

Lecture 1: Variational Problems I

Note Title

12/30/2014

★ Newton's Equations of Motion

We'll use the following notation:

$$\mathbf{r} = \mathbf{r}(t) = (x_1, \dots, x_n)^T \in \mathbb{R}^n$$

$$= (x_1(t), \dots, x_n(t))^T$$

a point (or particle) in $\mathbb{R}^n$
it's a position vector.

$$\mathbf{e}_i = (0, 0, \dots, 0, \underset{\uparrow}{1}, 0, \dots, 0)^T \in \mathbb{R}^n$$

The i th standard (or canonical)
basis vector of $\mathbb{R}^n$

$$\dot{\mathbf{r}} = \frac{d\mathbf{r}}{dt} = (\dot{x}_1, \dots, \dot{x}_n)^T$$

the time derivative, i.e.,
the velocity vector of the particle.

For $n=3$, we often use $(x, y, z)^T$
instead of $(x_1, x_2, x_3)^T$, and
 $\mathbf{i}, \mathbf{j}, \mathbf{k}$ instead of $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$.

Consider a particle of mass m with
its position $\mathbf{r}(t) \in \mathbb{R}^3$ moving under
an external force $\mathbf{F} \in \mathbb{R}^3$.

Then, it satisfies the following famous Newton's Equation of Motion:

$$m \ddot{\mathbf{r}} = \mathbf{F}$$

$\ddot{\mathbf{r}}$ acceleration
 $\mathbf{F}$ inertial force

$\Rightarrow$ Can solve for $\mathbf{r}(t)$ given $\mathbf{r}(0)$ & $\dot{\mathbf{r}}(0)$.

Now, let's consider a special force

$$\mathbf{F} = -\nabla V, \quad V = V(x, y, z).$$

$$= \left(-\frac{\partial V}{\partial x}, -\frac{\partial V}{\partial y}, -\frac{\partial V}{\partial z} \right)^T$$

a potential function (or potential energy)

Now suppose a particle moves under such a potential field with no other external forces.

Suppose we do not know Newton's eqn. What can we say?

$\perp$ Leibniz's viewpoint!

Def. The kinetic energy of a particle is

$$T := \frac{1}{2} m |\dot{\mathbf{r}}|^2, \quad |\cdot| \text{ is the Euclidean norm.}$$

The total energy of the particle is

$$E := T + V$$

If $E \equiv \text{const. indep. of time } t$,
 then its field is called **conservative**.
 (i.e., Conservation of energy)

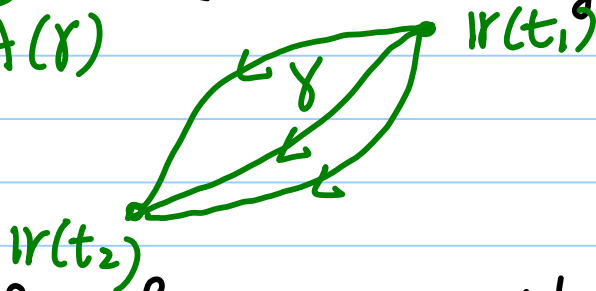
Q: Show that if the particle satisfies
 Newton's eqn of Motion, $m \ddot{\mathbf{r}} = -\nabla V$,
 then it's a conservative field.

Def. The **action** of the particle along
 a continuous path γ starting at
 $\mathbf{r}(t_1)$ and arriving at $\mathbf{r}(t_2)$, $t_1 \leq t_2$
 is defined as

$$A := \int_{t_1}^{t_2} (\underbrace{T - V}_{=: L}) dt = \int_{t_1}^{t_2} L dt$$

$\underbrace{\quad}_{\uparrow}$ depends on a path γ ,
 So $A = A(\gamma)$

the **Lagrangian**
 (or the Lagrange fcn)



$\Rightarrow$ Euler & Lagrange thought
 an actual path = $\arg \min_{\gamma \in \Gamma} A(\gamma)$

Γ : a set of **admissible** curves
 or more precisely,

$$\Gamma = \left\{ \gamma(t) = (x(t), y(t), z(t))^T, t_1 \leq t \leq t_2 \mid \gamma(t_i) = \mathbf{r}(t_i), i=1,2, x, y, z \in C^1[t_1, t_2] \right\}$$

Remark: Under the conservative field,

$$E = T + V = \text{const.}, \text{ say } C.$$

$$\text{So, } L = T - V = 2T - C$$

$\Rightarrow \arg \min_{\gamma \in \Gamma} A(\gamma)$ amounts to

$$\arg \min_{\gamma \in \Gamma} \int_{t_1}^{t_2} T \, dt$$

minimization of
the time integral
of the kinetic energy

$\Rightarrow$ "Nature always minimizes action."
"God made the world in the most
economical way."

See: Hildebrandt & Tromba:
The Parsimonious Universe,
Springer, 1996.

• Principle of Least Action

(Leibniz 1696; Maupertuis 1746;
Euler 1744; Lagrange 1760;
Hamilton 1834; ...)

A particle with mass m and potential energy $V(r)$ takes, in the time interval $[t_1, t_2]$, the path γ^* s.t.

$$\gamma^* = \arg \min_{\gamma \in \Gamma} A(\gamma) = \arg \min_{\gamma \in \Gamma} \int_{t_1}^{t_2} (T - V) dt$$

Let's check this!

$$r(t) = (x(t), y(t), z(t))^T$$

$$(*) \quad A = \int_{t_1}^{t_2} \left\{ \frac{m}{2} (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(x, y, z) \right\} dt$$

$$\text{Let } r(t_i) = P_i = (x_i, y_i, z_i)^T, \quad i=1, 2.$$

As Euler & Lagrange did, let's derive the necessary cond. for a path γ^* to be a minimizer of (*).

$$\text{Let } \gamma^* = \gamma^*(t) = (x^*(t), y^*(t), z^*(t))^T$$

$t \in [t_1, t_2]$, to be such a minimizer and assume $\gamma^* \in \Gamma$.

γ^* is said to be an **extremal**.

Now consider a path slightly deviating from γ^* , i.e., $\gamma_\varepsilon^* = (x^*(t) + \varepsilon \xi(t), y^*(t), z^*(t))^T \in \Gamma$

$$\text{So, } \xi(t_i) = 0, \quad i=1, 2.$$

γ_ε^* is called a **variation** of γ^* .

$\Rightarrow$ Hence the name **Calculus of variations**.

$$A(\gamma_\varepsilon^*) = A(\varepsilon) = \int_{t_1}^{t_2} \left\{ \frac{m}{2} ((\dot{x}^* + \varepsilon \dot{\xi})^2 + \dot{y}^{*2} + \dot{z}^{*2}) - V(x^* + \varepsilon \xi, y^*, z^*) \right\} dt$$

Since $\gamma^* = \gamma_0^*$ is the extremal, minimizer,

we must have $\left. \frac{dA}{d\varepsilon} \right|_{\varepsilon=0} = 0$

$\delta A := \left. \frac{dA}{d\varepsilon} \right|_{\varepsilon=0}$ is called the **first variation** of A .

$$\begin{aligned} \delta A &= \left. \frac{d}{d\varepsilon} \int_{t_1}^{t_2} \left\{ \frac{m}{2} (\dot{x}^{*2} + 2\varepsilon \dot{x}^* \dot{\xi} + \varepsilon^2 \dot{\xi}^2 + \dot{y}^{*2} + \dot{z}^{*2}) - V(x^* + \varepsilon \xi, y^*, z^*) \right\} dt \right|_{\varepsilon=0} \\ &= \int_{t_1}^{t_2} \left(m \dot{x}^* \dot{\xi} - \xi \frac{\partial V}{\partial x}(x^*, y^*, z^*) \right) dt \\ &\stackrel{\text{Int. by Parts}}{=} \underbrace{m \dot{x}^* \xi}_{=0 \text{ since } \xi(t_i)=0} \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \left(m \ddot{x}^* \xi + \xi \frac{\partial V}{\partial x} \right) dt = 0 \end{aligned}$$

$$\Rightarrow \int_{t_1}^{t_2} \xi \left(m \ddot{x}^* + \frac{\partial V}{\partial x} \right) dt = 0$$

This must hold for all admissible $\xi(t)$, i.e., $\xi \in \Gamma_0 := \Gamma$ with $P_1 = P_2 = 0$

By **the Fundamental Lemma of the**

Calculus of Variations (to be discussed later), we must have

$$m \ddot{x}^* + \frac{\partial V}{\partial x}(x^*, y^*, z^*) = 0$$

$$\text{i.e., } m \ddot{x}^* = - \frac{\partial V}{\partial x}(x^*, y^*, z^*)$$

Similarly, by the variation w.r.t. y^* & z^* , we recover Newton's eqn. of Motion:

$$m \ddot{\mathbf{r}}(t) = -\nabla V(\mathbf{r}) !!$$

Note that $\delta A = 0$ is just a necessary Cond., just like the usual calculus. The sufficient Cond. for the minimizer involves the second variation, and we won't discuss in this course. See, e.g., Gelfand & Fomin if you want to know the detail.

It's interesting to note that the only necessity, i.e., $\delta A = \left. \frac{dA}{d\varepsilon} \right|_{\varepsilon=0} = 0$,

was sufficient to derive Newton's eqn. of Motion! We really didn't have to find an absolute minimizer. Just finding a **stationary path** was sufficient for N. eqn.

Def. If a path $\gamma \in \Gamma$ satisfies $\delta A = 0$, then such γ is called a **stationary path**, and we say the action takes on a **stationary value**.

Hamilton generalized such Principle of Least Action for the mechanics/dynamics of continua/body and reached:

Hamilton's Principle of Stationary Action (1834-35):

A mechanical system with the kinetic energy T & the potential energy V behaves within time $t \in [t_1, t_2]$ for a given initial & end position s.t.
 $A = \int_{t_1}^{t_2} L dt$ assumes a stationary value.