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Lipschitz Continuity of inf-Projections*

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Dedicated to: This paper is dedicated to my long time friend and colleague Elijah (Lucien) Polak who has from time-to-time worried about inf-projection being locally Lipschitz continuous.

Abstract. It is shown that local epi-sub-Lipschitz continuity of the function-valued mapping associated with a perturbed optimization problem yields the local Lipschitz continuity of the inf-projections (= marginal functions, = infimal functions). The use of the theorem is illustrated by considering perturbed nonlinear optimization problems with linear constraints.

Keywords: set-valued mapping, function-valued mapping, Lipschitz continuity, sub-Lipschitz continuity, marginal function, infimal function

1. Introduction

Let's consider the following mathematical programming problem:

$$\begin{aligned} \min \quad & f_0(x) \\ \text{so that} \quad & f_i(x) \le 0, \quad i = 1, \dots, s, \\ & f_i(x) = 0, \quad i = s+1, \dots, m, \\ & x \in C \subset \mathbb{R}^n. \end{aligned}$$

In order to study the stability of the solution(s) and of the optimal value of such a problem, one relies usually on the embedding of this problem in a family of perturbed mathematical programming problems:

$$\begin{aligned} \min \quad & f_0(u, x) \\ \text{so that} \quad & f_i(u, x) \le 0, \quad i = 1, \dots, s, \\ & f_i(u, x) = 0, \quad i = s+1, \dots, m, \\ & x \in C \subset \mathbb{R}^n. \end{aligned}$$

where $u \in U \subset \mathbb{R}^m$ is the parameter inducing the perturbations, and such that with $u = 0$ one recovers the originally stated problem.

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For our purposes, it will be convenient to identify optimization problems with extended real-valued functions and analyze the dependence of these functions on perturbations. So, our given problem becomes

$$\min \quad f(x), x \in \mathbb{R}^n$$

where

$$f(x) = \begin{cases} f_0(x) & \text{if } f_i(x) \leq 0, i = 1, \ldots, s, \; f_i(x) = 0, i = s+1, \ldots, m, x \in C, \\ \infty & \text{otherwise.} \end{cases}$$

And, the family of perturbed problems becomes

$$\{\min f(u, x), x \in \mathbb{R}^n \mid u \in U\}$$

where

$$f(u, x) = \begin{cases} f_0(u, x) & \text{if } f_i(u, x) \leq 0, i = 1, \ldots, s, \; f_i(u, x) = 0, \\ & \quad i = s+1, \ldots, m, x \in C, \\ \infty & \text{otherwise.} \end{cases}$$

Let

$$p(u) = \inf_{x \in \mathbb{R}^n} f(x, u), \qquad P(u) = \operatorname*{arg\,min}_{x \in \mathbb{R}^n} f(x, u).$$

One refers to p as the *inf-projection* of the bivariate function f and to P as the *argmin-mapping* (associated with f). In general, p is an extended real-valued function, i.e., $p : \mathbb{R}^m \to \bar{\mathbb{R}}$, with

- $p(0)$ the optimal value of the given problem,
- $\operatorname{dom} p = \{u \in U \mid \inf_x f(u, x) < \infty\}$, its *effective domain*, the subset of perturbations $u \in U$ for which the corresponding mathematical programming problems are feasible,
- $p(u) = -\infty$ if the corresponding mathematical program is unbounded.

Also, $P : U \rightrightarrows \mathbb{R}^n$, is a set-valued mapping with

- $P(0)$ the set, possibly empty, of optimal solutions of the given problem,
- $\operatorname{dom} P = \{u \in U \mid \operatorname{arg\,min}_x f(u, x) \neq \emptyset\}$, its *effective domain*, the subset of perturbations $u \in U$ for which the infimum of the corresponding mathematical programming problems are actually attained.

Stability issues have to do with the continuity properties of the function p and the set-valued mapping P in a neighborhood of 0 relative to U. In view of the fact that one will always restrict the attention to dom p or dom P, one may as well work with $U = \mathbb{R}^m$, and this will be our framework henceforth. Section 2 provides a quick overview of what's known about the continuity of p and P and lays down the tools that will be needed to obtain local Lipschitz continuity of p in Section 3. In Section 4, we apply the results of Section 3 to analyze nonlinear programs with linear constraints.

2. The variational setting

The material of this section is from the book on 'Variational Analysis' [2]. It provides a brief introduction to set convergence, epi-convergence and culminates in a result about the continuity of inf-projections.

Convergence of a sequence of sets is usually described in terms of the inner and outer limits. In order to handle statements about sequences and subsequences, it's always convenient to work notation (involving subsets of $\mathbb{N}$):

$$\begin{aligned}\mathcal{N}_\infty &:= \{N \subset \mathbb{N} \mid \mathbb{N}\backslash N \text{ finite}\}\\ &= \{\text{subsequences of } \mathbb{N} \text{ containing all } \nu \text{ beyond some } \bar{\nu}\},\\ \mathcal{N}_\infty^\# &:= \{N \subset \mathbb{N} \mid N \text{ infinite}\} = \{\text{all subsequences of } \mathbb{N}\}.\end{aligned}$$

$\mathcal{N}_\infty$ is called the Fréchet filter on $\mathbb{N}$ and $\mathcal{N}^\#$ is the associated 'grill'.

Definition 2.1 (inner and outer limits). For a sequence $\{C^\nu\}_{\nu\in\mathbb{N}}$ of subsets of $\mathbb{R}^n$, the outer limit is the set

$$\limsup_{\nu\to\infty} C^\nu := \left\{x \mid \exists N \in \mathcal{N}_\infty^\#, \exists x^\nu \in C^\nu(\nu \in N) \text{ with } x^\nu \underset{N}{\to} x\right\}$$

while the inner limit is the set

$$\liminf_{\nu\to\infty} C^\nu := \left\{x \mid \exists N \in \mathcal{N}_\infty, \exists x^\nu \in C^\nu(\nu \in N) \text{ with } x^\nu \underset{N}{\to} x\right\}.$$

The limit of the sequence exists if the outer and inner limit sets are equal:

$$\lim_{\nu\to\infty} C^\nu := \limsup_{\nu\to\infty} C^\nu = \liminf_{\nu\to\infty} C^\nu.$$

Set convergence can also be characterized in terms of a metric $\boldsymbol{d}$ defined on the hyperspace cl-sets$_{\neq\emptyset}(\mathbb{R}^n)$, the space of all nonempty, closed subsets of $\mathbb{R}^n$ [1]. To define this metric, for every $\rho \in \mathbb{R}_+ = [0,\infty)$ and pair of nonempty sets C and D, let

$$\begin{aligned}\boldsymbol{d}_\rho(C,D) &:= \max_{|x|\le\rho} |d_C(x) - d_D(x)|,\\ \hat{\boldsymbol{d}}_\rho(C,D) &:= \inf\{\eta \ge 0 \mid C\cap\rho\mathbb{B} \subset D+\eta\mathbb{B},\ D\cap\rho\mathbb{B}\subset C+\eta\mathbb{B}\},\end{aligned}$$

where in particular $\boldsymbol{d}_0(C, D) = |d_C(0) - d_D(0)|$. Let

$$\boldsymbol{d}(C, D) := \int_0^\infty \boldsymbol{d}_\rho(C, D) e^{-\rho} d\rho,$$

to which one refers as the (*integrated*) *set distance*.

Theorem 2.2 (*quantification of set convergence*, [2, Theorems 4.36 and 4.42]). *For each $\rho \geq 0$, $\boldsymbol{d}_\rho$ is a pseudo-metric on the space* cl-sets$_{\neq\emptyset}(\mathbb{R}^n)$, *but $\hat{\boldsymbol{d}}_\rho$ is not. Both families $\{\boldsymbol{d}_\rho\}_{\rho\geq 0}$ and $\{\hat{\boldsymbol{d}}_\rho\}_{\rho\geq 0}$ characterize set convergence: for any $\bar{\rho} \in \mathbb{R}_+$, one has*

$$\begin{aligned} C = \lim_{\nu\to\infty} C^\nu &\Leftrightarrow \boldsymbol{d}_\rho(C^\nu, C) \to 0 \quad \text{for all } \rho \geq \bar{\rho} \\ &\Leftrightarrow \hat{\boldsymbol{d}}_\rho(C^\nu, C) \to 0 \quad \text{for all } \rho \geq \bar{\rho}. \end{aligned}$$

Moreover, $C = \lim_{\nu\to\infty} C^\nu \Leftrightarrow \boldsymbol{d}(C^\nu, C) \to 0$.

The better known Pompeiu-Hausdorff distance $\boldsymbol{d}_\infty(C, D) := \sup_{x\in\mathbb{R}^n} |d_C(x) - d_D(x)|$ doesn't characterize set convergence. As is clear from its definition and the preceding theorem, $\boldsymbol{d}_\infty(C^\nu, C) \to 0$ induces a more restrictive notion of convergence on cl-sets$_{\neq\emptyset}(\mathbb{R}^n)$.

Definition 2.3 (continuous mapping). A set-valued mapping $S : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is continuous at $\bar{u}$ if $S(\bar{u}) = \lim_{u\to\bar{u}} S(u)$, or equivalently, in view of separability, if for all sequences $u^\nu \to \bar{u}$, $S(\bar{u}) = \lim_\nu S(u^\nu)$. The mapping S is said to be continuous relative to a set $U \subset \mathbb{R}^m$ if for all $\bar{u} \in U$, one has $S(\bar{u}) = \lim_\nu S(u^\nu)$ for all sequence $u^\nu \underset{U}{\to} \bar{u}$.

It follows immediately from Theorem 2.2:

Proposition 2.4 (*continuity with respect to the set distance metric*). *A set-valued mapping $S : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is continuous at $\bar{u}$ if and only it for all $\rho \in \mathbb{R}_+$*:

$$\lim_{u\to\bar{u}} \boldsymbol{d}_\rho(S(u), S(\bar{u})) \to 0.$$

Definition 2.5 (locally bounded mappings). A set-valued mapping $S : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is locally bounded at $\bar{u}$ if for some neighborhood $U \in \mathcal{N}(\bar{u})$, the set $S(U) \subset \mathbb{R}^n$ is bounded. It's locally bounded if this holds for every $\bar{u} \in \mathbb{R}^m$.

Proposition 2.6 (*bounded images*, [2, Proposition 5.15]). *A mapping $S : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is locally bounded if and only if $S(B)$ is bounded for every bounded set $B \subset \mathbb{R}^m$.*

Let's now introduce the notion of a *function-valued mapping* from a space U to the space

fcns(X) := collection of all extended-real-valued functions on X.

Such a mapping assigns to each $u \in U$, a function defined on X that has values in $\bar{\mathbb{R}}$. There is a one-to-one correspondence between such mappings $u \mapsto f(u, \cdot) : U \to \mathrm{fcns}(X)$ and bivariate functions $(u, x) \mapsto f(u, x) : U \times X \to \bar{\mathbb{R}}$; $f(u, \cdot)$ denotes the function that assigns to x the value $f(u, x)$. This is the framework we shall adapt to study the dependence of p and P on perturbations. The function-valued mapping viewpoint has the advantage of bringing out the 'dynamic' qualities of the dependence. Continuity of such mappings comes in many flavors. We shall be interested in epi-continuity.

Definition 2.7 (epi-convergence). A sequence $\{f^\nu\}_{\nu\in\mathbb{N}}$ of extended real-valued functions defined on $\mathbb{R}^n$ epi-converges to a function $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ if their epigraphs $\{\mathrm{epi}\, f^\nu \subset \mathbb{R}^{n+1}\}_{\nu\in\mathbb{N}}$ converge, as subsets of $\mathbb{R}^{n+1}$, to epi f. One then writes $f^\nu \xrightarrow{e} f$. Thus,

$$f^\nu \xrightarrow{e} f \Leftrightarrow \mathrm{epi}\, f^\nu \to \mathrm{epi}\, f.$$

Definition 2.8 (epi-continuity of function-valued mappings). For $f : \mathbb{R}^m \times \mathbb{R}^n \to \bar{\mathbb{R}}$, the function-valued mapping $u \mapsto f(u, \cdot)$ is epi-continuous at $\bar{u}$ if

$$f(u, \cdot) \xrightarrow{e} f(\bar{u}, \cdot) \text{ as } u \to \bar{u};$$

i.e., the functions $f(u, \cdot)$ epi-converge to $f(\bar{u}, \cdot)$. This means that the set valued-mapping $S_f : \mathbb{R}^m \rightrightarrows \mathbb{R}^{n+1}$ is continuous at $\bar{u}$ where S_f is the epigraphical mapping associated with f; $S_f(u) = \mathrm{epi}\, f(u, \cdot)$.

Definition 2.9 (level boundedness). The function $f : \mathbb{R}^m \times \mathbb{R}^n \to \bar{\mathbb{R}}$ is said to be level bounded in x locally uniformly if, for each $\alpha \in \mathbb{R}$, the mapping

$$u \mapsto \mathrm{lev}_\alpha f(u, \cdot) = \{x \mid f(u, x) \le \alpha\}$$

is locally bounded.

Level boundedness (locally uniformly) guarantees the existence of solutions and epi-convergence guarantees the convergence of these solutions!

Theorem 2.10 (*continuity of inf-projections*, [2, Theorems 1.17 and 7.41])**.** *For $f : \mathbb{R}^m \times \mathbb{R}^n \to \bar{\mathbb{R}}$ proper, lsc, and such that $f(u, x)$ is level-bounded in x locally uniformly in u, let*

$$p(u) := \inf_x f(u, x), \quad P(u) := \arg\min_x f(u, x).$$

(a) *The mapping P, which is compact-valued with* dom P = dom p, *is osc with respect to p-attentive convergence $\underset{p}{\to}$. Furthermore, it is locally bounded relative to any set in $\mathbb{R}^m$ on which p is bounded from above.*
(b) *p is continuous relative to a set $U \subset \mathbb{R}^m$ when the function-valued mapping $u \mapsto f(u, \cdot)$ is epi-continuous at $\bar{u}$ relative to U. Another sufficient condition is that there exists $\bar{x} \in P(\bar{u})$ such that $f(\cdot, \bar{x})$ is continuous relative to U at $\bar{u}$.*

(c) *For P to be locally bounded and osc (not just in the p-attentive sense) relative to a set $U \subset \mathbb{R}^m$ at a point $\bar{u} \in U$ with $P(\bar{u}) \neq \emptyset$, it suffices to have p be continuous relative to U at $\bar{u}$.*

Our major objective is find out how far the hypotheses of this theorem need to be strengthened to obtain not just continuity but (local) Lipschitz continuity of inf-projections.

3. Lipschitz continuity

We now turn to Lipschitz continuity. The basic definitions come from [2, Chapter 9].

Definition 3.1 (Lipschitz and sub-Lipschitz continuity). A mapping $S : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is Lipschitz continuous on U, a subset of $\mathbb{R}^m$, if it is nonempty-closed-valued on U and there exists $\kappa \in \mathbb{R}_+$, a Lipschitz constant, such that

$$\boldsymbol{d}_\infty(S(x'), S(x)) \leq \kappa |x' - x| \quad \text{for all } x, x' \in U,$$

or in equivalent geometric form,

$$S(x') \subset S(x) + \kappa |x' - x| \mathbb{B} \quad \text{for all } x, x' \in U.$$

A mapping $S : \mathbb{R}^d \rightrightarrows \mathbb{R}^n$ is sub-Lipschitz continuous on $U \subset \mathbb{R}^n$ if it is nonempty-closed-valued on U for each $\rho \in \mathbb{R}_+$ there is a ρ-Lipschitz constant $\kappa \in \mathbb{R}_+$ such that

$$\boldsymbol{d}_\rho(S(u), S(u')) \leq \kappa |u' - u| \quad \forall u, u' \in U,$$

and hence also

$$S(u') \cap \rho\mathbb{B} \subset S(u) + \kappa |u' - u| \mathbb{B} \quad \forall u, u' \in U.$$

The mapping S is said to be locally sub-Lipschitz continuous if for every $\bar{u} \in \operatorname{dom} S$ there exists a neighborhood $U \in \mathcal{N}(\bar{u})$ such that S is sub-Lipschitz continuous on $U \cap \operatorname{dom} S$.

It's obvious from this definition and the characterization of continuity in terms of the ρ-set distances that sub-Lipschitz continuity relative to a subset $U \subset \operatorname{dom} S$ implies continuity relative to U. And, local sub-Lipschitz continuity actually implies continuity.

Example 3.2 (local sub-Lipschitz continuity). As can be expected the following continuous mapping, see figure 1: $S : \mathbb{R} \rightrightarrows \mathbb{R}$ with

$$S(u) = \begin{cases} [1, \infty) & \text{if } u \notin [-1, 1], \\ [(-2u - u^2)^{\frac{1}{2}}, \infty) & \text{if } u \in (-1, 0), \\ [(2u - u^2)^{\frac{1}{2}}, \infty) & \text{if } u \in [0, 1), \end{cases}$$

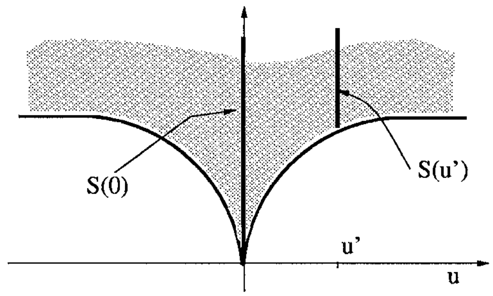

Figure 1. A mapping that isn't locally sub-Lipschitz continuous.

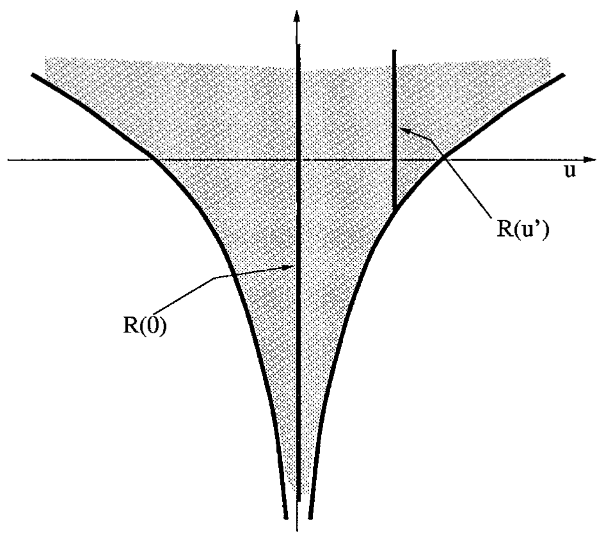

Figure 2. A locally sub-Lipschitz mapping continuous.

isn't locally sub-Lipschitz continuous at 0. On the other hand, the following continuous mapping $R : \mathbb{R} \rightrightarrows \mathbb{R}$, see figure 2,

$$R(u) = \begin{cases} [\ln |u|, \infty) & \text{if } u \neq 0, \\ \mathbb{R} & \text{when } u = 0, \end{cases}$$

that isn't locally bounded at 0, is locally sub-Lipschitz continuous at 0.

Detail. Indeed, if $u \in (-\eta, \eta)$ for some $\eta > 0$ and given any $\rho \geq 0$, S locally sub-Lipschitz continuous at 0 would mean that one should be able to find $\kappa \in \mathbb{R}_+$ such that

$$[0, \rho] \subset [-\kappa u + (2u - u^2)^{\frac{1}{2}}, \infty), \quad \forall u \in (0, \eta).$$

This would require that $\kappa \geq \lim_{u\searrow 0}(2u^{-1}-1)^{\frac{1}{2}} = \infty$. On the other hand for R, pick any $\eta > 0$ and $\rho \in \mathbb{R}_+$. As long as

$$\kappa \geq \max[e^{\rho-1}, \eta^{-1}(\rho + \ln \eta)] \text{ one has } R(0) \cap [-\rho, \rho] \subset R(u) + (\kappa|u|)[-1, 1].$$

Thus, R is locally sub-Lipschitz continuous at 0 as well as at any other point.

Definition 3.3 (epi-sub-Lipschitz continuity). A function-valued mapping $u \mapsto f(u,\cdot) : \mathbb{R}^m \to \mathrm{fcns}(\mathbb{R}^n)$ is epi-sub-Lipschitz continuous on U if the epigraphical mapping $S_f : \mathbb{R}^m \rightrightarrows \mathbb{R}^{n+1}$, with $S_f(u) = \mathrm{epi}\, f(u,\cdot)$, is sub-Lipschitz continuous on U. The function-valued mapping $u \mapsto f(u,\cdot)$ is locally epi-sub-Lipschitz continuous if the epigraphical mapping S_f is locally sub-Lipschitz continuous, i.e., for every $\bar{u} \in \mathrm{dom}\, S_f$ there is a neighborhood $U \in \mathcal{N}(\bar{u})$ such that the function-valued mapping is epi-sub-Lipschitz continuous on $U \cap \mathrm{dom}\, S_f$.

Note that the definition of epi-sub-Lipschitz continuity implies that $U \subset \mathrm{dom} S_f$, or in other words, that $\mathrm{dom}\, f(u,\cdot) \neq \emptyset$ for all $u \in U$. Moreover, $\mathrm{dom}\, S_f$ is also the effective domain of the function-valued mapping $u \mapsto f(u,\cdot)$.

It will be convenient to use the following definition of the unit ball in $\mathbb{R}^{n+1} : \mathbb{B}^+ = \mathbb{B} \times [-1, 1]$, where $\mathbb{B}$ is the unit (Euclidean) ball in $\mathbb{R}^n$. The sub-Lipschitz continuity of S_f on a set $U \subset \mathrm{dom} S_f$ can then be rephrased as follows: given any $\rho \in \mathbb{R}_+$ there exists $\kappa \in \mathbb{R}_+$ such that for all $\bar{u}, u' \in U$,

$$\text{whenever } (x, \alpha) \in S_f(\bar{u}) \text{ with } |x| \leq \rho,\ |\alpha| \leq \rho$$

one can find

$$(y, \beta) \in S_f(u') \quad \text{such that} \quad \begin{cases} |y - x| \leq \kappa|u' - \bar{u}|, \\ |\beta - \alpha| \leq \kappa|u' - \bar{u}|. \end{cases}$$

Theorem 3.4 (*local Lipschitz continuity of inf-projections*). *For $f : \mathbb{R}^m \times \mathbb{R}^n \to \bar{\mathbb{R}}$ proper, lsc, and such that $f(x, u)$ is level-bounded in x locally uniformly in u, let*

$$p(u) := \inf_x f(u, x).$$

Suppose that the function-valued mapping $u \mapsto f(u,\cdot) : \mathbb{R}^m \to \mathrm{fcns}(\mathbb{R}^n)$ is locally epi-sub-Lipschitz continuous. Then, p is locally Lipschitz continuous on its effective domain, $\mathrm{dom}\, p$.

Proof: As in Sections 1 and 2, let $P(u) = \arg\min f(u,\cdot)$. Given $\bar{u} \in \mathrm{dom}\, p$, let U a bounded neighborhood of $\bar{u}$ relative to $\mathrm{dom}\, p$ on which the function-valued mapping $u \mapsto f(u,\cdot)$ is epi-sub-Lipschitz continuous. In view of Theorem 2.10, $P(U) \subset \mathbb{R}^n$ and $p(U) \subset \mathbb{R}$ are bounded sets. Choose $\rho \in \mathbb{R}_+$ so that $\rho\mathbb{B} \supset P(U)$ and $[-\rho, \rho] \supset p(U)$.

For any $u, u' \in U$, let $x_u \in \operatorname{argmin} f(u, \cdot)$ and $x_{u'} \in \operatorname{argmin} f(u', \cdot)$ and then $p(u) = f(u, x_u)$, $p(u') = f(u', x_{u'})$; note that $x_u, x_{u'} \in \rho\mathbb{B}$ and $|p(u)| \le \rho$, $|p(u')| \le \rho$.

From the sub-Lipschitz continuity of S_f relative to U, it follows that given ρ, there exist $\kappa \in \mathbb{R}_+$ and $(y_{u'}, \beta_{u'}) \in S_f(u')$, $(y_u, \beta_u) \in S_f(u)$ such that

$$|\beta_{u'} - p(u)| \le \kappa|u' - u|, \quad |y_{u'} - x_u| \le \kappa|u' - u|$$
$$|\beta_u - p(u')| \le \kappa|u' - u|, \quad |y_u - x_{u'}| \le \kappa|u' - u|$$

One has

$$-\kappa|u' - u| \le \beta_{u'} - p(u) \le \kappa|u' - u| \quad \text{but also} \quad p(u') \le \beta_{u'}.$$

These inequalities imply:

$$p(u') - p(u) \le \kappa|u' - u|$$

Similarly,

$$-\kappa|u' - u| \le \beta_u - p(u') \le \kappa|u' - u| \quad \text{and} \quad p(u) \le \beta_u$$

imply

$$p(u) - p(u') \le \kappa|u' - u|.$$

Hence $|p(u') - p(u)| \le \kappa|u' - u|$. It follows that p is Lipschitz continuous relative to U. And consequently, locally Lipschitz continuous relative to dom p. □

Corollary 3.5 (*Lipschitz continuity of inf-projections*). *For $f : \mathbb{R}^m \times \mathbb{R}^n \to \bar{\mathbb{R}}$ proper, lsc, and such that $f(x, u)$ is level-bounded in x locally uniformly in u, let $p(u) := \inf_x f(u, x)$. Suppose that the function-valued mapping $u \mapsto f(u, \cdot) : \mathbb{R}^m \to$ fcns$(\mathbb{R}^n)$ is epi-Lipschitz continuous, i.e., the epigraphical mapping $S_f : \mathbb{R}^m \to \mathbb{R}^n$ is Lipschitz continuous ($S_f(u) =$* epi $f(u, \cdot)$). *Then, p is Lipschitz continuous on its effective domain,* dom p.

Proof: Actually, this is only a corollary in the sense that essentially the same proof applies, except that one doesn't have to 'localize' the argument. As before, let $P(u) = \operatorname{argmin} f(u, \cdot)$. Choose any $u, u' \in$ dom p and let $x_u \in P(u)$, $x_{u'} \in P(u')$. From the Lipschitz continuity of S_f, it follows that there exist $\kappa \in \mathbb{R}_+$ and $(y_{u'}, \beta_{u'}) \in S_f(u')$, $(y_u, \beta_u) \in S_f(u)$ such that

$$|\beta_{u'} - p(u)| \le \kappa|u' - u|, \quad |y_{u'} - x_u| \le \kappa|u' - u|$$
$$|\beta_u - p(u')| \le \kappa|u' - u|, \quad |y_u - x_{u'}| \le \kappa|u' - u|$$

One then proceeds as in the proof of the theorem, and one concludes that $|p(u') - p(u)| \le \kappa|u' - u|$, i.e., p is Lipschitz continuous with Lipschitz constant κ on dom p. □

One might expect that with the function-valued mapping $u \mapsto f(u, \cdot)$ locally, or possibly globally, sub-Lipschitz continuous one should be able to assert that whenever p takes on the value $-\infty$ a some point $\bar{u}$ in D, then it's identically $-\infty$ on dom p or at least on a neighborhood, relative to dom p, of $\bar{u}$. The following example dispels all such possibilities.

Example 3.6. Consider the function-valued mapping $u \mapsto f(u, \cdot) : \mathbb{R} \to \text{fcns}(\{0\})$ with

$$f(u, 0) = \begin{cases} \ln|u| & \text{if } u \neq 0; \\ -\infty & \text{if } u = 0; \\ \infty & \text{otherwise.} \end{cases}$$

One can also write f as follows: $f(u, 0) = \ln|u|$ with the understanding that $\ln 0 = -\infty$. The associated epigraphical mapping $S_f : \mathbb{R} \rightrightarrows \{0\} \times \mathbb{R}$ is then

$$S_f(u) = \begin{cases} \{0\} \times [\ln|u|, \infty) & \text{if } u \neq 0; \\ \{0\} \times \mathbb{R} & \text{for } u = 0. \end{cases}$$

This mapping is locally sub-Lipschitz continuous. But $p : \mathbb{R} \to \bar{\mathbb{R}}$ is not locally Lipschitz continuous.

Detail. Clearly the 'critical' point is at 0. Now, simply observe that S_f is the mapping R of Example 3.2 and $p(u) = \ln|u|$ if $u \neq 0$ and $p(0) = -\infty$. The functions $x \mapsto f(u, x)$ are level bounded but not locally uniformly in u. □

Example 3.7. Given that under epi-continuity of the set-valued mapping $u \mapsto f(u, \cdot)$ one is able to conclude that P, the argmin mapping is outer semicontinuous. One might hope that with sub-Lipschitz continuity, the mapping P might itself be sub-Lipschitz continuous. That this is not the case follows from the following simple example: Let

$$f(u, x) = \begin{cases} ux & \text{if } x \in [-1, 1], \\ \infty & \text{otherwise.} \end{cases}$$

Detail. The mapping $P : \mathbb{R} \to [-1, 1]$ with

$$P(u) = \begin{cases} 1 & \text{if } u < 0, \\ [-1, 1] & \text{if } u = 0, \\ -1 & \text{if } u > 0, \end{cases}$$

is clearly not locally sub-Lipschitz continuous at 0.

4. Examples

My initial motivation for this article came from certain questions that arose in stochastic programming. The (local) Lipschitz continuity of certain inf-projections can be exploited to obtain bounds on distances between the probability measures induced by stochastic programs. This, in turn, allows us to obtain error bounds for the distance between the solution of a stochastic program and of an approximating one obtained by replacing the given probability measure by an approximating measure [3].

Here, we are only going to consider the following nonlinear programming problem:

$$\min f_0(x), \quad Ax = b, \quad x \in D \subset \mathbb{R}^n,$$

where D is a polyhedral set and f_0 is a Lipschitz continuous function on a set that contains D.

To begin with, we are only going to be interested in perturbations that affect the linear constraints $Ax = b$. So, the bivariate function associated with the family of perturbed problems is

$$f(u, x) = \begin{cases} f_0(x) & \text{if } Ax = b - u, x \in D, \\ \infty & \text{otherwise.} \end{cases}$$

To apply Corollary 3.5, in addition to level boundedness in x locally uniformly in u, one needs to check if the function-valued mapping $u \mapsto f(u, \cdot)$ is epi-Lipschitz continuous. Let's begin with the level boundedness condition.

Of course, the function f_0 could be level bounded from which would immediately follow the level boundedness of f in x locally uniformly in u. This requirement could also follow from having the mapping

$$S : \mathbb{R}^m \rightrightarrows \mathbb{R}^n \quad \text{with } S(u) = \{x \in D \mid Ax = b - u\}$$

locally bounded. A necessary and sufficient condition is provided by Proposition 4.2; the following proposition is a key component of its proof as well as in obtaining epi-Lipschitz continuity.

Proposition 4.1 (*polyhedral graph-convex mappings*, [5]). *If $S : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is graph-convex but such that* gph S *is polyhedral in $\mathbb{R}^m \times \mathbb{R}^n$, then S is Lipschitz continuous on* dom S, *even if its values $S(u)$ are unbounded sets.*

Proof: A variant of the original proof can be found in [2, Example 9.35]. □

Let $\ker(A)$ denote the *kernel* of a matrix A, i.e., $\ker(A) = \{x \in \mathbb{R}^n | Ax = 0\}$; $\ker(A)$ is also called the *null space* of A. For a set $D \subset \mathbb{R}^n$, D^∞ will denote the horizon cone of D, which for a convex set D consists of all vectors x_d such that $x + \lambda x_d \in D$ for all $x \in D$ and $\lambda \in \mathbb{R}_+$, [2, Theorem 3.6].

Proposition 4.2 (*local boundedness of graph-polyhedral mappings*). *Consider the mapping $S : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ with $S(u) = \{x \in D \mid Ax = b - u\}$ where D is a (convex) polyhedral subset of $\mathbb{R}^n$ and A is a $m \times n$-matrix. Then, S is locally bounded if and only if $D^\infty \cap \ker(A) = \{0\}$.*

Proof: $S(u)$ is unbounded if and only if there is a half-line with $x_d \neq 0$ such that

$$\{x_0 + \lambda x_d \mid \lambda \in \mathbb{R}_+\} \subset D \cap \{x \mid Ax = b - u\},$$

and this occurs if and only if $x_d \in D^\infty \cap \ker(A)$. Moreover, the preceding also implies that whenever $S(u)$ is nonempty, $S(u) \subset S(u) + (ker(A) + D^\infty)$, and consequently, on dom S, $S(u)$ is bounded if and only if $\ker(A) \cap D^\infty = \{0\}$ [2, Theorem 3.5]. Thus, we are dealing with a set-valued mapping whose values are bounded and whose graph is a polyhedral convex set. In view of Proposition 4.1, this immediately implies that S is locally bounded. □

The epi-Lipschitz continuity of the mapping $u \mapsto f(u, x)$ also follows from Proposition 4.1. To show that the epigraphical mapping S_f is epi-Lipschitz continuous, it will suffice to show that if $u, u' \in$ dom S_f and $(x, \alpha) \in S_f(u)$, there exists a Lipschitz constant, say $\kappa \in \mathbb{R}_+$ and $(y', \beta') \in S_f(u')$ such that

$$|y' - x| \leq \kappa|u' - u|, \quad |\beta' - \alpha| \leq \kappa|u' - u|;$$

also here we use $\mathbb{B}^+ = \mathbb{B} \times [-1, 1]$ as the unit ball in $\mathbb{R}^{n+1}$ with $\mathbb{B}$ the (Euclidean) unit ball in $\mathbb{R}^n$. Proposition 4.1 implies that one can find $\kappa_S \in \mathbb{R}_+$, the Lipschitz constant associated with the mapping S, and $y' \in S(u)$ such that $|y' - x| \leq \kappa_S|u' - u|$. Lipschitz continuity of f_0, with Lipschitz constant κ_0, in turn implies

$$|f_0(y') - f_0(x)| \leq \kappa_0|y' - x| \leq \kappa|u' - u|$$

where $\kappa = \kappa_S(\max[1, \kappa_0])$. Now, $\alpha \geq f_0(x)$, so one can always find $\beta' \geq f_0(y')$ such that $|\beta' - \alpha| \leq |f_0(y') - f_0(x)|$, and thus not only

$$|y' - x| \leq \kappa|u' - u| \quad \text{but also} \quad |\beta' - \alpha| \leq \kappa|u' - u|.$$

From Corollary 3.5 and what precedes, one has that the inf-projection,

$$p(u) = \inf_x\{f_0(x) \mid Ax = b - u, x \in D\}$$

is Lipschitz continuous on dom p.

This is not a new result, cf. [6, Theorem 2]. Of course, the proof in [6] doesn't pass through checking epi-Lipschitz continuity.

To follow up, let's consider a more involved perturbation scheme which is closer to that encountered when dealing with stochastic programming problems. Again, we deal with the nonlinear programming problem:

$$\min f_0(x), \quad Ax = b, \quad x \in D \subset \mathbb{R}^n.$$

But this time, let's consider perturbations that affect both the constraints and the objective:

$$f(u,x) = \begin{cases} f_0(u_1,x) & \text{if } Ax = b - u_2, x \in D, \\ \infty & \text{otherwise.} \end{cases}$$

where $u = (u_1, u_2) \in \mathbb{R}^d (d > m)$ and the bivariate function f_0 is such that $f_0(0, x) = f_0(x)$.

Assumption 4.3. The function $f_0 : \mathbb{R}^d \times \mathbb{R}^n \to \bar{\mathbb{R}}$ is locally Lipschitz continuous on $\mathbb{R}^d \times D$.

Level boundedness in x locally uniformly in u of f can be obtained under the same conditions as earlier: either f_0 is level-bounded in x locally uniformly in u or one has that $\ker(A) \cap D^\infty = \{0\}$ which yields the local boundedness of the feasibility mapping $u \mapsto S(u) = \{x \in D \mid Ax = b - u_2\}$. (Of course, in certain situations one might have to invoke the properties of the level sets of $f_0(u_1, \cdot)$ in combination with those of $S(u)$ to obtain level boundedness in x locally uniformly in u.) Let's proceed with the assumption that $\ker(A) \cap D^\infty = \{0\}$ which means that $S : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is locally bounded and Lipschitz continuous on its effective domain with Lipschitz constant κ_S, cf. Proposition 4.1.

Next, let's analyze the continuity properties of the mapping $u \mapsto f(u, \cdot)$, or equivalently of its epigraphical mapping S_f where $S_f(u) = \text{epi } f(u, \cdot)$. This time, we are going to restrict ourselves to a bounded subset U of dom $S_f(u)$. Let $u, u' \in U$ and

$$(x, \alpha) \in S_f(u) \cap \rho\mathbb{B}^+, \quad (x', \alpha') \in S_f(u') \cap \rho\mathbb{B}^+.$$

The Lipschitz continuity of the feasibility mapping S yields $y' \in S((u)')$ and $y \in S(u)$ such that

$$|y' - x| \le \kappa_S |u_2' - u_2|, \quad |y - x'| \le \kappa_S |u_2 - u_2'|.$$

This means that y' and y belong to $\tilde{\rho}\mathbb{B}$ with $\tilde{\rho} = \rho + \kappa_S(\text{diam } U)$. Since f_0 is locally Lipschitz continuous, for $U \times \tilde{\rho}\mathbb{B}$ there is Lipschitz constant κ_0 such that

$$|f((u_1)', y') - f(u_1, x)| \le \kappa_0 |((u_1)', y') - (u_1, x)| \le \kappa_U |u' - u|$$
$$|f(u_1, y) - f((u_1)', x')| \le \kappa_0 |((u_1)', y) - (u_1, x')| \le \kappa_U |u - u'|$$

where the last inequalities involving the (new) constant κ_U come form the preceding string of inequalities for $|y' - x|$ and $|y' - x|$. There remains simply to observe that because $\alpha \ge f(u, x)$ there always exists $\beta' \ge f(u', y')$ such that $|\beta' - \alpha| \le |f(u', y') - f(u, x)|$. And for the same reasons one can find $\beta \in S_f(u)$ such that $|\beta - \alpha'| \le |f(u, y) - f(u', x')|$.

What we have shown is that S_f is sub-Lipschitz continuous on U with Lipschitz constant κ_U, or equivalently the mapping $u \mapsto f(u, \cdot)$ is epi-sub-Lipschitz on U. And from this argument, it follows that $u \mapsto f(u, \cdot)$ is locally sub-Lipschitz continuous (on its effective domain). Applying Theorem 3.4, then yields the local Lipschitz continuity of $p = \inf_x f(\cdot, x)$ on its effective domain. Let's summarize this as follows:

Proposition 4.4. *With $u = (u_1, u_2) \in \mathbb{R}^d$, let*

$$f((u_1, u_2), x) = \begin{cases} f_0(u_1, x) & \textit{if } Ax = b - u_2, x \in D, \\ \infty & \textit{otherwise}, \end{cases}$$

where D is a (convex) polyhedral subset of $\mathbb{R}^n$, A is a $m \times n$-matrix. Assume that $\ker(A) \cap D^\infty = \{0\}$ *and f_0 is locally Lipschitz continuous on $\mathbb{R}^d \times D$. Then, the inf-projection of f, i.e., $p = \inf_x f(\cdot, x)$, is locally Lipschitz continuous on* dom p, *its effective domain.*

Probably the 'simplest' case when the preceding result could be applied is when

$$f(u, x) = \begin{cases} \langle c + u_1, x \rangle & \text{if } Ax = b - u_2, x \geq 0, \\ \infty & \text{otherwise}, \end{cases}$$

i.e., when considering a linear programming problem with perturbations affecting both the objective and the constraints. In this situation, one could rely on the Basis Decomposition Theorem [4], applied to the primal and dual problems to obtain the local Lipschitz continuity. But as soon as the problem to be perturbed involves nonlinearity one has to rely on more comprehensive statements like Theorem 3.4 and its Corollary.

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