equations recover the entire KP system. Therefore, our system is a supersymmetric generalization of the usual KP system.

theorem 3.2. *For an arbitrary initial datum $S \in \Gamma_0$, there is a unique solution $S(t) \in \mathfrak{G}_0$ of the super KP system.*

Proof. By applying the super Birkhoff decomposition (3.4) to the operator $H \cdot S^{-1} \in \mathfrak{E}_0^\times$, we can find unique operators $S(t) \in \mathfrak{G}_0$ and $Y(t) \in \mathfrak{D}_0^\times$ such that

$$S(t)^{-1} \cdot Y(t) = H \cdot S^{-1} \ ,$$

i.e. $S(t) = Y(t) \cdot S \cdot H^{-1}$. Since the differential form $dS(t) \cdot S(t)^{-1}$ contains only negative order terms of δ, and since $dY(t) \cdot Y(t)^{-1}$ contains only positive order terms of δ in their coefficients, we have

$$\begin{aligned} dS(t) \cdot S(t)^{-1} &= dY(t) \cdot S \cdot H^{-1} \cdot S(t)^{-1} - Y(t) \cdot S \cdot H^{-1} \cdot dH \cdot H^{-1} \\ &= dY(t) \cdot Y(t)^{-1} - S(t) \cdot \Omega \cdot S(t)^{-1} \\ &= -\big(S(t) \cdot \Omega \cdot S(t)^{-1}\big)_- \end{aligned}$$

which is nothing but the super KP system. The uniqueness of the solution follows from the uniqueness of the super Birkhoff decomposition.

The above proof is exactly the same as that of [M3, Theorem 2.1], but this time it is far simpler. Just compare our equation with (2.25) of [M3]! The key point of the unique solvability is the super Birkhoff decomposition. Since this decomposition theorem is proved in its most general framework in [M3, Theorem 3.4], it applies to our new situation without any modification.

4. The super Krichever functor.

In order to study the geometric meaning of the super KP system (3.6), we need a super analogue of the Krichever map of Segal–Wilson [SW]. In the joint work with Rabin [MR], we have established the anti-equivalence of the super Krichever functor between the category of algebraic data consisting of points of the super Grassmannians and their stabilizers and the category of geometric data consisting of algebraic super curves and sheaves of super modules on them. In this section, we state the main theorem of [MR] in a more general framework. Only in this section, k can be a field of arbitrary characteristic.

For a point W of the super Grassmannian $G(\mu_0|\mu_1, 2\nu+1)$, we define the *maximal stabilizer* A_W of W by

$$A_W = \{a \in V \mid a \cdot W \subset W\}. \tag{4.1}$$

It is a super subalgebra of V and satisfies $A_W = (A_W)_0 \oplus (A_W)_1$ for $(A_W)_i = A_W \cap V_i$, $i = 0,\ 1$. Note that we have always $k \subset A_W$. If W is a generic point of the super Grassmannian, then the maximal stabilizer is just k. We say a super subalgebra $A = A_0 \oplus A_1 \subset A_W$ a *nontrivial stabilizer* of W if $A_0 \neq k$ and $A_1 \neq 0$. Since A is a super-commutative algebra, A_1 is a torsion free module over A_0.

Definition 4.1. *The category $\mathcal{S}(2\nu+1)$ is defined as follows:*

1. *An object of $\mathcal{S}(2\nu+1)$ is a pair (A, W) consisting of a point W of an arbitrary super Grassmannian of the fixed level $2\nu+1$ and its nontrivial stabilizer $A \subset A_W$;*
2. *A morphism among the objects is a pair*

$$(\alpha, \iota) : (A', W') \longrightarrow (A, W)$$

consisting of inclusion maps $\alpha : A' \hookrightarrow A$ and $\iota : W' \hookrightarrow W$.

We call an object of this category a Schur pair.

The *rank* of a Schur pair is the positive integer defined by

$$\text{rank}\ (A, W) = \text{rank}\ A = \frac{1}{2}\ G.C.D.\{\text{ord}\ a_0 \mid a_0 \in A_0\}\ . \tag{4.2}$$

If A has rank r and W is a point of $G(\mu_0|\mu_1, 2\nu+1)$, then we say (A, W) a Schur pair of rank r, index $\mu_0|\mu_1$, and level $2\nu+1$.

The geometric counterpart of the category of Schur pairs is the category $\mathcal{Q}(2\nu+1)$ of quintets.

Definition 4.2. *For a positive integer r and arbitrary integers μ_0, μ_1 and ν, the quintet of rank r, index $\mu_0|\mu_1$ and level $2\nu+1$ is a collection $(C, p, \pi, \mathcal{F}, \phi)$ of the following geometric data:*

(1) *$C = (C_{\mathrm{red}}, \mathcal{O}_C)$ is an irreducible complete algebraic super space of even-part dimension 1 defined over k, that means, C_{red} is a reduced irreducible complete algebraic curve over k and the structure sheaf is defined by $\mathcal{O}_C = \mathcal{O}_{C_{\mathrm{red}}} \oplus \mathcal{N}$, where $\mathcal{N}$ is a torsion free sheaf of rank one $\mathcal{O}_{C_{\mathrm{red}}}$-modules which has also a structure of the nilpotent algebra $\mathcal{N}^2 = 0$;*

(2) *$p \subset C$ is a divisor of C such that its reduced point p_{red} is a smooth k-rational point of C_{red};*

(3) *$\pi : U_o \to U_p$ is a morphism of formal super schemes, where $U_o = \mathrm{Spec}\ k\big[[z,\theta]\big]$ is the formal completion of the affine line $\mathbb{A}_k^{1|1}$ along the divisor $o = \{z = 0\}$ and $U_p = \mathrm{Spec}\ \widehat{\mathcal{O}}_{C_{\mathrm{red}}, p_{\mathrm{red}}} \widehat{\otimes} \mathcal{O}_C$ is the formal completion of C along the divisor p. We require that π is surjective, i.e. the corresponding ring homomorphism $\pi^* : \widehat{\mathcal{O}}_{C_{\mathrm{red}}, p_{\mathrm{red}}} \widehat{\otimes} \mathcal{O}_C \to k\big[[z,\theta]\big]$ is injective, the reduced morphism $\pi_{\mathrm{red}} : (U_o)_{\mathrm{red}} \to (U_p)_{\mathrm{red}}$ is an r-sheeted covering ramified at p_{red}, and that*

$$H^0(U_p, \mathcal{K}_{U_p}) \cap H^0(U_o, \mathcal{O}_{U_o}) = H^0(U_p, \mathcal{O}_{U_p})$$

as a subring of $H^0(U_o, \mathcal{K}_{U_o})$, where $\mathcal{K}_U$ denotes the sheaf of quotient rings of the structure sheaf $\mathcal{O}_U$ of the formal super scheme U;

(4) *$\mathcal{F} = \mathcal{F}_0 \oplus \mathcal{F}_1$ is a direct sum of torsion free sheaves $\mathcal{F}_0$ and $\mathcal{F}_1$ of $\mathcal{O}_{C_{\mathrm{red}}}$-modules of rank r such that*

$$\dim_k H^0(C, \mathcal{F}) - \dim_k H^1(C, \mathcal{F}) = \mu_0|\mu_1 .$$

We require that $\mathcal{F}$ has an $\mathcal{O}_C$-module structure which induces an injective homomorphism $\mathcal{N} \hookrightarrow \mathcal{H}om(\mathcal{F}_0, \mathcal{F}_1)$ and the zero homomorphism $\mathcal{N} \to 0 \in \mathcal{H}om(\mathcal{F}_1, \mathcal{F}_0)$;

(5) *$\phi : \mathcal{F}_{U_p} \xrightarrow{\sim} \pi_* \mathcal{O}_{U_o}(\nu)$ is an $\mathcal{O}_{U_p}$-module isomorphism, where $\mathcal{F}_{U_p}$ is the formal completion of $\mathcal{F}$ along the divisor $p \subset C$ and $\mathcal{O}_{U_o}(\nu) = \mathcal{O}_{U_o} \otimes \mathcal{O}_{(U_o)_{\mathrm{red}}}(\nu)$.*

Two quintets $(C, p, \pi_1, \mathcal{F}, \phi_1)$ and $(C, p, \pi_2, \mathcal{F}, \phi_2)$ are identified if we have

$$\begin{array}{ccc}
H^0(U_p, \mathcal{F}_{U_p}) & \xrightarrow[\sim]{\phi_1} & H^0(U_p, \pi_{1*}\mathcal{O}_{U_o}(\nu)) \\
{\scriptstyle \phi_2}\downarrow\wr & & \downarrow\wr \\
H^0(U_p, \pi_{2*}\mathcal{O}_{U_o}(\nu)) & \xrightarrow{\sim} & H^0(U_o, \mathcal{O}_{U_o}(\nu)) .
\end{array}$$

The reason why we call the above quintet having level $2\nu+1$ is because we have

$$H^0\big(U_o, \mathcal{O}_{U_o}(\nu)\big) = k\big[[z,\theta]\big] z^{-\nu} = V^{(2\nu+1)}.$$

Definition 4.3. *The category $\mathcal{Q}(2\nu+1)$ of quintets of level $2\nu+1$ is defined as follows:*

1. *An objects of $\mathcal{Q}(2\nu+1)$ is a quintet $(C, p, \pi, \mathcal{F}, \phi)$ of fixed level $2\nu+1$;*
2. *A morphism among quintets is a pair*

$$(\beta, \psi) : (C_1, p_1, \pi_1, \mathcal{F}_1, \phi_1) \longrightarrow (C_2, p_2, \pi_2, \mathcal{F}_2, \phi_2)$$

consisting of a morphism $\beta : C_1 \to C_2$ of algebraic super spaces and an $\mathcal{O}_C$-module homomorphism $\psi : \mathcal{F}_2 \to \beta_* \mathcal{F}_1$ of sheaves on C_2 such that

$$p_1 = \beta^{-1}(p_2), \tag{4.3}$$

$$\begin{CD} U_o @= U_o \\ @V{\pi_1}VV @VV{\pi_2}V \\ U_{p_1} @>{\widehat{\beta}}>> U_{p_2}, \end{CD} \tag{4.4}$$

i.e. $\pi_2 = \widehat{\beta} \circ \pi_1$, where $\widehat{\beta}$ is the morphism of formal super schemes induced by β, and

$$\begin{CD} \mathcal{F}_{2U_{p_2}} @>{\widehat{\psi}}>> \widehat{\beta}_* \mathcal{F}_{1U_{p_1}} \\ @V{\phi_2}V{\wr}V @V{\wr}V{\widehat{\beta}_*(\phi_1)}V \\ \pi_{2*}\mathcal{O}_{U_o}(\nu) @= \widehat{\beta}_* \pi_{1*} \mathcal{O}_{U_o}(\nu) \end{CD}, \tag{4.5}$$

where $\widehat{\psi}$ is the homomorphism of sheaves on U_{p_2} associated with ψ.

Theorem 4.4. *There is a contravariant functor*

$$\chi(2\nu+1) : \mathcal{Q}(2\nu+1) \xrightarrow{\sim} \mathcal{S}(2\nu+1)$$

which makes these categories anti-equivalent.

Remark. In [MR], we proved this theorem only when C is an algebraic super variety of dimension $1|1$. But the same argument which is based on the technique of [M4] can be applied to the current situation.

This functor is called the *super Krichever functor*. For every quintet of rank r, index $\mu_0|\mu_1$ and level $2\nu+1$, it assigns a Schur pair of the same rank, the same index and the same level by

$$\begin{cases} A = \pi^*\big((H^0(C \setminus p, \mathcal{O}_C)\big) \\ W = \phi\big(H^0(C \setminus p, \mathcal{F})\big) \end{cases}. \tag{4.6}$$

The super space C is a super variety, i.e. a super manifold with singularities, if and only if the odd part of the stabilizer A_1 is rank 1 over A_0. We proved in [MR] an interesting theorem which says that this condition is always satisfied for the maximal stabilizer A_W if it is nontrivial. Since the assignment $W \longmapsto (A_W, W)$ is canonical, every point W of the Grassmannian gives rise to a quintet consisting of an algebraic super curve if A_W is nontrivial.

Let $C = (C, \mathcal{O}_C)$ be an algebraic super space of even-part dimension 1. In this paper, we call a sheaf $\mathcal{F}$ on C a *vector bundle* of rank $r|0$ if there is a torsion free $\mathcal{O}_{C_{\text{red}}}$-module sheaf $\mathcal{F}_0$ on C_{red} such that $\mathcal{F} = \mathcal{F}_0 \otimes \mathcal{O}_C$. When C_{red} is nonsingular, our $\mathcal{F}$ coincides with the usual split vector bundle on C. But note that $\mathcal{F}$ is not locally free in general.

Proposition 4.5. *Let $(C, p, \pi, \mathcal{F}, \phi)$ be a quintet of rank 1 corresponding to a maximal Schur pair (A_W, W). Then $\mathcal{F}$ is a line bundle of rank $1|0$ if and only if $(A_W)_1 \cdot W_0 = W_1$.*

Proof. Following the construction of [M4, Section 3], the $(A_W)_0$-modules $(A_W)_1$, W_0 and W_1 determine the $\mathcal{O}_{C_{\text{red}}}$-module sheaves $\mathcal{N}$, $\mathcal{F}_0$ and $\mathcal{F}_1$. The condition $(A_W)_1 \cdot W_0 = W_1$ then implies $\mathcal{F}_1 = \mathcal{F}_0 \otimes \mathcal{N}$, which is equivalent with $\mathcal{F} = \mathcal{F}_0 \otimes \mathcal{O}_C$. This completes the proof.

Remark. The proposition does not hold in general if the quintet is not corresponding to the maximal Schur pair.

Since everything is defined on a field k, we can derive from the usual Riemann–Roch theorem the following (cf. [RSV]):

Theorem 4.6. *Let $(C, \mathcal{O}_C)$ be an algebraic super curve defined over k with the structure sheaf*

$$\mathcal{O}_C = \wedge^\bullet(\mathcal{N}) = \mathcal{O}_{C_{\text{red}}} \oplus \mathcal{N}$$

and let $\mathcal{F} = \mathcal{F}_0 \otimes \mathcal{O}_C$ be a vector bundle of rank $r|0$ on C. Then we have

$$\begin{aligned} &\dim_k H^0(C, \mathcal{F}) - \dim_k H^1(C, \mathcal{F}) \\ &\qquad = \big(\deg \mathcal{F} - r(g-1)\big) \big| \big(\deg \mathcal{F} + \deg \mathcal{N} - r(g-1)\big)\ , \end{aligned}$$

where $\deg \mathcal{F} = \deg \mathcal{F}_0$ and g is the (arithmetic) genus of C_{red}.

This theorem tells us that a quintet $(C, p, \pi, \mathcal{F}, \phi)$ consisting of an algebraic super curve of genus g with $\deg \mathcal{N} = n$ and a vector bundle $\mathcal{F}$ of rank r and degree $r(g-1)$ gives rise to a point on the super Grassmannian $G(0|n, 2\nu + 1)$. Therefore, no single super Grassmannian can handle all the algebraic super curves. In particular, since a super conformal structure on an algebraic super curve comes from a special line bundle $\mathcal{N}$ of degree $g-1$, the super Grassmannian $G(0|0,0)$ or $G(0|0,-1)$ is not the right space to study universal moduli of super conformal structures.

We can interpret both the super KP system of Manin and Radul [ManR] and our super KP system of (3.6) as dynamical systems on the super Grassmannian $G(0|0,-1)$ through the super Sato correspondence. It becomes clear for us now why nobody could ever discover a connection between the super conformal structures and the Manin-Radul super KP system. Because there is no such relation!

However, our theory is good enough from purely mathematical point of view, because it gives an interesting characterization of the Jacobian varieties of arbitrary algebraic super curves, as we are going to see in the next section.

5. A characterization of the Jacobians of super curves.

In this section, we define the set of super-commuting vector fields on the super Grassmannians and show that every finite-dimensional integral manifold of this flows has a natural structure of the Jacobian variety of an algebraic super curve. Since every Jacobian can be obtained in this way, what we are giving is a characterization of the Jacobian varieties of arbitrary algebraic super curves. It is also shown that if we restrict these flows on the big cell of the super Grassmannian of index $0|0$ and level -1, then they coincide with the flows which are defined by the super KP system of Section 3. In the nonsupersymmetric situation of [M4, Section 6], we defined a quotient space of the Grassmannians in order to deal with the generalized KP flows in the coordinate–free manner. But it is impossible to define a corresponding quotient space of the super Grassmannians rigorously in the infinite-dimensional supersymmetric situation we are working with. Our idea is to define the *vertical* vector fields on the Grassmannians and to study the *horizontal* integral manifolds of a super-commuting vector fields, so that we can avoid defining the quotient spaces. Of course the method we are presenting here can be also used for the nonsupersymmetric case.

Let us consider the super-commutative subalgebra

$$K = k\big((\partial^{-1})\big) \oplus k\big((\partial^{-1})\big)\frac{\partial}{\partial\xi} \tag{5.1}$$

of E. Since

$$\frac{\partial}{\partial\xi}\cdot\delta^n = (\delta - \xi\delta^2)\cdot\delta^n = \delta^{n+1} + (-1)^n \begin{bmatrix} n \\ 1 \end{bmatrix}\delta^{n+1} - (-1)^n\delta^{n+2}\xi \ ,$$

$\frac{\partial}{\partial\xi}$ maps V_1 to 0 and z^m to $z^m\theta$. Therefore, the operator action of elements of K on V is equal to the V action on itself by multiplication. In this sense, we can identify V with the subalgebra K of E by the bijection

$$K \longrightarrow \rho(K) = V \ . \tag{5.2}$$

We denote $K^{(1)} = K \cap E^{(1)}$. The identification (5.2) gives $K^{(1)} = V^{(1)}$.

Every $P \in E$ defines an element $\Phi_W(P) \in \mathrm{Hom}(W, V/W)$ through the action on V:

$$\Phi_W(P) : W \hookrightarrow V \xrightarrow{P} V \to V/W \ .$$

Since $\mathrm{Hom}(W, V/W)$ is the tangent space of the super Grassmannian at the point W,

$$\Phi(P) : G(\mu_0|\mu_1, 2\nu + 1) \ni W \longmapsto \Phi_W(P) \in \mathrm{Hom}(W, V/W)$$

gives a vector field. We call $\Phi(K^{(1)})$ the set of *vertical* vector fields on the super Grassmannians. For every $P \in E$, the vertical component of the vector field $\Phi(P)$ is given by $\Phi\big(\rho(P^{(1)})\big)$ using the identification of (5.2), where $P^{(1)}$ is the image of the projection

$$E \ni P \longmapsto P^{(1)} \in E^{(1)} .$$

We denote by

$$\Phi^{+}(P) = \Phi(P) - \Phi\big(\rho(P^{(1)})\big) \tag{5.3}$$

the *horizontal* vector field defined by $P \in E$.

Definition 5.1. *Let F be a super-commutative subalgebra of E. A horizontal integral manifold of the super-commuting flows $\Phi(F)$ is a nonsingular subvariety M of $G(\mu_0|\mu_1, 2\nu + 1)$ such that the tangent space $T_W M$ of M at every point $W \in M$ coincides with the set*

$$\{\Phi^{+}_W(P) = \Phi_W(P) - \Phi_W\big(\rho(P^{(1)})\big) \mid P \in F\}$$

as a subspace of $\mathrm{Hom}(W, V/W)$.

We call the super-commuting flows $\Phi(K)$ on the super Grassmannian the *Jacobian flows*, and its horizontal integral manifolds the *orbit* of the Jacobian flows.

The *one-parameter subgroup* of the vector field $\Phi(\zeta^{-n})$ is given by the infinite order super differential operator $\exp(t_n\zeta^{-n}) \in \mathfrak{D}_0^{\times}$, where ζ^{-n} represents the super differential operators of (3.1). It acts formally on the super Grassmannians by $W \longmapsto \exp(t_n\zeta^{-n}) \cdot W$, and on the Schur pair by

$$\begin{aligned}(A, W) \longmapsto &(\exp(t_n\zeta^{-n}) \cdot A \cdot \exp(-t_n\zeta^{-n}),\ \exp(t_n\zeta^{-n}) \cdot W)\\ &= (A, \exp(t_n\zeta^{-n}) \cdot W) ,\end{aligned}$$

where the last equality holds because $\exp(t_n\zeta^{-n})$ is a pure even operator which commutes with $A \subset V = K$. Therefore, the action of one-parameter subgroup $\exp(t_n\zeta^{-n})$ on a point W of the super Grassmannians preserves the maximal stabilizer A_W, and hence preserves the geometric data (C, p, π) of the corresponding quintet. Thus the Jacobian flows correspond to infinitesimal deformations of the sheaf $\mathcal{F}$ and its local information ϕ.

Definition 5.2. *The Jacobian variety* $\mathrm{Jac}(C)$ *of an algebraic super space* C *of even-part dimension* 1 *defined over* k *is the quotient module*

$$\mathrm{Jac}(C) = H^1(C, \mathcal{O}_C)/H^1(C, \mathbb{Z})$$

of cohomologies, where $\mathbb{Z} \subset \mathcal{O}_{C_{\mathrm{red}}}$ *is the sheaf of constant functions which are considered to be even.*

It is obvious from the definition that the reduced points of the Jacobian variety is given by

$$\mathrm{Jac}_{\mathrm{red}}(C) = \mathrm{Jac}(C_{\mathrm{red}}) = H^1(C_{\mathrm{red}}, \mathcal{O}_{C_{\mathrm{red}}})/H^1(C_{\mathrm{red}}, \mathbb{Z}) \ ,$$

which is the Jacobian variety of the algebraic curve C_{red}.

Remark. Unlike the usual situation, our Jacobian variety is not isomorphic in general to the Picard variety of the algebraic super space.

The following is the main theorem of this paper.

Theorem 5.3. *Every finite-dimensional orbit of the Jacobian flows on the super Grassmannian* $G(\mu_0|\mu_1, 2\nu+1)$ *is canonically isomorphic to the Jacobian variety of an algebraic super variety of dimension* $1|1$. *Conversely, every Jacobian variety of an algebraic super variety of dimension* $1|1$ *is obtained in this way. Therefore, a super manifold is the Jacobian variety of an algebraic super variety of dimension* $1|1$ *if and only if it can be a finite dimensional orbit of the Jacobian flows defined on the super Grassmannians.*

Proof. Let M be a finite-dimensional orbit of the Jacobian flows, $W \in M$ be a point and $(C, p, \pi, \mathcal{F}, \phi)$ be the quintet corresponding to the maximal Schur pair (A_W, W). The tangent space $T_W M$ of M at W is spanned by $\Phi_W^+(P)$ for $P \in K$, where Φ_+ is the map of (5.3). Since $V = K$, we have $\mathrm{Ker}\Phi_W^+ = A_W + K^{(1)}$. Therefore,

$$T_W M \cong V/(A_W + V^{(1)}) \ .$$

The finite-dimensionality of the orbit thus implies that A_W has rank 1. Take an arbitrary element $a_0 \in (A_W)_0$ of positive order and define

$$(A_W)_\infty = \{a_0^{-n} b \mid n \geq 0, b \in A_W \text{ and } \mathrm{ord}\, a_0^{-n} b \leq 1\}.$$

Then we can show that the completion of $(A_W)_\infty$ with respect to the adic topology is equal to $V^{(1)}$. Therefore, we obtain

$$T_W M \cong V/(A_W + V^{(1)}) \cong H^1(C, \mathcal{O}_C) \tag{5.4}$$

by the same argument of [M4, Section 3]. Since A_W does not change along the orbit M, (5.4) implies that M is covered by the vector space $H^1(C, \mathcal{O}_C)$. On the other hand, [M4, Theorem 6.3] shows that the reduced part of the orbit is isomorphic to the Jacobian

$$\mathrm{Jac}(C_{\mathrm{red}}) = H^1(C_{\mathrm{red}}, \mathcal{O}_{C_{\mathrm{red}}})/H^1(C_{\mathrm{red}}, \mathbb{Z}) \ .$$

Therefore, as a super manifold, we obtain that

$$M = \mathrm{Jac}(C) = H^1(C, \mathcal{O}_C)/H^1(C, \mathbb{Z}) \ .$$

In order to prove the converse, let C be an arbitrary algebraic super variety of dimension $1|1$ and p_{red} be a nonsingular point of the reduced algebraic curve C_{red}. Choose an arbitrary local coordinate (z, θ) of C around p_{red} and define the divisor by $p = \{z = 0\} \subset C$. We supply $\pi =$ identity, $\mathcal{F} = \mathcal{O}_C$ and $\phi =$ identity. Then $(C, p, \pi, \mathcal{F}, \phi)$ becomes a quintet of rank 1 and determines a Schur pair (A, W) of rank 1. Certainly, the orbit of the Jacobian flows starting at W is finite-dimensional and is isomorphic to the Jacobian variety of C. This completes the proof.

Theorem 5.4. *The Jacobian flows on the big cell $G^+(0|0, -1)$ coincide with the vector fields given by the super KP system of (3.6) through the super Sato correspondence of Theorem 2.3.*

Proof. Let $S^{-1} \in \Gamma_0$ be the initial datum of the super KP system and $W = S^{-1}\rho(D) \in G^+(0|0, -1)$ be the corresponding point of the Grassmannian. The time evolution of the super KP system is given by the super Birkhoff decomposition

$$S(t)^{-1} \cdot Y(t) = \exp(\sum_{n \geq 0} t_n \cdot \zeta^{-n}) \cdot S^{-1}, \tag{5.5}$$

where $S(t)$ is the solution and $Y(t) \in \mathfrak{D}_0^\times$. If we apply the both hand sides of (5.5) to $\rho(D) \in G^+(0|0, -1)$, then we have

$$S(t)^{-1}\rho(D) = \exp(\sum_{n \geq 0} t_n \cdot \zeta^{-n}) \cdot W \ , \tag{5.6}$$

since $Y(t)$ stabilizes $\rho(D)$ because of Lemma 2.1. Let us differentiate the both hand sides of (5.6) with respect to the parameter t_n and set $t = 0$. Then we see that the vector field $\frac{\partial}{\partial t_n}$ at W corresponds to the vector field

$$W \hookrightarrow V \xrightarrow{\zeta^{-n}} V \to V/W$$

obtained by the multiplication of the element $\zeta^{-n} \in K$. This completes the proof.

Remark. In particular, the differentiation $\frac{\partial}{\partial t_n}$ of (3.6) applied to S satisfies the super commutation relation

$$\begin{cases} \left[\frac{\partial}{\partial t_{2m}}, \frac{\partial}{\partial t_n}\right] = \frac{\partial}{\partial t_{2m}} \cdot \frac{\partial}{\partial t_n} - \frac{\partial}{\partial t_n} \cdot \frac{\partial}{\partial t_{2m}} = 0 \\ \left[\frac{\partial}{\partial t_{2m+1}}, \frac{\partial}{\partial t_{2n+1}}\right] = \frac{\partial}{\partial t_{2m+1}} \cdot \frac{\partial}{\partial t_{2n+1}} + \frac{\partial}{\partial t_{2n+1}} \cdot \frac{\partial}{\partial t_{2m+1}} = 0 \end{cases}$$

for arbitrary m and n which follows from $[\zeta^{-m},\ \zeta^{-n}] = 0$. Therefore, our super KP system of (3.6) is completely integrable in the category of partial super differential equations.

Thus every finite-dimensional solution of the super KP system gives rise to the Jacobian variety of an algebraic super curve $(C, \mathcal{O}_C)$, where $\mathcal{O}_C = \wedge^\bullet(\mathcal{N})$ is given by a line bundle $\mathcal{N}$ of degree 0.

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