GEOMETRIC CLASSIFICATION OF $\mathbb{Z}_2$-COMMUTATIVE ALGEBRAS OF SUPER DIFFERENTIAL OPERATORS

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Abstract. A complete classification of all supercommutative algebras of super differential operators is established in terms of graded algebraic varieties and vector bundles on them. A geometric interpretation of all the known supersymmetric KP equations is also given in terms of vector fields on a new noncommutative Grassmannian.

0. Introduction.

The purpose of this paper is to give a geometric classification of all the supercommutative algebras consisting of super differential operators.

A geometric classification theorem of commutative algebras of ordinary differential operators was established in [M3]. The result is, roughly speaking, that there is a bijective correspondence between the set of isomorphism classes of commutative algebras of ordinary differential operators, and the set of isomorphism classes of geometric *quintets* $(C, p, \pi, \mathcal{F}, \phi)$ consisting of an algebraic curve C of an arbitrary genus g, a smooth point $p \in C$, a local covering π of C ramified at p, a semi-stable torsion-free sheaf $\mathcal{F}$ over C of an arbitrary rank r and degree $r(g-1)$, and a local isomorphism ϕ of $\mathcal{F}$ with a free sheaf defined near p. (One can think of ϕ as a local trivialization of $\mathcal{F}$ around p.) Thus a commutative algebra of ordinary differential operators has enormously rich geometric information. All semi-stable torsion-free sheaves satisfying the cohomology vanishing

$$H^0(C, \mathcal{F}) = H^1(C, \mathcal{F}) = 0 \tag{0.1}$$

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show up in the above correspondence.

As we have described in the introduction of [M3], there is a long history in the study of the problem of classifying commuting ordinary differential operators. The success of solving this historical problem, together with the new trends coming from mathematical physics, promotes generalizations of the theory. There are three different directions one can study:

(1) Classification of *quasicommuting* pairs of ordinary differential operators.
(2) Classification of commutative algebras consisting of partial differential operators.
(3) Classification of supercommutative algebras of super differential operators.

We say two operators P and Q quasicommute if $[P,Q]=1$. The problem (1) of finding all solutions of the equation $[P,Q]=1$ is motivated by the recent research in mathematical physics, in particular, 2-dimensional quantum gravity and matrix models. A geometric classification theorem of the solutions of this equation, sometimes called *string equation*, has been established by Schwarz [Sc1] by generalizing the theory of [M3]. Surprisingly, the moduli space of the solutions of string equation has a much similar structure of the moduli spaces appearing in the classification of commuting operators.

The second direction, study of commuting partial differential operators, is very difficult. No direct generalization of the theory of ordinary differential operators works. One needs a completely new approach to this problem. Of course there are a lot of examples of commuting partial differential operators, except for the trivial ones like tensor products of ordinary differential operators. These examples come from the theory of *completely integrable systems*. Let

$$L=\frac{d}{dx}+u_2\left(\frac{d}{dx}\right)^{-1}+u_3\left(\frac{d}{dx}\right)^{-2}+\cdots$$

be an ordinary pseudodifferential operator with coefficients in $k[[x]]$, where k is an arbitrary field of characteristic zero. We denote by L^n_+ the differential operator part of the n-th power of L. The Kadomtsev-Petviashvili system (the KP system) is a system of equations of the form

$$\left[\frac{\partial}{\partial t_m}-L^m_+,\frac{\partial}{\partial t_n}-L^n_+\right]=0\ ,$$

which is nothing but the commutativity condition of partial differential operators. It is shown in [M1] that the coefficients u_i of every finite-dimensional solution L are given by the Riemann theta functions associated with a Jacobian variety. Thus the KP system gives a set of commuting partial differential operators defined on a Jacobian variety. Indeed, if the time-evolution parameters are represented on a Jacobian variety by

$$t_j=\sum_i h_{ij}z_i\ ,$$

where $z_1, z_2, \cdots, z_n$ are coordinates on the Jacobian variety, then

$$\sum_j h_{ij}\left(\frac{\partial}{\partial t_j} - L_+^j\right), \qquad i = 1, 2, \cdots, n \tag{0.2}$$

gives commuting partial differential operators defined on the Jacobian variety. Therefore, for every n, we can construct a nontrivial example of a set of n commuting partial differential operators. Since these operators are algebraically independent, they form a polynomial ring of n variables, which is nothing but the coordinate ring of the vector space $H^1(C, \mathcal{O}_C)$ of an algebraic curve C. Although the algebra thus obtained from (0.2) is isomorphic to

$$k\left[\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \cdots, \frac{\partial}{\partial z_n}\right],$$

it is not *equal* to the trivial polynomial ring. From the point of view of the cohomology theory of [M1, Section 2], the commuting partial differential operators of (0.2) are in a complementary position to the ring of commuting ordinary differential operators; the latter forms the affine coordinate ring of the algebraic curve C.

Recently, Nakayashiki [N1], [N2] has started a systematical and very interesting study of commuting partial differential operators which are defined on arbitrary Abelian varieties. At this moment, we still do not know what else will show up on our list of commuting partial differential operators. Further developments are necessary for the final solution to the problem (2).

In this paper, we study supercommutative super differential operators, and solve the problem (3). This problem is also closely related with the study of supersymmetric quantum field theories.

Although supersymmetric generalizations have been established for such theories like the KP theory, the Krichever construction, and the characterization of Jacobian varieties ([KL], [MaR], [M2], [M4], [MR], [R]), nothing has been known until now about supersymmetric counterpart of the theory of geometric classification of commuting ordinary differential operators.

The super Grassmannian of [M4] and [MR], which was effectively used in the study of Jacobian varieties of supercurves, turns out not to be the right object we need in order to solve the problem (3).

In giving a supersymmetric generalization of the KP theory, Manin and Radul [MaR] introduced the notion of super pseudodifferential operators. Let (x, ξ) be a coordinate of the affine superspace $\mathbb{A}_k^{1|1}$ of dimension $1|1$ over k, where x is the usual coordinate of $\mathbb{A}_k^1$ and ξ satisfies $\xi^2 = 0$. A super derivation operator acting on the coordinate ring $k[x, \xi]$ is defined by

$$\delta = \frac{\partial}{\partial \xi} + \xi \frac{\partial}{\partial x}.$$

An expression

$$a_n\delta^n + a_{n-1}\delta^{n-1} + a_{n-2}\delta^{n-2} + \cdots + a_0 + a_{-1}\delta^{-1} + \cdots \tag{0.3}$$

with coefficients in $R = k[[x]] \oplus k[[x]]\xi$ is called a super pseudodifferential operator. If $a_m = 0$ for all negative m, then we call it a super differential operator. Let E denote the set of all super pseudodifferential operators, and D the set of super differential operators. We also use the notation $E^{(-1)}$ for the set of all super pseudodifferential operators (0.3) with $a_m = 0$ for all nonnegative m. Then we have a natural decomposition

$$E = D \oplus E^{(-1)} .$$

Our goal is a geometric classification of all supercommutative subalgebras of D.

Let us review what we did for the case of ordinary differential operators. We consider a commutative algebra B of ordinary differential operators. With a suitable coordinate change, we can assume without loss of generality that B has a *monic* operator, that is, an operator whose leading coefficient is 1. We say B_1 and B_2 are *isomorphic* if there is a function $f \in k[[x]]$ such that

$$B_1 = f^{-1} \cdot B_2 \cdot f .$$

The key point of the theory, due to I. Schur, is that for every such B, we can always find a function $f \in k[[x]]$ and a monic pseudodifferential operator

$$S = 1 + s_1 \left(\frac{d}{dx}\right)^{-1} + s_2 \left(\frac{d}{dx}\right)^{-2} + s_3 \left(\frac{d}{dx}\right)^{-3} + \cdots \tag{0.4}$$

of order 0, such that

$$A = S^{-1} \cdot f^{-1} \cdot B \cdot f \cdot S \subset k((\partial^{-1})) ,$$

where $\partial = \frac{d}{dx}$. Thus all the information of a commutative algebra B of ordinary differential operator is encoded in the pair (A, S) consisting of a subalgebra

$$A \subset k((\partial^{-1}))$$

and an operator S of (0.4).

In order to obtain geometric information of the pair (A, S), we need yet another machinery involving the infinite-dimensional Grassmannian. Let us consider the quotient space E/Ex of E by the left maximal ideal Ex generated by x. The quotient space has a natural structure of a field. Actually, we can identify

$$E/Ex = k((\partial^{-1})) .$$

The set D of differential operators becomes $k[\partial]$ by the natural projection

$$\rho : E \longrightarrow E/Ex .$$

The Grassmannian we need, $G(0)$, is the set of all vector subspaces W of E/Ex which are commensurable with $\rho(D) = k[\partial]$. Since E acts on E/Ex from the left, E acts on the Grassmannian, too. Then we use the one-to-one correspondence between the big cell of the Grassmannian and the set of operators (0.4):

$$S \longmapsto S^{-1}\rho(D) = W \in G(0) \ .$$

Because of this correspondence, our pair (A, S) becomes a pair (A, W), which we called a *Schur pair* in [M3]. Then the relation $B \cdot D \subset D$ gives

$$(S^{-1} \cdot f^{-1} \cdot B \cdot f \cdot S) \cdot S^{-1}D \subset S^{-1}D \ .$$

Applying the projection ρ to the above, we obtain

$$A \cdot W \subset W \ .$$

At this point, it is easy to see how we construct from a Schur pair the geometric quintet we discussed in the beginning: the algebra A defines an affine curve $C \setminus \{p\}$, the inclusion $A \subset k((\partial^{-1}))$ gives a local covering π, the A-module W gives a torsion-free sheaf $\mathcal{F}$, and the inclusion $W \subset k((\partial^{-1}))$ determines a local isomorphism ϕ.

We want to follow the same path in the supersymmetric case. This time, we use an operator

$$S = 1 + s_1\delta^{-1} + s_2\delta^{-2} + s_3\delta^{-3} + \cdots \ . \tag{0.5}$$

We can show, after a suitable coordinate change and conjugation by a function $f \in k[[x]]$, that there exists an operator S of (0.5) for every supercommutative algebra B of super differential operators, such that

$$S^{-1} \cdot B \cdot S \subset k((\partial^{-1})) \oplus k((\partial^{-1}))\left(\xi\frac{\partial}{\partial\xi} - \frac{\partial}{\partial\xi}\xi\right) \oplus k((\partial^{-1}))\xi \oplus k((\partial^{-1}))\frac{\partial}{\partial\xi} \ . \tag{0.6}$$

We also have a natural isomorphism

$$E/Ex \cong k((\partial^{-1})) \oplus k((\partial^{-1}))\left(\xi\frac{\partial}{\partial\xi} - \frac{\partial}{\partial\xi}\xi\right) \oplus k((\partial^{-1}))\xi \oplus k((\partial^{-1}))\frac{\partial}{\partial\xi} \ . \tag{0.7}$$

Thus we need to construct a suitable theory of Grassmannian based on the vector space appearing in the above. It has to be emphasized that the Grassmannian we obtain in this paper is not the super Grassmannian of [M4]. It is a much larger object, on which one can represent all the known supersymmetric KP systems as a simple system of vector fields.

Using our *noncommutative* Grassmannian, a new type of Schur pairs is introduced, and their geometric counterpart is studied. With these preparations, we can state our main theorem: *a supercommutative algebra of super differential operators is in one-to-one correspondence with a geometric quintet* $(C,\ p,\ \pi,\ \mathcal{F},\ \phi)$ *consisting of a* $\mathbb{Z}_2$*-graded variety* C *of reduced dimension 1, a divisor* $p \subset C$*, a local covering* π *of* C

near p, an $\mathcal{O}_C$-module sheaf $\mathcal{F}$ satisfying the condition described below, and a local isomorphism ϕ.

The sheaf $\mathcal{F}$ satisfies that there is a semi-stable torsion-free sheaf $\mathcal{F}_0$ on the reduced variety C_{red} satisfying the same cohomology vanishing of (0.1), such that

$$\mathcal{F} = \mathcal{F}_0 \otimes gl(1|1) \ ,$$

where $gl(1|1)$ is the algebra of superlinear transformations of $\mathbb{A}_k^{1|1}$.

This paper is organized as follows. In Section 1, we give the definition of super pseudodifferential operators, and give the proof of the fact (0.6). The Grassmannian based on the space (0.7) is studied in Section 2. In Section 3, we establish the equivalence between the algebraic data of Schur pairs and the geometric quintets consisting of graded varieties and vector bundles on them. The main classification theorem is proved in Section 4. We also give a unified, geometric picture of the various super KP systems in this section.

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1. $\mathbb{Z}_2$-commuting super differential operators.

Throughout this paper, k denotes an arbitrary field of characteristic zero. Whenever we say *graded* in this paper, it means $\mathbb{Z}_2$-graded, and it is equivalent with *supersymmetric* in physics language. The graded commutator, or the *supercommutator*, acting on a graded k-algebra $X = X_0 \oplus X_1$ is defined by

$$\begin{cases} [x,y] = xy - yx = -\ [y,x] & \text{if } x \in X_0 \\ [x,y] = xy + yx = [y,x] & \text{if } x \in X_1 \text{ and } y \in X_1 \ . \end{cases}$$

We extend this definition k-linearly to the general element of X. We say $x \in X$ and $y \in X$ are $\mathbb{Z}_2$-commuting, or *supercommuting*, if $[x,y] = 0$. A subalgebra $Y \subset X$ is said to be a graded subalgebra, or a *super* subalgebra, if

$$Y = (Y \cap X_0) \oplus (Y \cap X_1) \ .$$

A graded subalgebra Y of X is $\mathbb{Z}_2$-commutative, or *supercommutative*, if $[y, y'] = 0$ for all y, $y' \in Y$.

In this section, we first define the algebra E of formal super pseudodifferential operators following Manin-Radul [MaR]. The set D of super differential operators forms a subalgebra of E. We then study $\mathbb{Z}_2$-commutative subalgebras of D, and determine their algebraic structure.

Our super pseudodifferential operators have coefficients in the $\mathbb{Z}_2$-commutative algebra

$$R = k[[x, \xi]] = k[[x]] \oplus k[[x]]\xi = R_0 \oplus R_1$$

of formal power series in an even variable x and an odd variable ξ. These variables satisfy $x \cdot \xi = \xi \cdot x$ and $\xi^2 = 0$. An element of R_0 (resp. R_1) is called a homogeneous element of degree 0 (resp. degree 1). We define the graded derivation operator δ by

$$\delta = \frac{\partial}{\partial \xi} + \xi \frac{\partial}{\partial x} .$$

It acts on the ring R by the graded Leibniz rule:

$$\delta(ab) = \delta(a) \cdot b + (-1)^{\tilde{a}} a \delta(b) ,$$

where a is a homogeneous element of R of $\mathbb{Z}_2$-degree $\tilde{a}$, and b is an arbitrary element of R. We have $\delta^2 = \frac{\partial}{\partial x}$. We call an expression

$$P = \sum_{m=0}^{\infty} a_m \delta^{n-m} \tag{1.1}$$

a *super pseudodifferential operator* with coefficients in R if $a_m \in R$. The *order* of P is defined to be n only when $0 \neq a_0 \in R_0$. The operator P of (1.1) is said to be *monic* if $a_0 = 1$, and *normalized* if $a_0 = 1$ and $a_1 = 0$. The set of all super pseudodifferential operators with coefficients in R is denoted by E. For an arbitrary integer ν and a nonnegative integer i, we define the graded binomial coefficients following [MaR] by

$$\begin{bmatrix} \nu \\ i \end{bmatrix} = \begin{cases} 0 & \text{if } 0 \le \nu < i \text{ or } (\nu, i) \equiv (0,1) \bmod 2 \\ \binom{[\frac{\nu}{2}]}{[\frac{i}{2}]} & \text{otherwise,} \end{cases}$$

where $[\alpha]$ denotes the largest integer not greater than α. The set E of super pseudodifferential operators has a graded algebra structure introduced by the generalized graded Leibniz rule:

$$\delta^{\nu} \cdot f = \sum_{i=0}^{\infty} (-1)^{\tilde{f} \cdot (\nu - i)} \begin{bmatrix} \nu \\ i \end{bmatrix} f^{[i]} \delta^{\nu - i} ,$$

where ν is an arbitrary integer, f is a homogeneous element of R of degree $\tilde{f}$, and $f^{[i]} = \delta^i(f)$.

Let $E^{(n)}$ denote the set of all super pseudodifferential operators of the form of (1.1). We have a natural filtration

$$\cdots \supset E^{(n+1)} \supset E^{(n)} \supset E^{(n-1)} \supset \cdots \tag{1.2}$$

of E which satisfies

$$\bigcup_{n\in\mathbb{Z}} E^{(n)} = E \quad \text{and} \quad \bigcap_{n\in\mathbb{Z}} E^{(n)} = \{0\} \ .$$

Thus E is a complete topological space. Let us define

$$(1.3) \qquad \begin{aligned} E_0 &= \{\sum_\nu f_\nu \delta^\nu \mid \tilde{f}_{2\nu} = 0 \text{ and } \tilde{f}_{2\nu+1} = 1\} \ , \\ E_1 &= \{\sum_\nu f_\nu \delta^\nu \mid \tilde{f}_{2\nu} = 1 \text{ and } \tilde{f}_{2\nu+1} = 0\} \ . \end{aligned}$$

Then $E = E_0 \oplus E_1$. An element of E_0 (resp. E_1) is called a *homogeneous-even* (resp. *homogenous-odd*) operator.

Symbolically, we can write $E = R((\delta^{-1}))$, where $k((x))$ is the standard notation for the field of fractions of the power series ring $k[[x]]$. Since $\frac{\partial}{\partial x} = \delta^2$ and $\frac{\partial}{\partial \xi} = \delta - \xi\delta^2$, we have

$$\begin{aligned} E = R((\delta^{-1})) &= R((\delta^{-2})) \oplus R((\delta^{-2}))(\delta - \xi\delta^2) \\ &= R((\partial^{-1})) \oplus R((\partial^{-1}))\frac{\partial}{\partial\xi} \ , \end{aligned}$$

where $\partial = \frac{\partial}{\partial x}$.

A rich geometric information is hidden in the set D of super differential operators. We call an element $P = \sum_\nu a_\nu \delta^\nu \in E$ a *super differential operator* if $a_\nu = 0$ for all negative ν.

Our goal of this paper is to give a geometric description of $\mathbb{Z}_2$-commutative graded subrings of D. Let us take an arbitrary odd parameter λ satisfying $\lambda^2 = 0$, and consider $\lambda \cdot D$. Certainly, this ring is a $\mathbb{Z}_2$-commutative ring of super differential operators, but it is not interesting at all. In order to avoid this trivial situation, we make the following assumption:

Assumption 1.4. *All the $\mathbb{Z}_2$-commutative subrings $B \subset D$ we consider in this paper satisfy the following conditions:*

(1) *B is a $\mathbb{Z}_2$-commutative graded k-subalgebra of D containing k.*
(2) *B has a normalized homogeneous-even operator of order $2n$ for some $n > 0$.*

Condition (2) is not a technical restriction. In fact, as we see in the below, we can always transform a monic operator into a normalized one by a simple coordinate change, and the monicness condition itself is not a strong restriction as explained in [M3]. For a subalgebra B of (1.4), we say that B is *nontrivial* if $B_1 = B \cap E_1 \neq 0$. Note that $B_0 = B \cap E_0 \neq k$ because of the above (2).

Lemma 1.5. *Every monic, homogeneous-even super pseudodifferential operator P of order $2n$ can be transformed into a normalized operator by a suitable coordinate transformation.*

Proof. Let

$$P = \delta^{2n} + a_1\delta^{2n-1} + a_2\delta^{2n-2} + \cdots .$$

Since $a_1 \in R_1$, it is of the form

$$a_1 = c\xi$$

with some $c \in k$. Therefore, the operator has the following form:

$$P = \partial^n + c\xi\frac{\partial}{\partial\xi}\partial^{n-1} + a_2\partial^{n-1} + \cdots .$$

Now define a coordinate transformation by

$$\begin{cases} x = y \\ \xi = e^{cy/n}\eta \, . \end{cases}$$

Then we have

$$\begin{aligned} \frac{\partial}{\partial y} &= \frac{\partial x}{\partial y}\frac{\partial}{\partial x} + \frac{\partial \xi}{\partial y}\frac{\partial}{\partial \xi} \\ &= \frac{\partial}{\partial x} + \frac{c}{n}\xi\frac{\partial}{\partial \xi} \, . \end{aligned}$$

Since

$$\left(\frac{\partial}{\partial y}\right)^n = \partial^n + c\xi\frac{\partial}{\partial\xi}\partial^{n-1} + \cdots ,$$

P has the desired normalized form in the new coordinate system. This completes the proof.

Remark. We remark here that the above lemma is true for any odd element a_1 in a more general context. In our paper, however, we restrict ourselves to the consideration of operators defined over k.

Let $B \subset D$ be a subalgebra satisfying (1.4). Then it has a normalized homogeneous even element P of order $2n$:

$$P = \delta^{2n} + 0\cdot\delta^{2n-1} + a_2\delta^{2n-2} + \cdots .$$

It is easy to see that if we define $f = exp\big(-\frac{1}{n}\int a_2 dx\big)$, then we have

$$f^{-1}\cdot P\cdot f = \delta^{2n} + 0\cdot\delta^{2n-1} + 0\cdot\delta^{2n-2} + a_3\delta^{2n-3} + \cdots .$$

By a simple computation, one can obtain

Lemma 1.6. *For a homogeneous-even super pseudodifferential operator $Q = f^{-1} \cdot P \cdot f$ of the form*

$$Q = \delta^{2n} + 0 \cdot \delta^{2n-1} + 0 \cdot \delta^{2n-2} + a_3 \delta^{2n-3} + \cdots ,$$

there is a monic homogeneous-even operator

$$S = 1 + s_1 \delta^{-1} + s_2 \delta^{-2} + \cdots$$

of order 0 such that

$$S^{-1} \cdot Q \cdot S = \delta^{2n} .$$

The proof is given by an easy modification of [M2, Proposition 2.2]. Two $\mathbb{Z}_2$-commutative algebras B and B' of (1.4) are said to be *isomorphic* if there is an even element $f \in R_0$ such that

$$B' = f^{-1} \cdot B \cdot f .$$

Let us define

$$A = S^{-1} \cdot f^{-1} \cdot B \cdot f \cdot S .$$

Since every element of B commutes with P, everything in A commutes with δ^{2n}, and hence with $\frac{\partial}{\partial x}$. Therefore, A is a $\mathbb{Z}_2$-commutative subalgebra of

$$k((\partial^{-1})) \oplus k((\partial^{-1})) \left(\xi \frac{\partial}{\partial \xi} - \frac{\partial}{\partial \xi} \xi \right) \oplus k((\partial^{-1}))\xi \oplus k((\partial^{-1})) \frac{\partial}{\partial \xi} .$$

Let $A = A_0 \oplus A_1$, where A_0 (resp. A_1) is the homogenous-even (resp. homogenous-odd) part of A. Then we have

$$\begin{cases} A_0 \subset k((\partial^{-1})) \oplus k((\partial^{-1})) \left(\xi \frac{\partial}{\partial \xi} - \frac{\partial}{\partial \xi} \xi \right) \\ A_1 \subset k((\partial^{-1}))\xi \oplus k((\partial^{-1})) \frac{\partial}{\partial \xi} . \end{cases}$$

Now let us assume that our algebra B is nontrivial. Then $A_0 \neq k$ and $A_1 \neq 0$. Since $\xi \frac{\partial}{\partial \xi}$ and $\frac{\partial}{\partial \xi} \xi$ do not $\mathbb{Z}_2$-commute with ξ nor $\frac{\partial}{\partial \xi}$, A_0 has no element proportional to $\left(\xi \frac{\partial}{\partial \xi} - \frac{\partial}{\partial \xi} \xi \right)$. Similarly, since ξ and $\frac{\partial}{\partial \xi}$ do not $\mathbb{Z}_2$-commute one another, A_1 should be contained either in $k((\partial^{-1}))\xi$ or in $k((\partial^{-1})) \frac{\partial}{\partial \xi}$. Thus we have established the following:

Theorem 1.7. *Let B be a nontrivial $\mathbb{Z}_2$-commutative graded subalgebra of D satisfying (1.4). Then there exist an invertible even element $f \in R_0$ and a homogenous-even operator*

$$S = 1 + s_1 \delta^{-1} + s_2 \delta^{-2} + \cdots$$

such that we have either

$$A = S^{-1} \cdot f^{-1} \cdot B \cdot f \cdot S \subset k((\partial^{-1})) \oplus k((\partial^{-1}))\xi ,$$

or

$$A = S^{-1} \cdot f^{-1} \cdot B \cdot f \cdot S \subset k((\partial^{-1})) \oplus k((\partial^{-1}))\frac{\partial}{\partial \xi} \ .$$

With this theorem, we have determined the algebraic structure of a $\mathbb{Z}_2$-commutative algebras of super differential operators. For every B, we define its *rank* by

$$\text{rank } B = \text{rank } A = \frac{1}{2}\ G.C.D.\{\text{ord } a_0 \mid a_0 \in A_0\} \ . \tag{1.8}$$

The rank of B is always a positive divisor of n of (1.4).

2. The noncommutative Grassmannian.

In order to extract the geometric information of a $\mathbb{Z}_2$-commutative algebra of super differential operators, we need an intermediate step involving an infinite-dimensional Grassmannian. For the usual commuting ordinary differential operators, we used in [M3] the Grassmannian introduced by Sato [Sa], and established the complete geometric classification of commutative algebras of ordinary differential operators in terms of the Krichever functor. We then introduced the most natural supersymmetric generalization of the Grassmannian and the Krichever functor in [M4] and [MR]. The super Grassmannian was used effectively to establish the characterization of Jacobian varieties of the algebraic supercurves in [M4].

However, it turns out that the supersymmetric machinery of [M4] and [MR] is not suitable for our purpose of this paper. Of course one could extend the supersymmetric theory further so that it could provide a geometric framework for classifying our algebra B, but then such a theory would be rather ugly from the point of view of fanctoriality. Instead of going into such a direction, we present in this section a different idea of using a *noncommutative* Grassmannian.

Let us start with recalling the definition of the algebra E of super pseudodifferential operators:

$$E = (k[[x]] \oplus k[[x]]\xi)((\partial^{-1})) \oplus (k[[x]] \oplus k[[x]]\xi)((\partial^{-1}))\frac{\partial}{\partial \xi} \ .$$

The algebra generated by the four elements ξ, $\frac{\partial}{\partial\xi}$, $\xi\frac{\partial}{\partial\xi}$, and $\frac{\partial}{\partial\xi}\xi$ over k is denoted by $gl(1|1)$, which is the algebra of *superlinear* transformations of the $1|1$-dimensional affine superspace $\mathbb{A}_k^{1|1}$. One can introduce a matrix representation of $gl(1|1)$ by

$$\xi = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \ , \quad \frac{\partial}{\partial\xi} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \ , \quad \xi\frac{\partial}{\partial\xi} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \ , \quad \frac{\partial}{\partial\xi}\xi = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \ .$$

Using this algebra, we can give yet another presentation of E:

$$E = R_0((\partial^{-1})) \otimes_k gl(1|1) \ ,$$

where $R_0 = k[[x]]$. In this expression, we have

$$D = R_0[\partial] \otimes gl(1|1) \quad \text{and} \quad E^{(-1)} = R_0[[\partial^{-1}]]\partial^{-1} \otimes gl(1|1)\ .$$

There is a natural left R-module direct sum decomposition

$$E = D \oplus E^{(-1)}\ . \tag{2.1}$$

Let

$$\rho : E \longrightarrow E/Ex = V$$

be the natural projection, where Ex is the left ideal of E generated by x. We have a natural identification

$$\begin{aligned} V = E/Ex &= k((\partial^{-1})) \oplus k((\partial^{-1}))\left(\xi\frac{\partial}{\partial\xi} - \frac{\partial}{\partial\xi}\xi\right) \oplus k((\partial^{-1}))\xi \oplus k((\partial^{-1}))\frac{\partial}{\partial\xi} \\ &= k((\partial^{-1})) \otimes gl(1|1)\ . \end{aligned}$$

The direct sum decomposition (2.1) of E descends to V:

$$\begin{aligned} V &= V_+ \oplus V^{(-1)} \\ &= \rho(D) \oplus \rho(E^{(-1)}) \\ &= \big(k[\partial] \otimes gl(1|1)\big) \oplus \big(k[[\partial^{-1}]]\partial^{-1} \otimes gl(1|1)\big)\ . \end{aligned}$$

The Grassmannian we use in this paper is the following:

Definition 2.2. *The noncommutative Grassmannian of index μ, which is denoted by $G(\mu)$, is the set of all right $gl(1|1)$ submodules W of V such that the natural projection*

$$\gamma_W : W \longrightarrow V/V^{(-1)}$$

satisfies that

$$\mathrm{rank}_{gl(1|1)}\mathrm{Ker}\ \gamma_W - \mathrm{rank}_{gl(1|1)}\mathrm{Coker}\ \gamma_W = \mu\ .$$

The big cell $G^+(0)$ of the noncommutative Grassmannian is the set of right $gl(1|1)$ submodules $W \subset V$ such that the projection γ_W is an isomorphism over $gl(1|1)$.

Our noncommutative Grassmannian is the set of all right $gl(1|1)$ submodules $W \subset V$ which differ from $\rho(D)$ by finite rank μ over $gl(1|1)$. Thus the noncommutative Grassmannian has the *base point* $\rho(D) \in G(0)$. We note here that E acts on V naturally from the left. Every element of E gives rise to a vector field on the Grassmannian. In order to see this, let us recall that the tangent space of $G(\mu)$ at W is given by

$$T_W G(\mu) = \mathrm{Hom}_{gl(1|1)}(W, V/W)\ ,$$

where $\mathrm{Hom}_{gl(1|1)}(W, V/W)$ denotes the set of all right $gl(1|1)$-homomorphisms of W into V/W. An operator $P \in E$ defines

$$\Phi_W(P) : W \hookrightarrow V \xrightarrow{P} V \to V/W \ ,$$

which is an element of $\mathrm{Hom}_{gl(1|1)}(W, V/W)$ because P acts on V from the left. Therefore, $P \in E$ determines a vector field on the noncommutative Grassmannian by

$$\Phi(P) : G(\mu) \ni W \longmapsto \Phi_W(P) \in \mathrm{Hom}_{gl(1|1)}(W, V/W) = T_W G(\mu) \ .$$

Following the argument of [M3, Section 7] almost literally, we can establish two theorems which play the key role in this paper:

Theorem 2.3. *A super pseudodifferential operator $P \in E$ is a super differential operator (i.e. $P \in D$) if and only if P stabilizes the base point of the noncommutative Grassmannian: $\Phi_{\rho(D)}(P) = 0$.*

Theorem 2.4. *There is a natural bijective correspondence between the group Γ_0 of monic homogeneous-even super pseudodifferential operators of order 0 and the big cell $G^+(0)$:*

$$\Gamma_0 \ni S \longmapsto S^{-1}\rho(D) \in G^+(0) \ .$$

With these preparation, let us go back to the final stage we reached at the end of Section 1. We have a $\mathbb{Z}_2$-commutative algebra $B \subset D$ and a prescribed element $P \in B$. Then Theorem 1.7 tells us that there are an even element $f \in R_0$ and an operator $S \in \Gamma_0$ such that

$$A = S^{-1} \cdot f^{-1} \cdot B \cdot f \cdot S$$

satisfies either $A \subset k((\partial^{-1})) \oplus k((\partial^{-1}))\frac{\partial}{\partial\xi}$ or $A \subset k((\partial^{-1})) \oplus k((\partial^{-1}))\xi$. Since $B \subset D$, we have $B \cdot D \subset D$. The fact that $f \cdot D = D$ implies that $(f^{-1} \cdot B \cdot f) \cdot D \subset D$. Therefore, we have

$$(S^{-1} \cdot f^{-1} \cdot B \cdot f \cdot S) \cdot S^{-1}D \subset S^{-1}D \ .$$

It means that the algebra A *stabilizes* the point $W = S^{-1}\rho(D)$ of the big cell $G^+(0)$ of the Grassmannian:

$$A \cdot W \subset W \ .$$

It should be noted that the data (A, W) are not uniquely determined by the algebra B. Actually, we transformed $P \in B$ into ∂^n in Section 1, but that is not necessary in order to make A a subset of V. In fact, every operator of E commutes with ∂ if it commutes with any nonconstant element of $k((\partial^{-1}))$. It motivates us to define the following:

Definition 2.5. *A homogeneous-even zeroth order operator $T \in E$ is said to be admissible if it is invertible and satisfies that*

$$T^{-1} \cdot \partial \cdot T \in k((\partial^{-1})) .$$

It is easy to show that an admissible operator T has the following form:

$$T = e^{c_0 x} \left(\sum_{m=0}^{\infty} a_m(x) \partial^{-m} + \sum_{n=0}^{\infty} b_n(x) \xi \frac{\partial}{\partial \xi} \partial^{-n-1} \right) ,$$

where $a_n(x)$ and $b_n(x)$ are polynomials in x of degree less than or equal to n with constant coefficients, and c_0 is an arbitrary constant. In order for T to be invertible, we need $a_0 \neq 0$.

Definition 2.6. *A Schur pair is a pair (A, W) consisting of a graded k-subalgebra $A = A_0 \oplus A_1$ of $k((\partial^{-1})) \oplus k((\partial^{-1}))\xi$ or $k((\partial^{-1})) \oplus k((\partial^{-1}))\frac{\partial}{\partial \xi}$, and a point W of the big cell of the noncommutative Grassmannian, satisfying that $A_0 \neq k$, $A_1 \neq 0$, and that*

$$A \cdot W \subset W .$$

Two Schur pairs (A, W) and (A', W') are said to be isomorphic if there is an admissible operator T such that

$$\begin{cases} A' = T^{-1} \cdot A \cdot T \\ W' = T^{-1} W \end{cases}$$

With these terminology, we can state

Theorem 2.7. *There is a bijective correspondence between the set of isomorphism classes of the $\mathbb{Z}_2$-commutative algebras of super differential operators of (1.4), and the set of isomorphism classes of Schur pairs of (2.6).*

Proof. Let B_1 be the algebra of (1.4), and chose $f_1 \in R_0$ and $S_1 \in \Gamma_0$ as in Theorem 1.7. Then the corresponding Schur pair is given by

$$(A_1, W_1) = (S_1^{-1} \cdot f_1^{-1} \cdot B_1 \cdot f_1 \cdot S_1,\ S_1^{-1} \rho(D)) .$$

Take another algebra B_2 isomorphic to B_1. Then it gives a different Schur pair

$$(A_2, W_2) = (S_2^{-1} \cdot f_2^{-1} \cdot B_2 \cdot f_2 \cdot S_2,\ S_2^{-1} \rho(D)) .$$

Since $B_2 = f^{-1} \cdot B_1 \cdot f$ for some $f \in R_0$, we have

$$A_2 = S_2^{-1} \cdot f_2^{-1} \cdot f^{-1} \cdot f_1 \cdot S_1 \cdot A_1 \cdot S_1^{-1} \cdot f_1^{-1} \cdot f \cdot f_2 \cdot S_2 .$$

Therefore, $T = S_1^{-1} \cdot f_1^{-1} \cdot f \cdot f_2 \cdot S_2$ is an admissible operator. Note that we have $W_2 = T^{-1} \cdot W_1$. Thus (A_1, W_1) and (A_2, W_2) are isomorphic.

Now let us show the converse. So take a Schur pair (A, W). There is an $S \in \Gamma_0$ corresponding to W such that $W = S^{-1}\rho(D)$. Since $A \cdot W \subset W$, we have

$$(S \cdot A \cdot S^{-1}) \cdot \rho(D) \subset \rho(D) \ .$$

But this means that the algebra

$$B = S \cdot A \cdot S^{-1}$$

stabilizes the base point $\rho(D)$. Therefore, B consists of super differential operators. It is easy to see that B satisfies (1.4). Let us choose an admissible operator T, and consider the Schur pair $(T^{-1} \cdot A \cdot T,\ T^{-1} \cdot W)$ isomorphic to the original one. The point $T^{-1} \cdot W$ of the big cell corresponds to an operator

$$a_0^{-1} \cdot S \cdot T \in \Gamma_0 \ ,$$

where $a_0 \in R_0$ is the leading term of T. This pair gives rise to an algebra

$$(a_0^{-1} \cdot S \cdot T) \cdot T^{-1} \cdot A \cdot T \cdot (T^{-1} \cdot S^{-1} \cdot a_0) = a_0^{-1} \cdot B \cdot a_0 \ ,$$

which is isomorphic to B. This completes the proof.

This theorem tells us that the Schur pair (A, W) possesses all the information that a $\mathbb{Z}_2$-commutative algebra of super differential operators has. In order to extract its geometric information, we need a suitable generalization of the Krichever functor of [M3].

3. The noncommutative version of the Krichever correspondence.

In the usual case of ordinary differential operators, a commutative algebra of ordinary differential operators corresponds to a set of geometric data consisting of an algebraic curve and a semi-stable vector bundle on it. However, in our case of super differential operators, a $\mathbb{Z}_2$-commutative graded subalgebra of D *does not* correspond to a set of data consisting of an algebraic supercurve and a super vector bundle on it. The geometric data appearing naturally in our case are rather different from what we have dealt with in [MR], and this is the topic of this section. We have established in the last section that an algebra B of (1.4) is in one-to-one correspondence with a Schur pair (A, W). In this section, firstly, we define a category of Schur pairs incorporating the notion of isomorphism in a more natural way. We also define a geometric category of algebraic curves and vector bundles on them in a suitable way to our current situation, and then establish antiequivalence of these categories.

In order to define a category of Schur pairs, we have to relax the condition of the Schur pairs given in Definition 2.6. We call (A, W) a Schur pair of *rank* r and *index* μ, if (i) A is a $\mathbb{Z}_2$-commutative subalgebra of $V = k((\partial^{-1})) \otimes gl(1|1)$ such that $A_0 = A \cap k((\partial^{-1})) \supset k$, $A_0 \neq k$, $A_1 = A \cap k((\partial^{-1}))\xi \oplus k((\partial^{-1}))\frac{\partial}{\partial\xi} \neq 0$, and

$$r = \text{rank } A = \frac{1}{2}\, G.C.D.\{\text{ord } a_0 \mid a_0 \in A_0\} ;$$

and (ii) $W \in G(\mu)$. (Note that we have defined ord ∂ = ord $\delta^2 = 2$.)

The *Category of Schur pairs*, which we denote by $\mathcal{S}$, has a Schur pair of an arbitrary positive rank and an arbitrary index as its object. A morphism between two Schur pairs (A, W) and (A', W') is a set (α, ι) of *twisted inclusions*

$$\left\{ \begin{array}{l} \alpha : T^{-1} \cdot A \cdot T \hookrightarrow A' \\ \iota : T^{-1}W \hookrightarrow W' . \end{array} \right. \tag{3.1}$$

Note here that we are not requiring that W is a point of the big cell any more. We also note that A is a subalgebra of either $k((\partial^{-1})) \oplus k((\partial^{-1}))\xi$ or $k((\partial^{-1})) \oplus k((\partial^{-1}))\frac{\partial}{\partial\xi}$ as in Section 1. If $A \subset k((\partial^{-1})) \oplus k((\partial^{-1}))\xi$ and $A' \subset k((\partial^{-1})) \oplus k((\partial^{-1}))\frac{\partial}{\partial\xi}$ (or the other way around), then there is no morphism between the Schur pairs (A, W) and (A', W').

Next, let us define the category of the geometric objects corresponding to the Schur pairs. The category $\mathcal{Q}$ consists of a *quintet* $(C,\ p,\ \pi,\ \mathcal{F},\ \phi)$ of an arbitrary positive *rank* r and an arbitrary *index* $\mu_0|\mu_1$, where

(3.2) $C = (C_{\text{red}}, \mathcal{O}_C)$ is a graded algebraic variety of even-part dimension 1 defined over k such that C_{red} is a reduced irreducible complete algebraic curve over k, and the structure sheaf is given by $\mathcal{O}_C = \mathcal{O}_{C_{\text{red}}} \oplus \mathcal{N}$, where $\mathcal{N}$ is a nonzero torsion-free sheaf of $\mathcal{O}_{C_{\text{red}}}$-modules. We require that $\mathcal{O}_C$ is a $\mathbb{Z}_2$-commutative subalgebra of $\mathcal{O}_{C_{\text{red}}} \otimes gl(1|1)$. It follows from this requirement that $\mathcal{N}$ is a nilpotent algebra: $\mathcal{N}^2 = 0$.

(3.3) $p \subset C$ is a divisor of C such that its reduced point p_{red} is a smooth k-rational point of C_{red}.

(3.4) $\pi : U_0 \to U_{\text{red},p_{\text{red}}}$ is a morphism of formal schemes, where $U_0 \cong \text{Spec } k[[\partial^{-1}]]$ is the formal completion of the affine line $\mathcal{A}^1_k$ at the origin, and $U_{\text{red},p_{\text{red}}}$ is the formal completion of the algebraic curve C_{red} at the point p_{red}. We require that π is an r-sheeted covering ramified at p_{red}.

(3.5) $\mathcal{F}$ is a sheaf of $\mathcal{O}_C$-modules defined on C. As an $\mathcal{O}_{C_{\text{red}}}$-module, $\mathcal{F}$ is torsion-free. It also has an $\mathcal{O}_{C_{\text{red}}} \otimes gl(1|1)$-module structure with generic rank r, and satisfies that

$$\text{rank}_{gl(1|1)} H^0(C_{\text{red}}, \mathcal{F}) - \text{rank}_{gl(1|1)} H^1(C_{\text{red}}, \mathcal{F}) = \mu .$$

(3.6) ϕ is a sheaf isomorphism

$$\phi : \mathcal{F} \otimes_{\mathcal{O}_{C_{\text{red}}}} \mathcal{O}_{U_{\text{red},p'_{\text{red}}}} \xrightarrow{\sim} \pi_* \mathcal{O}_{U_0}(-1) \otimes gl(1|1) .$$

Two quintets $(C, p, \pi_1, \mathcal{F}, \phi_1)$ and $(C, p, \pi_2, \mathcal{F}, \phi_2)$ are identified if the diagram

$$\begin{tikzcd}
H^0(U_{\mathrm{red},p_{\mathrm{red}}}, \mathcal{F} \otimes \mathcal{O}_{U_{\mathrm{red},p_{\mathrm{red}}}}) \arrow[r, "\phi_1", "\sim"'] \arrow[d, "\phi_2"', "\wr"] & H^0(U_{\mathrm{red},p_{\mathrm{red}}}, \pi_{1*}\mathcal{O}_{U_0}(-1) \otimes gl(1|1)) \arrow[d, "\wr"] \\
H^0(U_{\mathrm{red},p_{\mathrm{red}}}, \pi_{2*}\mathcal{O}_{U_0}(-1) \otimes gl(1|1)) \arrow[r, "\sim"] & H^0(U_0, \mathcal{O}_{U_0}(-1) \otimes gl(1|1)) \ .
\end{tikzcd}$$

commutes.

A morphism

$$(\beta, \psi) : (C', p', \mathcal{F}', \pi', \phi') \longrightarrow (C, p, \mathcal{F}, \pi, \phi)$$

of two quintets consists of a morphism $\beta : C' \to C$ of graded algebraic varieties and a homomorphism $\psi : \mathcal{F} \to \beta_* \mathcal{F}'$ of sheaves on C satisfying the following conditions:

(3.7) The divisor p' is the pull-back of p: $p' = \beta^{-1}(p)$.

(3.8) There exists a formal scheme isomorphism $h : U_0 \xrightarrow{\sim} U_0$ and a nonzero constant $c \in k^*$ such that

$$\begin{tikzcd}
U_0 \arrow[r, "ch"] \arrow[d, "\pi'"'] & U_0 \arrow[d, "\pi"] \\
U_{\mathrm{red},p'_{\mathrm{red}}} \arrow[r, "\widehat{\beta}"] & U_{\mathrm{red},p_{\mathrm{red}}} ,
\end{tikzcd}$$

where $\widehat{\beta}$ is the morphism of formal schemes determined by β.

(3.9) There is an $\mathcal{O}_{U_0}$-module isomorphism $\zeta : \mathcal{O}_{U_0}(-1) \xrightarrow{\sim} (ch)_* \mathcal{O}_{U_0}(-1)$ such that

$$\begin{tikzcd}
\mathcal{F} \otimes \mathcal{O}_{U_{\mathrm{red},p_{\mathrm{red}}}} \arrow[r, "\widehat{\psi}"] \arrow[d, "\phi"'] & \widehat{\beta}_* \mathcal{F}' \otimes \mathcal{O}_{U'_{\mathrm{red},p'_{\mathrm{red}}}} \arrow[d, "\widehat{\beta}_*(\phi')"] \\
\pi_* \mathcal{O}_{U_0}(-1) \otimes gl(1|1) \arrow[r, "\pi_*(\zeta)"] & \pi_*(ch)_* \mathcal{O}_{U_0}(-1) \otimes gl(1|1) \arrow[d, equal] \\
 & \widehat{\beta}_* \pi'_* \mathcal{O}_{U_0}(-1) \otimes gl(1|1) \ ,
\end{tikzcd}$$

where $\widehat{\psi}$ is the homomorphism of sheaves on $U_{\mathrm{red},p_{\mathrm{red}}}$ defined by ψ.

Our graded variety is rather different from what is called the *supermanifold*. Actually, our variety C has no supermanifold structure, even locally, in general, because of the nilpotency of $\mathcal{N}$ of (3.2).

Now we can state the main theorem of this section:

Theorem 3.10. *There is a fully-faithful contravariant functor, which we call the graded Krichever functor, between the categories $\mathcal{Q}$ and $\mathcal{S}$;*

$$\chi : \mathcal{Q} \xrightarrow{\sim} \mathcal{S} \ .$$

Proof. First of all, we identify

$$U_0 = \mathrm{Spec}\; k[[\partial^{-1}]] \; .$$

This identification together with the isomorphism of (3.6) makes

$$W = \phi\big(H^0(C_{\mathrm{red}} \setminus \{p_{\mathrm{red}}\}, \mathcal{F})\big)$$

a $gl(1|1)$ submodule of V. The condition of the Euler characteristic of $\mathcal{F}$ imposed in (3.5) dictates that W is a point of the noncommutative Grassmannian $G(\mu)$.

We define the algebra A by

$$A = \pi^*\big((H^0(C_{\mathrm{red}} \setminus \{p_{\mathrm{red}}\}, \mathcal{O}_C)\big) \; .$$

The definition of π given in (3.4) makes A a subalgebra of V. It is easy to see that (A, W) becomes a Schur pair of rank r and index μ.

Thus we have defined a map

$$\chi : (C, p, \pi, \mathcal{F}, \phi) \longmapsto (A, W) \; ,$$

which gives a functor of $\mathcal{Q}$ into $\mathcal{S}$. This is really a cohomology functor, and therefore its fanctoriality and the correspondence between the morphisms can be established using the cohomology theory along the line of [M3, Theorem 4.6].

In order to show that χ is fully-faithful, we have to give its inverse. The construction of C_{red} from A_0 is exactly the same as in [M3, Section 3]. The A_0-module A_1 gives the nilpotent sheaf $\mathcal{N}$, and the A-module W determines $\mathcal{F}$. The embedding of A_0 in the larger ring $k((\partial^{-1}))$ defines the local covering map π. Similarly, the inclusion $W \hookrightarrow V$ gives ϕ. Because of the fanctoriality, one can establish the antiequivalence of $\mathcal{Q}$ and $\mathcal{S}$. This completes the proof.

4. The main theorem.

We are almost ready to state the main classification theorem of this paper now. Before doing that, we need a couple of more notations.

We denote by $\mathcal{B}$ the set of isomorphism classes of $\mathbb{Z}_2$-commutative algebras B of (1.4). Let us recall that two algebras B and B' are said to be isomorphic if there is a function $f \in R_0$ such that $B' = f^{-1} \cdot B \cdot f$. An element of $\mathcal{B}$ corresponds bijectively to an isomorphism class of certain Schur pairs by Theorem 2.7. And by Theorem 3.10, an isomorphism class of Schur pairs corresponds, again bijectively, to an isomorphism class of quintets.

So let us denote by $\mathcal{M}^+(0)$ the set of isomorphism classes of quintets (C, p, π, $\mathcal{F}$, ϕ) such that the sheaf $\mathcal{F}$ satisfies that

$$H^0(C_{\mathrm{red}}, \mathcal{F}) = H^1(C_{\mathrm{red}}, \mathcal{F}) = 0 \; . \tag{4.1}$$

Because of (3.5), there is a sheaf $\mathcal{F}_0$ of torsion-free $\mathcal{O}_{C_{\mathrm{red}}}$-modules of rank r such that

$$\mathcal{F} \cong \mathcal{F}_0 \otimes gl(1|1) .$$

The cohomology vanishing of (4.1) then implies that $\mathcal{F}_0$ is a semi-stable sheaf of degree $r(g-1)$, where g is the arithmetic genus of C_{red} [M3, Proposition 3.8].

By the graded Krichever functor χ, an element of $\mathcal{M}^+(0)$ corresponds to an isomorphism class of a Schur pair (A, W) with $W \in G^+(0)$. Note that the action $T^{-1}W$ of an admissible operator T on the Grassmannian stabilizes the index and the big cell of the Grassmannian.

Thus we have established the geometric classification of $\mathbb{Z}_2$-commutative algebras of super differential operators:

Theorem 4.2. *There is a natural bijective correspondence between $\mathcal{B}$ and $\mathcal{M}^+(0)$:*

$$\mu : \mathcal{B} \xrightarrow{\sim} \mathcal{M}^+(0) .$$

In the rest of this section, we first study maximal elements of $\mathcal{B}$, and then study various super KP systems in terms of vector fields on the noncommutative Grassmannian.

An element of $\mathcal{B}$ is said to be *maximal* if it is a maximal element with respect to the twisted inclusion relation

$$B' \subset f^{-1} \cdot B \cdot f$$

for some $f \in R_0$. We call a Schur pair (A, W) *maximal* if A is a maximal $\mathbb{Z}_2$-commutative stabilizer of W, i.e.,

$$A = \{a \in V \mid a \cdot W \subset W\} ,$$

where

$$V = k((\partial^{-1})) \oplus k((\partial^{-1})) \left(\xi \frac{\partial}{\partial \xi} - \frac{\partial}{\partial \xi} \xi \right) \oplus k((\partial^{-1}))\xi \oplus k((\partial^{-1})) \frac{\partial}{\partial \xi} .$$

If $B \in \mathcal{B}$ is maximal, then the Schur pair (A, W) corresponding to B is maximal, and vice versa.

The algebra $gl(1|1)$ has an involution

$$\sigma : gl(1|1) \xrightarrow{\sim} gl(1|1)$$

defined by interchanging ξ and $\frac{\partial}{\partial \xi}$. The involution σ induces involutions in E, in $\mathcal{B}$, and in the categories $\mathcal{Q}$ and $\mathcal{S}$. If B is maximal, so is $\sigma^{-1}B$. If B corresponds to (A, W) with $A \subset k((\partial^{-1})) \oplus k((\partial^{-1}))\xi$, then $\sigma^{-1}B$ corresponds to $(\sigma^{-1}A, W)$, and we have $\sigma^{-1}A \subset k((\partial^{-1})) \oplus k((\partial^{-1}))\frac{\partial}{\partial \xi}$ this time. Certainly A and $\sigma^{-1}A$ are isomorphic as a graded algebra. Therefore, B and $\sigma^{-1}B$ are also isomorphic, but it is not so obvious from the Manin-Radul description of the super differential operators.

Thus the maximal elements of $\mathcal{B}$ are in two-to-one correspondence with the points of the big cell of the noncommutative Grassmannian, but, of course, this correspondence is not surjective.

A maximal Schur pair satisfies an interesting property (cf. [MR, Theorem 1.3]):

Proposition 4.3. *Let (A, W) be a maximal Schur pair, and $A = A_0 \oplus A_1$ the decomposition of A into its even part and odd part. Then A_1 is an A_0-module of rank one.*

Proof. Let r be the rank of (A, W). Then for every large $n \in \mathbb{N}$, A_0 has an element whose leading term is ∂^{nr} [M3, Proposition 3.2]. Since σ stabilizes W, $(\sigma^{-1}A, W)$ gives the other maximal Schur pair having the same W. Certainly, we have $\sigma^{-1}A = A_0 \oplus \sigma^{-1}A_1$. Therefore, we can assume without loss of generality that $A_1 \subset k((\partial^{-1}))\xi$.

Take a nonzero element $a\xi \in A_1$, where $a = a(\partial) \in k((\partial^{-1}))$. In order to show that A_1 has rank one over A_0, it suffices to show that

$$\dim_k A_1/a\xi A_0 < +\infty \ .$$

Since $a\frac{\partial}{\partial\xi} \in \sigma^{-1}A$, it stabilizes W. Thus $\left[a\frac{\partial}{\partial\xi},\ A\right] \subset \sigma^{-1}A$, and hence

$$\left[a\frac{\partial}{\partial\xi},\ A_1\right] \subset A_0$$

because of the maximality of A. Note that we have

$$\left[\frac{\partial}{\partial\xi},\ \xi\right] = 1 \ .$$

Applying the operation $\left[a\frac{\partial}{\partial\xi},\ \cdot\ \right]$ to the inclusion relation $a\xi A_0 \subset A_1$, we obtain

$$a^2 A_0 \subset \left[a\frac{\partial}{\partial\xi},\ A_1\right] \subset A_0 \ .$$

But since A_0/a^2A_0 is finite over k, so is $A_1/a\xi A_0$. This completes the proof.

It follows from this proposition that the graded variety C of the quintet corresponding to a maximal Schur pair is an algebraic supercurve of dimension $1|1$, because its structure sheaf satisfies

$$\mathcal{O}_C = \mathcal{O}_{C_{\text{red}}} \oplus \mathcal{N} \cong \wedge^\bullet(\mathcal{N})$$

with a torsion-free rank 1 sheaf $\mathcal{N}$ of $\mathcal{O}_{C_{\text{red}}}$-modules. If further C_{red} is nonsingular, then C is nothing but a supermanifold of dimension $1|1$ in the sense of [Ma]. The sheaf $\mathcal{F}$ is then a super vector bundle of rank $r|r$. It is an abuse of terminology, but we can safely call $\mathcal{F}$ a super vector bundle over C even if C_{red} is singular, because $\mathcal{F}$ is indeed torsion-free over $\mathcal{O}_{C_{\text{red}}}$ in any case. With these terminology, we have the following:

Theorem 4.4. *Every maximal element of $\mathcal{B}$ is in one-to-one correspondence with a quintet $(C,\, p,\, \pi,\, \mathcal{F},\, \phi) \in \mathcal{M}^+(0)$ such that C is an algebraic supercurve of dimension $1|1$ over k, p is a $0|1$-dimensional divisor of C, and $\mathcal{F}$ is a super vector bundle over C of rank $r|r$.*

Finally, let us study relations of our theory with the various known super KP systems of [KL], [MaR], [M4], and [R]. As we have noted in Section 2, every element P of E defines a vector field $\Phi(P)$ on the noncommutative Grassmannian, and hence on the big cell $G^+(0)$. We can interpret the vector field in terms of a differential equation of a monic homogeneous-even super pseudodifferential operator $S \in \Gamma_0$ of order 0 by Theorem 2.4. In the notation of [M2], the equation is given by

$$\frac{\partial S}{\partial t_P} = -\left(S \cdot P \cdot S^{-1}\right)_- \cdot S\ ,$$

where Q_- represents the $E^{(-1)}$-part of $Q \in E$ following the decomposition of (2.1), and t_P is the parameter corresponding to the vector field $\Phi(P)$. It is obvious from the equation that if P is in $E^{(-1)}$, then the equation can be integrated immediately:

$$S(t_P) = S \cdot e^{-t_P P}\ . \tag{4.5}$$

Therefore, the equation is interesting only when $P \in D$.

If we have a commutative set $K \subset D$ of super differential operators, then it induces a commutative system of vector fields on the Grassmannian, and a compatible system of nonlinear partial differential equations on $S \in \Gamma_0$. If we take K to be supercommutative, then the corresponding system is compatible in the supersymmetric sense.

The super KP system that Rabin [R] and I [M4] discovered independently is given by taking

$$K = k[\partial] \oplus k[\partial]\frac{\partial}{\partial\xi} \subset D\ ,$$

which is a system of maximally compatible nonlinear super partial differential equations on $S \in \Gamma_0$. Applying the involution σ, one obtains yet another compatible system:

$$\sigma^{-1}K = k[\partial] \oplus k[\partial]\xi\ .$$

The compatible system introduced by Manin and Radul [MaR] is the one which mixes these two systems. With the ingenious infinite sums, their system remains compatible.

It has been shown by Schwarz [Sc2] that if one takes

$$\begin{aligned} K &= k[\partial] \oplus k[\partial]\left(\xi\frac{\partial}{\partial\xi} - \frac{\partial}{\partial\xi}\xi\right) \oplus k[\partial]\xi \oplus k[\partial]\frac{\partial}{\partial\xi} \\ &= k[\partial] \otimes gl(1|1) \subset D\ , \end{aligned}$$

then it gives the Kac-van de Leur super KP system [KL], which is no longer compatible as a system of super partial differential equations. From the point of view of analysis, the Kac-van de Leur system is not natural because of the noncompatibility, but from the algebraic point of view, it is natural because of the universality. Namely, it generates every flow on the Grassmannian $G(0)$ that does not have a simple integral of (4.5).

Each of these systems defines a *deformation theory* of $\mathcal{B}$ and $\mathcal{Q}$. In particular, these deformations can be described in terms of geometric deformations of the graded algebraic variety C and the sheaf $\mathcal{F}$ on it. All of these deformations preserve the reduced curve C_{red}, but the graded structure $\mathcal{O}_C$ as well as the sheaf $\mathcal{F}$ are deformed. For a compatible deformations, we can talk about the orbit of the flows, which gives a deformation space. However, these deformation spaces do not have a simple geometric description, such as *Jacobian varieties*, in general. Our new noncommutative Grassmannian provides a systematic and unified picture for all the known supersymmetric KP systems.

But if one wants to study *geometry* of Jacobian varieties of supercurves, then the current picture we have established in this paper is not useful. For that purpose, one needs the super Grassmannian and the Jacobian flows of [M4]. On the other hand, now we can understand why the machinery of [M4] does not provide any clear picture for the Kac-van de Leur system. Actually, we can see only half of the possible deformations in terms of the super Grassmannian picture.

The deformations of an algebra B by these flows can be interpreted as *isospectral* deformations, because A_0 represents the spectral structure of B. It is easy to see that the Kac-van de Leur flows give all possible isospectral deformations of super differential operators in this sense.

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