NON-COMMUTATIVE MATRIX INTEGRALS AND REPRESENTATION VARIETIES OF SURFACE GROUPS IN A FINITE GROUP[1]

MOTOHICO MULASE[2] AND JOSEPHINE T. YU[3]

Abstract. A graphical expansion formula for non-commutative matrix integrals with values in a finite-dimensional real or complex von Neumann algebra is obtained in terms of ribbon graphs and their non-orientable counterpart called Möbius graphs. The contribution of each graph is an invariant of the topological type of the surface on which the graph is drawn. As an example, we calculate the integral on the group algebra of a finite group. We show that the integral is a generating function of the number of homomorphisms from the fundamental group of an arbitrary closed surface into the finite group. The graphical expansion formula yields a new proof of the classical theorems of Frobenius, Schur and Mednykh on these numbers.

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0. Introduction

The purpose of this paper is to establish Feynman diagram expansion formulas for non-commutative matrix integrals over a finite-dimensional real or complex von Neumann algebra. An interesting case is the real or complex group algebra of a finite group. Using the graphical expansion formulas, we give a new proof of the classical formulas for the number of homomorphisms from the fundamental group of a closed surface into a finite group, expressing the number in terms of irreducible representations of the finite group. Indeed, our integrals are generating functions for the cardinality of the *representation variety* of a surface group in a finite group.

The non-commutative matrix integrals of this article have their origin in *random matrix theory* (cf. [1, 7, 32, 51, 52]), and include real symmetric, complex hermitian, and quaternionic self-adjoint matrix integrals as a special case for a *simple* von Neumann algebra. Recently a surprising relation between random matrices and random permutations was discovered in [2], and was further studied from various points of view including representation theory of symmetric groups (cf. [3, 4, 8, 11, 25, 40, 41, 42]). Our theory exhibits yet another connection between matrix-type integrals and representation theory of finite groups.

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Let A be a finite-dimensional complex von Neumann algebra with the adjoint operation $*: A \to A$ and a linear map $\langle\ \rangle : A \to \mathbb{C}$ called the *trace*. The algebra A has a positive definite hermitian inner product defined by

$$\langle a, b\rangle = \langle ab^*\rangle$$

for $a, b \in A$. Let us choose an orthonormal basis $\{e_1, \dots, e_N\}$ for A with respect to the hermitian form, where $N = \dim A$. A *ribbon graph* is a graph with a cyclic order given at every vertex to incident half-edges. Recall that every ribbon graph Γ defines a unique closed oriented surface S_Γ on which Γ is drawn and gives a cell-decomposition. Let $g(\Gamma)$ and $f(\Gamma)$ denote the genus of S_Γ and the number of 2-cells, or faces, of the cell-decomposition, respectively. The Feynman diagram expansion formula we establish is the following:

$$\log \int_{\{x\in A \mid x=x^*\}} \exp\left(-\frac{1}{2}\langle x^2\rangle\right) \exp\left(\sum_{j=1}^{\infty} \frac{t_j}{j}\langle x^j\rangle\right) d\mu(x) = \sum_{\substack{\Gamma \text{ connected} \\ \text{ribbon graph}}} \frac{1}{|\mathrm{Aut}_R(\Gamma)|} A^{or}_{g(\Gamma),f(\Gamma)} \prod_j t_j^{v_j(\Gamma)}, \tag{0.1}$$

where $d\mu(x)$ is a normalized Lebesgue measure on the real vector subspace of A consisting of self-adjoint elements, $\mathrm{Aut}_R(\Gamma)$ is the ribbon graph automorphism group, and $v_j(\Gamma)$ is the number of j-valent vertices of the connected ribbon graph Γ. The integral of (0.1) is considered as a generating function of integrals

$$\int_{\{x\in A \mid x=x^*\}} \exp\left(-\frac{1}{2}\langle x^2\rangle\right) \prod_j^{\text{finite}} \left\langle x^j\right\rangle^{v_j} d\mu(x)$$

for all finite sequences $(v_1, v_2, v_3, \dots)$ of positive integers. The contribution of the graph Γ in (0.1) is defined by

$$A^{or}_{g,f} = \sum_{\substack{i_1,\dots,i_g;\, j_1,\dots,j_g \\ h_1,\dots,h_{f-1}=1}}^{N} \left\langle e_{i_1} e_{j_1} e^*_{i_1} e^*_{j_1} \cdots e_{i_g} e_{j_g} e^*_{i_g} e^*_{j_g} \cdot e_{h_1} e^*_{h_1} \cdots e_{h_{f-1}} e^*_{h_{f-1}}\right\rangle .$$

We notice that the graph contribution $A^{or}_{g(\Gamma),f(\Gamma)}$ depends only on the *topological type* of the surface S_Γ, which is the genus of the surface and the number of 2-cells in its cell-decomposition. If we apply (0.1) to a simple von Neumann algebra $A = M(n, \mathbb{C})$, then the formula recovers the well-known graphical expansion formula for $n \times n$ hermitian matrix integrals found in many articles, including [6, 22, 27, 35, 40, 41, 43, 54]. The word *non-commutative matrix integral* in the title is justified because our von Neumann algebra can take the form $A = B \otimes M(n, \mathbb{C})$ with another von Neumann algebra B.

For a real von Neumann algebra A with a real valued trace, our expansion formula is more complicated. Let us recall the notion of *Möbius graph* introduced in [38]. It is the non-orientable counterpart of ribbon graphs. A Möbius graph Γ defines a unique unoriented surface S_Γ and gives a cell-decomposition. Every closed non-orientable surface S is obtained by removing k disjoint disks from a sphere S^2 and gluing a *cross-cap* to each hole. The number of cross-caps is the *cross-cap genus* of the surface, and its Euler characteristic is given by $\chi(S) = 2 - k$. Every ribbon graph is an orientable Möbius graph, but it has a different automorphism group reflecting the fact that orientation-reversing map is allowed.

Now the formula for a real von Neumann algebra is the following:

$$
\begin{aligned}
&\log \int_{\{x\in A \mid x=x^*\}} \exp\left(-\frac{1}{4}\,\langle x^2\rangle\right) \exp\left(\sum_{j=1}^{\infty} \frac{t_j}{2j}\,\langle x^j\rangle\right) d\mu(x) \\
&= \sum_{\substack{\Gamma \text{ connected orientable} \\ \text{Möbius graph}}} \frac{1}{|\mathrm{Aut}_M(\Gamma)|}\, A^{or}_{g(\Gamma),f(\Gamma)} \prod_j t_j^{v_j(\Gamma)} \\
&+ \sum_{\substack{\Gamma \text{ connected non-} \\ \text{orientable Möbius graph}}} \frac{1}{|\mathrm{Aut}_M(\Gamma)|}\, A^{nor}_{k(\Gamma),f(\Gamma)} \prod_j t_j^{v_j(\Gamma)}\,,
\end{aligned}
\tag{0.2}
$$

where

$$
A^{nor}_{k,f} = \sum_{i_1,\dots,i_k;\, h_1,\dots,h_{f-1}=1}^{N} \left\langle e_{i_1}^2 \cdots e_{i_k}^2 \cdot e_{h_1}^* e_{h_1} \cdots e_{h_{f-1}}^* e_{h_{f-1}} \right\rangle\,,
$$

$\mathrm{Aut}_M(\Gamma)$ is the automorphism group of a graph Γ as a Möbius graph, and $k(\Gamma)$ is the cross-cap genus of a non-orientable surface S_Γ. We notice the sharp contrast between $A^{or}_{g,f}$ and $A^{nor}_{k,f}$, which reflects a particular choice of a presentation of the fundamental group $\pi_1(S_\Gamma)$ of a closed surface S_Γ. Every simple finite-dimensional real von Neumann algebra is a full matrix algebra over either the reals $\mathbb{R}$ or quaternions $\mathbb{H}$. We recover the graphical expansion formulas for real symmetric and quaternionic self-adjoint matrix integrals of [9, 17, 38] from (0.2). An explicit computation is also carried out for real Clifford algebras [55].

Here we emphasize again that even though their expressions look dependent on a presentation of $\pi_1(S_\Gamma)$, the quantity $A^{or}_{g,f}$ is an invariant of an orientable surface of topological type (g, f), and $A^{nor}_{k,f}$ is an invariant of a non-orientable surface of topological type (k, f). When the von Neumann algebra A in our theory is simple, the invariants $A^{or}_{g,f}$ and $A^{nor}_{k,f}$ do not show any significance. The invariants become more interesting when the algebra is complicated. Now we notice that every finite-dimensional von Neumann algebra is semi-simple, and hence is decomposable into simple factors. When we apply the decomposition of A into simple factors in the integral of (0.1) or (0.2), due to the logarithm in front of the integral, it becomes the sum of the integral for each simple factor. Therefore, any topological invariant given as $A^{or}_{g,f}$ or $A^{nor}_{k,f}$ is computable in terms of simple ones.

This idea can be concretely carried out for the real or complex group algebra of a finite group G. Using the complex group algebra $\mathbb{C}[G]$, we obtain

$$
\begin{aligned}
&\log \int_{\{x\in\mathbb{C}[G] \mid x=x^*\}} \exp\left(-\frac{1}{2}\,\chi_{\mathrm{reg}}(x^2)\right) \exp\left(\sum_j \frac{t_j}{j}\chi_{\mathrm{reg}}(x^j)\right) d\mu(x) \\
&\qquad = \sum_{\substack{\Gamma \text{ connected} \\ \text{ribbon graph}}} \frac{1}{|\mathrm{Aut}_R\Gamma|} |G|^{\chi(S_\Gamma)-1} |\mathrm{Hom}(\pi_1(S_\Gamma), G)| \prod_j t_j^{v_j(\Gamma)}\,,
\end{aligned}
\tag{0.3}
$$

where $\chi(S_\Gamma)$ is the Euler characteristic of S_Γ, and χ_{reg} denotes the character of the regular representation of G on $\mathbb{C}[G]$ linearly extended to the whole algebra. Notice that the formula gives a generating function for the cardinality of the representation variety $\mathrm{Hom}(\pi_1(S), G)$ of a closed oriented surface S in the group G. With the real group algebra $\mathbb{R}[G]$ of G, we have

$$\log \int_{\{x\in\mathbb{R}[G]\mid x=x^*\}} \exp\left(-\frac{1}{4}\,\chi_{\mathrm{reg}}(x^2)\right)\exp\left(\sum_j \frac{t_j}{2j}\chi_{\mathrm{reg}}(x^j)\right)d\mu(x) = \sum_{\substack{\Gamma \text{ connected}\\ \text{Möbius graph}}} \frac{1}{|\mathrm{Aut}_M\Gamma|}|G|^{\chi(S_\Gamma)-1}|\mathrm{Hom}(\pi_1(S_\Gamma),G)| \prod_j t_j^{v_j(\Gamma)}\ . \tag{0.4}$$

Surprisingly, the RHS of (0.4) has the same expression as in (0.3), with the only difference being replacing ribbon graphs with Möbius graphs. These generating functions were reported in an earlier paper [39].

Let G be a finite group and $\hat{G}$ the set of equivalence classes of complex irreducible representations of G. The most fundamental formula in representation theory of finite groups is the one that expresses the order of the group in terms of a square sum of the dimensions of irreducible representations of G:

$$|G| = \sum_{\lambda\in\hat{G}} (\dim\lambda)^2\ . \tag{0.5}$$

The formula follows from the decomposition of the group algebra into irreducible fctors:

$$\mathbb{C}[G] = \bigoplus_{\lambda\in\hat{G}} \mathrm{End}(\lambda)\ . \tag{0.6}$$

In 1978, Mednikh [31] discovered a remarkable generalization of the classical formula (0.5):

$$\sum_{\lambda\in\hat{G}} (\dim\lambda)^{\chi(S)} = |G|^{\chi(S)-1}|\mathrm{Hom}(\pi_1(S),G)|\ , \tag{0.7}$$

where S is a compact Riemann surface. When $S = S^2$, (0.7) reduces to (0.5). Note that (0.6) is a von Neumann algebra isomorphism. Thus the integral of (0.3) over the self-adjoint elements of $\mathbb{C}[G]$ becomes the sum of hermitian matrix integrals. It is now easy to see that evaluation of the integral of (0.3) using (0.6) yields Mednykh's formula (0.7).

For a non-orientable surface S, the formula for the number of representations of $\pi_1(S)$ involves more detailed information on irreducible representations of G. Using the Frobenius-Schur indicator of irreducible characters [15], we decompose the set of complex irreducible representations $\hat{G}$ into the union of three disjoint subsets, corresponding to real, complex, and quaternionic irreducible representations:

$$\begin{aligned} \hat{G}_1 &= \left\{\lambda\in\hat{G}\ \middle|\ \frac{1}{|G|}\sum_{w\in G}\chi_\lambda(w^2) = 1\right\}\ ;\\ \hat{G}_2 &= \left\{\lambda\in\hat{G}\ \middle|\ \frac{1}{|G|}\sum_{w\in G}\chi_\lambda(w^2) = 0\right\}\ ;\\ \hat{G}_4 &= \left\{\lambda\in\hat{G}\ \middle|\ \frac{1}{|G|}\sum_{w\in G}\chi_\lambda(w^2) = -1\right\}\ . \end{aligned} \tag{0.8}$$

The suffix $1, 2$ or 4 indicates the dimension of the base field $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, respectively. In the fundamental paper of Frobenius and Schur [15] published in 1906, we find

$$\sum_{\lambda\in\hat{G}_1} (\dim\lambda)^{\chi(S)} + \sum_{\lambda\in\hat{G}_4} (-\dim\lambda)^{\chi(S)} = |G|^{\chi(S)-1}|\mathrm{Hom}(\pi_1(S),G)|\ . \tag{0.9}$$

It is somewhat strange that a formula for non-orientable surfaces was known much earlier than its orientable counterpart. Actually, Frobenius and Schur obtained the formula as a counting formula for the number of group elements satisfying $x_1^2 \cdots x_k^2 = 1$, but no relation to surface topology was in their motivation. If we take $S = \mathbb{R}P^2$, then the formula reduces to the well-known formula [24, 47]

$$\sum_{\lambda \in \hat{G}_1} \dim \lambda - \sum_{\lambda \in \hat{G}_4} \dim \lambda = \text{ the number of involutions of } G\ .$$

The formula (0.9) immediately follows from the generating function (0.4) and the decomposition of $\mathbb{R}[G]$ into simple factors, which include real, complex, and quaternionic matrix algebras.

In a beautiful paper of Pierre van Moerbeke [52], we see the list of matrix-type integrals and the nonlinear integrable systems that characterize the integrals as functions on the potential. The simple von Neumann algebra integrals all fit into his scope. More general von Neumann algebra integrals of this article can be considered as a multi-matrix model with trivial interaction terms between matrices. They can be also interpreted as a matrix integral over an algebra different from $\mathbb{R}, \mathbb{C}$ and $\mathbb{H}$. In either point of view, we do not have any clear picture on the relation between our formulas (0.1), (0.2) and integrable equations. Since the generating functions for the Hurwitz numbers and the Gromov-Witten invariants of the Riemann surfaces are proven to satisfy integrable systems [41, 42], the integrability of the von Neumann algebra integrals seems to pose a condition on the structure of the algebra. However, the present article does not address this question.

The study of the volume of the representation variety $\mathrm{Hom}(\pi_1(S), G)$ of a surface group in a compact connected simply connected semi-simple Lie group is carried out by many authors including Witten [53], Gross-Taylor [18], and Liu [28, 29, 30]. Although their focus was on the moduli space $\mathcal{M}(S, G)$ of flat G-connections on a closed surface S, through a relation

$$\mathcal{M}(S, G) = \frac{\mathrm{Hom}(\pi_1(S), G)}{G/Z(G)}\ ,$$

the study of moduli spaces is equivalent to that of representation varieties. Here $Z(G)$ is the center of the group G that acts trivially on the representation variety through conjugation. It is interesting to note that exactly the same formulas (0.7) and (0.9) hold for a compact Lie group if the infinite sum of LHS converges absolutely and the cardinality is interpreted as the *volume* of the variety in an appropriate sense. The naive extension of the method of this article does not work for the case of Lie groups, however, because the von Neumann algebra involved becomes infinite-dimensional and the integration (0.3) makes no sense.

The present paper is organized as follows. We review the notion of ribbon graphs and *Möbius* graphs in Section 1. Then in Section 2, we compute Feynman diagram expansion of integrals over a finite-dimensional complex von Neumann algebra in terms of ribbon graphs. Since integrals over a real von Neumann algebra behave differently, they are treated separately in Section 3. The generating functions for the number of representations of surface groups in a finite group are given in Section 4. As an application, we give a new proof of the formulas of Frobenius-Schur and Mednykh.

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1. Ribbon graphs and Möbius graphs

In this section, we review basic facts about graphs drawn on an orientable or non-orientable surface. Definition of the automorphism groups of these graphs is crutial when we use them to compute non-commutative matrix integrals. Graphs on an oriented surface are called *ribbon* or *fat* graphs. We refer to [6, 21, 22, 27, 40, 41, 42, 43, 54] for the use of ribbon/fat graphs in the study of moduli spaces of Riemann surfaces and related topics. The double line notation was first introduced in [50], generalizing the graphical expansion idea of [12]. Graphs on complex algebraic curves were studied from the quite different point of view of Grothendieck's *dessins d'enfants* [5, 20, 45, 46]. A relation between Strebel differentials [16, 49] and dessins d'enfants was studied in [37]. The terminology *Möbius graph* was introduced in [38] for a graph on a surface that is not oriented in order to avoid possible confusion, since ribbon or fat graphs are usually oriented. Graphs on an orientable or non-orientable surface are also called *maps*. Maps were studied mainly in the context of map coloring theorems [19, 44].

For a detailed treatment of ribbon graphs, we refer to the references cited above. Only a brief description is given here. A graph $\Gamma = (\mathcal{V}, \mathcal{E}, \iota)$ consists of a set of vertices $\mathcal{V}$, a set of edges $\mathcal{E}$, and an incidence relation

$$\iota : \mathcal{E} \longrightarrow (\mathcal{V} \times \mathcal{V})/\mathfrak{S}_2.$$

Following [37], let us introduce the *edge refinement* $\Gamma_{\mathcal{E}}$ of a graph Γ, which is the original graph together with a two-valent vertex (the midpoint) chosen from each edge. A *half-edge* of Γ is an edge of its edge refinement. A *ribbon graph* is a graph with a cyclic order assigned at each vertex to the set of half-edges incident to the vertex. When a cyclic order is given, a vertex can be placed on an oriented plane, and half-edges incident to the vertex can be represented by double lines. The orientation of the plane gives an orientation of the ribbon-like structure, and its boundaries inherit a compatible orientation (see Figure 1.1).

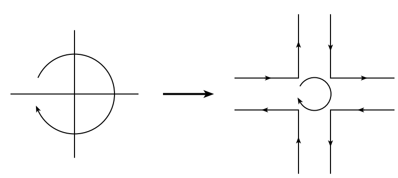

FIGURE 1.1. A vertex with a cyclic order given to incident half-edges. It is placed on a plane with the clockwise orientation. The half-edges become crossroads with a compatible orientation at the boundary.

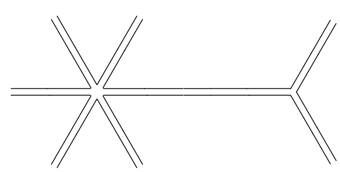

FIGURE 1.2. A ribbon graph is obtained by connecting cyclically ordered vertices with a ribbon like edge preserving the orientation.

A topological realization of a ribbon graph Γ is obtained by connecting these half-edges in an orientation-compatible manner. Since each boundary has a well-defined orientation, we can attach an oriented disk to the boundary and form a compact oriented surface S_Γ. Let $f(\Gamma)$ denote the number of disks attached. This number is uniquely determined by the ribbon graph structure of a graph. The attached disks, together with the vertices and edges of Γ, form a *cell-decomposition* of the surface S_Γ. The genus of the surface is determined by the formula for the Euler characteristic

$$\chi(S_\Gamma) = 2 - 2g(S_\Gamma) = v(\Gamma) - e(\Gamma) + f(\Gamma), \tag{1.1}$$

where $v(\Gamma) = |\mathcal{V}|$ is the number of vertices and $e(\Gamma) = |\mathcal{E}|$ the number of edges of Γ.

Conversely, if a connected graph Γ is drawn on an oriented surface S in a way that $S \setminus \Gamma$ is the union of disjoint open disks, then Γ is a ribbon graph that defines a cell-decomposition of S. The cyclic order of half-edges incident to a vertex is determined by the orientation of the surface (see Figure 1.3). Obviously we have $S = S_\Gamma$.

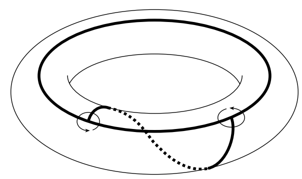

FIGURE 1.3. A graph drawn on an oriented surface. At each vertex, the orientation of the surface determines a cyclic order of the edges incident to the vertex.

Definition 1.1 ([37])**.** Let Γ be a ribbon graph. The group $\mathrm{Aut}_R\Gamma$ of automorphisms of Γ consists of graph automorphisms of the edge refinement $\Gamma_{\mathcal{E}}$ that preserves the cyclic order at each vertex of Γ.

In a ribbon graph, an edge connects two oriented vertices in the orientation-compatible manner. If we connect vertices without paying attention to the orientation, then we obtain a *Möbius* graph. An edge connecting two oriented vertices is *not twisted* if the connection is consistent with the orientation, and is *twisted* otherwise. Thus a double twist is the same as no twist. A new operation allowed in a Möbius graph that preserves the Möbius graph structure is a *vertex flip* at a vertex. This operation reverses the cyclic order assigned at the vertex, and twists every half-edge incident to the vertex (see Figure 1.4). If an edge is incident to a vertex and forms a loop, then the vertex flip at this vertex does not change the twist of the edge.

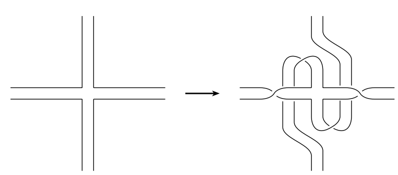

FIGURE 1.4. A vertex flip operation. It reverses the cyclic order at a vertex, and gives an extra twist to each half-edge incident to the vertex.

We can formalize the definition of a Möbius graph in the following way.

Definition 1.2. A *Möbius graph* is the equivalence class of ribbon graphs with a $\mathbb{Z}/2\mathbb{Z}$-color assigned to each edge. Two edge-colored ribbon graphs are equivalent if one is obtained from the other by a sequence of vertex flip operations. A vertex flip reverses the cyclic order of a vertex and the color of each half-edge incident to it. The group $\mathrm{Aut}_M\Gamma$ of automorphisms of a Möbius graph consists of graph automorphisms of the edge refinement of the underlying graph Γ that preserve the equivalence class of the edge-colored ribbon graph.

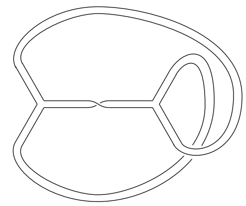

Figure 1.5. A Möbius graph.

A topological realization of a Möbius graph is the realization of the $\mathbb{Z}/2\mathbb{Z}$-color of each edge as a twist or non-twist. Each boundary component of a Möbius graph is a circle, without any consistent orientation. By attaching an open disk to each boundary circle, a Möbius graph gives rise to a closed surface without orientation. Let us denote this surface by S_Γ and by $f(\Gamma)$ the number of disks, as before. We note that Γ defines a cell-decomposition of S_Γ. Every closed non-orientable surface is constructed by removing k disks from a sphere and attaching a cross-cap at each hole. The number k is the *cross-cap genus* of the surface, and the Euler characteristic of the surface is given by $2-k$. If the surface S_Γ is non-orientable, then we have

$$\chi(S_\Gamma) = 2 - k(S_\Gamma) = v(\Gamma) - e(\Gamma) + f(\Gamma). \tag{1.2}$$

A ribbon graph Γ is also a Möbius graph. If Γ and its flip Γ^t (the graph obtained by applying the vertex flip operation at every vertex simultaneously) is the same ribbon graph, then we have

$$|\mathrm{Aut}_M\Gamma| = 2\ |\mathrm{Aut}_R\Gamma|. \tag{1.3}$$

Otherwise, Γ and Γ^t are different ribbon graphs but the same as a Möbius graph, and we have

$$\mathrm{Aut}_M\Gamma \cong \mathrm{Aut}_R\Gamma. \tag{1.4}$$

2. Integrals over a finite-dimensional complex von Neumann algebra

In this section we define the integrals over a finite-dimensional complex von Neumann algebra that we study, and establish their graphical expansion formulas in terms of ribbon graphs.

Definition 2.1. A finite-dimensional *complex von Neumann algebra* is a finite-dimensional $\mathbb{C}$-algebra with a conjugate-linear anti-isomorphism $* : A \to A$ and a $\mathbb{C}$-linear map called *trace* $\langle\ \rangle : A \to \mathbb{C}$ that satisfy the following conditions for every $a, b \in A$:

$$\begin{aligned} (a^*)^* &= a \\ (ab)^* &= b^* a^* \\ \langle a^* \rangle &= \overline{\langle a \rangle} \\ \langle ab \rangle &= \langle ba \rangle \\ \langle 1 \rangle &= 1 \\ \langle aa^* \rangle &> 0\ , \qquad a \neq 0\ . \end{aligned} \tag{2.1}$$

If A is an $\mathbb{R}$-algebra with a real valued trace, then it is called a *real* von Neunamm algebra.

To avoid confusion, we only deal with complex von Neumann algebras in this section. Real ones are considered in Section 3. A finite-dimensional von Neumann algebra A is a real vector space with a non-degenerate hermitian inner product defined by

$$\langle a, b \rangle = \langle ab^* \rangle \ . \tag{2.2}$$

As usual, an invertible linear transformation of A that preserves the hermitian form is called a unitary transformation. We denote by $U(A)$ the group of unitary transformations of A. A real vector subspace of A consisting of self-adjoint elements

$$\mathcal{H}_A = \{a \in A \mid a^* = a\} \tag{2.3}$$

is of our particular interest. We note that $\mathcal{H}_A$ is invariant under the conjugation action of $U(A)$. Let us denote by dx the translation invariant Lebesgue measure of the real vector space $\mathcal{H}_A$ that is also invariant under the conjugation action of $U(A)$. We notice that the quadratic form $\langle x^2 \rangle$ is positive definite on the space $\mathcal{H}_A$ of self-adjoint elements. We denote by

$$d\mu(x) = \frac{dx}{\int_{\mathcal{H}_A} e^{-\frac{1}{2}\langle x^2 \rangle}\, dx} \tag{2.4}$$

the normalized Lebesgue measure on $\mathcal{H}_A$.

Our subject of study is the following integral

$$\prod_{j=1}^{n} \frac{1}{v_j! \cdot j^{v_j}} \int_{\mathcal{H}_A} e^{-\frac{1}{2}\langle x^2 \rangle} \prod_{j=1}^{n} \langle x^j \rangle^{v_j} d\mu(x) \tag{2.5}$$

for every n-tuple of positive integers $(v_1, \ldots, v_n) \in \mathbb{N}^n$, $n = 0, 1, 2, \cdots$. The constant factor in front of the integral is placed for a combinatorial reason explained later in this section. To consider a generating function of these integrals, it is more convenient to introduce

$$\mathbf{e}(v_1, v_2, v_3, \cdots) = \sum_{j \geq 1} j v_j \tag{2.6}$$

and the sum of the integrals over all elements of

$$\mathbb{N}^\infty = \varinjlim_n \mathbb{N}^n = \{(v_1, v_2, v_3, \ldots) \mid v_j = 0 \ \text{ for } \ j >> 0\}$$

with a fixed value of $\mathbf{e}(v_1, v_2, v_3, \cdots)$. Notice that for every finite value of n,

$$(v_1, v_2, v_3, \ldots) \in \mathbb{N}^n \qquad \text{if} \qquad \mathbf{e}(v_1, v_2, v_3, \cdots) \leq n \ .$$

Thus let us define

$$Z_A^{\mathbb{C}}(t_1, t_2, t_3, \ldots) = \sum_{n=0}^{\infty} \sum_{\substack{(v_1, v_2, v_3, \ldots) \in \mathbb{N}^\infty \\ \mathbf{e}(v_1, v_2, v_3, \cdots) = n}} \prod_{j \geq 1}^{\text{finite}} \frac{t_j^{v_j}}{v_j! j^{v_j}} \int_{\mathcal{H}_A} e^{-\frac{1}{2}\langle x^2 \rangle} \prod_{j \geq 1}^{\text{finite}} \langle x^j \rangle^{v_j} d\mu(x) \ , \tag{2.7}$$

where $t_1, t_2, t_3, \ldots$ are expansion parameters carrying the weight

$$\deg t_j = j \ . \tag{2.8}$$

The monomial $\prod_j t_j^{v_j}$ for every $(v_1, v_2, v_3, \cdots)$ satisfying $\mathbf{e}(v_1, v_2, v_3, \cdots) = n$ has weighted homogeneous degree n by (2.6) and (2.8). Hence (2.7) is an infinite sum of weighted homogeneous polynomials of degree n for every $n \geq 0$.

Symbolically, we can write the generating function in an integral form

$$Z_A^{\mathbb{C}}(t) = \int_{\mathcal{H}_A} e^{-\frac{1}{2}\langle x^2 \rangle} e^{\sum_{j=1}^{\infty} \frac{t_j}{j} \langle x^j \rangle^{v_j}} d\mu(x) \ . \tag{2.9}$$

As an actual integral, (2.9) is ill defined because of the infinite sum in the exponent. There is a way to make it well-defined so that (2.7) is a rigorous asymptotic expansion of (2.9). Since we do not employ this point of view in this paper, we refer to [35, 36] for more detail, and work on the expansion form only.

Let $\{e_1, \dots, e_N\}$ be an orthonormal basis for A with respect to the hermitian form (2.2), where $N = \dim_{\mathbb{C}} A$. Since

$$\langle e_i, e_j \rangle = \langle e_i e_j^* \rangle = \langle e_j^* (e_i^*)^* \rangle = \langle e_j^*, e_i^* \rangle \ ,$$

$\{e_1^*, \dots, e_N^*\}$ also forms an orthonormal basis for A. For every $a \in A$ we have

$$a = \sum_{j=1}^{N} \langle a, e_j \rangle \ e_j = \sum_{j=1}^{N} \langle a, e_j^* \rangle \ e_j^* \ . \tag{2.10}$$

Equivalently,

$$\langle a, b \rangle = \sum_{j=1}^{N} \langle a, e_j \rangle \ \langle e_j, b \rangle = \sum_{j=1}^{N} \langle a, e_j^* \rangle \ \langle e_j^*, b \rangle \tag{2.11}$$

holds for every $a, b \in A$.

Lemma 2.2. *Choose two elements*

$$x = \sum_{i=1}^{N} x_i \ e_i \qquad \textit{and} \qquad y = \sum_{i=1}^{N} y_i \ e_i$$

of A, and consider $e^{\langle xy \rangle}$ as a function in $2N$ variables

$$(x_1, \dots, x_N, y_1, \dots, y_N) \in \mathbb{C}^{2N} \ .$$

With respect to the differential operator

$$\frac{\partial}{\partial y} = \sum_{i=1}^{N} \frac{\partial}{\partial y_i} \ e_i^* \ , \tag{2.12}$$

we have

$$\frac{\partial}{\partial y} \ e^{\langle xy \rangle} = x \ e^{\langle xy \rangle} \ . \tag{2.13}$$

In particular,

$$\left\langle \left(\frac{\partial}{\partial y} \right)^j \right\rangle^m \ e^{\langle xy \rangle} = \langle x^j \rangle^m e^{\langle xy \rangle} \tag{2.14}$$

for every $j, m > 0$.

Proof. By definition,

$$\begin{aligned}
\frac{\partial}{\partial y}\, e^{\langle xy\rangle} &= \sum_i \frac{\partial}{\partial y_i}\, e_i^* \exp\left(\left\langle \sum_j x_j\ e_j \sum_k y_k\ e_k \right\rangle\right) \\
&= \sum_i \sum_j x_j\ \langle e_j e_i\rangle\ e_i^*\ e^{\langle xy\rangle} \\
&= \sum_j x_j \sum_i \langle e_j, e_i^*\rangle\ e_i^*\ e^{\langle xy\rangle} \\
&= \sum_j x_j\ e_j\ e^{\langle xy\rangle} \\
&= x\ e^{\langle xy\rangle}\ .
\end{aligned}$$

Using the linearity of the trace and (2.13) repeatedly, we obtain (2.14). □

Lemma 2.3. *Let A be a finite-dimensional complex von Neumann algebra. Then we have the following Laplace transform formula for (2.5):*

$$\int_{\mathcal{H}_A} e^{-\frac{1}{2}\langle x^2\rangle} \prod_{j=1}^{n} \langle x^j\rangle^{v_j} d\mu(x) = \prod_{j=1}^{n} \left\langle \left(\frac{\partial}{\partial y}\right)^j \right\rangle^{v_j} e^{\frac{1}{2}\langle (y+y^*)^2\rangle}\Bigg|_{y=0}\ . \tag{2.15}$$

Proof. For $y \in A$ of Lemma 2.2, its adjoint is given by

$$y^* = \sum_{i=1}^{N} \overline{y_i}\ e_i^*\ .$$

Note that $\partial y^*/\partial y = 0$ since $\partial \overline{y_i}/\partial y_j = 0$ for any i and j. Now Lemma 2.2 yields

$$\int_{\mathcal{H}_A} e^{-\frac{1}{2}\langle x^2\rangle} \langle x^j\rangle^m d\mu(x) = \left\langle \left(\frac{\partial}{\partial y}\right)^j \right\rangle^m \int_{\mathcal{H}_A} e^{-\frac{1}{2}\langle x^2\rangle} e^{\langle x(y+y^*)\rangle} d\mu(x)\Bigg|_{y=0}\ .$$

Since $y + y^* \in \mathcal{H}_A$ and $d\mu(x)$ is a translational invariant measure, we have

$$\begin{aligned}
\int_{\mathcal{H}_A} e^{-\frac{1}{2}\langle x^2\rangle} e^{\langle x(y+y^*)\rangle} d\mu(x) &= \int_{\mathcal{H}_A} e^{-\frac{1}{2}\langle (x-(y+y^*))^2\rangle} e^{\frac{1}{2}\langle (y+y^*)^2\rangle} d\mu(x) \\
&= e^{\frac{1}{2}\langle (y+y^*)^2\rangle}\ .
\end{aligned}$$

Eqn. (2.15) follows from these formulas. □

In the same way as in the proof of Lemma 2.2, we obtain

$$\begin{aligned}
\frac{\partial}{\partial y}\ e^{\frac{1}{2}\langle (y+y^*)^2\rangle} &= \sum_i \frac{\partial}{\partial y_i}\ e_i^*\ e^{\frac{1}{2}\langle (\sum_j y_j\ e_j + \sum_j \overline{y_j}\ e_j^*)^2\rangle} \\
&= \sum_i \langle (y+y^*)\ e_i\rangle\ e_i^*\ e^{\frac{1}{2}\langle (y+y^*)^2\rangle} \\
&= \sum_i \langle (y+y^*), e_i^*\rangle\ e_i^*\ e^{\frac{1}{2}\langle (y+y^*)^2\rangle} \\
&= (y+y^*)\ e^{\frac{1}{2}\langle (y+y^*)^2\rangle}\ .
\end{aligned}$$

In particular,

$$\frac{\partial}{\partial y_i} e^{\frac{1}{2}\langle (y+y^*)^2\rangle} = \langle (y+y^*), e_i^*\rangle \; e^{\frac{1}{2}\langle (y+y^*)^2\rangle} \ ,$$

and hence

$$\begin{aligned} \frac{\partial}{\partial y_i}\frac{\partial}{\partial y_j} e^{\frac{1}{2}\langle (y+y^*)^2\rangle}\Big|_{y=0} &= \frac{\partial}{\partial y_i}\langle (y+y^*), e_j^*\rangle\Big|_{y=0} \\ &= \langle e_i, e_j^*\rangle \\ &= \langle e_i e_j\rangle \ . \end{aligned} \tag{2.16}$$

Our purpose is to compute

$$\sum_{\substack{(v_1,v_2,v_3,\dots)\in\mathbb{N}^\infty \\ \mathbf{e}(v_1,v_2,v_3,\cdots)=n}} \prod_{j\geq 1}^{\text{finite}} \frac{1}{v_j!\cdot j^{v_j}} \left\langle \left(\frac{\partial}{\partial y}\right)^j \right\rangle^{v_j} e^{\frac{1}{2}\langle (y+y^*)^2\rangle}\Big|_{y=0} \ . \tag{2.17}$$

To this end, first we observe

$$\left\langle \left(\frac{\partial}{\partial y}\right)^j \right\rangle = \sum_{i_1,\dots,i_j} \frac{\partial}{\partial y_{i_1}} \cdots \frac{\partial}{\partial y_{i_j}} \left\langle e_{i_1}^* \cdots e_{i_j}^* \right\rangle \ . \tag{2.18}$$

Notice that $\left\langle e_{i_1}^* \cdots e_{i_j}^* \right\rangle$ is invariant under cyclic permutations. For every factor (2.18) of (2.17), let us assign a j-valent vertex with j half-edges incident to it, with a cyclic order of these half-edges. Every half-edge corresponds to an index i_k, $k = 1, \cdots, j$, and we assign $e_{i_k}^* \in A$ to this half-edge. We then assign $\left\langle e_{i_1}^* \cdots e_{i_j}^* \right\rangle$ to this vertex (see Figure 2.1).

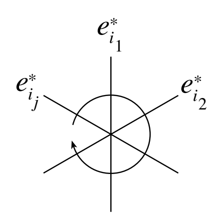

FIGURE 2.1. A j-valent vertex with a cyclic order given to incident half-edges. Each half-edge is labeled by i_k, and an element $e_{i_k}^*$ is assigned.

For every $j = 1, 2, 3, \cdots$, we draw v_j j-valent vertices with $\left\langle e_{i_1}^* \cdots e_{i_j}^* \right\rangle$ assigned. Every vertex has j degrees of freedom coming from cyclic rotations. This redundancy is compensated by the factor j^{v_j} in (2.17). The redundancy of permuting v_j vertices of the same valence j is compensated by $v_j!$ in (2.17). To indicate the effect of (2.16), we connect two half-edges according to the paired differentiation. We notice that since we set $y = 0$ after differentiation, no term in (2.17) survives unless all differentiations are paired as in (2.16). When we connect a half-edge labeled by i_k at one vertex with another half-edge labeled by h_ℓ, we assign $\langle e_{i_k} e_{h_\ell}\rangle$ to this edge (see Figure 2.2). This quantity is called the *propagator* of the edge. Notice that the propagator is symmetric

$$\langle e_{i_k} e_{h_\ell}\rangle = \langle e_{h_\ell} e_{i_k}\rangle \ , \tag{2.19}$$

and hence we do not have any particular *direction* on our edge.

Here we have to be cautious when two vertices are connected. For example, the connection described in Figure 2.2 preserves the cyclic orders of two vertices. An edge is connecting two vertices in the *orientation-preserving* manner if the cyclic orders of the two vertices agree

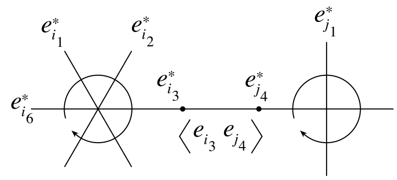

FIGURE 2.2. A half-edge labeled by i_3 of the left vertex is connected with a half-edge labeled by j_4 of the vertex at the right. A propagator $\langle e_{i_3} e_{j_4} \rangle$ is assigned to this edge.

when the edge is shrunk to a point and the two vertices are put together. Otherwise, the edge is *orientation-reversing*. All connections we make in this section should be orientation-preserving. When all half-edges are paired and connected in the orientation-preserving manner, we obtain a *ribbon graph* Γ. It is easy to see that the compensation of rotations around each vertex and permutations of vertices of the same valence leads to the factor of $1/|\mathrm{Aut}_R(\Gamma)|$ coming from the automorphism of Γ [36, 37]. The quantity v_j represents the number of j-valent vertices of Γ by the construction. Thus

$$v(\Gamma) = \sum_j v_j \qquad \text{and} \qquad e(\Gamma) = \frac{1}{2} \sum_j j v_j \tag{2.20}$$

represent the total number of vertices and edges of Γ, respectively. Notice the combinatorial constraint

$$2e(\Gamma) = \mathbf{e}(v_1, v_2, v_3, \dots) \ ,$$

which comes from the fact that unless every half-edge is paired with another one to form an edge of a ribbon graph, the corresponding contribution of $(v_1, v_2, v_3, \dots)$ in the sum of (2.17) is 0. Summarizing, we have

Proposition 2.4. *Let $RG(e)$ denote the set of all ribbon graphs, may or may not be connected, consisting of a total of e edges. The number of j-valent vertices of $\Gamma \in RG(e)$ is denoted by $v_j = v_j(\Gamma)$. For each j-valent vertex of Γ, let us assign $\langle e^*_{i_1} \cdots e^*_{i_j} \rangle$. For every edge we assign $\langle e_{i_k} e_{h_\ell} \rangle$, so that the incidence relation is consistent with the relation described above, namely, this edge connects the half-edge labeled by i_k of a vertex to the half-edge labeled by h_ℓ from another vertex, which could be the same vertex. Let A^{or}_Γ denote the sum with respect to all indices of the product of all contributions from vertices and edges. Then we have*

$$\begin{aligned}
&\sum_{\substack{(v_1,v_2,v_3,\dots)\in\mathbb{N}^\infty \\ \mathbf{e}(v_1,v_2,v_3,\cdots)=2e}} \int_{\mathcal{H}_A} e^{-\frac{1}{2}\langle x^2\rangle} \prod_{j\geq 1}^{\text{finite}} \frac{1}{v_j! \cdot j^{v_j}} \langle x^j \rangle^{v_j} \, d\mu(x) \\
&= \sum_{\substack{(v_1,v_2,v_3,\dots)\in\mathbb{N}^\infty \\ \mathbf{e}(v_1,v_2,v_3,\cdots)=2e}} \prod_{j\geq 1}^{\text{finite}} \frac{1}{v_j! \cdot j^{v_j}} \left\langle \left(\frac{\partial}{\partial y}\right)^j \right\rangle^{v_j} e^{\frac{1}{2}\langle (y+y^*)^2\rangle} \Bigg|_{y=0} \\
&= \sum_{\Gamma\in RG(e)} \frac{1}{|\mathrm{Aut}_R(\Gamma)|} A^{or}_\Gamma \ ,
\end{aligned} \tag{2.21}$$

where $\mathrm{Aut}_R(\Gamma)$ is the automorphism group of ribbon graph Γ defined in Section 1.

Therefore, to evaluate the integral, it suffices to calculate A^{or}_Γ for each ribbon graph Γ. A key fact is the following.

Lemma 2.5. *Let Γ be a connected ribbon graph with two or more vertices, and E an edge of Γ incident to two distinct vertices. Then the contribution of the graph A^{or}_Γ is invariant under the edge-contraction:*

$$A^{or}_\Gamma = A^{or}_{\Gamma/E}\ , \tag{2.22}$$

where Γ/E denotes the ribbon graph obtained by shrinking E to a point in Γ and joining the two incident vertices together.

Remark. This invariance is found in many literatures including [53]. Witten uses the invariance to calculate quantum Yang-Mills theory over a Riemann surface by approximation through lattice gauge theory. It appears also in [38].

Proof. Let V_1 and V_2 be the two vertices of Γ incident to E. The contribution from V_1 can be written as $\langle ae_i^*\rangle$ and that from V_2 as $\langle e_j^* b\rangle$, where a and b are products of the basis elements e_k^* of the von Neumann algebra A. The invariance of the edge contraction is local, and comes down to the following computation:

$$\begin{aligned}\sum_{i,j}\langle ae_i^*\rangle\langle e_ie_j\rangle\langle e_j^*b\rangle &= \sum_{i,j}\langle a,e_i\rangle\langle e_i,e_j^*\rangle\langle e_j^*,b^*\rangle\\ &= \langle a,b^*\rangle\\ &= \langle ab\rangle\ .\end{aligned} \tag{2.23}$$

The quantity $\langle ab\rangle$ is exactly the contribution of the new vertex obtained by joining V_1 and V_2. □

Every connected ribbon graph Γ gives rise to an oriented surface S_Γ whose Euler characteristic is determined by

$$\chi(S_\Gamma) = 2 - 2g(S_\Gamma) = v(\Gamma) - e(\Gamma) + f(\Gamma)\ .$$

The graph defines a cell-decomposition of S_Γ. The topological type of Γ is (g,f), the genus of the surface and the number of faces of its cell-decomposition. Note that since the edge contraction operation decreases $v(\Gamma)$ and $e(\Gamma)$ by one and preserves the number of faces, the topological type is preserved. A theorem of topology states that if Γ_1 and Γ_2 are two connected ribbon graphs with the same topological type, then by consecutive applications of edge contraction and its inverse operation (edge expansion), Γ_1 can be brought to Γ_2 [23]. Therefore, to compute A^{or}_Γ, we can use our favorite graph of the same topological type, for example, a graph of Figure 2.3.

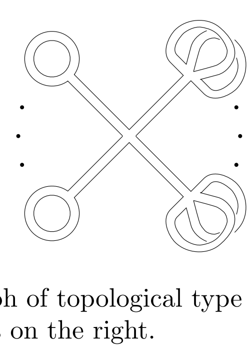

FIGURE 2.3. A standard graph of topological type (g,f). It has $f-1$ tadpoles on the left, and g bi-petal flowers on the right.

Proposition 2.6. *Let $A^{or}_{g,f}$ denote the contribution of the standard graph of Figure 2.3. Then*

$$A^{or}_{g,f} = \sum_{\substack{i_1,\dots,i_g;j_1,\dots,j_g\\ h_1,\dots,h_{f-1}}}^{N} \left\langle e_{i_1}e_{j_1}e^*_{i_1}e^*_{j_1}\cdots e_{i_g}e_{j_g}e^*_{i_g}e^*_{j_g}\cdot e_{h_1}e^*_{h_1}\cdots e_{h_{f-1}}e^*_{h_{f-1}}\right\rangle . \tag{2.24}$$

Proof. By definition,

$$\begin{aligned} A^{or}_{g,f} = \sum_{\substack{i_1,\dots,i_g;j_1,\dots,j_g\\ a_1,\dots,a_g;b_1,\dots,b_g\\ k_1,\dots,k_{f-1};h_1,\dots,h_{f-1}}}^{N} &\left\langle e^*_{a_1}e^*_{b_1}e^*_{i_1}e^*_{j_1}\cdots e^*_{a_g}e^*_{b_g}e^*_{i_g}e^*_{j_g}\cdot e^*_{h_1}e^*_{k_1}\cdots e^*_{h_{f-1}}e^*_{k_{f-1}}\right\rangle \\ &\times \langle e_{a_1}e_{i_1}\rangle\langle e_{b_1}e_{j_1}\rangle\cdots\langle e_{a_g}e_{i_g}\rangle\langle e_{b_g}e_{j_g}\rangle\cdot\langle e_{h_1}e_{k_1}\rangle\cdots\langle e_{h_{f-1}}e_{k_{f-1}}\rangle .\end{aligned}$$

Using cyclic invariance of the trace and (2.11), the desired formula (2.24) follows. □

The generating function $Z^{\mathbb{C}}_A(t)$ of (2.7) is expanded in terms of all ribbon graphs, connected or non-connected. Since $Z^{\mathbb{C}}_A(0)=1$, the formal logarithm is well-defined for $Z^{\mathbb{C}}_A(t)$. The graphical expansion of $\log Z^{\mathbb{C}}_A(t)$ then consists of connected ribbon graphs.

Theorem 2.7. *The graphical expansion of the logarithm of the generating function $Z^{\mathbb{C}}_A(t)$ of (2.7) is given by*

$$\begin{aligned}\log Z^{\mathbb{C}}_A(t) &= \log\int_{\mathcal{H}_A} e^{-\frac{1}{2}\langle x^2\rangle}e^{\sum_j \frac{t_j}{j}\langle x^j\rangle}d\mu(x)\\ &= \sum_{\substack{\Gamma\ \textit{connected}\\ \textit{ribbon graph}}} \frac{1}{|\mathrm{Aut}_R(\Gamma)|}A^{or}_{g(\Gamma),f(\Gamma)}\prod_j t_j^{v_j(\Gamma)} .\end{aligned} \tag{2.25}$$

Recall the graph theoretic formulas (2.20). If we change t_j to βt_j, then the graph expansion receives an extra factor of $\beta^{v(\Gamma)}$ in the contribution from Γ. If we change $e^{-\frac{1}{2}\langle x^2\rangle}$ to $e^{-\frac{\alpha}{2}\langle x^2\rangle}$ with a positive real number α, then a change of variable $x\mapsto x/\sqrt{\alpha}$ produces a factor $\alpha^{-e(\Gamma)}$ to the Γ-contribution. Therefore,

$$\log\int_{\mathcal{H}_A} e^{-\frac{\alpha}{2}\langle x^2\rangle}e^{\beta\sum_j \frac{t_j}{j}\langle x^j\rangle}d\mu_\alpha(x) = \sum_{\substack{\Gamma\ \text{connected}\\ \text{ribbon graph}}} \frac{\alpha^{-e(\Gamma)}\beta^{v(\Gamma)}}{|\mathrm{Aut}_R(\Gamma)|}A^{or}_{g(\Gamma),f(\Gamma)}\prod_j t_j^{v_j(\Gamma)} , \tag{2.26}$$

where the normalized Lebesgue measure is adjusted for $e^{-\frac{\alpha}{2}\langle x^2\rangle}$. An important example of Theorem 2.7 is a hermitian matrix integral.

Example 2.1. Let us apply (2.25) to a complex matrix algebra $A = M(n,\mathbb{C})$. The $*$-operation on this algebra is the matrix adjoint, and

$$\langle X\rangle = \frac{1}{n}\,\mathrm{tr}X \tag{2.27}$$

is the normalized trace. The space of self-adjoint elements is the set of hermitian matrices:

$$\mathcal{H}_{M(n,\mathbb{C})} = \mathcal{H}_{n,\mathbb{C}} .$$

As an orthonormal basis, we use $\{\sqrt{n}e_{ij}\}$, where

$$e_{ij} = \left[\delta_{i\alpha}\delta_{j\beta}\right]_{\alpha,\beta}$$

is the $n \times n$ elementary matrix which has 1 at its ij entry and 0 everywhere else. Then we have

$$\log \int_{\mathcal{H}_{n,\mathbb{C}}} e^{-\frac{1}{2}\operatorname{tr}(X^2)} e^{\sum_j \frac{t_j}{j}\operatorname{tr}(X^j)} d\mu(X) = \sum_{\substack{\Gamma \text{ connected} \\ \text{ribbon graph}}} \frac{1}{|\operatorname{Aut}_R(\Gamma)|} n^{f(\Gamma)} \prod_j t_j^{v_j(\Gamma)} . \tag{2.28}$$

Indeed, the computation of $M(n,\mathbb{C})^{or}_{g,f}$ is just evaluating the trace of the identity matrix I. Each tadpole contributes

$$\sum_{i,j} e_{ij} e^*_{ij} = \sum_{i,j} e_{ii} = n \cdot I ,$$

while each bi-petal flower contributes

$$\sum_{i,j,k,\ell} e_{ij} e_{k\ell} e_{ji} e_{\ell k} = \sum_{i,j} e_{ii} e_{jj} = I .$$

Therefore, we have

$$M(n,\mathbb{C})^{or}_{g,f} = n^{2g} n^{2(f-1)} \langle I \rangle = n^{-v+e+f} .$$

Eqn. (2.28) follows from (2.27) and (2.26). Another useful form of hermitian matrix integral is

$$\log \int_{\mathcal{H}_{n,\mathbb{C}}} e^{-\frac{n}{2}\operatorname{tr}(X^2)} e^{n\sum_j \frac{t_j}{j}\operatorname{tr}(X^j)} d\mu(X) = \sum_{\substack{\Gamma \text{ connected} \\ \text{ribbon graph}}} \frac{1}{|\operatorname{Aut}_R(\Gamma)|} n^{\chi(S_\Gamma)} \prod_j t_j^{v_j(\Gamma)} , \tag{2.29}$$

which also follows from (2.26).

Eqn. (2.28) is due to [6] and has been used by many authors in the study of hermitian matrix integrals [22, 35, 40, 41, 42, 43, 54]. In Section 4, we give another example of the general formula, where we consider $A = \mathbb{C}[G]$.

3. Integrals over a real von Neumann algebra

For a finite-dimensional real von Neumann algebra A, the corresponding formulas become quite different. Since its trace is real valued, the hermitian inner product is real symmetric:

$$\langle a, b\rangle = \langle ab^*\rangle = \langle ba^*\rangle = \langle b, a\rangle .$$

The integral we wish to evaluate is

$$\sum_{\substack{(v_1,v_2,v_3,\cdots)\in\mathbb{N}^\infty \\ \mathbf{e}(v_1,v_2,v_3,\cdots)=n}} \int_{\mathcal{H}_A} e^{-\frac{1}{4}\langle x^2\rangle} \prod_{j\geq 1}^{\text{finite}} \frac{1}{v_j! \cdot (2j)^{v_j}} \langle x^j\rangle^{v_j} d\mu(x) \tag{3.1}$$

for every $n \geq 0$ with respect to a different normalized Lebesgue measure

$$d\mu(x) = \frac{dx}{\int_{\mathcal{H}_A} e^{-\frac{1}{4}\langle x^2\rangle} dx} . \tag{3.2}$$

The generating function for (3.1) is given by

$$\begin{aligned} Z^{\mathbb{R}}_A(t) &= \int_{\mathcal{H}_A} e^{-\frac{1}{4}\langle x^2\rangle} e^{\sum_{j=1}^\infty \frac{t_j}{2j}\langle x^j\rangle} \, d\mu(x) \\ &= \sum_{n=0}^\infty \sum_{\substack{(v_1,v_2,v_3,\cdots)\in\mathbb{N}^\infty \\ \mathbf{e}(v_1,v_2,v_3,\cdots)=n}} \prod_{j\geq 1}^{\text{finite}} \frac{1}{v_j! \cdot (2j)^{v_j}} \int_{\mathcal{H}_A} e^{-\frac{1}{4}\langle x^2\rangle} \prod_{j\geq 1}^{\text{finite}} \langle x^j\rangle^{v_j} d\mu(x) . \end{aligned} \tag{3.3}$$

Lemma 3.1. *Let A be a real von Neumann algebra. Then*

$$\int_{\mathcal{H}_A} e^{-\frac{1}{4}\langle x^2\rangle} \prod_{j=1}^{n} \langle x^j \rangle^{v_j} d\mu(x) = \prod_{j=1}^{n} \left\langle \left(\frac{\partial}{\partial y}\right)^j \right\rangle^{v_j} e^{\frac{1}{4}\langle (y+y^*)^2\rangle}\Bigg|_{y=0} . \tag{3.4}$$

Proof. The adjoint of the element $y \in A$ of Lemma 2.2 is given by

$$y^* = \sum_{i-1}^{N} y_i \, e_i^* .$$

We note here that $y_i \in \mathbb{R}$ and the orthonormal basis $\{e_1, \ldots, e_N\}$ for A is a real basis. Unlike the complex case, we have $\langle xy^*\rangle = \langle x^* y\rangle$, and hence

$$\frac{\partial}{\partial y} \, e^{\langle xy^*\rangle} = \frac{\partial}{\partial y} \, e^{\langle x^* y\rangle} = x^* e^{\langle xy^*\rangle}$$

from Lemma 2.2. Therefore, for $x \in \mathcal{H}_A$,

$$\frac{\partial}{\partial y} \, e^{\langle x(y+y^*)/2\rangle} = \frac{x+x^*}{2} e^{\langle x(y+y^*)/2\rangle} = x \, e^{\langle x(y+y^*)/2\rangle} .$$

The completion of the square is modified to

$$\begin{aligned}\int_{\mathcal{H}_A} e^{-\frac{1}{4}\langle x^2\rangle} e^{\langle x(y+y^*)/2\rangle} d\mu(x) &= \int_{\mathcal{H}_A} e^{-\frac{1}{4}\langle (x-(y+y^*))^2\rangle} e^{\frac{1}{4}\langle (y+y^*)^2\rangle} d\mu(x) \\ &= e^{\frac{1}{4}\langle (y+y^*)^2\rangle} .\end{aligned}$$

The rest of the proof is the same as the complex case. □

To compute the RHS of (3.4), we first note

$$\begin{aligned}\frac{\partial}{\partial y} \, e^{\frac{1}{4}\langle (y+y^*)^2\rangle} &= \frac{1}{2} \sum_i \frac{\partial}{\partial y_i} \, e_i^* \, e^{\frac{1}{4}\langle (\sum_j y_j \, e_j + \sum_j y_j \, e_j^*)^2\rangle} \\ &= \frac{1}{2} \sum_i \langle (y+y^*)(e_i + e_i^*)\rangle \, e_i^* \, e^{\frac{1}{4}\langle (y+y^*)^2\rangle} \\ &= \frac{1}{2} \sum_i \langle (y+y^*), (e_i + e_i^*)\rangle \, e_i^* \, e^{\frac{1}{4}\langle (y+y^*)^2\rangle} \\ &= (y+y^*) \, e^{\frac{1}{4}\langle (y+y^*)^2\rangle} .\end{aligned}$$

In particular,

$$\frac{\partial}{\partial y_i} e^{\frac{1}{4}\langle (y+y^*)^2\rangle} = \frac{1}{2} \, \langle (y+y^*), (e_i + e_i^*)\rangle \, e^{\frac{1}{4}\langle (y+y^*)^2\rangle} ,$$

and hence

$$\begin{aligned}\frac{\partial}{\partial y_i} \frac{\partial}{\partial y_j} e^{\frac{1}{4}\langle (y+y^*)^2\rangle}\bigg|_{y=0} &= \frac{1}{2} \, \frac{\partial}{\partial y_i} \langle (y+y^*), (e_j + e_j^*)\rangle \bigg|_{y=0} \\ &= \frac{1}{2} \, \langle (e_i + e_i^*), e_j + e_j^*)\rangle \\ &= \langle e_i e_j^*\rangle + \langle e_i e_j\rangle \\ &= \delta_{ij} + \langle e_i e_j\rangle .\end{aligned} \tag{3.5}$$

This formula has an extra term δ_{ij} compared to (2.16). To compute the graphical expansion of

$$\sum_{\substack{(v_1,v_2,v_3,\cdots)\in\mathbb{N}^\infty \\ \mathbf{e}(v_1,v_2,v_3,\cdots)=n}} \prod_{j\geq 1}^{\text{finite}} \frac{1}{v_j!\cdot(2j)^{v_j}} \left\langle \left(\frac{\partial}{\partial y}\right)^j \right\rangle^{v_j} e^{\frac{1}{4}\langle (y+y^*)^2\rangle}\bigg|_{y=0}, \tag{3.6}$$

we proceed as before and assign a cyclically ordered j-valent vertex to each factor $\langle(\partial/\partial y)^j\rangle$ of the differentiation, assign the vertex contribution $\langle e_{i_1}^* \cdots e_{i_j}^*\rangle$ to it, and place the vertex consistently on an oriented plane with the clockwise orientation. When a pair of differentiation is applied, because of (3.5), there are now two choices: straight connection as in the ribbon graph case Figure 2.2 with a propagator $\langle e_i e_j\rangle$ assigned to the edge, or connection with a twisted edge carrying a propagator $\langle e_i e_j^*\rangle = \delta_{ij}$ as in Figure 3.1.

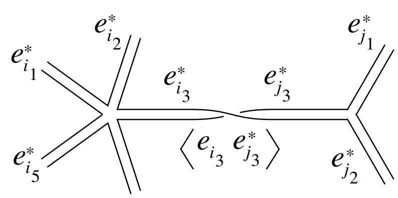

FIGURE 3.1. A half-edge labeled by i_3 of the left vertex is connected to a half-edge labeled by j_3 of the vertex at the right by a twisted edge. A propagator $\langle e_{i_3} e_{j_3}^*\rangle$ is assigned to this edge.

If the straight connection is used at every edge, then the resulting graph is a ribbon graph as before. Otherwise, we obtain a Möbius graph Γ with some twisted edges. It has to be noted that the existence of twisted edges does not necessarily mean that the Möbius graph is non-orientable. For a Möbius graph Γ thus obtained, let us define the graph contribution $A_\Gamma^{\mathbb{R}}$ as the sum with respect to all indices of the product of all vertex contributions $\langle e_{i_1}^* \cdots e_{i_j}^*\rangle$ and the product of all propagators, where $\langle e_i e_j\rangle$ is chosen for a straight edge and $\langle e_i e_j^*\rangle$ is chosen for a twisted edge.

The reality condition of the trace provides another invariance:

$$\left\langle e_{i_1}^* e_{i_2}^* \cdots e_{i_{j-1}}^* e_{i_j}^* \right\rangle = \left\langle e_{i_j} e_{i_{j-1}} \cdots e_{i_2} e_{i_1} \right\rangle . \tag{3.7}$$

This equality brings an equivalence relation into the set of Möbius graphs. To identify it, let us observe the following:

Lemma 3.2. *Let us denote by $\langle e_i e_j^{\pm *}\rangle$ either $\langle e_i e_j\rangle$ or $\langle e_i e_j^*\rangle$, and use $\langle e_i e_j^{\mp *}\rangle$ to indicate the other propagator. Then*

$$\sum_{i_1,\dots,i_j} \left\langle e_{i_1}^* \cdots e_{i_j}^* \right\rangle \langle e_{i_1} e_{h_1}^{\pm *}\rangle \cdots \langle e_{i_j} e_{h_j}^{\pm *}\rangle = \sum_{i_1,\dots,i_j} \left\langle e_{i_j}^* \cdots e_{i_1}^* \right\rangle \langle e_{i_1} e_{h_1}^{\mp *}\rangle \cdots \langle e_{i_j} e_{h_j}^{\mp *}\rangle . \tag{3.8}$$

Proof. Using the contraction formula (2.11), the LHS is equal to $\left\langle e_{h_1}^{\pm *} \cdots e_{h_j}^{\pm *} \right\rangle$. Similarly, the RHS is equal to $\left\langle e_{h_j}^{\mp *} \cdots e_{h_1}^{\mp *} \right\rangle$. Because of the reality condition (3.7), these are actually the same. □

Notice that the equation (3.8) is exactly the vertex flip operation of Figure 1.4. This allows us to define the graph contribution $A_\Gamma^{\mathbb{R}}$ slightly differently: it is the sum with respect to all indices of the product of all vertex contributions $\langle e_{i_1}^* \cdots e_{i_j}^*\rangle$ each of which has a cyclic order that is determined according to the cyclic order of the vertex, and the product of all propagators of edges determined by their twist. The extra redundancy of the vertex flip is compensated with the factor $(2j)^{v_j}$ in front of (3.6), which is the order of the product of dihedral groups acting on the vertices through rotations and flips. In a parallel way with Proposition 2.4, we have thus established:

Proposition 3.3. *Let $MG(e)$ be the set of all Möbius graphs consisting of e edges. For each j-valent vertex of Γ, let us assign $\langle e^*_{i_1} \cdots e^*_{i_j}\rangle$, where the cyclic order of the product is determined by the cyclic order of the vertex. For every edge we assign a propagator $\langle e_{i_k} e_{h_\ell}\rangle$ if the edge is straight and $\langle e_{i_k} e^*_{h_\ell}\rangle$ if it is twisted. The incidence relation should be consistent with the labeling of half-edges, namely, the edge labeled with $i_k h_\ell$ connects the half-edge labeled by i_k of a vertex to the half-edge labeled by h_ℓ from another vertex. Let $A^{\mathbb{R}}_\Gamma$ denote the sum with respect to all indices of the product of all contributions from vertices and edges. Then*

$$
\begin{aligned}
&\sum_{\substack{(v_1,v_2,v_3,\cdots)\in\mathbb{N}^\infty \\ \mathbf{e}(v_1,v_2,v_3,\cdots)=2e}} \int_{\mathcal{H}_A} e^{-\frac{1}{4}\langle x^2\rangle} \prod_{j\geq 1}^{\text{finite}} \frac{\langle x^j\rangle^{v_j}}{v_j!\cdot(2j)^{v_j}} d\mu(x) \\
&= \sum_{\substack{(v_1,v_2,v_3,\cdots)\in\mathbb{N}^\infty \\ \mathbf{e}(v_1,v_2,v_3,\cdots)=2e}} \prod_{j\geq 1}^{\text{finite}} \frac{1}{v_j!\cdot(2j)^{v_j}} \left\langle \left(\frac{\partial}{\partial y}\right)^j\right\rangle^{v_j} e^{\frac{1}{4}\langle (y+y^*)^2\rangle}\Bigg|_{y=0} \\
&= \sum_{\Gamma\in MG(e)} \frac{1}{|\mathrm{Aut}_M(\Gamma)|} A^{\mathbb{R}}_\Gamma ,
\end{aligned}
\tag{3.9}
$$

where $\mathrm{Aut}_M(\Gamma)$ is the automorphism group of Γ as a Möbius graph.

If Γ is orientable, then a series of vertex flip operations makes Γ a ribbon graph, and for such a graph, $A^{\mathbb{R}}_\Gamma = A^{or}_\Gamma$. Although the von Neumann algebra A is real, we can use the same definition of A^{or}_Γ as in Proposition 2.4 for a real A. Its invariance with respect to the topological type of the orientable surface S_Γ is the same as before. Even a Möbius graph Γ is non-orientable, we still have the following:

Lemma 3.4. *Let Γ be a connected Möbius graph with two or more vertices, and E an edge of Γ incident to two distinct vertices. Then the contribution of the graph $A^{\mathbb{R}}_\Gamma$ is invariant under the edge-contraction:*

$$A^{\mathbb{R}}_\Gamma = A^{\mathbb{R}}_{\Gamma/E} . \tag{3.10}$$

Proof. Let V_1 and V_2 be the two vertices incident to the edge E. If E is not twisted, then the same argument of Lemma 2.5 applies. If the edge is twisted, then first apply a vertex flip operation to V_2 and untwist E. Then the situation is the same as before, and we can contract the edge, joining V_1 and V_2 together. We give the cyclic oder of V_1 as the new cyclic order to the newly created vertex. □

It is known [23] that the set of all connected Möbius graphs with f faces drawn on a closed non-orientable surface of cross-cap genus k is *connected* with respect to the edge contraction and edge expansion moves. (These moves are called *diagonal flips* in [23].) Therefore, we can compute the invariant $A^{\mathbb{R}}_\Gamma$ for a non-orientable Möbius graph again by choosing our favorite graph. If we use a graph of Figure 3.2, then for every non-orientable

Möbius graph of topological type (k, f), the graph contribution is equal to

$$A^{nor}_{k,f} = \sum_{\substack{i_1,\ldots,i_k \\ h_1,\ldots,h_{f-1}}} \left\langle (e^*_{i_1})^2 \cdots (e^*_{i_k})^2 \cdot e_{h_1} e^*_{h_1} \cdots e_{h_{f-1}} e^*_{h_{f-1}} \right\rangle . \tag{3.11}$$

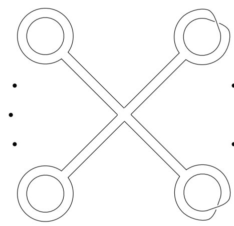

FIGURE 3.2. A standard graph for a non-orientable surface of topological type (k, f). It has $f-1$ tadpoles on the left, and k twisted tadpoles on the right.

Now we have

Theorem 3.5. *The graphical expansion of the logarithm of the generating function $Z^{\mathbb{R}}_A(t)$ of (3.3) associated with a real von Neumann algebra A is given by*

$$\begin{aligned} \log Z^{\mathbb{R}}_A(t) &= \log \int_{\mathcal{H}_A} e^{-\frac{1}{4}\langle x^2\rangle} e^{\sum_j \frac{t_j}{2j}\langle x^j\rangle} d\mu(x) \\ &= \sum_{\substack{\Gamma \textit{ connected orientable} \\ \textit{Möbius graph}}} \frac{1}{|\mathrm{Aut}_M(\Gamma)|} A^{or}_{g(\Gamma),f(\Gamma)} \prod_j t_j^{v_j(\Gamma)} \\ &\quad + \sum_{\substack{\Gamma \textit{ connected non-} \\ \textit{orientable Möbius graph}}} \frac{1}{|\mathrm{Aut}_M(\Gamma)|} A^{nor}_{k(\Gamma),f(\Gamma)} \prod_j t_j^{v_j(\Gamma)} . \end{aligned} \tag{3.12}$$

For future convenience, we also record

$$\begin{aligned} &\log \int_{\mathcal{H}_A} e^{-\frac{\alpha}{4}\langle x^2\rangle} e^{\beta \sum_j \frac{t_j}{2j}\langle x^j\rangle} d\mu(x) \\ &= \sum_{\substack{\Gamma \text{ connected orientable} \\ \text{Möbius graph}}} \frac{\alpha^{-e(\Gamma)}\beta^{v(\Gamma)}}{|\mathrm{Aut}_M(\Gamma)|} A^{or}_{g(\Gamma),f(\Gamma)} \prod_j t_j^{v_j(\Gamma)} \\ &\quad + \sum_{\substack{\Gamma \text{ connected non-} \\ \text{orientable Möbius graph}}} \frac{\alpha^{-e(\Gamma)}\beta^{v(\Gamma)}}{|\mathrm{Aut}_M(\Gamma)|} A^{nor}_{k(\Gamma),f(\Gamma)} \prod_j t_j^{v_j(\Gamma)} \end{aligned} \tag{3.13}$$

for a positive α and any β.

As an example of these formulas, let us consider the case when A is a simple algebra. This time, it is isomorphic to either $M(n, \mathbb{R})$ or $M(n, \mathbb{H})$.

Example 3.1. Let $A = M(n, \mathbb{R})$. Then as in Example 2.1, we can use $\{\sqrt{n} e_{ij}\}$ as our orthonormal basis. The space $\mathcal{H}_A$ of self-adjoint elements is the set of all real symmetric matrices $\mathcal{H}_{n,\mathbb{R}}$. Since elementary matrices are defined over the reals, we immediately see

$$A^{or}_{g,f} = n^{-v+e+f}$$

as before. To calculate $A^{nor}_{k,f}$, we note that

$$(e_{i_1 j_1})^2 \cdots (e_{i_k j_k})^2 = 0$$

unless all $2k$ indices are the same, and if they are the same, then the result is e_{ii}. Thus the sum of all these products is the identity matrix I. The contribution from the tadpoles of Figure 3.2 is the same as in Example 2.1, so we have

$$\log \int_{\mathcal{H}_{n,\mathbb{R}}} e^{-\frac{1}{4}\mathrm{tr}(X^2)} e^{\sum_j \frac{t_j}{2j}\mathrm{tr}(X^j)} d\mu(X) = \sum_{\substack{\Gamma \text{ connected} \\ \text{Möbius graph}}} \frac{1}{|\mathrm{Aut}_M(\Gamma)|}\, n^{f(\Gamma)} \prod_j t_j^{v_j(\Gamma)} \,, \tag{3.14}$$

or more conveniently,

$$\log \int_{\mathcal{H}_{n,\mathbb{R}}} e^{-\frac{n}{4}\mathrm{tr}(X^2)} e^{n\sum_j \frac{t_j}{2j}\mathrm{tr}(X^j)} d\mu(X) = \sum_{\substack{\Gamma \text{ connected} \\ \text{Möbius graph}}} \frac{1}{|\mathrm{Aut}_M(\Gamma)|}\, n^{\chi(S_\Gamma)} \prod_j t_j^{v_j(\Gamma)} \,. \tag{3.15}$$

These formulas are well known (see for example, [9, 17]).

Example 3.2. This time let us choose $A = M(n, \mathbb{H})$. As a real basis for quaternions, we use

$$e^0 = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \quad e^1 = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}, \quad e^2 = \begin{pmatrix} i & \\ & -i \end{pmatrix}, \quad e^3 = \begin{pmatrix} & -i \\ -i & \end{pmatrix}. \tag{3.16}$$

The adjoint operation $(e^\mu)^*$ on these 2×2 matrices is the same as the conjugate transposition $(e^\mu)^\dagger$. A real basis for $M(n,\mathbb{H})$ is given by

$$e_{ij}^\nu = e_{ij} \otimes e^\nu, \qquad i,j = 1,\dots,n; \quad \nu = 0,\dots,3\,.$$

The normalized trace is defined by

$$\langle e_{ij}^\nu \rangle = \frac{1}{2n}\,\mathrm{tr}_{n\times n}(e_{ij}) \cdot \mathrm{tr}_{2\times 2}(e^\nu)$$

for the basis elements and $\mathbb{R}$-linearly extended to all matrices. We notice that $\langle\ \rangle$ is real valued because the 2×2 trace has value 0 for imaginary quaternionic units. With respect to the normalized trace, $\{\sqrt{n} e_{ij}^\nu\}$ is an orthonormal basis. The $*$-operation with respect the basis is given by

$$(e_{ij}^\nu)^* = e_{ji} \otimes (e^\nu)^\dagger\,.$$

The space of self-adjoint elements $\mathcal{H}_A = \mathcal{H}_{n,\mathbb{H}}$ consists of self-adjoint quatermionic matrices of size $n \times n$, and is spanned by $e_{ij} \otimes e^\nu + e_{ji} \otimes (e^\nu)^\dagger$. Note that the diagonal entries of a self-adjoint matrix are spanned by $e^\nu + (e^\nu)^\dagger$, and hence are real. Thus we have a real linear map

$$\mathrm{tr}_{n\times n} : \mathcal{H}_{n,\mathbb{H}} \longrightarrow \mathbb{R}\,.$$

Since $e^\nu (e^\nu)^\dagger = e^0$, we have

$$\sum_{i,j,\nu} e_{ij}^\nu (e_{ij}^\nu)^* = 4nI \otimes e^0\,.$$

Similarly, $e^0 = (e^0)^2 = -(e^1)^2 = -(e^2)^2 = -(e^3)^2$, hence

$$\sum_{i,j,\nu} (e_{ij}^\nu)^2 = -2I \otimes e^0\,.$$

To compute $e_{ij}^\mu e_{k\ell}^\nu (e_{ij}^\mu)^* (e_{k\ell}^\nu)^*$, we use

$$e^\mu e^\nu (e^\mu)^\dagger (e^\nu)^\dagger = \begin{cases} -e^0 & \mu,\nu > 0,\ \mu \neq \nu \\ e^0 & \text{otherwise}\,. \end{cases} \tag{3.17}$$

Therefore, of the 16 combinations, 6 cases are equal to $-e^0$ and 10 are equal to e^0. Thus

$$\sum_{\mu,\nu} e^\mu e^\nu (e^\mu)^\dagger (e^\nu)^\dagger = 4e^0 ,$$

and altogether, we have

$$\sum_{i,j,k,\ell,\mu,\nu} e^\mu_{ij} e^\nu_{k\ell} (e^\mu_{ij})^* (e^\nu_{k\ell})^* = 4I \otimes e^0 .$$

From all the above, we calculate

$$M(n,\mathbb{H})^{or}_{g,f} = (4n^2)^g (4n^2)^{f-1} \langle I \otimes e^0 \rangle = (2n)^{-v+e+f}$$
$$M(n,\mathbb{H})^{nor}_{k,f} = (-2n)^k (4n^2)^{f-1} \langle I \otimes e^0 \rangle = (-1)^k (2n)^{-v+e+f} .$$

Note that $(-1)^k = (-1)^{\chi(S_\Gamma)}$. Combining these computations with (3.13) and using the $n \times n$ trace of $M(n,\mathbb{H})$, we finally obtain

$$\log \int_{\mathcal{H}_{n,\mathbb{H}}} e^{-\frac{1}{2}\mathrm{tr}(X^2)} e^{\sum_j \frac{t_j}{j}\mathrm{tr}(X^j)} d\mu(X) = \sum_{\substack{\Gamma \text{ connected} \\ \text{Möbius graph}}} \frac{(-1)^{\chi(S_\Gamma)}}{|\mathrm{Aut}_M(\Gamma)|} (2n)^{f(\Gamma)} \prod_j t_j^{v_j(\Gamma)} , \tag{3.18}$$

or equivalently,

$$\log \int_{\mathcal{H}_{n,\mathbb{H}}} e^{-n\mathrm{tr}(X^2)} e^{2n\sum_j \frac{t_j}{j}\mathrm{tr}(X^j)} d\mu(X) = \sum_{\substack{\Gamma \text{ connected} \\ \text{Möbius graph}}} \frac{1}{|\mathrm{Aut}_M(\Gamma)|} (-2n)^{\chi(S_\Gamma)} \prod_j t_j^{v_j(\Gamma)} . \tag{3.19}$$

These results are in agreement with recently established formulas found in [38].

4. Generating functions for the number of representations of surface groups

Let us now turn our attention to the case of the complex group algebra $A = \mathbb{C}[G]$ of a finite group G. The $*$-operation is defined by

$$* : \mathbb{C}[G] \ni x = \sum_{w\in G} x(w)\cdot w \longmapsto x^* = \sum_{w\in G} \overline{x(w)}\cdot w^{-1} \in \mathbb{C}[G]. \tag{4.1}$$

As the trace, we use

$$\langle\ \rangle = \frac{1}{|G|}\chi_{\mathrm{reg}}, \tag{4.2}$$

where χ_{reg} is the character of the regular representation of G on $\mathbb{C}[G]$, linearly extended to the whole group algebra. The self-adjoint condition $x^* = x$ means $x(w^{-1}) = \overline{x(w)}$, and we have $\mathcal{H}_{\mathbb{C}[G]} = \mathbb{R}^{|G|}$ as a real vector space. A natural orthonormal basis for $\mathbb{C}[G]$ is the group G itself, since we have

$$\langle uv^* \rangle = \frac{1}{|G|}\chi_{\mathrm{reg}}(uv^{-1}) = \begin{cases} 1 & u = v \\ 0 & \text{otherwise} . \end{cases}$$

It is because the normalized trace on G takes value 1 only when the group element is the identity and 0 otherwise. Recall that

$$\pi_1(S) = \left\langle a_1, b_1, \dots, a_g, b_g \;\middle|\; a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} = 1 \right\rangle ,$$

where S is an orientable surface of genus g. With respect the orthonormal basis, we immediately see

$$\begin{aligned}\mathbb{C}[G]^{or}_{g,f} &= \sum_{u_i,v_i,w_j\in G} \left\langle u_1v_1u_1^{-1}v_1^{-1}\cdots u_gv_gu_g^{-1}v_g^{-1}\cdot w_1w_1^{-1}\cdots w_{f-1}w_{f-1}^{-1}\right\rangle \\ &= |G|^{f-1}\,|\mathrm{Hom}(\pi_1(S),G)|\ .\end{aligned} \tag{4.3}$$

Summarizing these facts and changing the constant factors using (2.26), we obtain a generating function of the number of homomorphisms from the fundamental group of an orientable surface into the finite group.

Theorem 4.1. *Let G be a finite group. The following integral over the self-adjoint elements of the complex group algebra $\mathbb{C}[G]$ gives the generating function for the cardinality of the representation variety of an orientable surface group in G:*

$$\begin{aligned}&\log\int_{\mathcal{H}_{\mathbb{C}[G]}} \exp\left(-\frac{1}{2}\,\chi_{\mathrm{reg}}(x^2)\right)\exp\left(\sum_j \frac{t_j}{j}\chi_{\mathrm{reg}}(x^j)\right)d\mu(x) \\ &\qquad = \sum_{\substack{\Gamma\ \text{connected}\\ \text{ribbon graph}}} \frac{1}{|\mathrm{Aut}_R\Gamma|}|G|^{\chi(S_\Gamma)-1}|\mathrm{Hom}(\pi_1(S_\Gamma),G)|\prod_j t_j^{v_j(\Gamma)}\ .\end{aligned} \tag{4.4}$$

Note that we have a von Neumann algebra isomorphism

$$\mathbb{C}[G]\cong\bigoplus_{\lambda\in\hat{G}}\mathrm{End}(V_\lambda)\ , \tag{4.5}$$

which decomposes the character of the regular representation into the sum of irreducible characters:

$$\chi_{\mathrm{reg}} = \sum_{\lambda\in\hat{G}}(\dim\lambda)\ \chi_\lambda = \sum_{\lambda\in\hat{G}}(\dim\lambda)\ \mathrm{tr}_\lambda\ ,$$

where $\dim\lambda$ is the dimension of $\lambda\in\hat{G}$ and χ_λ is its character. Therefore, using (2.29) for each irreducible factor, we calculate

$$\begin{aligned}&\log\int_{\mathcal{H}_{\mathbb{C}[G]}} \exp\left(-\frac{1}{2}\,\chi_{\mathrm{reg}}(x^2)\right)\exp\left(\sum_j \frac{t_j}{j}\chi_{\mathrm{reg}}(x^j)\right)d\mu(x) \\ &= \log\int_{\mathcal{H}_{\mathbb{C}[G]}}\prod_{\lambda\in\hat{G}}\exp\left(-\frac{\dim\lambda}{2}\,\mathrm{tr}_\lambda(x^2)\right)\exp\left(\dim\lambda\sum_j\frac{t_j}{j}\mathrm{tr}_\lambda(x^j)\right)d\mu_\lambda(x) \\ &= \sum_{\lambda\in\hat{G}}\log\int_{\mathcal{H}_{\dim\lambda,\mathbb{C}}}\exp\left(-\frac{\dim\lambda}{2}\,\mathrm{tr}_\lambda(x^2)\right)\exp\left(\dim\lambda\sum_j\frac{t_j}{j}\mathrm{tr}_\lambda(x^j)\right)d\mu_\lambda(x) \\ &= \sum_{\substack{\Gamma\ \text{connected}\\ \text{ribbon graph}}}\frac{1}{|\mathrm{Aut}_R\Gamma|}\sum_{\lambda\in\hat{G}}(\dim\lambda)^{\chi(S_\Gamma)}\prod_j t_j^{v_j(\Gamma)}\ ,\end{aligned} \tag{4.6}$$

where $d\mu_\lambda$ is the normalized Lebesgue measure on the space of $\dim\lambda\times\dim\lambda$ hermitian matrices. Comparing the two expressions (4.4) and (4.6), we recover Mednykh's formula (0.7):

$$\sum_{\lambda\in\hat{G}}(\dim\lambda)^{\chi(S)} = |G|^{\chi(S)-1}|\mathrm{Hom}(\pi_1(S),G)|\ .$$

Remark. Another proof of Mednykh's formula is found in [13], which uses Chern-Simons gauge theory with a finite gauge group. Burnside asked a related question on p. 319 (§ 238, Ex. 7) of his textbook [10]. The formula for genus 1 case is found in Frobenius [14] of 1896. We refer to [48] for the relation of these formulas to combinatorics. An excellent historical account on this and Frobenius-Schur formula (0.9) is found in [26].

Now consider the real group algebra $\mathbb{R}[G]$. For a non-orientable surface of cross-cap genus k, we know

$$\pi_1(S) = \left\langle a_1, \dots, a_k \;\middle|\; a_1^2 \cdots a_k^2 = 1 \right\rangle .$$

Therefore,

$$\begin{aligned} \mathbb{R}[G]^{nor}_{k,f} &= \sum_{u_i, w_j \in G} \left\langle u_1^2 \cdots u_k^2 \cdot w_1^{-1} w_1 \cdots w_{f-1}^{-1} w_{f-1} \right\rangle \\ &= |G|^{f-1} \left|\mathrm{Hom}(\pi_1(S), G)\right| . \end{aligned} \tag{4.7}$$

Our general formula (3.13) yields

Theorem 4.2. *Let G be a finite group. The following integral over the space of self-adjoint elements of the real group algebra $\mathbb{R}[G]$ gives the generating function for the number of homomorphisms from the fundamental group of a closed surface into G, $|\mathrm{Hom}(\pi_1(S), G)|$, for all S, including orientble and non-orientable surfaces.*

$$\begin{aligned} &\log \int_{\mathcal{H}_{\mathbb{R}[G]}} e^{-\frac{1}{4}\chi_{\mathrm{reg}}(x^2)} e^{\frac{1}{2}\sum_j \frac{t_j}{j}\chi_{\mathrm{reg}}(x^j)} d\mu(x) \\ &\qquad = \sum_{\substack{\Gamma \text{ connected} \\ \text{Möbius graph}}} \frac{1}{|\mathrm{Aut}_M \Gamma|} |G|^{\chi(S_\Gamma)-1} |\mathrm{Hom}(\pi_1(S_\Gamma), G)| \prod_j t_j^{v_j(\Gamma)} . \end{aligned} \tag{4.8}$$

Recall that the real group algebra $\mathbb{R}[G]$ decomposes into simple factors according to the three types of irreducible representations (0.8). Notice that $\hat{G}_1$ consists of complex irreducible representations of G that are defined over $\mathbb{R}$. A representation in $\hat{G}_2$ is not defined over $\mathbb{R}$, and its character is not real-valued. Thus the complex conjugation acts on the set $\hat{G}_2$ without fixed points. Let $\hat{G}_{2+}$ denote a half of $\hat{G}_2$ such that

$$\hat{G}_{2+} \cup \overline{\hat{G}_{2+}} = \hat{G}_2 . \tag{4.9}$$

A complex irreducible representation of G that belongs to $\hat{G}_4$ admits a skew-symmetric bilinear form. In particular, its dimension (over $\mathbb{C}$) is even. Now we have a von Neumann algebra isomorphism

$$\mathbb{R}[G] \cong \bigoplus_{\lambda \in \hat{G}_1} \mathrm{End}_{\mathbb{R}}(\lambda^{\mathbb{R}}) \oplus \bigoplus_{\lambda \in \hat{G}_{2+}} \mathrm{End}_{\mathbb{C}}(\lambda) \oplus \bigoplus_{\lambda \in \hat{G}_4} \mathrm{End}_{\mathbb{H}}(\lambda^{\mathbb{H}}) , \tag{4.10}$$

where $\lambda^{\mathbb{R}}$ is a real irreducible representation of G that satisfies $\lambda = \lambda^{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$. The representation space $\lambda^{\mathbb{H}}$ is a $(\dim \lambda)/2$-dimensional vector space defined over $\mathbb{H}$ for $\lambda \in \hat{G}_4$ such that its image under the natural injection

$$\mathrm{End}_{\mathbb{H}}(\lambda^{\mathbb{H}}) \longrightarrow \mathrm{End}_{\mathbb{C}}(\lambda) \tag{4.11}$$

coincides with the image of

$$\rho_\lambda : \mathbb{R}[G] \longrightarrow \mathrm{End}_{\mathbb{C}}(\lambda) ,$$

where ρ_λ is the representation of $\mathbb{R}[G]$ corresponding to $\lambda \in \hat{G}$. The injective algebra homomorphism (4.11) is defined by the 2×2 matrix representation of the quaternions

(3.16). The algebra isomorphism (4.10) gives a formula for the character of the regular representation on $\mathbb{R}[G]$:

$$\chi_{\mathrm{reg}} = \sum_{\lambda\in\hat{G}_1} (\dim\lambda)\,\chi_\lambda + \sum_{\lambda\in\hat{G}_{2+}} (\dim\lambda)\,(\chi_\lambda + \overline{\chi_\lambda}) + \sum_{\lambda\in\hat{G}_4} 2(\dim\lambda)\cdot \mathrm{trace}_{\lambda^{\mathbb{H}}}\,, \tag{4.12}$$

where in the last term the character is given as the trace of quaternionic $(\dim\lambda)/2\times(\dim\lambda)/2$ matrices. Notice that if $\lambda\in\hat{G}_2$, then for every $x=x^*\in\mathcal{H}_{\mathbb{R}[G]}$, we have

$$\chi_\lambda(x) = \overline{\chi_\lambda}(x) = \mathrm{tr}_{\dim\lambda}(\rho_\lambda(x))$$

since $\rho_\lambda(x)$ is a hermitian matrix of size $\dim\lambda\times\dim\lambda$.

The integration (4.8) can be carried out using (2.29), (3.15) and (3.19) with the decomposition (4.10) and (4.12). The result is

$$\begin{aligned}
&\log\int_{\mathcal{H}_{\mathbb{R}[G]}} e^{-\frac14\chi_{\mathrm{reg}}(x^2)} e^{\frac12\sum_j \frac{t_j}{j}\chi_{\mathrm{reg}}(x^j)} d\mu(x)\\
&= \sum_{\lambda\in\hat{G}_1} \log\int_{\mathcal{H}_{\dim\lambda,\mathbb{R}}} e^{-\frac{\dim\lambda}{4}\mathrm{tr}(x^2)} e^{\dim\lambda\sum_j\frac{t_j}{2j}\mathrm{tr}(x^j)} d\mu_\lambda(x)\\
&\quad + \sum_{\lambda\in\hat{G}_2} \log\int_{\mathcal{H}_{\dim\lambda,\mathbb{C}}} e^{-\frac{\dim\lambda}{4}\mathrm{tr}(x^2)} e^{\dim\lambda\sum_j\frac{t_j}{2j}\mathrm{tr}(x^j)} d\mu_\lambda(x)\\
&\quad + \sum_{\lambda\in\hat{G}_4} \log\int_{\mathcal{H}_{\dim\lambda/2,\mathbb{H}}} e^{-\frac{\dim\lambda}{2}\mathrm{tr}(x^2)} e^{\dim\lambda\sum_j\frac{t_j}{j}\mathrm{tr}_{\dim\lambda/2}(x^j)} d\mu_\lambda(x)\\
&= \sum_{\substack{\Gamma\text{ connected}\\ \text{Möbius graph}}} \frac{1}{|\mathrm{Aut}_M(\Gamma)|}\sum_{\lambda\in\hat{G}_1}(\dim\lambda)^{\chi(S_\Gamma)}\prod_j t_j^{v_j(\Gamma)}\\
&\quad + \sum_{\substack{\Gamma\text{ connected orientable}\\ \text{Möbius graph}}} \frac{1}{|\mathrm{Aut}_M(\Gamma)|}\sum_{\lambda\in\hat{G}_2}(\dim\lambda)^{\chi(S_\Gamma)}\prod_j t_j^{v_j(\Gamma)}\\
&\quad + \sum_{\substack{\Gamma\text{ connected}\\ \text{Möbius graph}}} \frac{1}{|\mathrm{Aut}_M(\Gamma)|}\sum_{\lambda\in\hat{G}_4}(-\dim\lambda)^{\chi(S_\Gamma)}\prod_j t_j^{v_j(\Gamma)}\\
&= \sum_{\substack{\Gamma\text{ connected orientable}\\ \text{Möbius graph}}} \frac{1}{|\mathrm{Aut}_M(\Gamma)|}\sum_{\lambda\in\hat{G}}(\dim\lambda)^{\chi(S_\Gamma)}\prod_j t_j^{v_j(\Gamma)}\\
&\quad + \sum_{\substack{\Gamma\text{ connected non-}\\ \text{orientable Möbius graph}}} \frac{1}{|\mathrm{Aut}_M(\Gamma)|}\sum_{\lambda\in\hat{G}_1}(\dim\lambda)^{\chi(S_\Gamma)}\prod_j t_j^{v_j(\Gamma)}\\
&\quad + \sum_{\substack{\Gamma\text{ connected non-}\\ \text{orientable Möbius graph}}} \frac{1}{|\mathrm{Aut}_M(\Gamma)|}\sum_{\lambda\in\hat{G}_4}(-\dim\lambda)^{\chi(S_\Gamma)}\prod_j t_j^{v_j(\Gamma)}\,.
\end{aligned} \tag{4.13}$$

Notice that the sum over orientable Möbius graphs recovers Mednykh's formula (0.7) again, because the Euler characteristic $\chi(S)$ is even for an orientable surface. From the sum over non-orientable Möbius graphs, we obtain the formula of Frobenius-Schur (0.9) of [15]:

$$\sum_{\lambda\in\hat{G}_1}(\dim\lambda)^{\chi(S)} + \sum_{\lambda\in\hat{G}_4}(-\dim\lambda)^{\chi(S)} = |G|^{\chi(S)-1}|\mathrm{Hom}(\pi_1(S),G)|\,.$$

Note that the $\hat{G}_2$ component has no contribution in this formula. This is due to the fact that graphical expansion of a complex hermitian matrix integral contains only orientable ribbon graphs.

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Department of Mathematics, University of California, Davis, CA 95616–8633
E-mail address: mulase@math.ucdavis.edu

Department of Mathematics, University of California, Berkeley, CA 94720-3840
E-mail address: jyu@math.berkeley.edu