A GENERATING FUNCTION OF THE NUMBER OF HOMOMORPHISMS FROM A SURFACE GROUP INTO A FINITE GROUP[†]

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ABSTRACT. A generating function of the number of homomorphisms from the fundamental group of a compact oriented or non-orientable surface without boundary into a finite group is obtained in terms of an integral over a real group algebra. We calculate the number of homomorphisms using the decomposition of the group algebra into irreducible factors. This gives a new proof of the classical formulas of Frobenius, Schur, and Mednykh.

0. INTRODUCTION

Let S be a compact oriented or non-orientable surface without boundary, and $\chi(S)$ its Euler characteristic. The subject of our study is a generating function of the number $|\mathrm{Hom}(\pi_1(S), G)|$ of homomorphisms from the fundamental group of S into a finite group G. We give a generating function in terms of a *non-commutative integral* Eqn.(2.7) or Eqn.(3.2), according to the orientability of S. The idea of such integrals comes from random matrix theory. Our integrals can be thought of as a generalization of real symmetric, complex hermitian, and quaternionic self-adjoint matrix integrals.

The graphical expansion methods for real symmetric [7, 14] and complex hermitian [4] matrix integrals are generalized in [31] for quaternionic self-adjoint matrix integrals. The technique developed in [31] is further generalized in [32] to the integrals over matrices with values in non-commutative $*$-algebras. In this article we consider 1×1 matrix integrals over group algebras. Surprisingly, the graphical expansion of the integral gives a generating function of $|\mathrm{Hom}(\pi_1(S), G)|$ for all closed surfaces.

1. COUNTING FORMULAS

Computation of our generating functions Eqn.(2.7) and Eqn.(3.2) yields a new proof of the following classical counting formulas:

Theorem 1.1 (Mednykh [24]). *Let G be a finite group of order $|G|$, and $\hat{G}$ the set of all complex irreducible representations of G. By V_λ we denote the irreducible representation parameterized by $\lambda \in \hat{G}$. Then for every compact Riemann surface S, we have*

$$(1.1) \qquad \sum_{\lambda \in \hat{G}} (\dim V_\lambda)^{\chi(S)} = |G|^{\chi(S)-1} \cdot |\mathrm{Hom}(\pi_1(S), G)| \,,$$

where $\chi(S)$ is the Euler characteristic of the surface S.

Remark 1.2. For a surface of genus 0, Eqn.(1.1) reduces to the classical formula

$$\sum_{\lambda \in \hat{G}} (\dim V_\lambda)^2 = |G| \,.$$

Since $\dim V_\lambda$ is a divisor of $|G|$, we note that the expression

$$|G|^{-\chi(S)} \sum_{\lambda \in \hat{G}} (\dim V_\lambda)^{\chi(S)} = |\mathrm{Hom}(\pi_1(S), G)|/|G|$$

is an integer for a surface of positive genus.

For a non-orientable surface, there is another formula:

Theorem 1.3 (Frobenius-Schur [13]). *Let us decompose $\hat{G}$ into three disjoint subsets according to the Frobenius-Schur indicator:*

$$\hat{G}_1 = \left\{ \lambda \in \hat{G} \,\middle|\, \frac{1}{|G|} \sum_{\gamma \in G} \chi_\lambda(\gamma^2) = 1 \right\}$$

(1.2)
$$\hat{G}_2 = \left\{ \lambda \in \hat{G} \,\middle|\, \frac{1}{|G|} \sum_{\gamma \in G} \chi_\lambda(\gamma^2) = 0 \right\}$$

$$\hat{G}_3 = \left\{ \lambda \in \hat{G} \,\middle|\, \frac{1}{|G|} \sum_{\gamma \in G} \chi_\lambda(\gamma^2) = -1 \right\}.$$

Then we have

(1.3) $$\sum_{\lambda \in \hat{G}_1} (\dim_{\mathbb{C}} V_\lambda)^{\chi(S)} + \sum_{\lambda \in \hat{G}_3} (- \dim_{\mathbb{C}} V_\lambda)^{\chi(S)} = |G|^{\chi(S)-1} \cdot |\mathrm{Hom}(\pi_1(S), G)|,$$

where S is now an arbitrary compact non-orientable surface without boundary.

Remark 1.4. If we take $S = \mathbb{R}P^2$, then $\chi(\mathbb{R}P^2) = 1$ and $\pi_1(\mathbb{R}P^2) = \mathbb{Z}/2\mathbb{Z}$, and the formula reduces to a well-known formula [17, 37]

$$\sum_{\lambda \in \hat{G}_1} \dim_{\mathbb{C}} V_\lambda - \sum_{\lambda \in \hat{G}_3} \dim_{\mathbb{C}} V_\lambda = \text{ the number of involutions in } G.$$

In particular, if every complex irreducible representation of G is defined over $\mathbb{R}$, then $\hat{G} = \hat{G}_1$ and the same formula (1.1) holds for an arbitrary closed surface S, orientable or non-orientable. This class of groups includes symmetric groups $\mathfrak{S}_n$.

2. Orientable Case

The generating function of $|\mathrm{Hom}(\pi_1(S), G)|$ for an oriented surface S comes from analysis of hermitian matrix integrals. The prototype of the asymptotic expansion formula is

(2.1) $$\log \int_{\mathcal{H}_{N,\mathbb{C}}} e^{-\frac{1}{2}N\mathrm{tr}X^2} e^{N \sum_{j \geq 3} \frac{t_j}{j} \mathrm{tr}X^j} d\mu(X) = \sum_{\substack{\Gamma \text{ connected} \\ \text{Ribbon graph}}} \frac{1}{|\mathrm{Aut}_R \Gamma|} N^{\chi(S_\Gamma)} \prod_{j \geq 3} t_j^{v_j(\Gamma)}$$

for an $N \times N$ hermitian matrix integral [4], where $\mathcal{H}_{N,\mathbb{C}}$ denotes the space of hermitian matrices of size N, and $e^{-\frac{N}{2}\mathrm{tr}X^2} d\mu(X)$ the normalized probability measure on $\mathcal{H}_{N,\mathbb{C}} = \mathbb{R}^{N^2}$. A *ribbon graph* is a graph with a cyclic order assigned to each vertex. Equivalently, it is a graph Γ that is drawn on a closed oriented surface S such that the complement $S \setminus \Gamma$ is the disjoint union of open disks. We use $f(\Gamma)$ to denote the number of these disks (or *faces*), and $v_j(\Gamma)$ the number of j-valent vertices of Γ. Let

$$v(\Gamma) = \sum_j v_j(\Gamma) \qquad \text{and} \qquad e(\Gamma) = \frac{1}{2} \sum_j j v_j(\Gamma)$$

be the number of vertices and edges of Γ, respectively. Then the genus $g(S)$ of the surface S is given by a formula for Euler characteristic

$$\chi(S) = 2 - 2g(S) = v(\Gamma) - e(\Gamma) + f(\Gamma).$$

The automorphism group $\mathrm{Aut}_R(\Gamma)$ of a ribbon graph Γ consists of automorphisms of the cell-decomposition of S that is determined by the graph. We refer to [30] for precise definition of ribbon graphs and their automorphism groups.

The asymptotic formula is an equality in the ring

$$\mathbb{Q}(N)[[t_3, t_4, t_5, \dots]]$$

of formal power series in an infinite number of variables t_3, t_4, t_5, $\dots$, with coefficients in the field $\mathbb{Q}(N)$ of rational functions in N. The size N of matrices is considered as a variable here. The topology of this formal power series ring is the Krull topology defined by setting $\deg t_j = j$. The monomial $\prod_j t_j^{v_j(\Gamma)}$ is a finite product for each connected ribbon graph Γ, and has degree $2e(\Gamma)$. The matrix integral in LHS of Eqn.(2.1) is meaningful only as an asymptotic series. Using the asymptotic technique developed in [28, 29], we can obtain a well-defined formal power series in $\mathbb{Q}(N)[[t_3, t_4, t_5, \dots]]$ from the calculation of the integral. First we consider a truncation of variables $(t_3, t_4, \dots, t_{2m})$ for some m. If we replace the integral with

$$Z(m) = \int_{\mathcal{H}_{N,\mathbb{C}}} e^{-\frac{1}{2}N\mathrm{tr}X^2} e^{N\sum_{j=3}^{2m} \frac{t_j}{j}\mathrm{tr}X^j} d\mu(X) \,,$$

then it converges and defines a holomorphic function on

$$(t_3, \dots, t_{2m-1}, t_{2m}) \in \mathbb{C}^{2m-3} \times \{t_{2m} \in \mathbb{C} \mid Re(t_{2m}) < 0\} \,.$$

The holomorphic function has a unique asymptotic series expansion at

$$(t_3, \dots, t_{2m-1}, t_{2m}) = 0 \,.$$

We can show that the terms of degree n in the asymptotic expansion of $Z(m)$ are stable for every $m > n$. Therefore, $\lim_{m\to\infty} Z(m)$ is convergent in the Krull topology of $\mathbb{Q}(N)[[t_3, t_4, t_5, \dots]]$, and determines a well-defined element. LHS of Eqn.(2.1) is the logarithm of this limit.

Let us now consider a finite-dimensional $*$-algebra A together with a linear map called *trace*

$$\langle \, \rangle : A \longrightarrow \mathbb{C}$$

satisfying that $\langle ab \rangle = \langle ba \rangle$. A typical example is the group algebra $A = \mathbb{C}[G]$ of a finite group G. We define a $*$-operation by

$$* : \mathbb{C}[G] \ni x = \sum_{\gamma \in G} x^\gamma \cdot \gamma \longmapsto x^* = \sum_{\gamma \in G} \overline{x^\gamma} \cdot \gamma^{-1} \in \mathbb{C}[G] \,.$$

As a trace, we use the character of the regular representation χ_{reg}, by linearly extending it to the whole group algebra. Let $\mathcal{H}_{\mathbb{C}[G]}$ denote the real vector subspace of $\mathbb{C}[G]$ consisting of self-adjoint elements. We need a Lebesgue measure on $\mathcal{H}_{\mathbb{C}[G]}$.

The self-adjoint condition $x^* = x$ means $x^{\gamma^{-1}} = \overline{x^\gamma}$. Let us decompose the group G into the disjoint union of three subsets

$$(2.2) \qquad\qquad G = G_I \cup G_+ \cup G_- \,,$$

where $G_I = \{\gamma \in G \mid \gamma^2 = 1\}$ is the set of involutions of G, and G_+ and G_- are chosen so that

$$(2.3) \qquad\qquad \begin{cases} G \setminus G_I = G_+ \cup G_- \\ *(G_+) = G_- \,. \end{cases}$$

Every self-adjoint element of $\mathbb{C}[G]$ is written as

$$(2.4) \qquad x = \sum_{\gamma \in G_I} x^\gamma \cdot \gamma + \frac{1}{\sqrt{2}} \sum_{\gamma \in G_+} \left(y^\gamma \cdot (\gamma + \gamma^{-1}) + i z^\gamma \cdot (\gamma - \gamma^{-1}) \right),$$

where x^γ, y^γ, and z^γ are real numbers. Thus we have $\mathcal{H}_{\mathbb{C}[G]} = \mathbb{R}^{|G|}$ as a real vector space. We see from Eqn.(2.4) that if x is self-adjoint, then

$$(2.5) \qquad \frac{1}{|G|} \chi_{\mathrm{reg}}(x^2) = \sum_{\gamma \in G_I} (x^\gamma)^2 + \sum_{\gamma \in G_+} \left((y^\gamma)^2 + (z^\gamma)^2 \right).$$

This is a non-degenerate quadratic form on $\mathbb{R}^{|G|}$. Let dx denote the Lebesgue measure on $\mathcal{H}_{\mathbb{C}[G]}$ that is invariant under the Euclidean transformations with respect to the quadratic form (2.5). A *normalized* Lebesgue measure is defined by

$$(2.6) \qquad d\mu(x) = \frac{dx}{\int_{\mathcal{H}_{\mathbb{C}[G]}} \exp\left(-\frac{1}{2}\chi_{\mathrm{reg}}(x^2)\right) \, dx}.$$

Theorem 2.1. *As a formal power series in infinitely many variables t_3, t_4, t_5, ..., we have an equality*

$$(2.7) \quad \log \int_{\mathcal{H}_{\mathbb{C}[G]}} \exp\left(-\frac{1}{2}\,\chi_{\mathrm{reg}}(x^2)\right) \exp\left(\sum_{j \geq 3} \frac{t_j}{j} \chi_{\mathrm{reg}}(x^j)\right) d\mu(x)$$

$$= \sum_{\substack{\Gamma \text{ connected ribbon} \\ \text{graph with valence} \geq 3}} \frac{1}{|\mathrm{Aut}_R\Gamma|} |G|^{\chi(S_\Gamma)-1} |\mathrm{Hom}(\pi_1(S_\Gamma), G)| \prod_{j \geq 3} t_j^{v_j(\Gamma)},$$

where S_Γ is the oriented surface determined by a ribbon graph Γ.

Proof. In the computation of the integral, it is easier to use a normalized trace function $\langle \rangle = \frac{1}{|G|}\chi_{\mathrm{reg}}$ instead of χ_{reg}. The formula to be established is then

$$(2.8) \quad \log \int_{\mathcal{H}_{\mathbb{C}[G]}} \exp\left(-\frac{1}{2}\langle x^2 \rangle\right) \exp\left(\sum_{j \geq 3} \frac{t_j}{j} \langle x^j \rangle\right) d\mu(x)$$

$$= \sum_{\substack{\Gamma \text{ connected ribbon} \\ \text{graph with valence} \geq 3}} \frac{1}{|\mathrm{Aut}_R\Gamma|} |G|^{f(\Gamma)-1} |\mathrm{Hom}(\pi_1(S_\Gamma), G)| \prod_{j \geq 3} t_j^{v_j(\Gamma)},$$

where $f(\Gamma)$ is the number of faces in the surface S_Γ.

First we expand the exponential factor:

$$\exp\left(\sum_{j \geq 3} \frac{t_j}{j} \langle x^j \rangle\right) = \sum_{v_3, v_4, v_5, \dots} \prod_{j \geq 3} \left(\frac{t_j^{v_j}}{v_j! j^{v_j}} \langle x^j \rangle^{v_j} \right).$$

The coefficient of $\prod_j t_j^{v_j}$ in the asymptotic expansion is then

$$(2.9) \qquad \prod_{j \geq 3} \frac{1}{v_j! j^{v_j}} \int_{\mathcal{H}_{\mathbb{C}[G]}} \exp\left(-\frac{1}{2}\langle x^2 \rangle\right) \prod_{j \geq 3} \langle x^j \rangle^{v_j} d\mu(x).$$

We note that (2.9) is a convergent integral. Let $y = \sum_{\gamma \in G} y^\gamma \cdot \gamma \in \mathbb{C}[G]$ be a variable running on the group algebra, and write $y^\gamma = u^\gamma + i w^\gamma$ as the sum of real and imaginary

parts. Now introduce

$$(2.10) \qquad \frac{\partial}{\partial y} = \sum_{\gamma \in G} \frac{\partial}{\partial y^\gamma} \cdot \gamma^{-1},$$

where

$$\frac{\partial}{\partial y^\gamma} = \frac{1}{2} \left(\frac{\partial}{\partial u^\gamma} - i \frac{\partial}{\partial w^\gamma} \right) .$$

Lemma 2.2. *For every $j > 0$ and $n \geq 0$, we have*

$$(2.11) \qquad \left\langle \left(\frac{\partial}{\partial y} \right)^j \right\rangle^n e^{\langle x(y+y^*) \rangle} = \langle x^j \rangle^n \cdot e^{\langle x(y+y^*) \rangle} .$$

Proof. Since the y derivative of y^* is zero, we can ignore y^* in the formula.

$$\begin{aligned}
\left\langle \left(\frac{\partial}{\partial y} \right)^j \right\rangle e^{\langle xy \rangle} &= \sum_{\gamma_1, \ldots, \gamma_j} \frac{\partial}{\partial y^{\gamma_1}} \cdots \frac{\partial}{\partial y^{\gamma_j}} \langle \gamma_1^{-1} \cdots \gamma_j^{-1} \rangle e^{\sum_\gamma x^{\gamma^{-1}} y^\gamma} \\
&= \sum_{\gamma_1, \ldots, \gamma_j} x^{\gamma_1^{-1}} \cdots x^{\gamma_j^{-1}} \langle \gamma_1^{-1} \cdots \gamma_j^{-1} \rangle e^{\langle xy \rangle} \\
&= \langle x^j \rangle e^{\langle xy \rangle}.
\end{aligned}$$

Applying this computation n times, we obtain (2.11). $\qquad\square$

Using the completion of square, we have

$$(2.12) \qquad -\frac{1}{2} \langle x^2 \rangle + \langle x(y+y^*) \rangle = -\frac{1}{2} \langle (x - (y+y^*))^2 \rangle + \frac{1}{2} \langle (y+y^*)^2 \rangle.$$

Thus

$$(2.13)$$

$$\begin{aligned}
\int_{\mathcal{H}_{\mathbb{C}[G]}} \exp\left(-\frac{1}{2} \langle x^2 \rangle \right) \prod_{j \geq 3} \langle x^j \rangle^n d\mu(x) &= \left\langle \left(\frac{\partial}{\partial y} \right)^j \right\rangle^n \int_{\mathcal{H}_{\mathbb{C}[G]}} e^{-\frac{1}{2} \langle x^2 \rangle + \langle x(y+y^*) \rangle} d\mu(x) \bigg|_{y=0} \\
&= \left\langle \left(\frac{\partial}{\partial y} \right)^j \right\rangle^n \int_{\mathcal{H}_{\mathbb{C}[G]}} e^{-\frac{1}{2} \langle (x-(y+y^*))^2 \rangle + \frac{1}{2} \langle (y+y^*)^2 \rangle} d\mu(x) \bigg|_{y=0} \\
&= \left\langle \left(\frac{\partial}{\partial y} \right)^j \right\rangle^n e^{\frac{1}{2} \langle (y+y^*)^2 \rangle} \bigg|_{y=0} ,
\end{aligned}$$

where we used the fact that $y + y^* \in \mathcal{H}_{\mathbb{C}[G]}$ and the translational invariance of the Lebesgue measure $d\mu(x)$.

Lemma 2.3. *We have*

$$(2.14) \qquad \frac{\partial}{\partial y} e^{\frac{1}{2} \langle (y+y^*)^2 \rangle} = (y + y^*) e^{\frac{1}{2} \langle (y+y^*)^2 \rangle} .$$

Proof.

$$\begin{aligned}
\frac{\partial}{\partial y} e^{\frac{1}{2} \langle (y+y^*)^2 \rangle} &= \sum_\gamma \frac{\partial}{\partial y^\gamma} \cdot \gamma^{-1} \exp\left(\frac{1}{2} \sum_\gamma (y^\gamma + \overline{y^{\gamma^{-1}}})(y^{\gamma^{-1}} + \overline{y^\gamma}) \right) \\
&= \sum_\gamma (y^{\gamma^{-1}} + \overline{y^\gamma}) \gamma^{-1} e^{\frac{1}{2} \langle (y+y^*)^2 \rangle} \\
&= (y + y^*) e^{\frac{1}{2} \langle (y+y^*)^2 \rangle} .
\end{aligned}$$

$\square$

The application of powers of $\langle (\partial/\partial y)^j \rangle$ to $e^{\frac{1}{2}\langle (y+y^*)^2 \rangle}$ produces zero contribution unless two differentiations are paired because of the restriction $y = 0$ at the end. Therefore, (2.13) counts the number of pairs of differential operators in the product. Since the trace $\langle\ \rangle$ is invariant under cyclic permutations, let us represent each $\langle (\partial/\partial y)^j \rangle$ as a j-valent vertex of a ribbon graph. Two vertices are connected if the differentiations from the vertices are paired one another. Since y^* is killed by the differentiation and

$$(2.15) \qquad \langle \cdots \frac{\partial}{\partial y} \cdots \rangle \langle \cdots y \cdots \rangle = \sum_{\gamma \in G} \langle \cdots \gamma \cdots \rangle \langle \cdots \gamma^{-1} \cdots \rangle ,$$

the edge connecting the vertices comes with an assignment of a group element γ and γ^{-1} on the two half-edges. With a factor $1/\prod_j v_j! j^{v_j}$, the integral (2.9) is equal to

$$\sum_{\substack{\Gamma \text{ ribbon graph} \\ v_j(\Gamma) = v_j}} \frac{1}{|\mathrm{Aut}_R \Gamma|} \mu_\Gamma(G) \prod_j t_j^{v_j} ,$$

where $\mu_\Gamma(G)$ is the number of assignments of group elements to each half-edge of a ribbon graph Γ subject to the following conditions:

Condition 1. If half-edges E_+ and E_- form an edge E of Γ and a group element w is assigned to E_+, then w^{-1} is assigned to E_-;

Condition 2. At every vertex, the product of all group elements assigned to half-edges incident to the vertex according to the cyclic order of the vertex is equal to 1.

The second condition comes from the fact

$$\langle w_1 w_2 \cdots w_j \rangle = \begin{cases} 1 & w_1 w_2 \cdots w_j = 1 \\ 0 & \text{oherwise} , \end{cases}$$

that appears as a j-valent vertex of (2.15).

Lemma 2.4. *The quantity $\mu_\Gamma(G)$ is a topological invariant of a compact oriented surface with a fixed number of marked points (or the faces of its cell-decomposition).*

Proof. This follows from the invariance of $\mu_\Gamma(G)$ under an edge contraction and edge insertion [31]. When an edge that is not a loop is contracted, the configuration of group elements on the new graph still satisfies Conditions 1 and 2. If the edge is inserted back, then we know exactly what group element has to be assigned to each half-edge, due to Condition 2. This proves the lemma. $\square$
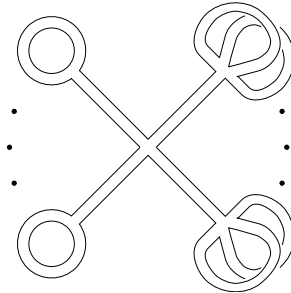

FIGURE 2.1. A standard graph for a closed oriented surface of genus g with f marked points, or faces. It has $f-1$ tadpoles on the left and g bi-petal flowers on the right.

As in [31], we can use a standard graph for each topology to calculate the number $\mu_\Gamma(G)$. If we use Figure 2.1 as our standard graph Γ, then we immediately see that the number of configurations of group elements on this graph is

$$|G|^{f(\Gamma)-1}|\mathrm{Hom}(\pi_1(S_\Gamma), G)| \,,$$

where the tadpoles of the graph have the contribution of $|G|^{f(\Gamma)-1}$ in the computation.

This completes the proof of the expansion formula (2.8), and by adjusting the constant factor in the exponential function of the integrand, we establish Theorem 2.1. $\qquad\square$

Note that we have a $*$-algebra isomorphism

$$(2.16) \qquad \mathbb{C}[G] \cong \bigoplus_{\lambda \in \hat{G}} \mathrm{End}(V_\lambda) \,,$$

which decomposes the character of the regular representation into the sum of irreducible characters:

$$\chi_{\mathrm{reg}} = \sum_{\lambda \in \hat{G}} (\dim V_\lambda)\chi_\lambda = \sum_{\lambda \in \hat{G}} N_\lambda \,\mathrm{tr}_{V_\lambda} \,,$$

where $N_\lambda = \dim V_\lambda$ and χ_λ is its character. Therefore,

$$\log \int_{\mathcal{H}_{\mathbb{C}[G]}} \exp\left(-\frac{1}{2}\,\chi_{\mathrm{reg}}(x^2)\right) \exp\left(\sum_{j\geq 3} \frac{t_j}{j}\chi_{\mathrm{reg}}(x^j)\right) d\mu(x)$$

$$(2.17) \qquad = \log \int_{\mathcal{H}_{\mathbb{C}[G]}} \prod_{\lambda \in \hat{G}} \exp\left(-\frac{N_\lambda}{2}\,\mathrm{tr}_{V_\lambda}(x^2)\right) \exp\left(N_\lambda \sum_{j\geq 3} \frac{t_j}{j}\mathrm{tr}_{V_\lambda}(x^j)\right) d\mu_\lambda(x)$$

$$= \sum_{\lambda \in \hat{G}} \log \int_{\mathcal{H}_{N_\lambda,\mathbb{C}}} \exp\left(-\frac{N_\lambda}{2}\,\mathrm{tr}_{V_\lambda}(x^2)\right) \exp\left(N_\lambda \sum_{j\geq 3} \frac{t_j}{j}\mathrm{tr}_{V_\lambda}(x^j)\right) d\mu_\lambda(x)$$

$$= \sum_{\substack{\Gamma \text{ connected ribbon} \\ \text{graph with valence} \geq 3}} \frac{1}{|\mathrm{Aut}_R\Gamma|} \sum_{\lambda \in \hat{G}} (\dim V_\lambda)^{\chi(S_\Gamma)} \prod_{j\geq 3} t_j^{v_j(\Gamma)} \,,$$

where $d\mu_\lambda$ is the normalized Lebesgue measure on the space of $N_\lambda \times N_\lambda$ hermitian matrices. Comparing the two expressions (2.7) and (2.17), we obtain Eqn.(1.1).

3. Non-orientable Case

For the non-orientable case, the prototype integral is a matrix integral over $N \times N$ real symmetric matrices [7, 14]:

$$(3.1) \qquad \log \int_{\mathcal{H}_{N,\mathbb{R}}} e^{-\frac{1}{4}N\mathrm{tr}X^2} e^{\frac{N}{2}\sum_j \frac{t_j}{j}\mathrm{tr}X^j} d\mu(X) = \sum_{\substack{\Gamma \text{ connected} \\ \text{Möbius graph}}} \frac{1}{|\mathrm{Aut}\Gamma|} N^{\chi(S_\Gamma)} \prod_j t_j^{v_j(\Gamma)} \,,$$

where what we call a *Möbius graph* is a graph drawn on a closed surface, orientable or non-orientable, defining a cell-decomposition of the surface. Its automorphism is an automorphism of the cell-decomposition of the surface that is determined by the graph, but this time we allow orientation-reversing automorphisms.

Every compact non-orientable surface without boundary is obtained by removing k disjoint disks from a sphere S^2 and glue k *cross-caps* back into the holes. The number of cross-caps is called the *cross-cap genus* of the non-orientable surface. If S_Γ is non-orientable, then its cross-cap genus k is determined by

$$\chi(S_\Gamma) = 2 - k = v(\Gamma) - e(\Gamma) + f(\Gamma) \,,$$

where again by $f(\Gamma)$ we denote the number of disjoint open disks in $S_\Gamma \setminus \Gamma$.

The space $\mathcal{H}_{N,\mathbb{R}}$ of $N \times N$ real symmetric matrices is a real vector space of dimension $N(N+1)/2$, and $d\mu(X)$ is the normalized Lebesgue measure of this space. We note that the coefficients of the integral in (3.1) are different from (2.1), reflecting the fact that a dihedral group naturally acts on a vertex of a Möbius graph.

We can generalize the matrix integral (3.1) to an integral over the real group algebra $\mathbb{R}[G]$, which is a $*$-algebra with χ_{reg} as a trace function.

Theorem 3.1.

$$(3.2) \quad \log \int_{\mathcal{H}_{\mathbb{R}[G]}} e^{-\frac{1}{4}\chi_{\mathrm{reg}}(x^2)} e^{\frac{1}{2}\sum_j \frac{t_j}{j}\chi_{\mathrm{reg}}(x^j)} d\mu(x)$$

$$= \sum_{\substack{\Gamma \ connected \\ M\"{o}bius \ graph}} \frac{1}{|\mathrm{Aut}\Gamma|} |G|^{\chi(S_\Gamma)-1} |\mathrm{Hom}(\pi_1(S_\Gamma), G)| \prod_j t_j^{v_j(\Gamma)} \,,$$

where the integral is taken over the space of self-adjoint elements of $\mathbb{R}[G]$, and S_Γ is the orientable or non-orientable surface determined by a Möbius graph Γ.

For the proof of this expansion formula, we refer to [32]. Recall that the real group algebra $\mathbb{R}[G]$ decomposes into simple factors according to the three types of irreducible representations (1.2). First we note that $\hat{G}_1$ consists of complex irreducible representations of G that are defined over $\mathbb{R}$. A representation in $\hat{G}_2$ is not defined over $\mathbb{R}$, and its character is not real-valued. Thus the complex conjugation acts on the set $\hat{G}_2$ without fixed points. Let $\hat{G}_{2+}$ denote a half of $\hat{G}_2$ such that

$$(3.3) \qquad\qquad \hat{G}_{2+} \cup \overline{\hat{G}_{2+}} = \hat{G}_2 \,.$$

Finally, a complex irreducible representation of G that belongs to $\hat{G}_3$ admits a skew-symmetric bilinear form. In particular, its dimension (over $\mathbb{C}$) is even. Now we have a $*$-algebra isomorphism

$$(3.4) \qquad \mathbb{R}[G] \cong \bigoplus_{\lambda \in \hat{G}_1} \mathrm{End}_{\mathbb{R}}(V_\lambda^{\mathbb{R}}) \oplus \bigoplus_{\lambda \in \hat{G}_{2+}} \mathrm{End}_{\mathbb{C}}(V_\lambda) \oplus \bigoplus_{\lambda \in \hat{G}_3} \mathrm{End}_{\mathbb{H}}(V_\lambda^{\mathbb{H}}) \,,$$

where $V_\lambda^{\mathbb{R}}$ is a real irreducible representation of G that satisfies $V_\lambda = V_\lambda^{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$. The space $V_\lambda^{\mathbb{H}}$ is a $(\dim_{\mathbb{C}} V_\lambda)/2$-dimensional vector space defined over $\mathbb{H}$ such that its image under the natural injection

$$(3.5) \qquad \mathrm{End}_{\mathbb{H}}(V_\lambda^{\mathbb{H}}) \longrightarrow \mathrm{End}_{\mathbb{C}}(V_\lambda)$$

coincides with the image of

$$\rho_\lambda : \mathbb{R}[G] \longrightarrow \mathrm{End}_{\mathbb{C}}(V_\lambda) \,,$$

where ρ_λ is the representation of G corresponding to $\lambda \in \hat{G}_3$. The injective algebra homomorphism Eqn.(3.5) is defined by

$$1 \longmapsto \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \quad i \longmapsto \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}, \quad j \longmapsto \begin{pmatrix} i & \\ & -i \end{pmatrix}, \quad k \longmapsto \begin{pmatrix} & -i \\ -i & \end{pmatrix} \,.$$

The algebra isomorphism Eqn.(3.4) gives a formula for the character of the regular representation on $\mathbb{R}[G]$:

$$(3.6) \quad \chi_{\mathrm{reg}} = \sum_{\lambda \in \hat{G}_1} (\dim_{\mathbb{R}} V_\lambda^{\mathbb{R}})\chi_\lambda + \sum_{\lambda \in \hat{G}_{2+}} (\dim_{\mathbb{C}} V_\lambda)(\chi_\lambda + \overline{\chi_\lambda}) + \sum_{\lambda \in \hat{G}_3} 2(\dim_{\mathbb{C}} V_\lambda) \cdot \mathrm{trace}_{V_\lambda^{\mathbb{H}}} \,,$$

where in the last term the character is given as the trace of quaternionic $(\dim_{\mathbb{C}} V_\lambda)/2 \times (\dim_{\mathbb{C}} V_\lambda)/2$ matrices. To carry out the non-commutative integral Eqn.(3.2), we need to

know the result for quaternionic self-adjoint matrix integrals. Fortunately, a recent paper [31] provides exactly this necessary formula:

$$(3.7) \qquad \log \int_{\mathcal{H}_{N,\mathbb{H}}} e^{N\mathrm{tr}X^2} e^{2N \sum_j \frac{t_j}{j} \mathrm{tr}X^j d\mu(X)} = \sum_{\substack{\Gamma \text{ connected} \\ \text{Möbius graph}}} \frac{1}{|\mathrm{Aut}\Gamma|} (-2N)^{\chi(S_\Gamma)} \prod_j t_j^{v_j(\Gamma)} ,$$

where $\mathcal{H}_{N,\mathbb{H}}$ is the space of $N \times N$ quaternionic self-adjoint matrices. We pay a particular attention to the negative sign in RHS of the formula and the factor $2N$. The extra factor 2 cancels with the size of the quaternionic matrices in $\mathrm{End}_\mathbb{H}(V_\lambda^\mathbb{H})$, which is half of the dimension of V_λ. The negative sign in RHS of (3.7) is the source of the negative sign in the second summation term of Eqn.(1.3).

The computation of the matrix integral of each factor of Eqn.(3.4) then establishes Eqn.(1.3). Note that the $\hat{G}_2$ component has no contribution in this formula. This is due to the fact that graphical expansion of a complex hermitian matrix integral contains only oriented ribbon graphs.

4. Algebraic Proof of the Counting Formula

The counting formulas (1.1) and (1.3) have an algebraic proof [13, 24] without going through the computation of matrix integrals. Although the proof is well known to experts in group theory [19], since it is not found in modern textbooks, we record it here to illuminate its relation to our graphical expansion formulas. The necessary backgrounds for the algebraic proof are found in Isaacs [17], Serre [37] and Stanley [38].

Let us first identify the group algebra

$$\mathbb{C}[G] = \left\{ x = \sum_{\gamma \in G} x(\gamma) \cdot \gamma \right\}$$

with the vector space $F(G)$ of functions on G. The *convolution product* of two functions $x(\gamma)$ and $y(\gamma)$ is defined by

$$(x * y)(w) = \sum_{\gamma \in G} x(w\gamma^{-1}) y(\gamma) ,$$

which makes $(F(G), *)$ an algebra isomorphic to the group algebra. In this isomorphism, the set of class functions $CF(G)$ corresponds to the center $Z\mathbb{C}[G]$ of $\mathbb{C}[G]$. According to the decomposition into simple factors Eqn.(2.16), we have an algebra isomorphism

$$Z\mathbb{C}[G] = \bigoplus_{\lambda \in \hat{G}} \mathbb{C} ,$$

where each factor $\mathbb{C}$ is the center of $\mathrm{End}V_\lambda$. The projection to each factor is given by an algebra homomorphism

$$pr_\lambda : Z\mathbb{C}[G] \ni x = \sum_{\gamma \in G} x(\gamma) \cdot \gamma \longmapsto pr_\lambda(x) = \frac{1}{\dim V_\lambda} \sum_{\gamma \in G} x(\gamma)\chi_\lambda(\gamma) \in \mathbb{C} .$$

Now let

$$(4.1) \qquad p_\lambda = \frac{\dim V_\lambda}{|G|} \sum_{\gamma \in G} \chi_\lambda(\gamma^{-1}) \cdot \gamma \in Z\mathbb{C}[G] .$$

The orthogonality of the irreducible characters shows that $pr_\lambda(p_\mu) = \delta_{\lambda\mu}$. Consequently, we have $p_\lambda p_\mu = \delta_{\lambda\mu} p_\lambda$, or more concretely,

$$\frac{\dim V_\lambda}{|G|} \sum_{s \in G} \chi_\lambda(s^{-1}) \cdot s \cdot \frac{\dim V_\mu}{|G|} \sum_{t \in G} \chi_\mu(t^{-1}) \cdot t$$

$$= \frac{\dim V_\lambda \cdot \dim V_\mu}{|G|^2} \sum_{w \in G} \left(\sum_{t \in G} \chi_\lambda((wt^{-1})^{-1}) \chi_\mu(t^{-1}) \right) \cdot w$$

$$= \delta_{\lambda\mu} \frac{\dim V_\lambda}{|G|} \sum_{w \in G} \chi_\lambda(w^{-1}) \cdot w .$$

Therefore, we have

$$\sum_{t \in G} \chi_\lambda(tw^{-1}) \chi_\mu(t^{-1}) = \frac{|G|}{\dim V_\mu} \delta_{\lambda\mu} \chi_\lambda(w^{-1}) ,$$

or in terms of the convolution product,

$$(4.2) \qquad \chi_\lambda * \chi_\mu = \frac{|G|}{\dim V_\mu} \delta_{\lambda\mu} \chi_\lambda .$$

Now let us consider

$$(4.3) \qquad \begin{cases} f_g(w) = \left| \{(a_1, b_1, a_2, b_2, \ldots, a_g, b_g) \in G^{2g} \mid a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} = w\} \right| \\ r_k(w) = \left| \{(a_1, a_2, \ldots, a_k) \in G^k \mid a_1^2 a_2^2 \cdots a_k^2 = w\} \right| . \end{cases}$$

Since these are class functions on the group G, they can be written as linear combinations of the irreducible characters $\{\chi_\lambda \mid \lambda \in \hat{G}\}$ of G.

Theorem 4.1. *For every $g \geq 1$ and $w \in G$, we have*

$$(4.4) \qquad f_g(w) = \sum_{\lambda \in \hat{G}} \left(\frac{|G|}{\dim_{\mathbb{C}} V_\lambda} \right)^{2g-1} \cdot \chi_\lambda(w) .$$

Corresponding to the non-orientable case, for every $k \geq 1$, we have

$$(4.5) \qquad r_k(w) = \sum_{\lambda \in \hat{G}_1} \left(\frac{|G|}{\dim_{\mathbb{C}} V_\lambda} \right)^{k-1} \cdot \chi_\lambda(w) - \sum_{\lambda \in \hat{G}_3} \left(-\frac{|G|}{\dim_{\mathbb{C}} V_\lambda} \right)^{k-1} \cdot \chi_\lambda(w) .$$

Proof. From the definition (4.3), we see that

$$f_{g_1 + g_2} = f_{g_1} * f_{g_2} \qquad \text{and} \qquad r_{k_1 + k_2} = r_{k_1} * r_{k_2} .$$

In particular,

$$f_g = \overbrace{f_1 * \cdots * f_1}^{g\text{-times}} \qquad \text{and} \qquad r_k = \overbrace{r_1 * \cdots * r_1}^{k\text{-times}} .$$

The general formulas now follow from the formulas for f_1 and r_1 that are found in [38] and [17], respectively, using (4.2). $\qquad \square$

Evaluating Eqns.(4.4) and (4.5) at $w = 1$, we obtain the counting formulas (1.1) and (1.3). The quantity g in the formula is of course the genus of an oriented surface, and k is the cross-cap genus of a non-orientable surface. The class function f_g (resp. r_k) gives the number $|\mathrm{Hom}(\pi_1(S), G)|$ when evaluated at $w = 1$ for an oriented (resp. non-orientable) surface S.

5. Remarks

Hermitian matrix integrals have been used effectively in the study of topology of moduli spaces of Riemann surfaces with marked points [15, 20, 36, 42]. These works rely on a relation between graphs and Riemann surfaces (cf. [39]). There is a burst of developments in this direction lately [33, 34, 35]. Real symmetric matrix integrals are used for the study of moduli spaces of real algebraic curves [14]. In a context of finite groups, a striking relation between random matrices and representation theory of symmetric groups is discovered in [1, 2, 6, 3, 9, 18, 33]. We have seen in this article that an extension of these matrix integrals shows yet another interesting relation between surface geometry and finite group theory.

There is a completely different proof of the counting formula Theorem 1.1 due to Freed and Quinn [11]. They use Chern-Simons gauge theory with a finite gauge group. To establish the formula, they construct a Chern-Simons gauge theory on each Riemann surface S. Interestingly, our approach does not start with a specific manifold, which is usually the space-time in physics. In a sense our non-commutative integral is a quantum field theory on a finite group *without* space-time, and surfaces appear in its Feynman diagram expansion. Yet the result shows that Chern-Simons gauge theory on a surface is mathematically equivalent to our non-commutative integrals over group algebras. To be more precise, the group algebra model is a generating function of Chern-Simons gauge theory with finite gauge group for *all* closed surfaces. The original algebraic proofs of the formulas using the convolution product found in [13, 24] are indeed related to the cut-and-paste construction of topological quantum field theory that reduces the construction of an invariant on a surface of genus g to surfaces of lower genera [11, 41].

A more recent development on counting formulas in finite groups is found in [23].

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