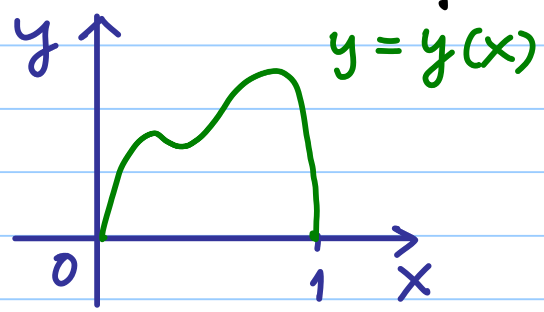
Lecture 4: The Isoperimetric Problem

Note Title 1/2/2015

☆ The Problem of Queen Dido

See the reference page for its rich history.



Find a curve $y=y(x)$ with $y(0)=y(1)=0$ s.t. the enclosed area

$$I=\int_0^1 y\,dx \longrightarrow \max!$$

subj. to $C=\int_0^1 \sqrt{1+y'^2}\,dx = \text{const., say, } \frac{\pi}{2}.$

The difference from the previous variational problem is the constraint (not in the form of B.C.)

See also P. Lax's nice short article based on non-variational approach!

A more general form is:

Find a fcn $y=y(x)$ s.t.

$$(*)\begin{cases} \underline{\text{B.C.}}: y(x_i)=y_i, \quad i=1,2. \\ \underline{\text{fcnl}}: I=\int_{x_1}^{x_2} f(x,y,y')\,dx \to \max \text{ (or min)} \\ \underline{\text{constr}}: C=\int_{x_1}^{x_2} g(x,y,y')\,dx = L \text{ (const.)} \end{cases}$$

☆ The E-L Eqn. for the Isoperimetric Problem in One Indep. Variable

again, we consider the solution $y=y^*(x)$ to $(*)$ and its variation. However, unlike the previous case, $y_\varepsilon(x)=y^*(x)+\varepsilon\eta(x)$, $\eta(x_i)=0$, $i=1,2$ does not work. $\leftarrow$ C must be kept const. $\equiv L$.

Adding $\varepsilon\eta$, C would change from L.

$\Rightarrow$ We need a two-param. variation to counter-balance this, i.e., consider

$$\begin{cases} y = y^*(x) + \varepsilon_1 \eta_1(x) + \varepsilon_2 \eta_2(x) \\ \eta_i(x_j) = 0, \quad i,j = 1,2. \end{cases}$$

Substitute this into $I = \int f$ & $C = \int g$ to get:

$$(**) \begin{cases} I(\varepsilon_1, \varepsilon_2)\big|_{\varepsilon_1=\varepsilon_2=0} = I(0,0) = E \text{ (extreme val.)} \\ C(\varepsilon_1, \varepsilon_2)\big|_{\varepsilon_1=\varepsilon_2=0} = C(0,0) = L \end{cases}$$

For E to be an extreme value, the Jacobian must vanish at $(\varepsilon_1, \varepsilon_2) = (0,0)$, i.e.,

$$\Delta := \frac{\partial(I,C)}{\partial(\varepsilon_1, \varepsilon_2)}\Bigg|_{\varepsilon_1=\varepsilon_2=0} = \begin{vmatrix} \frac{\partial I}{\partial \varepsilon_1}(0,0) & \frac{\partial I}{\partial \varepsilon_2}(0,0) \\ \frac{\partial C}{\partial \varepsilon_1}(0,0) & \frac{\partial C}{\partial \varepsilon_2}(0,0) \end{vmatrix} = 0$$

Why? If $\Delta \neq 0$ at $(\varepsilon_1, \varepsilon_2) = (0,0)$, then by the Implicit Fcn Thm, we could solve $(**)$ in the neighborhood of $(\varepsilon_1, \varepsilon_2) = (0,0)$ s.t.

$\varepsilon_i = \varepsilon_i(E, L)$, $i = 1,2$, which are continuous.

Now, if E is an extreme val. (i.e., min or max) of I, then we'll get $\varepsilon_1 = \varepsilon_2 = 0$ by assumption. But, since $\varepsilon_i = \varepsilon_i(E,L)$, $i=1,2$, we can get some $(\varepsilon_1, \varepsilon_2)$ for (E', L) with $E' < E$ or $E' > E$.

$\Rightarrow$ E cannot be an extreme val. #

Now, let's compute Δ!

Similarly to the previous variational problems, we can get:

$$\begin{cases} \frac{\partial I}{\partial \varepsilon_i}(0,0) = \int_{x_1}^{x_2} \eta_i \left(\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} \right) dx \\ \frac{\partial C}{\partial \varepsilon_i}(0,0) = \int_{x_1}^{x_2} \eta_i \left(\frac{\partial g}{\partial y} - \frac{d}{dx} \frac{\partial g}{\partial y'} \right) dx \end{cases} \quad i = 1,2.$$

Now, we assume that $y^*(x)$ is not a sol. of

$$\frac{\partial g}{\partial y} - \frac{d}{dx}\frac{\partial g}{\partial y'} = 0$$

$\Rightarrow$ This assump. is reasonable since y^* maximizes (or minimizes) I and it's unlikely to make C extreme simultaneously.

$\Rightarrow$ So, we can choose any admissible fcn $\eta_2(x)$ s.t. $\int_{x_1}^{x_2} \eta_2 \left(\frac{\partial g}{\partial y} - \frac{d}{dx}\frac{\partial g}{\partial y'}\right) dx \neq 0$.

Once we choose such η_2, let

$$\int_{x_1}^{x_2} \eta_2 \left(\frac{\partial g}{\partial y} - \frac{d}{dx}\frac{\partial g}{\partial y'}\right) dx =: \lambda_1 \neq 0$$

$$\int_{x_1}^{x_2} \eta_2 \left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'}\right) dx =: \lambda_2$$

Then,

$$\Delta = \begin{vmatrix} \int_{x_1}^{x_2} \eta_1 \left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'}\right) dx & \lambda_2 \\ \int_{x_1}^{x_2} \eta_1 \left(\frac{\partial g}{\partial y} - \frac{d}{dx}\frac{\partial g}{\partial y'}\right) dx & \lambda_1 \end{vmatrix}$$

$$= \lambda_1 \int_{x_1}^{x_2} \eta_1 \left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'}\right) dx - \lambda_2 \int_{x_1}^{x_2} \eta_1 \left(\frac{\partial g}{\partial y} - \frac{d}{dx}\frac{\partial g}{\partial y'}\right) dx$$

$$= 0.$$

By setting $\lambda := -\lambda_2/\lambda_1$, we have

$$\int_{x_1}^{x_2} \eta_1 \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} + \lambda\left(\frac{\partial g}{\partial y} - \frac{d}{dx}\frac{\partial g}{\partial y'}\right)\right] dx = 0$$

Using the F.L.C.V., we have the necessary cond. for the sol. to the isoperimetric problem (*):

$$\frac{\partial}{\partial y}(f + \lambda g) - \frac{d}{dx}\frac{\partial}{\partial y'}(f + \lambda g) = 0 \quad \text{or}$$

$$\frac{\partial h}{\partial y} - \frac{d}{dx}\frac{\partial h}{\partial y'} = 0 \quad \text{with } h = f + \lambda g$$

This is a form of the Lagrange multiplier!
λ is, at this point, an arbitrary parameter.
Yet, λ must be used to satisfy the constraint
$$C = \int_{x_1}^{x_2} g \, dx = L.$$

Let's check the Queen Dido problem.

$$f = y, \quad g = \sqrt{1+y'^2} \Rightarrow h = y + \lambda\sqrt{1+y'^2}$$

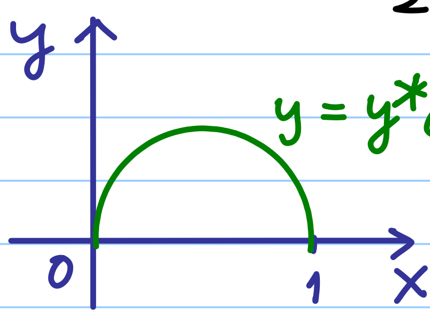
$$\frac{\partial h}{\partial y} = 1, \quad \frac{\partial h}{\partial y'} = \frac{\lambda y'}{\sqrt{1+y'^2}}$$

So, $\dfrac{\partial h}{\partial y} - \dfrac{d}{dx}\dfrac{\partial h}{\partial y'} = 0 \Leftrightarrow 1 - \dfrac{d}{dx}\left(\dfrac{\lambda y'}{\sqrt{1+y'^2}}\right) = 0$ — (***)

Notice that $\dfrac{d}{dx}\left(\dfrac{y'}{\sqrt{1+y'^2}}\right) = \dfrac{y''}{(1+y'^2)^{3/2}} = \kappa$ curvature of $y(x)$.

So, (***) $\Leftrightarrow \lambda\kappa = 1$ or
$$\kappa = \frac{1}{\lambda} = \text{const.}$$

This means that $y = y^*(x)$ is a semicircle with $L = \frac{\pi}{2}$.



$$y = y^*(x) = \sqrt{\tfrac{1}{4} - \left(x - \tfrac{1}{2}\right)^2} \qquad r = \frac{1}{2}$$

$$\kappa = \frac{1}{r} = 2, \text{ so } \lambda = \frac{1}{2}.$$

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Exercise: What happens if $L \neq \frac{\pi}{2}$?